BRAIDED HOPF ALGEBRAS OVER NON ABELIAN FINITE GROUPS

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Abstract. In the last years a new theory of Hopf algebras has begun to be developed: that of Hopf algebras in braided categories, or, briefly, braided Hopf algebras. This is a survey of general aspects of the theory with emphasis in ${}^H_H\mathcal{YD}$, the Yetter–Drinfeld category over H, where H is the group algebra of a non abelian finite group Γ. We discuss a special class of braided graded Hopf algebras from different points of view following Lusztig, Nichols and Schauenburg. We present some finite dimensional examples arising in an unpublished work by Milinski and Schneider.

Sinopsis. En los últimos años comenzó a ser desarrollada una nueva teoría de álgebras de Hopf en categorías trenzadas, o brevemente, álgebras de Hopf trenzadas. Presentamos aquí aspectos generales de la teoría con énfasis en ${}^H_H\mathcal{YD}$, la categoría de Yetter–Drinfeld sobre H, donde H es el álgebra de grupo de un grupo finito no abeliano Γ. Discutimos una clase especial de álgebras de Hopf trenzadas graduadas desde diferentes puntos de vista, siguiendo a Lusztig, Nichols y Schauenburg. Presentamos algunos ejemplos de dimensión finita que aparecen en un trabajo inédito de Milinski y Schneider.

0. Introduction and notations

0.1. Introduction.

The idea of considering Hopf algebras in braided categories (categories with a tensor product which is associative and "commutative") goes back to Milnor–Moore [MM65] and Mac Lane [ML63]. Hopf superalgebras, or $\mathbb{Z}/2$-graded Hopf algebras, were intensively studied in the work of Kac, Kostant, Berezin and others. In this case, the braiding c is symmetric: $c^2 = \mathrm{id}$. With the advent of quantum groups, it became clear that braided (non symmetric) categories have a rôle to play in several parts of algebra. This point of view was pioneered by Manin [Man88, p. 81], Majid [Maj95] (see also [Gur91]) and widely developed since then. One of its main applications is Lusztig's presentation of quantized enveloping algebras [Lus93].

Our motivation to study braided Hopf algebras is the so-called bosonization (or biproduct) construction, due to Radford [Rad85] and interpreted in the terms of braided categories by Majid [Maj94b]. More precisely, we are interested in a specific type of braided Hopf algebras. To explain the reason we recall a general principle from [AS]:

Let K be a Hopf algebra with coradical filtration

$$K_0 \subset K_1 \subset \dots .$$

If the coradical K_0 is a Hopf subalgebra (this happens for instance if K is pointed, in which case $H = K_0$ is a group algebra) then the associated graded space

$$\operatorname{gr} K = \bigoplus_{n \geq 0} K_n / K_{n-1} \qquad (K_{-1} = 0)$$

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has a graded Hopf algebra structure inherited from that of K. Moreover, since the inclusion $K_0 \hookrightarrow \operatorname{gr} K$ has a retraction $\operatorname{gr} K \to K_0$ of Hopf algebras, the inverse process to the bosonization construction makes the algebra of coinvariants $R = (\operatorname{gr} K)^{\operatorname{co} K_0}$ into a graded Hopf algebra in ${}^{K_0}_{K_0}\mathcal{YD}$ with trivial coradical. Conversely, if H is a group algebra, let R be a graded Hopf algebra in ${}^{H}_{H}\mathcal{YD}$ with trivial coradical (i.e., $R_0 = \mathbf{k}1$). Then the bosonization $R\#H$ is a pointed graded Hopf algebra with coradical isomorphic to H. It is then reasonable to expect that information one can give about graded Hopf algebras in ${}^{H}_{H}\mathcal{YD}$ can be translated to information about pointed Hopf algebras.

We say that a graded Hopf algebra $R = \bigoplus_{i\geq 0} R(i)$ in ${}^{H}_{H}\mathcal{YD}$ is a TOBA if $R(0)$ is the base field (and then the coradical is trivial by [Swe69, 11.1.1]), the space of primitive elements is exactly $R(1)$ and this space generates R (as an algebra). It is then proved that R is a TOBA iff $R\#H$ is a Hopf algebra of type one, in the sense of Nichols [Nic78]. An important example of TOBA is the quantum analog of the enveloping algebra of the nilpotent part of a Borel algebra, see [Lus93], [Sch96], [Ros95], [Ros92].

The article is organized as follows:

In section 1 we define and give examples of braided categories, Hopf algebras in braided categories and review the bosonization construction.

In section 2 we give duals and opposite algebras of braided Hopf algebras (a deep treatment of the subject can be found in [Maj95]). For finite Hopf algebras we define the space of integrals, and prove (following Takeuchi [Tak97]) that it is an invertible object in the category. This allows to state "braided" versions of several useful results concerning finite dimensional Hopf algebras (e.g. the bijectivity of the antipode).

In section 3 we concentrate on braided Hopf algebras in ${}^{H}_{H}\mathcal{YD}$ where $H = \mathbf{k}\Gamma$, the group algebra of a finite group Γ. We show that a TOBA R is determined, up to isomorphism, by the space of primitive elements $\mathcal{P}(R) = R(1)$. Moreover, given a Yetter-Drinfeld module V, we present three different constructions of a TOBA $\mathfrak{t}(V)$ such that its space of primitive elements is isomorphic to V. The first two constructions use quantum shuffles and universal properties, and are essentially contained in [Nic78], [Sch93], [Ros95], [Roz96], [Wor89]. The third construction, by means of a bilinear form, seems to be new. It is however inspired by [Lus93], [Sch93], [Ros95], [Mül98]. We finally discuss some explicit examples from [MS97] for Γ a symmetric or a dihedral group.

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0.2. **Notations.**
We shall work over a field $\mathbf{k}$. Sometimes we impose some hypothesis to the field, but in most of the article it may be any field. Tensor products and Homs are taken over $\mathbf{k}$ when not specified. We use the letters H, K for Hopf algebras over $\mathbf{k}$, and the letter R for braided Hopf algebras.

Given a Hopf algebra H, we use subindices $H_0 \subset H_1 \subset \dots$ to indicate the coradical filtration of H (see [Swe69]). In order to avoid confusion with this notation, a graded algebra shall be denoted by $H = \oplus_i H(i)$.

Given an algebra A, we denote by ${}_A\mathcal{M}$ the category of finite dimensional left A-modules and by ${}^{\infty}_{A}\mathcal{M}$ the category of all left A-modules; *ditto* for the categories of right modules $\mathcal{M}_A$ and $\mathcal{M}^{\infty}_{A}$.

Given a coalgebra C, we denote by ${}^{C}\mathcal{M}$ the category of left C-comodules. The same for $\mathcal{M}^C$.

Given a Hopf algebra H, we denote by ${}^{H}_{H}\mathcal{M}^{H}_{H}$ the category of Hopf bimodules over H, by ${}^{H}_{H}\mathcal{YD}$ the Yetter–Drinfeld category over H (of finite dimensional left YD modules) and by ${}^{H}_{H}\mathcal{YD}_{\infty}$ the category of all left YD modules. See 1.1.15 for the precise definitions.

Given a Hopf algebra H with bijective antipode, we denote by H^{op} the Hopf algebra with opposite multiplication, H^{cop} the Hopf algebra with opposite comultiplication, and H^{bop} the Hopf algebra $(H^{\mathrm{op}})^{\mathrm{cop}}$.

We use for coalgebras Sweedler notation without summation symbol: $\Delta(h) = h_{(1)} \otimes h_{(2)}$, and the same for comodules: $\delta(m) = m_{(-1)} \otimes m_{(0)}$ for left comodules and $\delta(m) = m_{(0)} \otimes m_{(1)}$ for right comodules.

If M is a $\mathbf{k}$-vector space, $m \in M$ and $f \in M^*$, we use either $f(m)$, $\langle f, m\rangle$ or $\langle m, f\rangle$ to denote the evaluation map.

1. Definitions and examples

1.1. Braided categories. Abelian braided categories.

We recall in this section the definitions of monoidal and braided categories. See [JS93] for a detailed treatment of the subject.

Definition 1.1.1. A *monoidal category* is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called *tensor product*), an object $\mathbf{1}$ of $\mathcal{C}$ (called *unit*) and natural isomorphisms

$$\begin{aligned} &a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W) && \text{(associativity constraint)},\\ &r_V : V \otimes \mathbf{1} \to V, \qquad l_V : \mathbf{1} \otimes V \to V && \text{(unit constraints)}, \end{aligned}$$

subject to the following conditions:

$$\begin{aligned} &((U \otimes V) \otimes W) \otimes X \to (U \otimes V) \otimes (W \otimes X) \to U \otimes (V \otimes (W \otimes X))\\ =\ &((U \otimes V) \otimes W) \otimes X \to (U \otimes (V \otimes W)) \otimes X \to U \otimes ((V \otimes W) \otimes X) \to U \otimes (V \otimes (W \otimes X)), \end{aligned}$$

$$\mathrm{id} = V \otimes W \to (V \otimes \mathbf{1}) \otimes W \to V \otimes (\mathbf{1} \otimes W) \to V \otimes W.$$

We shall assume in what follows that the associativity constraint is the identity morphism. This is possible thanks to [ML71], where the author proves that any monoidal category can be embedded in another monoidal category in which this is true.

Definition 1.1.2. A *braided monoidal category* (or briefly *braided category*) is a monoidal category $\mathcal{C}$ together with a natural isomorphism

$$c = c_{M,N} : M \otimes N \to N \otimes M$$

(called *braiding*) subject to the conditions

$$c_{M,N\otimes P} = (\mathrm{id}_N \otimes c_{M,P}) \circ (c_{M,N} \otimes \mathrm{id}_P), \tag{1.1.3}$$

$$c_{M\otimes N,P} = (c_{M,P} \otimes \mathrm{id}_N) \circ (\mathrm{id}_M \otimes c_{N,P}). \tag{1.1.4}$$

The category $\mathcal{C}$ is called *symmetric* if $c^2 = \mathrm{id}$, i.e., for all M, N in $\mathcal{C}$, $c_{N,M} c_{M,N} = \mathrm{id}_{M\otimes N}$.

There are some identities that can be proved from the axioms (and hence hold in any braided category). One of these identities is $lc = r$, that is,

$$(M \otimes \mathbf{1} \xrightarrow{c_{M,\mathbf{1}}} \mathbf{1} \otimes M \xrightarrow{l_M} M) = (M \otimes \mathbf{1} \xrightarrow{r_M} M).$$

Analogously $rc = l$.

Remark 1.1.5. To define when two monoidal (braided) categories are equivalent, it is necessary to know what a functor between monoidal (braided) categories is. Let $\mathcal{C}$ and $\mathcal{C}'$ be monoidal categories. A functor between them is a pair (F,η), where $F:\mathcal{C}\to\mathcal{C}'$ is a functor and η is a natural isomorphism $\eta:\otimes\circ F^2\to F\circ\otimes$ (that is, $\eta_{M,N}: FM\otimes FN\to F(M\otimes N)$) subject to the conditions

$$\begin{tikzcd} FM\otimes FN\otimes FP \arrow[r] \arrow[d] & F(M\otimes N)\otimes FP \arrow[d] \\ FM\otimes F(N\otimes P) \arrow[r] & F(M\otimes N\otimes P) \end{tikzcd} \quad \text{must commute,} \tag{1.1.6}$$

$$F(\mathbf{1}_{\mathcal{C}})=\mathbf{1}_{\mathcal{C}'},$$
$$\mathbf{1}\otimes FM\xrightarrow{l_{FM}}FM=\mathbf{1}\otimes FM\xrightarrow{\eta}F(\mathbf{1}\otimes M)\xrightarrow{F(l_M)}FM,$$
$$FM\otimes\mathbf{1}\xrightarrow{r_{FM}}FM=FM\otimes\mathbf{1}\xrightarrow{\eta}F(M\otimes\mathbf{1})\xrightarrow{F(r_M)}FM.$$

Observe that when the associativity constraint is not the identity, then (1.1.6) must be suitably modified. For braided categories the following diagram must also commute.

$$\begin{tikzcd} FM\otimes FN \arrow[r, "c_{FM,FN}"] \arrow[d, "\eta"'] & FN\otimes FM \arrow[d, "\eta"'] \\ F(M\otimes N) \arrow[r, "F(c_{M,N})"] & F(N\otimes M). \end{tikzcd}$$

Given a braided category $\mathcal{C}$, we shall denote by $\overline{\mathcal{C}}$ the braided category whose objects and morphisms are those of $\mathcal{C}$ but whose braiding is the inverse of that of $\mathcal{C}$. The axioms (1.1.3) and (1.1.4) are automatically verified for this category.

An important source of examples of braided categories is given by the quasitriangular bialgebras. Let H be a bialgebra. An element $\mathcal{R}\in H\otimes H$ is called a *triangular structure* for H if it is invertible (with respect to the usual product of $H\otimes H$) and verifies

$$\forall h\in H,\ \Delta^{\mathrm{op}}(h)=\mathcal{R}\Delta(h)\mathcal{R}^{-1}, \tag{1.1.7}$$
$$(\mathrm{id}\otimes\Delta)(\mathcal{R})=\mathcal{R}^{13}\mathcal{R}^{12}, \tag{1.1.8}$$
$$(\Delta\otimes\mathrm{id})(\mathcal{R})=\mathcal{R}^{13}\mathcal{R}^{23}. \tag{1.1.9}$$

In this case, the pair $(H,\mathcal{R})$ is called a *quasitriangular bialgebra* (QT bialgebra). If $\tau(\mathcal{R})=\mathcal{R}^{-1}$ (τ is the usual flip), then $(H,\mathcal{R})$ is called *triangular.* For $(H,\mathcal{R})$ a QT bialgebra, the category of left (right) H-modules ${}_H^\infty\mathcal{M}$ ($\mathcal{M}_H^\infty$) and the category of finite dimension left (right) H-modules ${}_H\mathcal{M}$ ($\mathcal{M}_H$) are braided, where

$$c_{M,N}(m\otimes n)=\mathcal{R}_2 n\otimes\mathcal{R}_1 m \qquad \text{for left modules,}$$
$$c_{M,N}(m\otimes n)=n\mathcal{R}_1\otimes m\mathcal{R}_2 \qquad \text{for right modules.}$$

Equation (1.1.7) is equivalent to c being a morphism of H-modules. Equations (1.1.8) and (1.1.9) are respectively equivalent to (1.1.3) and (1.1.4) in the case of left modules, and to (1.1.4) and (1.1.3) in the case of right modules. These categories are symmetric if $(H,\mathcal{R})$ is triangular.

