QUANDLE KNOT INVARIANTS ARE QUANTUM KNOT INVARIANTS

MATÍAS GRAÑA

Abstract. We prove that the knot invariants given by quandles can be recovered by invariants given by traces on braided vector spaces, i.e. quantum invariants.

1. Introduction

Up to my knowledge, *racks* have been considered for the first time in [FR] and *quandles* appeared in [CJKLS]. There is also a (co)homology theory presented in those papers. Independently, in [G1] I considered *crossed sets* and gave a (co)homology theory, which turns out to be exactly the same as the old one. The purpose for the introduction of racks and quandles in the former articles was to provide knot invariants. The purpose for the introduction of crossed sets in the later was to construct in a systematic way Yetter–Drinfeld modules over group algebras (that is, modules over the Drinfeld double of a group algebra). In the present article we prove that these modules, used as colors for knots and links, give the same invariants as those defined via quandles. This does not mean that quandle invariants should be disregarded. On the contrary, quandle invariants could be thought of as a discretization of the quantum invariants given by Yetter–Drinfeld modules over group algebras. This discretization is specially good for actual computations, and this explains the rather long series of successful articles by –among others– J. Scott Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito (see for instance the archive `math.GT` numbers `9806023, 9811084, 9903135, 9906115, 9909161, 0006115, 0006175, 0007141, 0008099, 0008118`). Furthermore, in that series of papers the authors use the third group of quandle cohomology to provide invariants of knotted surfaces. It is an interesting question whether these invariants can be computed from the Yetter–Drinfeld-category point of view.

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2. Racks, quandles and crossed sets. Homology theories.

The category of racks includes that of quandles, which in turn includes that of crossed sets.

Definition 2.1. A *crossed set* is a pair $(X, \triangleright)$, where X is a set and $\triangleright : X \times X \to X$ is a binary operation (which will be denoted by $x \triangleright y := \triangleright(x, y)$) verifying

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1. For any $x \in X$ the function $x \triangleright \bullet : X \to X$ is a bijection,
2. $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)\ \forall x, y, z \in X$,
3. $x \triangleright x = x\ \forall x \in X$,
4. $x \triangleright y = y \Leftrightarrow y \triangleright x = x$.

A pair $(X, \triangleright)$ is a *quandle* if it verifies the first three conditions. A pair $(X, \triangleright)$ is a *rack* if it verifies the first two conditions.

Remark 2.2. The usual notation is $x * y := y \triangleright x$. We use the $\triangleright$ notation in order to be coherent with [G1]. The complexes below should be read from right to left if one uses the $*$ notation.

Let $(X, \triangleright)$ be a rack. For $n \geq 0$, let $C_n(X) := \mathbb{Z}X^n = \mathbb{Z}(X \times \cdots \times X)$ (n times), X^0 is a singleton. For an abelian group A, let $C_n(X, A) = A \otimes_{\mathbb{Z}} C_n(X)$ and $C^n(X, A) = \mathrm{Hom}_{\mathbb{Z}}(C_n(X), A) \simeq \mathrm{Fun}(X^n, A)$. Consider then $C_*(X)$ as a complex, with boundary given by

$$\begin{aligned}
&\partial_1 = \partial_0 = 0\\
&\partial_{n+1} : C_{n+1}(X) \to C_n(X),\ (n > 1)\\
&\partial_{n+1}(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n} (-1)^{i-1} [(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1})\\
&\qquad\qquad - (x_1, \dots, x_{i-1}, x_i \triangleright x_{i+1}, \dots, x_i \triangleright x_{n+1})].
\end{aligned}$$

It is straightforward, and done for instance in [G2], to verify that this is actually a complex, i.e. $\partial^2 = 0$. Consider then $C_*(X, A)$ as a complex with the boundary $\partial^A = (\mathrm{id}_A \otimes \partial)$, and $C^*(X, A)$ as a co-complex with the coboundary $\delta = \mathrm{Hom}(\partial, A)$, that is,

$$\begin{aligned}
&\delta^0 = \delta^1 = 0\\
&\delta_n : C^n(X, A) \to C^{n+1}(X, A), (n > 1)\\
&\delta^n(f)(x_0, \dots, x_n) = \sum_{i=0}^{n-1} (-1)^i [f(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n)\\
&\qquad\qquad - f(x_0, \dots, x_{i-1}, x_i \triangleright x_{i+1}, \dots, x_i \triangleright x_n)].
\end{aligned}$$

From now on, we shall write the abelian group A multiplicatively. Then, the 2-cocycles are given by the functions $f : X \times X \to A$ such that

$$f(y, z) f(x, y \triangleright z) = f(x \triangleright y, x \triangleright z) f(x, z)\ \forall x, y, z \in X.$$

In [CJKS], the authors define a useful quotient complex of $C_*(X)$ for the case X being a quandle: let $C^D_*(X)$ be the subcomplex of $C_*(X)$ generated by all the n-tuples $(x_1, \dots, x_n)$ such that $\exists i < n$ with $x_i = x_{i+1}$ (the condition on X to be a quandle guarantees that this is indeed a subcomplex). Then one can take $C^Q_*(X) = C_*(X)/C^D_*(X)$, and define $C^Q_*(X, A) = A \otimes_{\mathbb{Z}} C^Q_*(X)$, $C^*_Q(X, A) = \mathrm{Hom}_{\mathbb{Z}}(C^Q_*(X), A)$.

Definition 2.3. For $(X, \triangleright)$ a rack, we put $H_*(X, A)$ and $H^*(X, A)$ the homologies of the complexes $C_*(X, A)$ and $C^*(X, A)$ respectively. If X is a quandle, we put $H^Q_*(X, A)$ and $H^*_Q(X, A)$ the homologies of the complexes $C^Q_*(X, A)$ and $C^*_Q(X, A)$ respectively.

3. Knot invariants and the main theorem

For *representations of knots* or *links* in the plane we mean the usual diagrammatic drawing of the knot or link in which undercrossing strings are broken. This is a planar diagram of an embedding of copies of circles S^1 in $\mathbb{R}^3$. A *virtual knot* or *link* is the more general notion in which one allows crossings where no string overpasses the other one, which are represented by a usual crossing with no broken lines. We call these *virtual crossings* (see Figure 1). By *components* of a virtual link we mean the representation of each of the copies of S^1 (thus the components may be not disjoint in $\mathbb{R}^2$, as they are the two components of the virtual link of the Figure 1).

An *oriented virtual knot* or *link* is a virtual knot in which there is defined an orientation for each component.

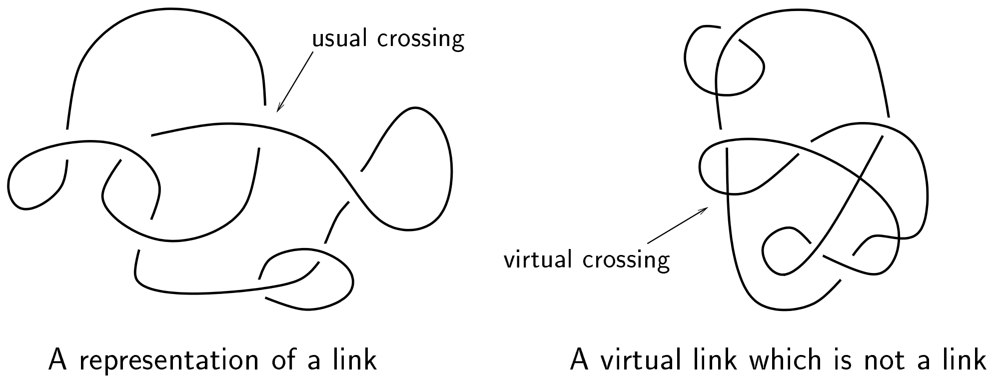

Figure 1

Definition 3.1. Let L be a virtual link and X be a rack. The *edges* of L are the components of $L - \{\text{double points of } L\}$. For the singular crossings we consider the opposed edges as a unique edge (for instance, the links of Figure 1 have 7 and 6 components). A *coloring of L by X* is a function $p : \{\text{edges of } L\} \to X$ such that in each double point both edges of the undercrossing component are related as in the left and middle sides of Figure 2.