The notion of QT bialgebra can be dualized to that of *co-quasitriangular bialgebras* (or briefly CQT bialgebras), for which the category of left (right) comodules is braided. Both notions can be generalized to quasi-bialgebras, to get QT *quasi-bialgebras* (CQT quasi-bialgebras) for which the categories of left (right) modules (left (right) comodules) are also braided (see [Dri90]). In these cases the associativity constraint is no longer the usual associativity for vector spaces, and the verification of the axioms becomes more tedious.

Definition 1.1.10. A monoidal category $\mathcal{C}$ is called *rigid* if every object has a left and right dual in it. That is, for every object M of $\mathcal{C}$ there exist ${}^*M$ and M^* objects of $\mathcal{C}$ and natural morphisms

$$\begin{aligned} br_M &: \mathbf{1} \to M \otimes M^*, \\ bl_M &: \mathbf{1} \to {}^*M \otimes M, \\ dr_M &: M^* \otimes M \to \mathbf{1}, \\ dl_M &: M \otimes {}^*M \to \mathbf{1}, \end{aligned}$$

subject to the conditions

$$\begin{aligned} \mathrm{id} &= M \xrightarrow{\mathrm{id}\otimes bl} M \otimes {}^*M \otimes M \xrightarrow{dl\otimes \mathrm{id}} \mathbf{1} \otimes M \xrightarrow{l} M, \\ \mathrm{id} &= M \xrightarrow{br\otimes \mathrm{id}} M \otimes M^* \otimes M \xrightarrow{\mathrm{id}\otimes dr} M \otimes \mathbf{1} \xrightarrow{r} M. \end{aligned} \tag{1.1.11}$$

Remark 1.1.12. The conditions on M^* and ${}^*M$ determine them up to isomorphism. We shall use in what follows the word "rigid" for a category in which the correspondences $M \mapsto M^*$ and $M \mapsto {}^*M$ are given by functors, which is true in the usual cases.

Definition 1.1.13. It is well known that the symmetric group in n elements $\mathbb{S}_n$ $(n \geq 2)$, can be presented by elementary transpositions $\tau_i = (i, i+1)$, $(1 \leq i < n)$ subject to the relations

$$\begin{aligned} \tau_i\tau_j &= \tau_j\tau_i && \text{if } |i-j| > 1, \\ \tau_i\tau_j\tau_i &= \tau_j\tau_i\tau_j && \text{if } |i-j| = 1, \\ \tau_i^2 &= 1 && \forall i. \end{aligned}$$

If we drop the last set of relations, we get the Artin *braid group*. To be precise, we define $\mathbb{B}_n$, $(n \geq 2)$ to be the group with generators σ_i, $(1 \leq i < n)$ subject to the relations

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{if } |i-j| > 1, \\ \sigma_i\sigma_j\sigma_i &= \sigma_j\sigma_i\sigma_j && \text{if } |i-j| = 1. \end{aligned}$$

This is an infinite group, and there is a projection map from $\mathbb{B}_n$ to $\mathbb{S}_n$ given by $\sigma_i \mapsto \tau_i$.

Remark 1.1.14. The group $\mathbb{S}_n$ acts naturally on n-fold tensor products in the category of vector spaces, or more generally, in the category of representations of a cocommutative Hopf algebra. In both cases the category is symmetric. When this does not happen, the group $\mathbb{S}_n$ has to be replaced by $\mathbb{B}_n$, as we now explain. Let $\mathcal{C}$ be a braided category and M an object of $\mathcal{C}$. Then $\mathbb{B}_n$ acts on $\underbrace{M \otimes \cdots \otimes M}_{n \text{ times}}$ via

$$\sigma_i \mapsto \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{i-1} \otimes c \otimes \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{n-i-1}.$$

This useful observation allows to translate several statements into drawings, and in fact many authors do use drawings to prove certain equalities. The axioms above can be viewed as rules to pass from one configuration to another.

Our main example of braided (rigid) category is the Yetter–Drinfeld category over a Hopf algebra:

Definition 1.1.15. Let H be a Hopf algebra over $\mathbf{k}$ with bijective antipode. We shall denote by ${}_H^H\mathcal{YD}$ the category of finite left *Yetter–Drinfeld modules* over H. That is, M is an object in ${}_H^H\mathcal{YD}$ if M is a left H-module, a left H-comodule, has finite dimension over $\mathbf{k}$ and

$$(hm)_{(-1)} \otimes (hm)_{(0)} = h_{(1)}m_{(-1)}\mathcal{S}h_{(3)} \otimes h_{(2)}m_{(0)}, \qquad \forall h \in H,\ m \in M.$$

${}_H^H\mathcal{YD}$ is a monoidal category with the usual tensor product over $\mathbf{k}$, where $\mathbf{1} = \mathbf{k}$ and associativity and unit constraints are the usual ones for vector spaces and where, for $M, N \in {}_H^H\mathcal{YD}$, $M \otimes N$ has the diagonal module and comodule structures given by

$$h(m \otimes n) = h_{(1)}m \otimes h_{(2)}n, \quad (m \otimes n)_{(-1)} \otimes (m \otimes n)_{(0)} = m_{(-1)}n_{(-1)} \otimes m_{(0)} \otimes n_{(0)}.$$

It is also a braided category, where the braiding is given by

$$c = c_{M,N} : M \otimes N \to N \otimes M, \quad c(m \otimes n) = m_{(-1)}n \otimes m_{(0)}.$$

It is immediate to see that c is an isomorphism, with inverse

$$(c_{N,M})^{-1} = (c^{-1})_{M,N} \quad m \otimes n \mapsto n_{(0)} \otimes \mathcal{S}^{-1}(n_{(-1)})m.$$

As with any braided category, we shall denote by ${}_H^H\overline{\mathcal{YD}}$ the same category as ${}_H^H\mathcal{YD}$ but with the inverse braiding. ${}_H^H\overline{\mathcal{YD}}$ is a Yetter–Drinfeld category (see 2.2.1). We prove rigidity of these categories in 2.1.1. We denote by ${}_H^H\mathcal{YD}_\infty$ the category of all (non necessarily finite dimensional) Yetter–Drinfeld modules over H. This is a braided category (with the braiding given by the same formula as in ${}_H^H\mathcal{YD}$) but it is not rigid.

If M is an object in ${}_H^H\mathcal{YD}$, we shall denote by $\Im M$ the object of ${}_H^H\mathcal{YD}$ with the same underlying vector space but with the structure given by

$$h \rightharpoonup m = \mathcal{S}^2(h)m, \quad \delta_{\Im M}(m) = \mathcal{S}^{-2}m_{(-1)} \otimes m_{(0)}.$$

Theorem 1.1.16 (Majid)**.** *Let H be a finite dimensional Hopf algebra. Let $\mathcal{D}(H) = H \bowtie H^{*\mathrm{op}}$ be the Drinfeld double of H, defined by $\mathcal{D}(H) = H \otimes H^*$ as a coalgebra, with multiplication and antipode given by (we denote here $fg = m(f \otimes g)$ in H^* rather than in $H^{*\mathrm{op}}$)*

$$\begin{aligned}(h \bowtie f)(h' \bowtie f') &= \langle f_{(1)}, h'_{(1)}\rangle\langle f_{(3)}, \mathcal{S}h'_{(3)}\rangle(hh'_{(2)} \bowtie f'f_{(2)}),\\ \mathcal{S}_{\mathcal{D}(H)}(h \bowtie f) &= (1 \bowtie \mathcal{S}^{-1}f)(\mathcal{S}h \bowtie \varepsilon) = (\mathcal{S}h_{(2)} \bowtie \mathcal{S}^{-1}f_{(2)})\langle f_{(1)}, \mathcal{S}h_{(1)}\rangle\langle f_{(3)}, h_{(3)}\rangle.\end{aligned}$$

We observe that H and $H^{*\mathrm{op}}$ are Hopf subalgebras of $\mathcal{D}(H)$. We have that $\mathcal{D}(H)$ is a QT Hopf algebra, and the category ${}_{\mathcal{D}(H)}\mathcal{M}$ of finite dimensional left $\mathcal{D}(H)$-modules is equivalent, as braided category, to ${}_H^H\mathcal{YD}$.

Proof. See [Mon93]. □

There is another category which appears naturally in the framework of Hopf algebras which is equivalent to ${}_H^H\mathcal{YD}$ and ${}_{\mathcal{D}(H)}\mathcal{M}$, namely ${}_H^H\mathcal{M}_H^H$. This is the category whose objects are H-bimodules and H-bicomodules, such that the structure morphisms $H \otimes M \to M$ and $M \otimes H \to M$ are bicomodule morphisms, taking in $M \otimes H$ and $H \otimes M$ the codiagonal structure (equivalently, the structure morphisms $M \to M \otimes H$ and $M \to H \otimes M$ are bimodule morphisms taking in $M \otimes H$ and $H \otimes M$ the diagonal structure). We take in this category tensor products over H (alternatively, we can take the monoidal structure given by cotensor products over H). This category has a braiding, namely

$$c_{M,N}(m \otimes n) = m_{(-2)}n_{(0)}\mathcal{S}(n_{(1)})\mathcal{S}(m_{(-1)}) \otimes m_{(0)}n_{(2)}.$$

The following result was independently found by Schauenburg and the first author.

Proposition 1.1.17. *The category ${}^H_H\mathcal{M}^H_H$ is equivalent as braided category to ${}^H_H\mathcal{YD}$. (Alternatively, the category with the monoidal structure given by cotensor products is also equivalent).*

Proof. (Sketch, see [Sch93, Satz 1.3.5], [Sch94] or [AD95, Appendix] for the details). Let M be in ${}^H_H\mathcal{M}^H_H$. By [Swe69, Th. 4.1.1], $M \simeq V \otimes H$ as a right module and right comodule, where $V = M^{\mathrm{co}\ H}$ and the right module and comodule structures of $V \otimes H$ are those of H. Let us identify V with $V \otimes 1$. We take the structure of V in ${}^H_H\mathcal{YD}$ by

$$h \rightharpoonup v = h_{(1)}v\mathcal{S}h_{(2)}, \quad \delta(v) = \delta_l(v) \quad (\delta_l : M \to H \otimes M \text{ is the structure morphism}).$$

Conversely, if V is a Yetter–Drinfeld module over H, then $V \otimes H$ is an object in ${}^H_H\mathcal{M}^H_H$ via

$$h(v \otimes g) = h_{(1)} \rightharpoonup v \otimes h_{(2)}g, \quad \delta(v \otimes g) = v_{(-1)}g_{(1)} \otimes (v_{(0)} \otimes g_{(2)}),$$

and H acts and coacts on the right only over H. □

1.2. Hopf algebras in braided categories (braided Hopf algebras).

Monoidal categories are the natural context to define algebras and coalgebras. If $\mathcal{C}$ is a monoidal category, we define an algebra in $\mathcal{C}$ to be a pair (A, m), where A is an object of $\mathcal{C}$, $m : A \otimes A \to A$ is a morphism in $\mathcal{C}$ and there exists a morphism $u : \mathbf{1} \to A$ such that

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m) : A \otimes A \otimes A \to A,$$
$$m \circ (u \otimes \mathrm{id}) \circ l_A^{-1} = \mathrm{id} = m \circ (\mathrm{id} \otimes u) \circ r_A^{-1} : A \to A.$$

We define dually a coalgebra in $\mathcal{C}$ to be a pair (C, Δ), where C is an object of $\mathcal{C}$, $\Delta : C \to C \otimes C$ is a morphism in $\mathcal{C}$ and there exists a morphism $\varepsilon : C \to \mathbf{1}$ such that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta : C \to C \otimes C \otimes C,$$
$$l_C \circ (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = r_C \circ (\mathrm{id} \otimes \varepsilon) \circ \Delta : C \to C.$$

In turn, braided categories are the natural context to define bialgebras and Hopf algebras.

Definition 1.2.1. Let $\mathcal{C}$ be a braided category. A *bialgebra in* $\mathcal{C}$ is a triple (R, m, Δ), where R is an object in $\mathcal{C}$ and there exist morphisms $u : \mathbf{1} \to R$ and $\varepsilon : R \to \mathbf{1}$ in such a way that (R, u, m) is an algebra in $\mathcal{C}$, (R, ε, Δ) is a coalgebra in $\mathcal{C}$, ε is an algebra morphism, and the usual compatibility between m and Δ is replaced by

$$\Delta m = (m \otimes m) \circ (\mathrm{id}_R \otimes c_{R,R} \otimes \mathrm{id}_R) \circ (\Delta \otimes \Delta).$$

If moreover there exists a morphism $\mathcal{S} : R \to R$ which is the inverse of the identity in the monoid $\mathrm{Hom}_{\mathcal{C}}(R, R)$ with the convolution product, then we say that R is a *Hopf algebra* in $\mathcal{C}$ and call $\mathcal{S}$ the *antipode* of R. We recall that this last definition can be stated in other words as

$$m(\mathcal{S} \otimes \mathrm{id})\Delta = u\varepsilon = m(\mathrm{id} \otimes \mathcal{S})\Delta : R \to R.$$

As in the classical case, the compatibility between the algebra and coalgebra structure can be alternatively stated saying that m is a morphism of coalgebras, or that Δ is a morphism of algebras, with the only difference that $R \otimes R$ is considered as an algebra with the product

$$m_{R\otimes R} = (m_R \otimes m_R) \circ (\mathrm{id}_R \otimes c_{R,R} \otimes \mathrm{id}_R)$$

or as a coalgebra with the coproduct

$$\Delta_{R\otimes R} = (\mathrm{id}_R \otimes c_{R,R} \otimes \mathrm{id}_R) \circ (\Delta_R \otimes \Delta_R).$$

As in the classical case, the antipode of a braided Hopf algebra twists multiplications and comultiplications. One should be careful to distinguish between c and c^{-1}. The precise equalities are the following ones.