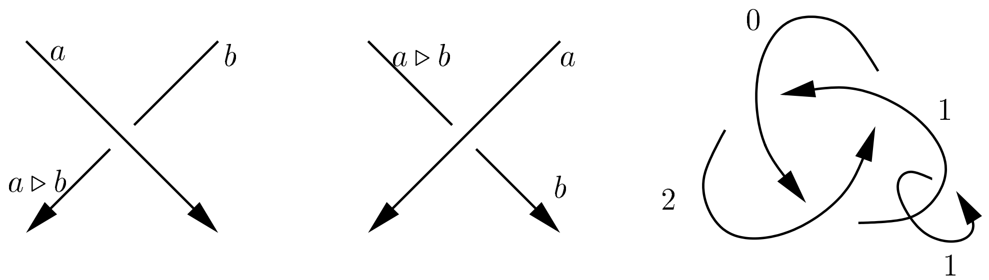

Figure 2. Coloring

In the right side of Figure 2 there is a coloring of a link with the rack $X = \{0, 1, 2\}$, $x \triangleright y = 2x - y \mod 3$ (this is the rack R_3 in the notation of [CJKLS]). There are 6 possible colorings of the link with this rack.

We give now the link invariants of [CJKLS].

Definition 3.2. Let L be a virtual link. Let X be a quandle, A be an abelian group (which, as said, will be written multiplicatively) and $f \in Z^2_Q(X, A)$. Consider $\mathbb{Z}A$ the group ring of A. For a coloring p of L by X, let $\tau_f(L, X; p)$ be defined as $\prod_{a,b} f(a, b)^{s(a,b)} \in A$, where (a, b) runs over all the non-singular crossings, taken as depicted in the left and middle sides of Figure 2; the exponents $s(a, b)$ are defined to be $+1$ if the crossing is "possitive", as in the left side of Figure 2; and to be -1 if the crossing is "negative", as in the middle side of Figure 2. That is, a crossing as in the left side of the Figure contributes as $f(a, b)$ to the product, while one as in the middle side contributes as $f(a, b)^{-1}$. In the example in the right side of Figure 2, we have $\tau_f(L, X; p) = f(0, 1)f(1, 2)f(2, 0)$ (notice that since f is a quandle cocycle, $f(1, 1) = 1 \in A$). Define then

$$\Phi_f(L, X) = \sum_p \tau_f(L, X; p) \in \mathbb{Z}A,$$

where the sum is taken over all possible colorings of L by X.

We go now to the other end of the bridge. There is a large literature on knot and link invariants defined via ribbon categories. Usually they are defined for framed links instead of links, since the modules are not invariant under the first Reidemeister move, depicted in Figure 3.

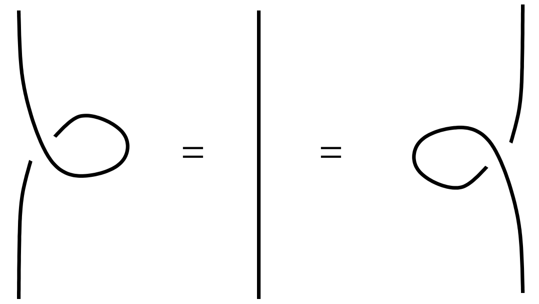

FIGURE 3. First Reidemeister move

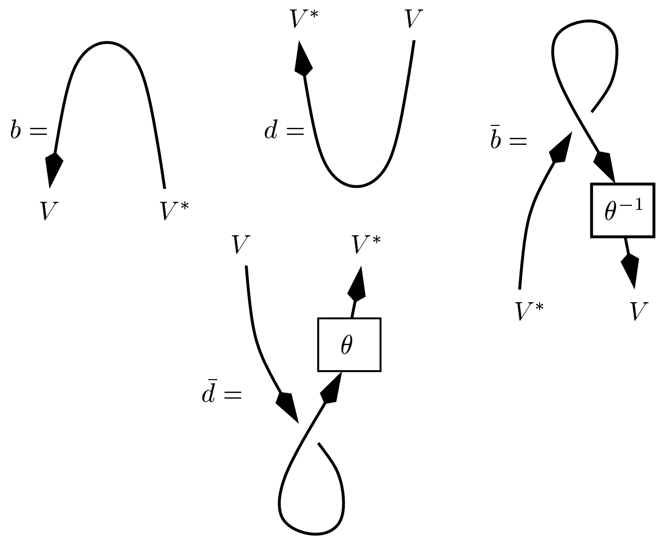

FIGURE 4. birth and death maps

In order to be self-contained, we add a very rough definition of a ribbon category (enough for the purpose of this article): A ribbon category $\mathcal{C}$ is a cateogry which is monoidal (there is a notion of tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with a unit $\mathbf{1}$ such that $\mathbf{1} \otimes V \simeq V \simeq V \otimes \mathbf{1}$ via natural isomorphisms), which is rigid (there exists V^* for any V and there are birth and death maps $b_V : \mathbf{1} \to V \otimes V^*$, $d_V : V^* \otimes V \to \mathbf{1}$ such that $(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V$ and $(d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}$), which has a braiding (there is a family of natural isomorphisms $c_{V,W} : V \otimes W \to W \otimes V$ for any two objects V, W of $\mathcal{C}$ such that the braid relation

$$(c_{W,Z} \otimes \mathrm{id}_V)(\mathrm{id}_W \otimes c_{V,Z})(c_{V,W} \otimes \mathrm{id}_Z) = (\mathrm{id}_Z \otimes c_{V,W})(c_{V,Z} \otimes \mathrm{id}_W)(\mathrm{id}_V \otimes c_{W,Z})$$

is verified), and which has a ribbon (a family of natural isomorphisms $\theta_V : V \to V$ such that $\theta_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}}$, $\theta_{V^*} = (\theta_V)^*$ and $\theta_{V \otimes W} = c_{W,V} c_{V,W} (\theta_V \otimes \theta_W)$). Our example of ribbon category will be that of the finite dimensional left Yetter–Drinfeld modules over a group algebra, ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$. These are finite dimensional left $\mathbf{k}\Gamma$-modules V which have a grading over the group, i.e., $V = \oplus_{g \in \Gamma} V^h$ in such a way that $g \cdot V^h = V^{ghg^{-1}}$. The tensor product has the diagonal and codiagonal structures: $g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w$ and $(V \otimes W)^g = \oplus_{hk=g} V^h \otimes W^k$. The braiding is given by $c(v \otimes w) = g \cdot w \otimes v$ for $v \in V^g$. The dual V^* is the sum $\oplus_g (V^g)^*$ with grading $(V^*)^h = (V^{h^{-1}})^*$ and the structure $< g \cdot f, v > = < f, g^{-1} \cdot v >$; the birth and death maps are $b(1) = \sum_i v_i \otimes v^i \in V \otimes V^*$ and $d(v^i \otimes v_j) = \delta_{ij}$ (here $\{v_i\}$ is a $\mathbf{k}$-basis of V and $\{v^i\}$ its dual basis). Finally, there are several possible twists for ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$, and we take the simplest one, which is given by $\theta_V(v) = g \cdot v$ for $v \in V^g$ (see [KR] for a characterization of all the twists).

Suppose that K is a framed oriented knot, let $\mathcal{C}$ be a ribbon category which is $\mathbf{k}$-linear. Consider then the coloring of K by an object V of $\mathcal{C}$ in such a way that an up arrow is colored by V^*, a down arrow by V, the crossing as in the left side of Figure 2 by the braiding, the crossing as in the middle side by the inverse of the braiding, and the birth and death maps are represented by the curves in Figure 4. The same diagrams but with the arrows pointing to the right correspond to the maps $\bar{b} = (\mathrm{id} \otimes \theta_V^{-1}) \circ c_{V,V^*} \circ b_V : \mathbf{k} \to V^* \otimes V$ and $\bar{d} = d_V \circ c_{V,V^*} \circ (\theta_V \otimes \mathrm{id}) : V \otimes V^* \to \mathbf{k}$ (see for instance [T] for the fully explained theory). Now, if L is the diagram of an oriented link with a framing; denote its components by $C_1, \dots, C_n$. Let $V_1, \dots, V_n$ be modules in $\mathcal{C}$. Color the component C_i by V_i. Then the diagram can be considered as a map from $\mathbf{k}$ to $\mathbf{k}$, thus an element of $\mathbf{k}$. This is a first quantum invariant of L, which of course depends on the choice of the V_i's. If $\mathcal{C}$ is a modular category (such a category has a finite set of preferred objects), one can define a linear combination of this last invariant over all combinations $V_1, \dots, V_n$ within the preferred objects. This is necessary for instance to define invariants of closed 3-manifolds. We take here another approach: we restrict the attention to a fixed object V of $\mathcal{C}$ and color each component by V. In this way one gets a well defined invariant of framed links. We shall denote this invariant by