Lemma 1.2.2. *Let R be a Hopf algebra in a braided category. Let us denote $m = m_R$, $\Delta = \Delta_R$, $c = c_{R,R}$. Then*

$$\mathcal{S}_R m = m(\mathcal{S}_R \otimes \mathcal{S}_R)c, \qquad \Delta\mathcal{S}_R = c(\mathcal{S}_R \otimes \mathcal{S}_R)\Delta.$$

If $\mathcal{S}$ is invertible with respect to composition, then

$$\mathcal{S}_R^{-1} m = m(\mathcal{S}_R^{-1} \otimes \mathcal{S}_R^{-1})c^{-1}, \qquad \Delta\mathcal{S}_R^{-1} = c^{-1}(\mathcal{S}_R^{-1} \otimes \mathcal{S}_R^{-1})\Delta.$$

Proof. Since m is a coalgebra morphism and $\mathcal{S}$ is the inverse of the identity in the monoid $\mathrm{Hom}_{\mathcal{C}}(R, R)$, we have that $\mathcal{S}m$ is the inverse of m in the monoid $\mathrm{Hom}_{\mathcal{C}}(R \otimes R, R)$. We have moreover that

$$\begin{aligned}
m * (m(\mathcal{S} \otimes \mathcal{S})c) &= m(m \otimes (m(\mathcal{S} \otimes \mathcal{S})c))(\mathrm{id} \otimes c \otimes \mathrm{id})(\Delta \otimes \Delta) \\
&= m(m \otimes m)(\mathrm{id} \otimes \mathrm{id} \otimes \mathcal{S} \otimes \mathcal{S})(\mathrm{id} \otimes c_{R,R\otimes R})(\Delta \otimes \Delta) \\
&= m(\mathrm{id} \otimes m)(\mathrm{id} \otimes m \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \mathcal{S} \otimes \mathcal{S})(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\mathrm{id} \otimes c)(\Delta \otimes \mathrm{id}) \\
&= m(m \otimes \mathrm{id})(\mathrm{id} \otimes \mathcal{S} \otimes \mathrm{id})(\Delta \otimes u\varepsilon) \\
&= u\varepsilon \otimes u\varepsilon = u_{R\otimes R}\varepsilon_{R\otimes R}.
\end{aligned}$$

whence the first equality.

The second equality in the first line follows dualizing the equality just proved. The second line follows immediately from the first using the naturality of c. ☐

Remark 1.2.3. Let us suppose that there exists a forgetful functor $U : \mathcal{C} \to \mathcal{C}'$ into some monoidal category $\mathcal{C}'$ in such a way that if $U(f)$ is an isomorphism then f is an isomorphism. Let H be a bialgebra in $\mathcal{C}$. UH is then an algebra and a coalgebra in $\mathcal{C}'$ (in general it is not a bialgebra). If there exists an antipode for UH in $\mathcal{C}'$ (i.e., if the identity morphism has an inverse in the monoid $\mathrm{Hom}_{\mathcal{C}'}(UH, UH)$) then there exists an antipode for H in $\mathcal{C}$, namely, $\mathcal{S}_{\mathcal{C}}$ is the morphism such that $U(\mathcal{S}_{\mathcal{C}}) = \mathcal{S}_{\mathcal{C}'}$, which exists by our hypothesis on U, as we now prove. Consider the morphism in $\mathcal{C}$ given by $F : H \otimes H \to H \otimes H$, $F = (\mathrm{id} \otimes m)(\Delta \otimes \mathrm{id})$. $U(F) \in \mathrm{End}_{\mathcal{C}'}(UH \otimes UH)$ is an isomorphism in $\mathcal{C}'$, whose inverse is $(\mathrm{id} \otimes m)(\mathrm{id} \otimes \mathcal{S}_{\mathcal{C}'} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})$. Let $T \in \mathrm{End}_{\mathcal{C}}(H \otimes H)$ be the inverse of F. The antipode is then given by the composition

$$\mathcal{S}_{\mathcal{C}} = \left(H \xrightarrow{r_H^{-1}} H \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes u} H \otimes H \xrightarrow{T} H \otimes H \xrightarrow{\varepsilon \otimes \mathrm{id}} \mathbf{1} \otimes H \xrightarrow{l_H} H \right).$$

In the usual cases, $\mathcal{C}'$ is the category of $\mathbf{k}$-vector spaces. The hypothesis is verified for instance when $\mathcal{C} = {}^H_H\mathcal{YD}$ for H a Hopf algebra with bijective antipode, or $\mathcal{C} = {}_H\mathcal{M}$ for H a QT bialgebra.

Let us recall the definition of $\Im M$ for $M \in {}^H_H\mathcal{YD}$ stated after the definition 1.1.15. It is immediate to see with the previous remark (or by direct computation) that if R is a Hopf algebra in ${}^H_H\mathcal{YD}$ then $\Im R$ is also a Hopf algebra.

1.3. **Bosonization.**

Let now H be a fixed Hopf algebra over $\mathbf{k}$ with bijective antipode. There is a one-to-one correspondence between Hopf algebras in ${}^H_H\mathcal{YD}$ and Hopf algebras A with morphisms of Hopf algebras

$$A \underset{p}{\overset{\iota}{\leftrightarrows}} H$$

such that $p\iota = \mathrm{id}_H$. This correspondence was found by Radford in [Rad85] and explained in these terms by Majid in [Maj94b]. We give the details here:

Let $A \underset{p}{\overset{\iota}{\leftrightarrows}} H$ be as above. Let $R = A^{\mathrm{co}H} = \mathrm{LKer}\, p = \{a \in A \mid (\mathrm{id} \otimes p)\Delta(a) = a \otimes 1\}$. It is immediate that this is a subalgebra of A, with the same unit. The counit of R is the restriction of that of A. We define the comultiplication, the antipode, the action and the coaction by

$$\begin{aligned}
\Delta_R(r) &= r_{(1)}(\iota\mathcal{S}_H(pr_{(2)})) \otimes r_{(3)},\\
\mathcal{S}_R(r) &= (\iota p(r_{(1)}))\mathcal{S}_A(r_{(2)}),\\
h \rightharpoonup r &= h_{(1)} r \mathcal{S} h_{(2)},\\
\delta(r) &= (p \otimes \mathrm{id})\Delta(r).
\end{aligned}$$

It is straightforward to see that these morphisms make R into a Hopf algebra in ${}^H_H\mathcal{YD}$.

Conversely, if R is a Hopf algebra in ${}^H_H\mathcal{YD}$, let $A = R\#H$ be the semidirect product algebra build from the action of H on R, and let

$$\begin{aligned}
&\Delta_A(r\#h) = (r_{(1)}\#r_{(2)(-1)}h_{(1)}) \otimes (r_{(2)(0)}\#h_{(2)}),\\
&\iota(h) = 1\#h, \qquad p(r\#h) = \varepsilon(r)h,\\
&\mathcal{S}_A(r\#1) = \iota(\mathcal{S}(r_{(-1)}))(\mathcal{S}_R(r_{(0)})\#1) = (\mathcal{S}(r_{(-1)}) \rightharpoonup \mathcal{S}_R(r_{(0)})\#\mathcal{S}(r_{(-2)})).
\end{aligned}$$

These morphisms make $R\#H$ into a Hopf algebra, and the constructions are mutually inverse. Majid calls $R\#H$ the "bosonization" of R.

1.4. **Examples of braided Hopf algebras.**

1.4.0. Let $H = \mathbf{k}$. The Yetter–Drinfeld category over H reduces in this case to the category of vector spaces over $\mathbf{k}$ (with trivial actions and coactions), and the braiding is just the usual flip $x \otimes y \mapsto y \otimes x$. A Hopf algebra in this category is just a (classic) Hopf algebra over $\mathbf{k}$.

1.4.1. Let N be a natural number, ξ a primitive N-root of unity in $\mathbf{k}$ and $A = T_{\xi,N}$ the Taft algebra of order N^2 over $\mathbf{k}$, which is generated as a $\mathbf{k}$ vector space by the elements $\{g^i x^j\}_{0 \le i,j \le N-1}$, with relations $g^N = 1$, $x^N = 0$, and $xg = \xi gx$. The comultiplication is given by $\Delta g = g \otimes g$, and $\Delta x = g \otimes x + x \otimes 1$. The antipode is given by $\mathcal{S}g = g^{-1}$, and $\mathcal{S}x = -g^{-1}x$. The counit, by $\varepsilon g = 1$, $\varepsilon x = 0$.

Let H be the group algebra of the cyclic group of N elements. We shall denote also by g a generator of this group. There is a morphism of Hopf algebras

$$\pi : A \to H, \quad \pi(g^i x^j) = g^i \delta_{j,0}.$$

This morphism has a section, namely

$$\iota : H \to A, \quad \iota(g^i) = g^i.$$

One sees that $R = A^{\mathrm{co}H}$ is isomorphic as an algebra to $\mathbf{k}[x]/(x^N)$. It has comultiplication $\Delta_R x = x \otimes 1 + 1 \otimes x$, counit $\varepsilon_R(x) = 0$, and antipode $\mathcal{S}_R x = -x$. The action and coaction of H over R are given by $g \rightharpoonup x = gxg^{-1} = \xi^{-1}x$, and $\delta(x) = g \otimes x$.

1.4.2. Let H be as before the group algebra of a cyclic group of order N with generator g. Let $A = \mathbf{h}(\xi, m) = \mathbf{k}{<}y, x, g{>} / \sim$ be the book algebra considered in [AS98]. It is the Hopf algebra with generators $\{x, y, g\}$ and relations

$$x^N = y^N = 0, \quad g^N = 1, \quad , gx = \xi xg, \quad gy = \xi^m yg, \quad xy = yx$$

and with comultiplication, antipode and counit given by

$$\Delta(x) = x \otimes g + 1 \otimes x, \quad \Delta(y) = y \otimes 1 + g^m \otimes y, \quad \Delta(g) = g \otimes g,$$
$$\mathcal{S}(x) = -xg^{-1}, \quad \mathcal{S}(y) = -g^{-m}y, \quad \mathcal{S}(g) = g^{-1}, \quad \varepsilon(x) = \varepsilon(y) = 0, \quad \varepsilon(g) = 1.$$

One can take here either $H = T_{\xi,N} = \mathbf{k}{<}x, g{>} / \sim$ or $H = \mathbf{k}{<}g{>}$. In the first case, $p(y) = 0$, $p(x) = x$, $p(g) = g$, and

$$\mathrm{LKer}(p) = \frac{\mathbf{k}[y]}{(y^N)}.$$

In the second case, let $\bar{x} = xg^{-1}$, and $p(y) = p(x) = 0$, $p(g) = g$. Then

$$\mathrm{LKer}(p) = \frac{\mathbf{k}{<}\bar{x}, y{>}}{(\bar{x}^N, y^N, \bar{x}y - \xi^m y\bar{x})}.$$

1.4.3. The preceding examples are particular cases of a wider class of braided Hopf algebras, which we now define. Suppose Γ is an abelian group. Let $g_1, \dots, g_n \in \Gamma$ and $\chi_1, \dots, \chi_n : \Gamma \to \mathbf{k}^\times$ characters. Suppose that for $i \neq j$ we have $\chi_i(g_j)\chi_j(g_i) = 1$. Let N_i be the order of $\chi_i(g_i)$, and $q_{i,j} = \chi_j(g_i)$. Let R be the algebra generated by elements $x_1, \dots, x_n$ with relations

$$x_i^{N_i} = 0 \qquad \forall i,$$
$$x_i x_j = q_{i,j} x_j x_i \quad \text{if } i \neq j.$$

Thus the set of monomials $\{x_1^{r_1} \cdots x_n^{r_n},\ 0 \leq r_i \leq N_i\}$ is clearly a basis for R. We define the action and coaction of H by

$$g \rightharpoonup (x_1^{r_1} \dots x_n^{r_n}) = \chi_1(g)^{r_1} \dots \chi_n(g)^{r_n} x_1^{r_1} \dots x_n^{r_n},$$
$$\delta(x_1^{r_1} \dots x_n^{r_n}) = g_1^{r_1} \dots g_n^{r_n} \otimes x_1^{r_1} \dots x_n^{r_n},$$

and the comultiplication, counit and antipode by

$$\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1,$$
$$\varepsilon x_i = 0,$$
$$\mathcal{S} x_i = -x_i.$$

Then R is a braided Hopf algebra over $\mathbf{k}\Gamma$. Following Manin, these Hopf algebras are called *quantum linear spaces*. Several classification results were obtained in [AS] from the study of these braided Hopf algebras, including the classification of pointed Hopf algebras of order p^3, p an odd prime.

1.4.4. Let $\mathfrak{g}$ be a complex finite dimensional simple Lie algebra, $\mathfrak{b}$ a Borel subalgebra, A the Cartan matrix of $\mathfrak{g}$. The Lusztig's algebras $\mathbf{f}$ and $'\mathbf{f}$ constructed from A are braided Hopf algebras in ${}^H_H\mathcal{YD}$, where H is the group algebra of a free abelian group. See example 3.2.22, or [Sch96], [Lus93] for the details. The bosonization of $\mathbf{f}$ is the quantized enveloping algebra $\mathbf{U}_q(\mathfrak{b})$ of $\mathfrak{b}$. Since Drinfeld showed how to obtain the quantized enveloping algebra $\mathbf{U}_q(\mathfrak{g})$ of $\mathfrak{g}$ from $\mathbf{U}_q(\mathfrak{b})$ via the double construction, we see that quantum groups can be derived in a conceptual way from the setting of braided Hopf algebras.