$$\Psi_V(L).$$

Notice that we do not need the category $\mathcal{C}$ anymore: we only use V, V^*, b_V, d_V, the braiding $c_{V,V}$ (the braidings $c_{V^*,V}$, c_{V,V^*} and c_{V^*,V^*} can be obtained from $c_{V,V}$), and the twist θ_V. If $\theta_V = \mathrm{id}_V$, then the invariant just defined is an invariant of oriented links (non-necessarily framed). That is, the invariant verifies the first Reidemeister move.

We construct now examples of spaces in ribbon categories such that $\theta_V = \mathrm{id}_V$. We follow [G1] for a moment. We fix $\mathbf{k}$ as an algebraically closed field of characteristic 0. Let G_∞ be the group of roots of unity of $\mathbf{k}$. As with any abelian group, we may consider rack (resp. quandle) cohomologies $H^*(X, G_\infty)$ and $H^*(X, \mathbf{k}^\times)$ (resp. $H^*_Q(X, G_\infty)$ and $H^*_Q(X, \mathbf{k}^\times)$) for any rack (resp. quandle) X. Suppose from now on that X is finite. For an element $f \in C^2(X, \mathbf{k}^\times)$, define the vector space $\mathbf{k}X$ with basis X, and the map $c^f \in \mathrm{End}(\mathbf{k}X \otimes \mathbf{k}X)$ by $c^f(a \otimes b) = f(a,b) a \triangleright b \otimes a$. Then we have

Proposition 3.3.

1. *c^f is a solution to the braid equation if and only if f is a 2-cocycle.*
2. *Modyfing f by a coboundary δg is the same as changing the basis $\{x \mid x \in X\}$ by $\{g(x)x \mid x \in X\}$.*
3. *If $f \in Z^2(X, G_\infty)$, there is a finite group Γ such that $\mathbf{k}X$ can be considered as a Yetter–Drinfeld module over $\mathbf{k}\Gamma$. Furthermore, the braiding of $\mathbf{k}X$ as a YD-module coincides with c^f.*
4. *If $f \in Z^2(X, \mathbf{k}^\times)$ then $(\mathbf{k}X, c^f)$ defined as above is also a Yetter–Drinfeld module over a group algebra for a (possibly) infinite group.*

Proof. For the first three items, see [G1, §3]. The last item is straightforward using the methods given there. $\square$

In the particular case in which $V = \mathbf{k}X$ with the braiding c^f, we shall denote the invariant $\Psi_V(L)$ by

$$\Psi_f(L, X) \in \mathbf{k}.$$

Consider then this case. Since V is an element of a Yetter–Drinfeld category over a group algebra, it is easy to put θ_V in terms of the rack and the cocycle: if $x \in X \subset V$, we have $\theta_V(x) = g \cdot x$, where $x \in (\mathbf{k}X)^g$. We can compute $g \cdot x$ from the equation $g \cdot x \otimes x = c(x \otimes x) = f(x, x)(x \triangleright x) \otimes x$. Thus

$$\theta_V(x) = f(x, x) x \triangleright x; \quad \theta_V(x) = x \text{ if } X \text{ is a quandle and } f \text{ is a quandle cocycle.} \tag{3.4}$$

For $V \in \mathcal{C}$, denote by T_V the usual flip $T_V(a \otimes b) = b \otimes a$; this is usually not a morphism in the category, but it is of course a well defined linear map. For a virtual link, we can extend the invariant just defined by using this flip for each degenerated crossing. We shall call also by $\Psi_f(L, X)$ this invariant if L is a virtual link and X, f are a quandle and a quandle 2-cocycle.