2. Duals and opposite algebras. Integrals

2.1. First results about duals.

Proposition 2.1.1. *Let H be a Hopf algebra over $\mathbf{k}$ with bijective antipode. Then ${}^H_H\mathcal{YD}$ is a braided rigid category.*

Proof. We have seen that ${}^H_H\mathcal{YD}$ is braided. We shall prove now rigidity. Let M be an object of ${}^H_H\mathcal{YD}$. Let $\{{}_\alpha m\}_{\alpha\in A}$ be a basis of M as a $\mathbf{k}$-vector space, and $\{{}^\alpha m\}_{\alpha\in A}$ its dual basis. We shall omit the summation symbol in any formula with the occurrence of the element $\sum_{\alpha\in A} {}_\alpha m\otimes {}^\alpha m$. We take ${}^*M$ and M^* to be the dual of M as $\mathbf{k}$-vector spaces, with the following structure:

$$\left.\begin{aligned}(h\cdot f)(m) &= f(\mathcal{S}(h)m),\\ f_{(-1)}\otimes f_{(0)} &= \mathcal{S}^{-1}({}_\alpha m_{(-1)})\otimes f({}_\alpha m_{(0)}){}^\alpha m\end{aligned}\right\}\quad \text{for } M^*,$$

and

$$\left.\begin{aligned}(h\cdot f)(m) &= f(\mathcal{S}^{-1}(h)m),\\ f_{(-1)}\otimes f_{(0)} &= \mathcal{S}({}_\alpha m_{(-1)})\otimes f({}_\alpha m_{(0)}){}^\alpha m\end{aligned}\right\}\quad \text{for } {}^*M.$$

The morphisms br, bl, dr, dl are the canonical morphisms

$$\begin{aligned} br &= \iota : \mathbf{k}\to M\otimes M^* & 1&\mapsto {}_\alpha m\otimes {}^\alpha m,\\ bl &= \iota : \mathbf{k}\to {}^*M\otimes M & 1&\mapsto {}^\alpha m\otimes {}_\alpha m,\\ dr &= \mathrm{ev} : M^*\otimes M\to \mathbf{k} & f\otimes m&\mapsto f(m),\\ dl &= \mathrm{ev} : M\otimes {}^*M\to \mathbf{k} & m\otimes f&\mapsto f(m),\end{aligned}$$

which are morphisms in ${}^H_H\mathcal{YD}$ with respect to the above defined structures. It is immediate that these morphisms satisfy equations (1.1.11). ☐

Let $\mathcal{C}$ be any braided rigid category. We recall that this means (for us) that there exist functors $M\mapsto M^*$ and $M\mapsto {}^*M$. These functors are inverse to each other via canonical isomorphisms since ${}^*(M^*)\simeq ({}^*M)^*\simeq M$. Indeed, it is clear that $(M, br_M : \mathbf{1}\to M\otimes M^*, dr_M : M^*\otimes M\to \mathbf{1})$ satisfies the axioms of left dual for M^*, and $(M, bl_M : \mathbf{1}\to {}^*M\otimes M, dl_M : M\otimes {}^*M\to \mathbf{1})$ satisfies the axioms of right dual for ${}^*M$. Moreover, the functors $M\mapsto M^*$ and $M\mapsto {}^*M$ are naturally isomorphic, as can be seen considering

$$M^*\xrightarrow{\mathrm{id}\otimes bl} M^*\otimes {}^*M\otimes M\xrightarrow{c\otimes\mathrm{id}} {}^*M\otimes M^*\otimes M\xrightarrow{\mathrm{id}\otimes dr} {}^*M,$$
$${}^*M\xrightarrow{br\otimes\mathrm{id}} M\otimes M^*\otimes {}^*M\xrightarrow{\mathrm{id}\otimes c^{-1}} M\otimes {}^*M\otimes M^*\xrightarrow{dl\otimes\mathrm{id}} M^*.$$

Thus, $M^{**} \simeq {}^*(M^*) \simeq M \simeq ({}^*M)^* \simeq {}^{**}M$ via natural isomorphisms. In the category ${}^H_H\mathcal{YD}$ these morphisms are given by

$$\begin{aligned} M &\to M^{**}, & m &\mapsto \mathcal{S}(m_{(-1)})m_{(0)},\\ M &\to {}^{**}M, & m &\mapsto \mathcal{S}^{-2}(m_{(-1)})m_{(0)}. \end{aligned}$$

The rather strange asymmetry between both morphisms comes from the fact that we use c^{-1} in the first one and c in the second one.

Definition 2.1.2. Let $\mathcal{C}$ be any rigid category, and M, N be objects of $\mathcal{C}$. Let $F : M \to N$ be a morphism in $\mathcal{C}$. We define the transposes of F as

$$F^* = N^* \to N^* \otimes \mathbf{1} \to N^* \otimes M \otimes M^* \xrightarrow{\mathrm{id} \otimes F \otimes \mathrm{id}} N^* \otimes N \otimes M^* \to \mathbf{1} \otimes M^* \to M^*,$$

$${}^*F = {}^*N \to \mathbf{1} \otimes {}^*N \to {}^*M \otimes M \otimes {}^*N \xrightarrow{\mathrm{id} \otimes F \otimes \mathrm{id}} {}^*M \otimes N \otimes {}^*N \to {}^*M \otimes \mathbf{1} \to {}^*M.$$

Remark 2.1.3. Most of the rigid categories we consider are subcategories of the category of **k**-vector spaces, and the duals are preserved by the forgetful functor (the maps br, bl, dr, dl are also preserved). When this happens, the maps F^* and ${}^*F$ coincide (via the forgetful functor) with the usual transpose map of F. We observe that this means that for $F : M \to N$ a morphism in ${}^H_H\mathcal{YD}$, the transpose as **k**-vector spaces $F^* : N^* \to M^*$ is a morphism in ${}^H_H\mathcal{YD}$.

Lemma 2.1.4. *Let $\mathcal{C}$ be any rigid category, and let $M, N \in \mathcal{C}$. There exist natural isomorphisms*

$$\begin{aligned} \phi^*_{M,N} &: M^* \otimes N^* \to (N \otimes M)^*,\\ {}^*\phi_{M,N} &: {}^*M \otimes {}^*N \to {}^*(N \otimes M). \end{aligned}$$

Proof. To prove that $M^* \otimes N^* \simeq (N \otimes M)^*$ it would be sufficient to prove that $M^* \otimes N^*$ satisfy (1.1.11) for certain morphisms br, bl, dr and dl, but in order to prove naturality it is necessary to give the explicit definition of ϕ^* and ${}^*\phi$.

$$\begin{aligned} \phi^* = M^* \otimes N^* &\xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes br_{N\otimes M}} (M^* \otimes N^*) \otimes (N \otimes M) \otimes (N \otimes M)^* \to\\ &M^* \otimes (N^* \otimes N) \otimes M^* \otimes (N \otimes M)^* \xrightarrow{\mathrm{id} \otimes dr_N \otimes \mathrm{id} \otimes \mathrm{id}} M^* \otimes M \otimes (N \otimes M)^* \xrightarrow{dr_M \otimes \mathrm{id}} (N \otimes M)^*. \end{aligned}$$

Analogously for ${}^*\phi$. The proof that ${}^*\phi$ and ϕ^* are natural is straightforward but tedious and we omit it. □

Lemma 2.1.5. *Let N, M be objects of $\mathcal{C}$. We have*

$$c^*_{M,N}\phi^*_{M,N} = \phi^*_{N,M}c_{M^*,N^*}.$$

Proof. First, we claim that

$$\begin{aligned} \mathbf{1} \to (M \otimes N) \otimes (M \otimes N)^* &\xrightarrow{c \otimes \mathrm{id}} (N \otimes M) \otimes (M \otimes N)^*\\ &= \mathbf{1} \to (N \otimes M) \otimes (N \otimes M)^* \xrightarrow{\mathrm{id} \otimes c^*} (N \otimes M) \otimes (N \otimes M)^* \end{aligned}$$

In fact, tensoring both sides with $(M \otimes N)$ on the right and composing with $dr_{(M\otimes N)}$ one gets c, whence the claim.

Second, we claim that

$$\begin{aligned} M^* \otimes N^* \otimes M \otimes N &\xrightarrow{\mathrm{id} \otimes c} M^* \otimes N^* \otimes N \otimes M \to M^* \otimes M \to \mathbf{1}\\ &= M^* \otimes N^* \otimes M \otimes N \xrightarrow{c \otimes \mathrm{id}} N^* \otimes M^* \otimes M \otimes N \to N^* \otimes N \to \mathbf{1}. \end{aligned}$$

In fact, both sides equal $M^* \otimes N^* \otimes M \otimes N \xrightarrow{\mathrm{id} \otimes c^{-1} \otimes \mathrm{id}} M^* \otimes M \otimes N^* \otimes N \to \mathbf{1}$. Thus,

$$
\begin{aligned}
&\phi^*_{N,M} c_{M^*,N^*} \\
&= M^* \otimes N^* \xrightarrow{c_{M^*,N^*}} N^* \otimes M^* \to N^* \otimes M^* \otimes (M \otimes N) \otimes (M \otimes N)^* \xrightarrow{dr \otimes \mathrm{id}} (M \otimes N)^* \\
&= M^* \otimes N^* \to M^* \otimes N^* \otimes (M \otimes N) \otimes (M \otimes N)^* \xrightarrow{c_{M^*,N^*} \otimes \mathrm{id} \otimes \mathrm{id}} \\
&\qquad\qquad N^* \otimes M^* \otimes (M \otimes N) \otimes (M \otimes N)^* \xrightarrow{dr \otimes \mathrm{id}} (M \otimes N)^* \\
&= M^* \otimes N^* \to M^* \otimes N^* \otimes (M \otimes N) \otimes (M \otimes N)^* \xrightarrow{\mathrm{id}^{\otimes 2} \otimes c_{M,N} \otimes \mathrm{id}} \\
&\qquad\qquad M^* \otimes N^* \otimes (N \otimes M) \otimes (M \otimes N)^* \xrightarrow{dr \otimes \mathrm{id}} (M \otimes N)^* \\
&= M^* \otimes N^* \to M^* \otimes N^* \otimes (N \otimes M) \otimes (N \otimes M)^* \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes (c_{M,N})^*} \\
&\qquad\qquad M^* \otimes N^* \otimes (N \otimes M) \otimes (M \otimes N)^* \xrightarrow{dr \otimes \mathrm{id}} (M \otimes N)^* \\
&= M^* \otimes N^* \to M^* \otimes N^* \otimes (N \otimes M) \otimes (N \otimes M)^* \xrightarrow{dr \otimes \mathrm{id}} (N \otimes M)^* \xrightarrow{(c_{M,N})^*} (M \otimes N)^* \\
&\qquad\qquad = (c_{M,N})^* \phi^*_{M,N}.
\end{aligned}
$$

□

2.2. **Equivalence of some Yetter–Drinfeld categories and dual Hopf algebras.**

In the setting of Yetter-Drinfeld categories, one often needs to pass from one category to another. This is usually possible. We give the corresponding functors. We recall from 1.1.5 the definition of a functor between braided categories.

Proposition 2.2.1. 1. *Let H be a Hopf algebra with bijective antipode. The following categories are equivalent as braided categories:*

$$(i)\ {}^H_H\mathcal{YD}, \quad (ii)\ {}^{H^{\mathrm{bop}}}_{H^{\mathrm{bop}}}\mathcal{YD}, \quad (iii)\ \mathcal{YD}^H_H, \quad (iv)\ \mathcal{YD}^{H^{\mathrm{bop}}}_{H^{\mathrm{bop}}}.$$

2. *If H is finite dimensional the preceding categories are equivalent to the following ones (as braided categories):*

$$(v)\ \mathcal{YD}^{H^*}_{H^*}, \quad (vi)\ \mathcal{YD}^{H^{*\mathrm{bop}}}_{H^{*\mathrm{bop}}}, \quad (vii)\ {}^{H^*}_{H^*}\mathcal{YD}, \quad (viii)\ {}^{H^{*\mathrm{bop}}}_{H^{*\mathrm{bop}}}\mathcal{YD}.$$

3. *The following categories are equivalent as braided categories if H is finite dimensional:*

$$(ix)\ {}^H_H\overline{\mathcal{YD}}, \quad (x)\ \mathcal{YD}^{H^{\mathrm{cop}}}_{H^{\mathrm{cop}}}.$$

Proof. For (1) and (2), let M be an object in ${}^H_H\mathcal{YD}$. We first prove that (i), (ii), (v), (vi) are equivalent. We take the structure

$$
\begin{array}{lll}
h \rightharpoonup^2 m = \mathcal{S}(h)m, & \delta^2(m) = \mathcal{S}^{-1}m_{(-1)} \otimes m_{(0)} & \text{for } {}^{H^{\mathrm{bop}}}_{H^{\mathrm{bop}}}\mathcal{YD}, \\
m \leftharpoonup^5 f = m_{(0)}\langle f, m_{(-1)}\rangle, & \delta^5(m) = {}_\alpha h m \otimes {}^\alpha h & \text{for } \mathcal{YD}^{H^*}_{H^*}, \\
m \leftharpoonup^6 f = m_{(0)}\langle f, \mathcal{S}^{-1}m_{(-1)}\rangle, & \delta^6(m) = {}_\alpha h m \otimes \mathcal{S}({}^\alpha h) & \text{for } \mathcal{YD}^{H^{*\mathrm{bop}}}_{H^{*\mathrm{bop}}}.
\end{array}
$$

It is not difficult to verify that these structures make M into objects in the stated categories and that preserve tensor products. In all these cases the natural isomorphism η of Remark 1.1.5 is the identity, i.e. $F(M \otimes N) = FM \otimes FN$.

We verify the compatibility with the braiding between ${}_H^H\mathcal{YD}$ and $\mathcal{YD}_{H^*}^{H^*}$. The others are analogous. Let c_1 and c_5 denote the braidings in the respective categories. Let M and N be objects in ${}_H^H\mathcal{YD}$. Then we have to prove that $Fc_1 = c_5F : M \otimes N \to N \otimes M$. Let $m \otimes n \in M \otimes N$, and denote by the same symbol the corresponding element in $FM \otimes FN = F(M \otimes N)$. Then

$$\begin{aligned} c_5(m \otimes n) &= \delta_0^3 n \otimes m\delta_1^3 n = {}_\alpha hn \otimes m \leftharpoonup^3 {}^\alpha h \\ &= {}_\alpha hn \otimes m_{(0)}\langle {}^\alpha h, m_{(-1)}\rangle = m_{(-1)}n \otimes m_{(0)} = c_1(m \otimes n). \end{aligned}$$

In an analogous way it can be proved that the categories (iii), (iv), (vii) and $(viii)$ are equivalent. We give the equivalence between (i) and (iii), which is more subtle since $\eta \neq \mathrm{id}$. Let M be an object in ${}_H^H\mathcal{YD}$. We define $\Re(M)$ in $\mathcal{YD}_H^H$ to be M as a $\mathbf{k}$-vector space, with the structure given by

$$m \leftharpoonup^3 h = \mathcal{S}^{-1}(h)m, \quad \delta^3 m = m_{(0)} \otimes \mathcal{S}m_{(-1)},$$

whence

$$\delta^3(m \leftharpoonup^3 h) = \delta_0^3 m \leftharpoonup^3 h_{(2)} \otimes \mathcal{S}(h_{(1)})\delta_1^3(m)h_{(3)}.$$

Observe that there is a natural isomorphism

$$\phi = \phi_{M,N} : \Re(M \otimes N) \to \Re N \otimes \Re M, \quad (m \otimes n) \mapsto n \otimes m.$$

We define

$$\eta_{M,N} = \phi_{M,N}^{-1} \circ c_{\Re M, \Re N} : \Re(M) \otimes \Re(N) \to \Re(M \otimes N),$$

that is,

$$\begin{aligned} \eta_{M,N}(m \otimes n) &= \phi_{M,N}^{-1}(\delta_0^5(n) \otimes m \leftharpoonup^5 \delta_1^5(n)) \\ &= \phi_{M,N}^{-1}(n_{(0)} \otimes \mathcal{S}^{-1}(\mathcal{S}n_{(1)})m) \\ &= \phi_{M,N}^{-1}(n_{(0)} \otimes n_{(1)}m) = n_{(1)}m \otimes n_{(0)} \in \Re(M \otimes N). \end{aligned}$$

It is straightforward to check that $(\Re, \eta)$ is a functor between braided categories. We verify for instance (1.1.6):

$$\begin{aligned} \eta_{M\otimes N,P} \circ (\eta_{M,N} \otimes \mathrm{id})(m \otimes n \otimes p) &= \eta_{M\otimes N,P}(n_{(-1)}m \otimes m_{(0)} \otimes p) \\ &= p_{(-1)}(n_{(-1)}m \otimes m_{(0)}) \otimes p_{(0)} \\ &= p_{(-2)}n_{(-1)}m \otimes p_{(-1)}n_{(0)} \otimes p_{(0)}, \\ \eta_{M,N\otimes P} \circ (\mathrm{id} \otimes \eta_{N,P})(m \otimes n \otimes p) &= \eta_{M,N\otimes P}(m \otimes p_{(-1)}np_{(0)}) \\ &= p_{(-4)}n_{(-1)}\mathcal{S}(p_{(-2)})p_{(-1)}m \otimes p_{(3)}n_{(0)} \otimes p_{(0)} \\ &= p_{(-2)}n_{(-1)}m \otimes p_{(-1)}n_{(0)} \otimes p_{(0)}. \end{aligned}$$

(3) Let M be an object in ${}_H^H\overline{\mathcal{YD}}$, and define

$$m \leftharpoonup^{10} h = \mathcal{S}^{-1}(h)m, \quad \delta^{10}(m) = m_{(0)} \otimes m_{(-1)}.$$

As before, it is straightforward to see that this is an object in $\mathcal{YD}_{H^{\mathrm{cop}}}^{H^{\mathrm{cop}}}$, and that the braiding is that of ${}_H^H\overline{\mathcal{YD}}$. □

We concentrate now on dual Hopf algebras. It would be possible to define the dual of a braided Hopf algebra in ${}_H^H\mathcal{YD}$ declaring R^* (resp. ${}^*R$) to be the right dual (resp. left dual) of R in ${}_H^H\mathcal{YD}$ with the algebra and coalgebra structure transposes of the coalgebra and algebra structures of R. This would fail to be a bialgebra in ${}_H^H\mathcal{YD}$ because the compatibility between multiplication and comultiplicatin does not transpose to the compatibility between the transpose operations. There are two ways to fix this problem. The first one is to take a kind of $R^{*\mathrm{bop}}$, which is a Hopf algebra in ${}^H_H\mathcal{YD}$ (as it is done for a general rigid braided category in [Maj94a]). The second one is to consider R^* (or ${}^*R$) as a Hopf algebra in ${}^{H^*}_{H^*}\mathcal{YD}$ and recover a Hopf algebra in ${}^H_H\mathcal{YD}$ via $\overline{\mathfrak{R}}$, the inverse functor to $\mathfrak{R}$. The natural way to see R^* as a Hopf algebra in ${}^{H^*}_{H^*}\mathcal{YD}$ is by means of the following construction: let

$$R\#H \underset{p}{\overset{\iota}{\leftrightarrows}} H$$

be the construction given in section 1.3. We can dualize it to get

$$R^* \otimes H^* \simeq (R\#H)^* \underset{p^*}{\overset{\iota^*}{\rightleftarrows}} H^*,$$

where the first is an isomorphism of vector spaces, and $\iota^* p^* = \mathrm{id}_{H^*}$. It is immediate that $\mathrm{LKer}(\iota^*) = R^* \otimes \varepsilon_H \subseteq R^* \otimes H^*$. This makes R^* into a Hopf algebra in ${}^{H^*}_{H^*}\mathcal{YD}$. One can get ${}^*R$ with the same procedure, but starting out with $\Im R$ instead of R.

We prefer instead of doing this the more categorical way: we define duals of a Hopf algebra in any rigid braided category as it is done by several authors (see for instance [Tak97]). For the special case of ${}^H_H\mathcal{YD}$, we get the above duals.

Definition 2.2.2. Let $\mathcal{C}$ be any braided rigid category and M, N objects of $\mathcal{C}$. Let ϕ^* and ${}^*\phi$ be the isomorphisms of 2.1.4. We define

$$\sigma^*_{M,N} = M^* \otimes N^* \xrightarrow{c^{-1}} N^* \otimes M^* \xrightarrow{\phi^*} (M \otimes N)^*,$$

and then we define the structure of R^* by

$$m_{R^*} = R^* \otimes R^* \xrightarrow{\sigma^*} (R \otimes R)^* \xrightarrow{\Delta^*} R^*,$$
$$\Delta_{R^*} = R^* \xrightarrow{m^*} (R \otimes R)^* \xrightarrow{(\sigma^*)^{-1}} R^* \otimes R^*,$$
$$\mathcal{S}_{R^*} = (\mathcal{S}_R)^*,\ u_{R^*} = (\varepsilon_R)^*,\ \varepsilon_{R^*} = (u_R)^*.$$

We define ${}^*R$ in the same manner, replacing the duals on the right by duals on the left.

Lemma 2.2.3. *These morphisms make R^* and ${}^*R$ into Hopf algebras in $\mathcal{C}$.*

Proof. We use lemma 2.1.5. It is straightforward to prove associativity, coassociativity and the axioms for unit, counit and antipode. We shall prove the compatibility between multiplication and comultiplication for R^*. The proof for ${}^*R$ is analogous. Let M, N, S and T be objects in $\mathcal{C}$. We denote

$$c_{2,2} : (M \otimes N \otimes S \otimes T) \to (S \otimes T \otimes M \otimes N) = c_{M\otimes N, S\otimes T},$$
$$\phi^*_{2,2} : (M \otimes N)^* \otimes (S \otimes T)^* \to (S \otimes T \otimes M \otimes N)^* = \phi^*_{M\otimes N, S\otimes T},$$
$$\phi^*_4 : (M^* \otimes N^* \otimes S^* \otimes T^*) \to (T \otimes S \otimes N \otimes M)^* = \phi^*_{2,2}(\phi^* \otimes \phi^*).$$

Let us observe that if f and g are morphisms then

$$(f^* \otimes g^*) = (\phi^*)^{-1}(g \otimes f)^* \phi^*.$$

We claim that

$$c(\phi^*_{2,2})^{-1}(\mathrm{id} \otimes c \otimes \mathrm{id})^* \phi^*_{2,2} c^{-1} = (\phi^* \otimes \phi^*)(c^{-1} \otimes c^{-1})(\mathrm{id} \otimes c \otimes \mathrm{id})(c \otimes c)((\phi^*)^{-1} \otimes (\phi^*)^{-1}).$$

This is true because

$$\begin{aligned}
&\phi^*_{2,2}c^{-1}(\phi^*\otimes\phi^*)(c^{-1}\otimes c^{-1})(\mathrm{id}\otimes c\otimes\mathrm{id})(c\otimes c)((\phi^*)^{-1}\otimes(\phi^*)^{-1})c(\phi^*_{2,2})^{-1}\\
&\quad=\phi^*_{2,2}(\phi^*\otimes\phi^*)[c_{2,2}^{-1}(c^{-1}\otimes c^{-1})(\mathrm{id}\otimes c\otimes\mathrm{id})(c\otimes c)c_{2,2}]((\phi^*)^{-1}\otimes(\phi^*)^{-1})(\phi^*_{2,2})^{-1}\\
&\quad=\phi^*_4(\mathrm{id}\otimes c\otimes\mathrm{id})(\phi^*_4)^{-1}=(\mathrm{id}\otimes c\otimes\mathrm{id})^*.
\end{aligned}$$

Hence

$$\begin{aligned}
\Delta_{R^*}m_{R^*}&=c(\phi^*)^{-1}m^*\Delta^*\phi^*c^{-1}=c(\phi^*)^{-1}(\Delta m)^*\phi^*c^{-1}\\
&=c(\phi^*)^{-1}\left[(m\otimes m)(\mathrm{id}\otimes c\otimes\mathrm{id})(\Delta\otimes\Delta)\right]^*\phi^*c^{-1}\\
&=c(\phi^*)^{-1}(\Delta\otimes\Delta)^*(\mathrm{id}\otimes c\otimes\mathrm{id})^*(m\otimes m)^*\phi^*c^{-1}\\
&=c(\Delta^*\otimes\Delta^*)(\phi^*_{2,2})^{-1}(\mathrm{id}\otimes c\otimes\mathrm{id})^*\phi^*_{2,2}(m^*\otimes m^*)c^{-1}\\
&=(\Delta^*\otimes\Delta^*)c(\phi^*_{2,2})^{-1}(\mathrm{id}\otimes c\otimes\mathrm{id})^*\phi^*_{2,2}c^{-1}(m^*\otimes m^*)\\
&=(\Delta^*\otimes\Delta^*)(\phi^*\otimes\phi^*)(c^{-1}\otimes c^{-1})(\mathrm{id}\otimes c\otimes\mathrm{id})(c\otimes c)((\phi^*)^{-1}\otimes(\phi^*)^{-1})(m^*\otimes m^*)\\
&=(m_{R^*}\otimes m_{R^*})(\mathrm{id}\otimes c\otimes\mathrm{id})(\Delta_{R^*}\otimes\Delta_{R^*})
\end{aligned}$$

□

Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$. We give the specific structure for R^* and ${}^*R$. The formulae are exactly the same for both algebras.

$$\begin{aligned}
\langle m(f\otimes g),r\rangle&=\langle f,r_{(2)(0)}\rangle\langle g,\mathcal{S}^{-1}(r_{(2)(-1)})r_{(1)}\rangle\\
&=\langle g_{(0)},r_{(2)}\rangle\langle\mathcal{S}^{-1}(g_{(-1)})f,r_{(1)}\rangle.\\
\langle f_{(1)},r\rangle\langle f_{(2)},s\rangle&=\langle f,mc(s\otimes r)\rangle=\langle f,(s_{(-1)}r)s_{(0)}\rangle\\
&=\langle\mathcal{S}^{-1}(f_{(2)(-1)})f_{(1)},s_{(-1)}r\rangle\langle f_{(2)(0)},s_{(0)}\rangle.\\
\langle\mathcal{S}f,r\rangle&=\langle f,\mathcal{S}r\rangle,\quad\langle 1_{R^*},r\rangle=\langle\varepsilon_R,r\rangle,\quad\langle\varepsilon_{R^*},f\rangle=\langle f,1_R\rangle.
\end{aligned}$$

The following result was found by many authors, see for instance [Tak97] or [BKLT97].

Proposition 2.2.4. *Let $\mathcal{C}$ be a braided monoidal category. As usual, we denote by $\overline{\mathcal{C}}$ the same category but with the inverse braiding, i.e. $c^{\overline{\mathcal{C}}}_{M,N}=(c^{\mathcal{C}}_{N,M})^{-1}$. Let R be a Hopf algebra in $\mathcal{C}$ whose antipode is an isomorphism. We define R^{op}, R^{cop} and R^{bop} by*

$$\begin{aligned}
&m_{R^{\mathrm{op}}}=m_R\circ c^{-1}_{R,R}, &&\Delta_{R^{\mathrm{op}}}=\Delta_R, &&\mathcal{S}_{R^{\mathrm{op}}}=\mathcal{S}_R^{-1},\\
&m_{R^{\mathrm{cop}}}=m_R, &&\Delta_{R^{\mathrm{cop}}}=c^{-1}_{R,R}\circ\Delta_R, &&\mathcal{S}_{R^{\mathrm{cop}}}=\mathcal{S}_R^{-1},\\
&m_{R^{\mathrm{bop}}}=m_R\circ c_{R,R}, &&\Delta_{R^{\mathrm{bop}}}=c^{-1}_{R,R}\circ\Delta_R, &&\mathcal{S}_{R^{\mathrm{bop}}}=\mathcal{S}_R.
\end{aligned}$$

and the other structure morphisms remain equal as those of R. Then R^{op} and R^{cop} are Hopf algebras in $\overline{\mathcal{C}}$, and R^{bop} is a Hopf algebra in $\mathcal{C}$.

Proof. The general proof is straightforward (and in fact very easy using drawings). We give a direct proof for the particular case of a Yetter–Drinfeld category. We prove the statement for R^{op}. The proof for R^{cop} is analogous, and for R^{bop} is the composition of the other two. Associativity is easy to prove. We check the compatibility between the multiplication and comultiplication:

$$\begin{aligned}
\Delta m^{\mathrm{op}}(r\otimes s) &= \Delta(s_{(0)}(\mathcal{S}^{-1}(s_{(-1)})r))\\
&= s_{(0)(1)}\left(s_{(0)(2)(-1)}\mathcal{S}^{-1}(s_{(-1)})r_{(1)}\right)\otimes s_{(0)(2)(0)}\left(\mathcal{S}^{-1}(s_{(-2)})r_{(2)}\right)\\
&= s_{(1)(0)}\left(s_{(2)(-1)}\mathcal{S}^{-1}(s_{(1)(-1)}s_{(2)(-2)})r_{(1)}\right)\otimes s_{(2)(0)}\left(\mathcal{S}^{-1}(s_{(1)(-2)}s_{(2)(-3)})r_{(2)}\right)\\
&= s_{(1)(0)}\left(\mathcal{S}^{-1}(s_{(1)(-1)})r_{(1)}\right)\otimes s_{(2)(0)}\left(\mathcal{S}^{-1}(s_{(2)(-1)})\mathcal{S}^{-1}(s_{(1)(-2)})r_{(2)}\right)\\
&= (m^{\mathrm{op}}\otimes m^{\mathrm{op}})\left(r_{(1)}\otimes s_{(1)(0)}\otimes \mathcal{S}^{-1}(s_{(1)(-1)})r_{(2)}\otimes s_{(2)}\right)\\
&= (m^{\mathrm{op}}\otimes m^{\mathrm{op}})(\mathrm{id}\otimes c^{-1}\otimes \mathrm{id})(\Delta\otimes\Delta)(r\otimes s).
\end{aligned}$$

It is straightforward, using 1.2.2, to check that $\mathcal{S}^{-1}$ verifies the axioms for the antipode. □

Remark 2.2.5. We note that the above definitions can be made for algebras or coalgebras in a braided category. Then, if A is an algebra in $\mathcal{C}$ it can be defined A^{op} as the same object as A with multiplication $m_{A^{\mathrm{op}}} = m_A c_{A,A}^{-1}$, and if C is a coalgebra in $\mathcal{C}$ it can be defined C^{cop} as the same object as C with comultiplication $\Delta_{C^{\mathrm{cop}}} = c_{C,C}^{-1}\circ\Delta_C$.