We are in position now to prove the main result:

Theorem 3.5. *Let L be a virtual link, X a finite quandle and A an abelian group. Let $f \in Z^2_Q(X, A)$ be a quandle 2-cocycle, and let $\chi : A \to G_\infty$ be a representation of A. Then*

$$\chi(\Phi_f(L, X)) = \Psi_{\chi f}(L, X). \tag{3.6}$$

Proof. First, observe that the right hand side of the equality is well defined: we have $\theta_V = \mathrm{id}_V$ by (3.4). Furthermore, $\bar{b}_V = c_{V,V} b_V = T_V b_V$; this follows from f being a quandle cocycle. Analogously, $\bar{d}_V = d_V T_V$.

Consider L as the closure of a virtual braiding L', say on n strands, as in Figure 5.

Now L' corresponds to an endomorphism α of $V^{\epsilon_1} \otimes \dots \otimes V^{\epsilon_n}$, where there are n tensorands, $\epsilon_i = \pm 1$ and we put $V^{-1} = V^*$; the distribution of the signs is the same as that of the up or down arrows at the top of L'. Then the right hand side of (3.6) coincides with the trace (usual trace, by the previous paragraph) of α. We denote by $\{x^{-1} \mid x \in X\}$ the basis of V^* dual to V. To compute the trace of α we must put all the combinations $x_1^{\epsilon_1}, \dots, x_n^{\epsilon_n}$ $(x_i \in X)$ on the top of L', compute α for it, evaluate the result on $x_1^{-\epsilon_1} \otimes \dots \otimes x_n^{-\epsilon_n}$ and then sum all the results. Now, by the form the braiding has, $\alpha(x_1^{\epsilon_1} \otimes \dots \otimes x_n^{\epsilon_n})$ is a scalar multiple of some element $(x_1')^{\epsilon_1} \otimes \dots \otimes (x_n')^{\epsilon_n}$. Now, this term does not count in the trace unless $(x_1')^{\epsilon_1} \otimes \dots \otimes (x_n')^{\epsilon_n} = x_1^{\epsilon_1} \otimes \dots \otimes x_n^{\epsilon_n}$, since we must evaluate it on $x_1^{-\epsilon_1} \otimes \dots \otimes x_n^{-\epsilon_n}$. Thus the terms which contribute to the sum are exactly those which give a coloring of L. Moreover (as it follows by direct computation for each of the eight possible non-virtual crossings) the scalar involved is the product

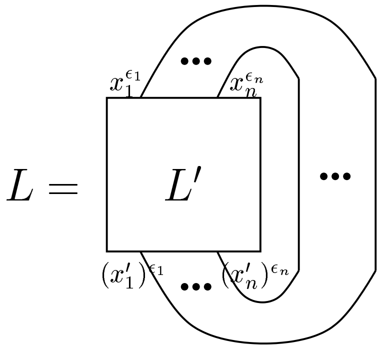

FIGURE 5. L as the closure of L'

of the $(\chi f(a,b))^{\pm 1}$ for (a,b) running on the non-virtual crossings; the signs depending on whether it is possitive or negative. This is exactly the left hand side of (3.6). □

Corollary 3.7. *Let L be a knot, X be a finite quandle, A be a group and f be a 2-quandle cocycle. Then $\Phi_f(L,X)$ can be recovered from quantum invariants.*

Proof. First, notice that we can restrict the attention to B, the subgroup of A generated by $f(a,b)$ for all $a,b \in X$, since $\Phi_f(L,X)$ lies in $\mathbb{Z}B$. This is a finitely generated abelian group because X is finite. Compute $\Psi_{\chi f}(L,X)$ for each character $\chi \in \mathrm{Hom}(B,\mathbf{k}^\times)$, these are quantum invariants. Then this family of elements in $\mathbf{k}$ determine $\Phi_f(L,X)$ thanks to the theorem. □

Remark 3.8. Proposition 3.3 part 2 explains why $\Psi(L,X)$ –and then $\Phi(L,X)$– is invariant under modifications by coboundaries: we simply compute the trace with another basis. This argument seems more clear than that of [CJKLS].

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MATÍAS GRAÑA
DEPTO. DE MATEMÁTICA
PAB I, FCEYN, CIUDAD UNIVERSITARIA
(1428) BUENOS AIRES, ARGENTINA

E-mail address: `matiasg@dm.uba.ar`