2.3. **Integrals.**

By classical results a finite dimensional Hopf algebra has a one dimensional space of left (resp. right) integrals. These results can be generalized to finite Hopf algebras in braided categories, as it is done in [Doi97], [Lyu95b, Lyu95a] or [Tak97] (we define an object in $\mathcal{C}$ to be *finite* if it has a (right) dual [1] in the sense of 1.1.10). In what follows $\mathcal{C}$ shall be a braided category which has equalizers. As we noted above, monoidal categories are the natural context to define algebras and coalgebras. Given an algebra A in a monoidal category it is routine to define a (left) A module: it is a pair $(M,\rightharpoonup)$ with M an object in the category and $\rightharpoonup\colon A\otimes M\to M$ which verifies

$$\begin{aligned}
&\rightharpoonup\circ(\mathrm{id}\otimes\rightharpoonup) = \rightharpoonup\circ(m\otimes\mathrm{id}) : A\otimes A\otimes M\to M &&\text{(associativity)},\\
&\rightharpoonup\circ(u\otimes\mathrm{id})\circ l_M^{-1} = \mathrm{id}: M\to M &&\text{(unitary)}.
\end{aligned}$$

Analogously is defined a right A-module. If C is a coalgebra in the category we define in a dual fashion a (left) C-comodule to be a pair (M,δ) with M an object in the category and $\delta: M\to C\otimes M$ which verifies

$$\begin{aligned}
&(\Delta\otimes\mathrm{id})\circ\delta = (\mathrm{id}\otimes\delta)\circ\delta : M\to C\otimes C\otimes M &&\text{(coassociativity)},\\
&l_M\circ(\varepsilon\otimes\mathrm{id})\circ\delta = \mathrm{id}: M\to M &&\text{(counitary)}.
\end{aligned}$$

Analogously is defined a right C-comodule. It is routine also to define a Hopf module in a braided category:

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[1]Takeuchi defines *dual* in a weaker form: he calls M^* *the dual of* M if there exists a morphism $M^*\otimes M\to\mathbf{1}$ with the universal property that for every morphism $f: X\otimes M\to\mathbf{1}$, there exists a unique morphism $F: X\to M^*$ such that $f = \left(X\otimes M\xrightarrow{F\otimes\mathrm{id}} M^*\otimes M\xrightarrow{\mathrm{ev}}\mathbf{1}\right)$. He define an object to be *finite* if it has a dual in the sense of 1.1.10. It is easy to see that if M has a dual M^* in the sense of 1.1.10 then M^* is the dual in the sense of Takeuchi. The terminology is consistent with the usual cases: for instance, if H is a QT bialgebra, every module in ${}_H^{\infty}\mathcal{M}$ has a dual in the sense of Takeuchi, but it is finite if and only if it is finite dimensional.

Definition 2.3.1. Let R be a Hopf algebra in $\mathcal{C}$. A *left R-Hopf module* is a triple $(M, \rightharpoonup, \delta)$, where $(M, \rightharpoonup)$ is a left R-module, (M, δ) is a left R-comodule and δ is a morphism of modules, where the structure of left R-module of $R \otimes M$ is given by

$$R \otimes R \otimes M \xrightarrow{\Delta_R \otimes \mathrm{id} \otimes \mathrm{id}} R \otimes R \otimes R \otimes M \xrightarrow{\mathrm{id} \otimes c \otimes \mathrm{id}} R \otimes R \otimes R \otimes M \xrightarrow{m_R \otimes \rightharpoonup} R \otimes M.$$

As in the classic case, δ is a module morphism iff $\rightharpoonup$ is a comodule morphism, where the comodule structure of $R \otimes M$ is given by

$$R \otimes M \xrightarrow{\Delta_R \otimes \delta} R \otimes R \otimes R \otimes M \xrightarrow{\mathrm{id} \otimes c \otimes \mathrm{id}} R \otimes R \otimes R \otimes M \xrightarrow{m_R \otimes \mathrm{id} \otimes \mathrm{id}} R \otimes R \otimes M.$$

The definition of right R-Hopf module is analogous.

Definition 2.3.2. Let C be a coalgebra in $\mathcal{C}$ with a unit $u : \mathbf{1} \to C$ which is a coalgebra morphism, and (M, δ) be a left C-comodule. We define the space of coinvariants by means of the equalizer

$$^{\mathrm{co}C}M = \mathrm{Eq}(M \underset{u \otimes \mathrm{id}}{\overset{\delta}{\rightrightarrows}} C \otimes M).$$

The fundamental theorem for Hopf modules can be modified to braided categories. Specifically,

Proposition 2.3.3. *Let $\mathcal{C}$ be a braided category, let R be a Hopf algebra in $\mathcal{C}$, and let ${}^R_R\mathcal{M}$ be the category of left R-Hopf modules. Then ${}^R_R\mathcal{M}$ is equivalent to $\mathcal{C}$ via*

$$V \in \mathcal{C} \to (R \otimes V, m_r \otimes \mathrm{id}, \Delta_R \otimes \mathrm{id}) \in {}^R_R\mathcal{M},$$
$$^{\mathrm{co}R}M \in \mathcal{C} \leftarrow M \in {}^R_R\mathcal{M}.$$

Proof. Mimic [Swe69, Th 4.1.1]. See also [BD95, 3.3] for the case when $\mathcal{C}$ is a braided category with split idempotents. □

This is the first step to prove the existence of non zero (left) integrals in a finite dimensional Hopf algebra, and is used by Takeuchi in [Tak97] in the same way. We follow now his work. If R is a finite Hopf algebra in $\mathcal{C}$, we define the structure of right R-Hopf module on R^* given by

$$\leftharpoondown := R^* \otimes R \xrightarrow{\mathrm{id} \otimes \mathcal{S} \otimes br} R^* \otimes R \otimes R \otimes R^* \xrightarrow{\mathrm{id} \otimes c \otimes \mathrm{id}} R^* \otimes R \otimes R \otimes R^* \xrightarrow{\mathrm{id} \otimes m \otimes \mathrm{id}} R^* \otimes R \otimes R^* \xrightarrow{dr \otimes \mathrm{id}} R^*$$

$$\delta := R^* \xrightarrow{\mathrm{id} \otimes br} R^* \otimes R \otimes R^* \xrightarrow{\mathrm{id} \otimes \Delta \otimes \mathrm{id}} R^* \otimes R \otimes R \otimes R^* \xrightarrow{\mathrm{id} \otimes c^{-1} \otimes \mathrm{id}} R^* \otimes R \otimes R \otimes R^* \xrightarrow{dr \otimes c} R^* \otimes R.$$

The proof that $(R^*, \leftharpoondown, \delta)$ is an R-Hopf module is straightforward.

From the other hand, let A be a coalgebra in $\mathcal{C}$, which is also a finite object. Then A^* is an algebra in $\mathcal{C}$, with multiplication m^* as in 2.2.2. Moreover, if $u : \mathbf{1} \to A$ is a coalgebra map, then (A^*, u^*) is an augmented algebra in $\mathcal{C}$. If $(M, \rightharpoonup)$ is a left A^*-module in $\mathcal{C}$, we define on M a structure of right A-comodule as follows:

$$\rho = M \xrightarrow{br \otimes \mathrm{id}} A \otimes A^* \otimes M \xrightarrow{\mathrm{id} \otimes \rightharpoonup} A \otimes M \xrightarrow{c} M \otimes A.$$

Then the invariants of M are defined by the equalizer $^{A^*}M = M^{\mathrm{co}A} = \mathrm{Eq}(\rho, \mathrm{id} \otimes u)$.

Let now R^* act on R^* on the left by multiplication. We define the integrals by

$$\mathcal{I}_\ell(R^*) = {}^{R^*}R^* = (R^*)^{\mathrm{co}R}.$$

Then the right coaction ρ coincides with δ, as can be seen tensoring both morphism on the left with R^*, and then composing with $(dr \otimes \mathrm{id})(\mathrm{id} \otimes c^{-1})$. The fundamental theorem on Hopf modules gives then

$$R^* \simeq \mathcal{I}_\ell(R^*) \otimes R.$$

Changing (R, R^*) by $({}^*R, R)$ we get

$$R \simeq \mathcal{I}_\ell(R) \otimes {}^*R,$$

and thus $R^* \simeq \mathcal{I}_\ell(R^*) \otimes \mathcal{I}_\ell(R) \otimes {}^*R \simeq \mathcal{I}_\ell(R^*) \otimes \mathcal{I}_\ell(R) \otimes R^*$. If the category has coequalizers, we can apply to this isomorphism the functor $- \otimes_{R^*} \mathbf{1}$ and we get

$$\mathbf{1} \simeq \mathcal{I}_\ell(R) \otimes \mathcal{I}_\ell(R^*),$$

which means that $\mathcal{I}_\ell(R)$ is an invertible object in $\mathcal{C}$. It is clear that the above construction can be made analogously to get right integrals

$$\mathcal{I}_r({}^*R) = {}^*R^{{}^*R} = {}^{\mathrm{co}R}({}^*R),$$

such that ${}^*R \simeq R \otimes \mathcal{I}_r({}^*R)$, $R \simeq R^* \otimes \mathcal{I}_r(R)$.

From the invertibility of the space of integrals, it is possible to deduce the existence of a distinguished grouplike in R^*, which reflects the action of R on the right over $\mathcal{I}_\ell(R)$. See [Tak97].

We want to compute now the defining equation for the space of integrals $\mathcal{I}_\ell(R)$ for R a Hopf algebra in ${}^H_H\mathcal{YD}$. Let $\{{}_\alpha r\}$, $\{{}^\alpha r\}$ be dual bases for R and ${}^*R$. We have

$$\begin{aligned}\rho(x) &= c(\mathrm{id} \otimes m_R)(br_{{}^*R} \otimes \mathrm{id})(x) = c(\mathrm{id} \otimes m_R)({}^\alpha r \otimes {}_\alpha r \otimes x)\\ &= c({}^\alpha r \otimes {}_\alpha r x) = {}^\alpha r_{(-1)}({}_\alpha r x) \otimes {}_\alpha r_{(0)}.\end{aligned}$$

We then have for $y \in R$

$$\begin{aligned}(\mathrm{id} \otimes y)(\rho x) &= {}^\alpha r_{(-1)}({}_\alpha r x)\langle {}^\alpha r_{(0)}, y\rangle = \mathcal{S}({}_\beta r_{(-1)})({}_\alpha r x)\langle {}^\alpha r, {}_\beta r_{(0)}\rangle\langle {}^\beta r, y\rangle\\ &= \mathcal{S}(y_{(-1)})({}_\alpha r x)\langle {}^\alpha r, y_{(0)}\rangle = \mathcal{S}(y_{(-1)})(y_{(0)}x).\end{aligned}$$

Therefore $x \in \mathcal{I}_\ell(R)$ iff

$$\mathcal{S}(y_{(-1)})(y_{(0)}x) = (\mathrm{id} \otimes y)(\rho x) = (\mathrm{id} \otimes y)(x \otimes \varepsilon) = \varepsilon(y)x \quad \forall y \in R. \tag{2.3.4}$$

Hence, if $x \in \mathcal{I}_\ell(R)$ we have

$$yx = y_{(-2)}\mathcal{S}(y_{(-1)})(y_{(0)}x) = y_{(-1)}\varepsilon(y_{(0)})x = \varepsilon(y)x \quad \forall y \in R. \tag{2.3.5}$$

Conversely, it is immediate to see (2.3.5) implies (2.3.4). Thus, for R a Hopf algebra in ${}^H_H\mathcal{YD}$ we have the well known equation

$$x \in \mathcal{I}_\ell(R) \Leftrightarrow yx = \varepsilon(y)x \quad \forall y \in R. \tag{2.3.6}$$

Furthermore, the inveribility of $\mathcal{I}_\ell(R)$ tells that it is a one dimensional Yetter-Drinfeld module.

Analogously, the defining equation for left integral elements in R^* is stated as

$$\lambda \in \mathcal{I}_\ell(R^*) \Leftrightarrow \langle \lambda, x\rangle 1 = \langle \lambda, x_{(2)(0)}\rangle \mathcal{S}^{-1}(x_{(2)(-1)})x_{(1)} \quad \forall x \in R.$$

We note that $\langle \lambda, x_{(2)(0)}\rangle \mathcal{S}^{-1}(x_{(2)(-1)})x_{(1)} = \langle \lambda_{(0)}, x_{(2)}\rangle \lambda_{(-1)}x_{(1)}$. Let now $\lambda \in \mathcal{I}_\ell(R^*)$, $\lambda \neq 0$. We have an isomorphism of Yetter–Drinfeld modules $R \simeq R^*$, $x \mapsto (\lambda \leftharpoondown x)$. Therefore we have the following nondegenerate bilinear form on R:

$$(x, y) = (\lambda \leftharpoondown x)(y) = \langle \lambda, (x_{(-1)}y)\mathcal{S}(x_{(0)})\rangle. \tag{2.3.7}$$

Takeuchi proves also that for a finite braided Hopf algebra the antipode is an isomorphism as we now sketch. Let $\mathcal{I} = \mathcal{I}_\ell(R^*)$. The isomorphism $R^* \simeq \mathcal{I} \otimes R$ is given by

$$\alpha = \left(\mathcal{I} \otimes R \to R^* \otimes R \xrightarrow{\leftharpoondown} R^*\right).$$

Note that, because of the definition of $\leftharpoondown$, α can be factorized as

$$\mathcal{I}\otimes R \xrightarrow{\mathrm{id}\otimes\mathcal{S}} \mathcal{I}\otimes R \xrightarrow{\beta} R^*,$$

which implies that $\mathcal{I}\otimes R \xrightarrow{\mathrm{id}\otimes S} \mathcal{I}\otimes R$ has a left inverse. Tensoring it with R on the left and composing with the isomorphism

$$R\otimes\mathcal{I}\otimes R \xrightarrow{c} \mathcal{I}\otimes R\otimes R \to R^*\otimes R,$$

we get that $R^*\otimes R \xrightarrow{\mathrm{id}\otimes\mathcal{S}} R^*\otimes R$ has a left inverse. Since $\mathbf{1}\xrightarrow{\varepsilon^*} R^* \xrightarrow{u^*}\mathbf{1}$ is the identity morphism, we can compose $(\mathrm{id}\otimes\mathcal{S})$ and its left inverse conveniently with $\varepsilon^*\otimes\mathrm{id}_R$ and $u^*\otimes\mathrm{id}_R$, and we get that $\mathcal{S}: R\to R$ has a left inverse. Since the same argument proves that $\mathcal{S}^*: R^*\to R^*$ has a left inverse, this implies that $\mathcal{S}$ has a right inverse also.

3. Braided Hopf algebras of type one

3.1. **Semisimplicity of Yetter–Drinfeld categories over group algebras.**

Let Γ be a finite group. Let H be the group algebra of Γ over $\mathbf{k}$, where $\mathbf{k}$ is an algebraically closed field whose characteristic does not divide the order of Γ. We prove that ${}_H^H\mathcal{YD}$ is a semisimple category, and give a complete description of the simple objects in terms of irreducible representations of some subgroups of Γ. This seems to be folklore; it can be found e.g. in [CR97, Prop 3.3] in the language of Hopf bimodules (see also [Cib97]). Thanks to 1.1.16, in order to give all the simple objects of ${}_H^H\mathcal{YD}$ it is enough to give a collection of mutually non isomorphic simple objects for which the sum of the squares of their dimensions is the dimension of $\mathcal{D}(H)$. It is known, in fact, that the double of a semisimple and cosemisimple Hopf algebra is semisimple, see [Mon93, Cor 10.3.13], but the argument there is not constructive, in the sense that it refers to Maschke's theorem.

We consider the conjugacy classes of Γ, and choose an element in each class, which gives a subset $\mathcal{Q}$ of Γ. For any $g\in\Gamma$ we denote by $\mathcal{O}_g=\{xgx^{-1}|x\in\Gamma\}$ the conjugacy class of g, and by $\Gamma_g=\{x\in\Gamma|xg=gx\}$ the isotropy subgroup of g.

Definition 3.1.1. Let $\rho:\Gamma_g\to\mathrm{End}(V)$ be an irreducible representation of Γ_g, and let

$$M(g,\rho) := \mathrm{Ind}_{\Gamma_g}^{\Gamma} V = \mathbf{k}\Gamma\otimes_{\mathbf{k}\Gamma_g} V.$$

For $v\in V$, $x\in\Gamma$, we denote by ${}^xv$ the element $x\otimes v\in M(g,\rho)$, and by ${}^xg$ the conjugate xgx^{-1}. We take for $M(g,\rho)$ the structure given by

$$\begin{aligned} &h\rightharpoonup {}^xv = {}^{hx}v \quad \text{(the induced structure)},\\ &\delta({}^xv) = {}^xg\otimes{}^xv,\end{aligned}$$

which makes $M(g,\rho)$ into an object of ${}_H^H\mathcal{YD}$. Observe that $\dim M(g,\rho)=[\Gamma:\Gamma_g]\times\dim(\rho)$.

Given a group G we denote as usually by $\widehat{G}$ the set of isomorphism classes of irreducible representations of G. We often denote a class in $\widehat{G}$ by a representative element.

Proposition 3.1.2. *The objects $M(g,\rho)$ are simple, and any simple object of ${}_H^H\mathcal{YD}$ is isomorphic to $M(g,\rho)$ for a unique $g\in\mathcal{Q}$ and a unique $\rho\in\widehat{\Gamma_g}$.*

Proof. Let $g \in \mathcal{Q}$, $\rho \in \widehat{\Gamma_g}$. Let $0 \neq W \subseteq M(g,\rho)$ be a Yetter–Drinfeld submodule. We have to prove that $W = M(g,\rho)$. Let E_g be a set of representatives of left coclasses of Γ modulo Γ_g, i.e. $\Gamma = \bigcup_{x \in E_g} x\Gamma_g$. Observe that $M(g,\rho) = \bigoplus_{x\in E_g} \mathbf{k}x \otimes V$ as vector spaces, where V is the space affording ρ, i.e. $\rho : \Gamma_g \to \operatorname{Aut}(V)$. Let $0 \neq v \in W$, $v = \sum_{x \in E_g} x \otimes v_x = \sum_{x\in E_g} {}^x(v_x)$. Let $p_x = \delta_{{}^x g} \in H^*$ be defined by $p_x(t) = \delta_{{}^x g,t}$. We have

$$\delta(v) = \sum_{x\in E_g} {}^x g \otimes {}^x(v_x) \in H \otimes W \Rightarrow {}^x(v_x) = (p_x \otimes \mathrm{id})(\delta(v)) \in W \quad \forall x \in E_g.$$

Now, as $v \neq 0$, we have $v_y \neq 0$ for some $y \in E_g$. Then $v_y = 1 \otimes v_y = y^{-1} \rightharpoonup ({}^y(v_y)) \in W$, but $\mathbf{k}\Gamma_g \rightharpoonup v_y = \mathbf{k}1 \otimes V$ because of the ireducibility of ρ, and then $\mathbf{k}1 \otimes V \subseteq W$. Thus

$$\forall x \in E_g,\ \forall v \in V, \quad {}^x v = (x \rightharpoonup v) \in (\Gamma \rightharpoonup W) \subseteq W,$$

whence $W = M(g,\rho)$.

Let now $h \in \mathcal{Q}$ and $\tau : \Gamma_h \to \operatorname{End}(V')$ be an irreducible representation of Γ_h. Define $M(h,\tau)$ as before. If $g \neq h$, it is immediate that $M(g,\rho) \not\simeq M(h,\tau)$ because $M(g,\rho)$ has elements of degree g and $M(h,\tau)$ does not. If $g = h$ and $\rho \not\simeq \tau$ then $M(g,\rho) \not\simeq M(h,\tau)$ because any isomorphism, being a morphism of comodules, restricts to an isomorphism between V and V'. Therefore, we have a collection of mutually non isomorphic objects $M(g,\rho)$ taking $g \in \mathcal{Q}$ and $\rho \in \widehat{\Gamma_g}$. These are all the irreducible objects in ${}^H_H\mathcal{YD}$, since

$$\begin{aligned}
\sum_{g\in\mathcal{Q}} \sum_{\rho\in\widehat{\Gamma_g}} (\dim M(g,\rho))^2 &= \sum_{g\in\mathcal{Q}} \sum_{\rho\in\widehat{\Gamma_g}} ([\Gamma:\Gamma_g]\dim\rho)^2 \\
&= \sum_{g\in\mathcal{Q}} ([\Gamma:\Gamma_g]^2 \sum_{\rho\in\widehat{\Gamma_g}} (\dim\rho)^2) = \sum_{g\in\mathcal{Q}} ([\Gamma:\Gamma_g])^2(\#\Gamma_g) \\
&= \sum_{g\in\mathcal{Q}} (\#\mathcal{O}_g)^2(\#\Gamma_g) = \sum_{g\in\mathcal{Q}} (\#\mathcal{O}_g)(\#\Gamma) = (\#\Gamma)\sum_{g\in\mathcal{Q}}(\#\mathcal{O}_g) = (\#\Gamma)(\#\Gamma) = \dim(\mathcal{D}(H))
\end{aligned}$$

□

Corollary 3.1.3. *If Γ is abelian, every Yetter–Drinfeld module over $\mathbf{k}\Gamma$ can be decomposed as a direct sum of YD modules of dimension* 1.

Proof. It is immediate, since $[\Gamma : \Gamma_g] = \dim\rho = 1$ for every $g \in \Gamma$ and $\rho \in \widehat{\Gamma_g}$. □

From now on R shall be a Hopf algebra in ${}^H_H\mathcal{YD}$. We shall denote by

$$\mathcal{P}(R) = \{x \in R \mid \Delta(x) = 1 \otimes x + x \otimes 1\}$$

the space of primitive elements of R.

Lemma 3.1.4. *$\mathcal{P}(R)$ is a Yetter–Drinfeld submodule of R.*

Proof. Consider the morphism from R to $R \otimes R$ given by $\mathrm{id}_R \otimes u_R + u_R \otimes \mathrm{id}_R$. It is a morphism in ${}^H_H\mathcal{YD}$, as well as Δ_R. Then $\mathcal{P}(R)$ is the equalizer of both morphisms, which is a submodule and a subcomodule of R. □

We recall from [Swe69] that the *coradical* of R is the sum of all its simple subcoalgebras. Since R is in particular a (classic) coalgebra, we can apply to R the machinery of the coradical filtration. We shall denote by R_0 the coradical of R. We are interested in braided Hopf algebras R such that $R_0 = \mathbf{k}1$ and $\mathcal{P}(R)$ generates R as an algebra. We give now some first results about such algebras, mainly for the case in which $\mathcal{P}(R)$ is an irreducible object, and postpone to the next section a more formal and general treatment.

Definition 3.1.5. Let R be a braided Hopf algebra in ${}^H_H\mathcal{YD}$. We say that R is an *ET-algebra* if $R_0 = \mathbf{k}1$ and $\mathcal{P}(R)$ generates R.

Proposition 3.1.6. *Let R be an ET-algebra such that $\mathcal{P}(R) = M(g,\rho)$ for some $g \in \Gamma$, $\rho \in \widehat{\Gamma_g}$. Then the bosonization $R\#H$ is an extension of the bosonization $R\#\mathbf{k}G$ by the group algebra $\mathbf{k}(\Gamma/G)$, where G is the subgroup of Γ generated by $\mathcal{O}_g$.*

Proof. Let V be the space affording ρ, i.e. $\rho : \Gamma_g \to \operatorname{Aut}(V)$. Observe first that G is normal, because if $h \in \Gamma$ and $g_1, \dots, g_n \in \mathcal{O}_g$ then ${}^h(g_1 \cdots g_n) = {}^hg_1 \cdots {}^hg_n \in G$. Observe now that $\delta(R) \subseteq \mathbf{k}G \otimes R$ because $\mathcal{P}(R)$ generates R, and then R can be considered as a $\mathbf{k}G$-module and a $\mathbf{k}G$-comodule. Furthermore, it is immediate that R is a Hopf algebra in ${}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$. Consequently, there exists an inclusion

$$A = R\#\mathbf{k}G \hookrightarrow R\#H = B.$$

Moreover, this inclusion is normal: let $h \in G,\ x \in \Gamma,\ r \in R,\ s = {}^tv \in M(g,\rho) \subset R$, where $t \in E_g$ ($=$ left coclasses Γ/Γ_g). Then

$$\begin{aligned}
\Delta_B(s) &= \Delta_B(s\#1) = (\mathrm{id} \otimes c \otimes \mathrm{id})((1 \otimes s + s \otimes 1) \otimes (1 \otimes 1)) = {}^tg \otimes s + s \otimes 1,\\
\mathcal{S}(s) &= \mathcal{S}(s\#1) = -({}^tg)^{-1}s,\\
\Delta_B(x) &= \Delta_B(1\#x) = (1\#x) \otimes (1\#x) = x \otimes x,
\end{aligned}$$

and thus

$$\begin{aligned}
\mathrm{Ad}_s(h) &= s_{(1)}h\mathcal{S}(s_{(2)}) = -{}^tgh({}^tg)^{-1}s + sh = -{}^{({}^tg)}hs + sh \in R\#\mathbf{k}G,\\
\mathrm{Ad}_s(r) &= s_{(1)}r\mathcal{S}(s_{(2)}) = -{}^tgr({}^tg)^{-1}s + sr = -({}^tg \rightharpoonup r)s + sr \in R\#\mathbf{k}G,\\
\mathrm{Ad}_x(h) &= x_{(1)}h\mathcal{S}(x_{(2)}) = xhx^{-1} = {}^xh \in R\#\mathbf{k}G,\\
\mathrm{Ad}_x(r) &= x_{(1)}r\mathcal{S}(x_{(2)}) = xrx^{-1} = x \rightharpoonup r \in R\#\mathbf{k}G.
\end{aligned}$$

The condition that $\mathcal{P}(R)$ generates R guarantees that $\forall s \in R,\ \mathrm{Ad}_s(R\#\mathbf{k}G) \subseteq R\#\mathbf{k}G$. We have therefore a sequence of Hopf algebras

$$\mathbf{k} \longrightarrow R\#\mathbf{k}G \longrightarrow R\#H \longrightarrow \mathbf{k}(\Gamma/G) \longrightarrow \mathbf{k}.$$

It is straightforward to see that this sequence fulfills the conditions of [AD95, 1.2.3], and then the sequence is exact. $\square$

Remark 3.1.7. The space $\mathcal{P}(R)$ may be a simple object in ${}^H_H\mathcal{YD}$, but may be decomposable when considered as an object in ${}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$. For instance, when G is abelian we know from corollary 3.1.3 that $\mathcal{P}(R)$ decomposes as a sum of objects of dimension 1.

Definition 3.1.8. We say that R *can be obtained from the abelian case* if its bosonization is an extension

$$\mathbf{k} \longrightarrow R\#\mathbf{k}\Gamma_1 \longrightarrow R\#H \longrightarrow \mathbf{k}\Gamma_2 \longrightarrow \mathbf{k}$$

where Γ_1 is abelian.

Lemma 3.1.9. *Let R be an ET-algebra such that $\mathcal{P}(R) = M(g,\rho)$ for some g, and $\rho \in \widehat{\Gamma_g}$. Let G be the subgroup generated by $\mathcal{O}_g$. If $\Gamma_g \triangleleft \Gamma$ then G is abelian, and thus R can be obtained from the abelian case.*

Proof. Let $t \in \mathcal{O}_g$, $t = {}^x g$. Then we have $\Gamma_t = {}^x\Gamma_g = x\Gamma_g x^{-1} = \Gamma_g$, which implies that $\Gamma_{t_1} = \Gamma_{t_2}$ for any $t_1, t_2 \in \mathcal{O}_g$. Thus any two elements in $\mathcal{O}_g$ commute, and hence the group generated by $\mathcal{O}_g$ is abelian. □

Example 3.1.10. The preceding lemma has the following application: if all the isotropy subgroups of Γ are normal, then any ET-algebra with an irreducible space of primitive elements can be obtained from the abelian case. This happens, for instance, for $\mathbb{D}_4$. Other examples are the groups such that every subgroup is normal. It is known that these groups are abelian, or of the form $H \times A$, where H is the quaternion group, i.e.

$$H = \{1, -1, i, -i, j, -j, k, -k |\ i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j\},$$

and A is an abelian group without elements of order 4 (see [Car56]).

Proposition 3.1.11. *Let R be an ET-algebra such that $\dim \mathcal{P}(R) = 2$. Then R can be obtained from the abelian case.*

Proof. Let $M = \mathcal{P}(R)$. We have $\dim M = 2$ and then there are three possibilities:

1. M is decomposable as $M = M(g_1, \chi_1) \oplus M(g_2, \chi_2)$ with g_i central in Γ and χ_i characters.
2. M is simple, and then $M = M(g, \rho)$ with $\Gamma_g = \Gamma$ and $\dim \rho = 2$, or
3. $M = M(g, \chi)$ with $[\Gamma : \Gamma_g] = 2$ and χ a character.

In the first case, let G be the group generated by g_1 and g_2. It is abelian because the g_i are central. The construction of proposition 3.1.6 can be made with this G.

In cases 2 and 3 we have $\Gamma_g = \Gamma$ or $[\Gamma : \Gamma_g] = 2$, and the result follows from the lemma above. □

3.2. **Bialgebras of type one.**

As we said before, we are interested in certain classes of braided Hopf algebras, which we define in this section. Most of the following can be made in any braided category, as it is done by Schauenburg in [Sch96]. Given a braided category $\mathcal{C}$, he is obliged to work in an $\mathbb{N}$-graded category $\mathcal{C}^{\mathbb{N}}$. To avoid these technicalities we shall work in ${}^H_H\mathcal{YD}$ and ${}^H_H\mathcal{YD}_\infty$.

Definition 3.2.1. A graded Hopf algebra in ${}^H_H\mathcal{YD}$ or ${}^H_H\mathcal{YD}_\infty$ is simply a Hopf algebra R in any of these categories such that $R = \bigoplus_n R(n)$ and

$$R(i)R(j) \subseteq R(i+j), \quad \Delta(R(k)) \subseteq \bigoplus_{i+j=k} R(i) \otimes R(j).$$

An important application of the existence of an integral is the following

Proposition 3.2.2 (Nichols)**.** *Let $R = \oplus_{i=0}^N R(i)$ be a graded Hopf algebra in ${}^H_H\mathcal{YD}$ (it is in particular finite dimensional), and suppose that $R(N) \neq 0$. Then $\dim R(i) = \dim R(N-i)$ $\forall i$.*

Proof. Since R is graded, it is clear that R^* is also a graded Hopf algebra in ${}^H_H\mathcal{YD}$. Let $\lambda \in R^*$ be a non zero left integral. We have then $\lambda = \sum_i \lambda_i$, where $\lambda_i \in R^*(i)$ is the component of degree i. It is immediate, looking at (2.3.6), that each λ_i is a left integral in R^*. Then, by the one dimensionality of the space of integrals, we have $\lambda = \lambda_j$ for some j. We recall now (see (2.3.7)) that λ defines a non degenerate bilinear form

$$(x, y) = \langle \lambda, (x_{(-1)}y)\mathcal{S}(x_{(0)})\rangle.$$

Since $R(i)$ and $R(k)$ are orthogonal if $i+k=j$, this map induces a non degenerate bilinear form between $R(i)$ and $R(j-i)$ for each i. Hence in particular we have that $R(i)=0\ \forall i>j$, and then $j=N$, whence the thesis. □

Definition 3.2.3. A *braided Hopf algebra of type one*, or briefly *TOBA*, [2] in ${}_H^H\mathcal{YD}$ or in ${}_H^H\mathcal{YD}_\infty$ is a graded Hopf algebra in any of these categories such that

$$R(0)\simeq \mathbf{k}, \tag{3.2.4}$$

$$\left(\oplus_{i\geq 1}R(i)\right)^2=\oplus_{i\geq 2}R(i), \tag{3.2.5}$$

$$\mathcal{P}(R)=R(1). \tag{3.2.6}$$

Remark 3.2.7. A graded bialgebra in ${}_H^H\mathcal{YD}$ or ${}_H^H\mathcal{YD}_\infty$ which satisfies these conditions is automatically a Hopf algebra, thanks to [Mon93, Lemma 5.2.10].

It is easy to see that if R is a TOBA then the unit and counit are respectively the canonical inclusion and canonical projection

$$u:\mathbf{k}=R(0)\hookrightarrow R,\quad \varepsilon:R\twoheadrightarrow R(0)=\mathbf{k}.$$

It is easy to see that in presence of (3.2.4) the condition (3.2.6) is equivalent to the condition

$$R_1=R_0\wedge R_0=R(0)\oplus R(1),$$

where $\wedge$ stands for the wedge product and $R_0\subset R_1\subset\dots$ stands for the coradical filtration of R (see [Swe69, Ch. 9]). Moreover, it is easy to see by induction that the condition (3.2.5) is equivalent to

$$(R(1))^n=R(n)\quad \forall n\geq 1,$$

which in presence of (3.2.6) can be stated by saying that $\mathcal{P}(R)$ generates R.

Example 3.2.8. Let $R=\mathbf{k}[x]/x^{(p^2)}$, where $\operatorname{char}\mathbf{k}=p$. The comultiplication is determined by imposing x to be a primitive element. This is a (usual) graded Hopf algebra which verifies (3.2.4) and (3.2.5), but not (3.2.6). Its dual is a graded Hopf algebra which verifies (3.2.4) and (3.2.6) but not (3.2.5). Another example is the tensor algebra $T(V)$ of a vector space V of dimension greater than 1, with comultiplication determined by $V\subseteq\mathcal{P}(T(V))$. This Hopf algebra satisfies (3.2.4) and (3.2.5) but not (3.2.6). Indeed, $\mathcal{P}(T(V))$ is the free Lie algebra generated by V.

It is not known whether a finite dimensional graded (braided) Hopf algebra satisfying (3.2.4) and (3.2.6) should satisfy (3.2.5) provided that $\operatorname{char}\mathbf{k}=0$. This was proved in [AS] in the case $\dim\mathcal{P}(R)=1$.

We give three ways to construct a TOBA. The second one is due to Nichols (see [Nic78]), from where we borrow the name. The first one can be seen as a rewriting of that of Nichols in the language of braided categories, and is due to Schauenburg (see [Sch96], see also [Ros95, Ros92], [BD97]). The last one is inspired in the work of Lusztig [Lus93] and is stated for the category ${}_H^H\mathcal{YD}_\infty$, where $H=\mathbf{k}\Gamma$ (Lusztig's algebras $\mathbf{f}$ and $'\mathbf{f}$ are braided Hopf algebras in a category of comodules). The approach of [Sch96] is in fact motivated by this work. We prefer to give the way of [Sch96] first because it seems more useful to us, and then give that of [Nic78] in the terms of [Sch96]. It is important to note that we work in ${}_H^H\mathcal{YD}$ and ${}_H^H\mathcal{YD}_\infty$, rather than in an $\mathbb{N}$-graded category.

[2] The Tobas are an aboriginal ethnic group living in the north of Argentina.

Let $n \in \mathbb{N},\ n \geq 2$. Let $\mathbb{S}_n$ and $\mathbb{B}_n$ be the symmetric and braid groups defined in 1.1.13. There is a projection $\mathbb{B}_n \twoheadrightarrow \mathbb{S}_n,\ \sigma_i \mapsto \tau_i$.

Let $x \in \mathbb{S}_n$. We denote by $\ell(x)$ the length of a minimal word in the alphabet $\{\tau_i |\ 1 \leq i < n\}$ which represents x. For $y \in \mathbb{B}_n$ we denote also by $\ell(y)$ the length of a minimal word in the alphabet $\{\sigma_i,\ \sigma_i^{-1} |\ 1 \leq i < n\}$ which represents y. There is a unique section $s : \mathbb{S}_n \to \mathbb{B}_n$ to the projection $\mathbb{B}_n \to \mathbb{S}_n$ such that $s(\tau_i) = \sigma_i$ and $s(ww') = s(w)s(w')$ whenever $\ell(w \cdot w') = \ell(w) + \ell(w')$. It is given by

$$(w = \tau_{i_1} \cdots \tau_{i_j}) \mapsto (\sigma_{i_1} \cdots \sigma_{i_j}) \quad \text{if } \ell(w) = j \qquad (\text{thus } \ell s = \ell). \tag{3.2.9}$$

It is clear that it is unique; it is proved in [CR94, 64.20] that it is well defined. Using this section, we define the $\mathbf{S}$ morphisms: let V be an object in ${}^H_H\mathcal{YD}$. As in remark 1.1.14, $\mathbb{B}_n$ acts on $V^{\otimes n}$. For $w \in \mathbb{B}_n$ we denote also by w the corresponding morphism given by this action. If $X \subseteq \mathbb{S}_n$, we define the morphism

$$\mathbf{S}_X : V^{\otimes n} \to V^{\otimes n}, \quad \mathbf{S}_X = \sum_{x \in X} s(x).$$

Let $k_1, \dots, k_j \in \mathbb{N}$ such that $k_1 + \cdots + k_j = n$. We denote by $X_{k_1,\dots,k_j} \subseteq \mathbb{S}_n$ the $(k_1, \dots, k_j)$-shuffle and $Y_{k_1,\dots,k_j} \subseteq \mathbb{S}_n$ the inverse of $X_{k_1,\dots,k_j}$, i.e.

$$\begin{aligned} X_{k_1,\dots,k_j} &= \{x \in \mathbb{S}_n |\ x^{-1}(k_1 + \cdots + k_i + 1) < \cdots < x^{-1}(k_1 + \cdots + k_{i+1})\ \forall i = 0, \dots, j-1\} \\ Y_{k_1,\dots,k_j} &= X^{-1}_{k_1,\dots,k_j} = \{x^{-1} |\ x \in X_{k_1,\dots,k_j}\} \end{aligned}$$

We then define $\mathbf{S}_{k_1,\dots,k_j} = \mathbf{S}_{X_{k_1,\dots,k_j}}$, $\mathbf{S}^n = \mathbf{S}_{1,1,\dots,1} = \mathbf{S}_{\mathbb{S}_n}$, and $\mathbf{T}_{k_1,\dots,k_j} = \mathbf{S}_{Y_{k_1,\dots,k_j}}$.

Then, for instance, $\mathbf{S}^2 = \mathrm{id} + c$, and

$$\mathbf{S}_{2,1} = \mathrm{id}_{V^{\otimes 3}} + \mathrm{id}_V \otimes c_{V,V} + (\mathrm{id}_V \otimes c_{V,V})(c_{V,V} \otimes \mathrm{id}_V) : V^{\otimes 3} \to V^{\otimes 2} \otimes V.$$

We observe that for $i + j = n$, $\mathbf{S}^n = (\mathbf{S}^i \otimes \mathbf{S}^j)\mathbf{S}_{i,j}$.

Definition 3.2.10. Let V be an object of ${}^H_H\mathcal{YD}$. We denote by $T^n(V) = V^{\otimes n}$ and by $T(V)$ the tensor object

$$T(V) = \mathbf{k} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

$T(V)$ is not an object of ${}^H_H\mathcal{YD}$, but an object of ${}^H_H\mathcal{YD}_\infty$. We consider on $T(V)$ two different bialgebra structures, which we denote by $A(V)$ and $C(V)$. Both are graded bialgebras in the sense of 3.2.1, and we denote

$$m = \bigoplus_{i,j} m_{i,j}, \quad \Delta = \bigoplus_{i,j} \Delta_{i,j},$$

where $m_{i,j} : A^i(V) \otimes A^j(V) \to A^{i+j}(V)$ and $\Delta_{i,j} : A^{i+j}(V) \to A^i(V) \otimes A^j(V)$ and the same for $C(V)$. We take for both $A(V)$ and $C(V)$ the unit and counit given by inclusion $\mathbf{k} \to T(V)$ and projection $T(V) \to \mathbf{k}$. We take on $A(V)$ the multiplication given by

$$m_{i,j} = \mathrm{id} : A^i(V) \otimes A^j(V) \to A^{i+j}(V).$$

There exists only one comultiplication making $A(V)$ into a bialgebra in ${}^H_H\mathcal{YD}_\infty$ with

$$\Delta_{1,0} = \mathrm{id} : V \to V \otimes \mathbf{k}, \quad \Delta_{0,1} = \mathrm{id} : V \to \mathbf{k} \otimes V,$$

which is given by

$$\Delta_{i,j} = \mathbf{S}_{i,j} : A^{i+j}(V) \to A^i(V) \otimes A^j(V).$$

Dually, we take on $C(V)$ the comultiplication given by

$$\Delta_{i,j} = \mathrm{id} : C^{i+j}(V) \to C^i(V) \otimes C^j(V).$$

There exists only one multiplication making $C(V)$ into a bialgebra in ${}^H_H\mathcal{YD}_\infty$ with

$$m_{0,1} = \mathrm{id} : \mathbf{k} \otimes V \to V,\ m_{1,0} = \mathrm{id} : V \otimes \mathbf{k} \to V,$$

which is given by

$$m_{i,j} = \mathbf{T}_{i,j} : C^i(V) \otimes C^j(V) \to C^{i+j}(V).$$

There exists only one morphism of bialgebras $\mathbf{S} : A(V) \to C(V)$ such that $\mathbf{S}|_{A^1} = \mathrm{id} : V \to V$. This is the graded morphism given by

$$\mathbf{S} = \oplus_n \mathbf{S}^n : A^n(V) \to C^n(V).$$

Let $B(V) = \bigoplus_n B^n(V)$ be the image $\mathbf{S}(A(V)) \subset C(V)$. This is a bialgebra in ${}^H_H\mathcal{YD}_\infty$ and is isomorphic to

$$A(V)/\ker(\mathbf{S}) = \bigoplus_n (A^n(V)/\ker(\mathbf{S}^n)).$$

We have then a graded bialgebra $B(V)$ in ${}^H_H\mathcal{YD}_\infty$ which, by construction, verifies (3.2.4) and (3.2.5). It also verifies (3.2.6) since, as a subbialgebra of $C(V)$, the comultiplication components $\Delta_{i,j} : B^{i+j} \to B^i \otimes B^j$ are injective for all $i, j \in \mathbb{N}$. Hence we have

Definition 3.2.11. Let $V \in {}^H_H\mathcal{YD}$, $\mathfrak{t}(V) := \bigoplus_n (A^n(V)/\ker \mathbf{S}^n) \subseteq C(V)$. It is a TOBA with $\mathcal{P}(\mathfrak{t}(V)) \simeq V$. As we noted in 3.2.7 it has an antipode. It is given by $\mathcal{S}(x) = -x \quad \forall x \in R(1)$ and it is extended by 1.2.2. The following proposition proves that a TOBA is fully determined by its space of primitive elements, and thus $\mathfrak{t}(V)$ can be defined alternatively by conditions (3.2.4)–(3.2.6) plus $\mathfrak{t}(V)(1) = V$.

Proposition 3.2.12. *Let R be a TOBA. Let $V = R(1)$, $p : A(V) \to R$ be the algebra surjection induced by the inclusion $V \hookrightarrow R$, and I be its kernel. Then $I = \ker(\mathbf{S})$.*

Proof. We prove first that $I \supseteq \ker(\mathbf{S})$. Since both I and $\ker(\mathbf{S})$ are homogeneous, we have to prove that $I_n \supseteq \ker(\mathbf{S}^n)$, where I_n is the homogeneous component of I of degree n. We proceed by induction.

For $n = 1$ there is nothing to prove, since $\mathbf{S}^1 = \mathrm{id}$. Let $p : A(V) \to R$ be the projection, and suppose that the inclusion is true for $m < n$. Let $x \in \ker(\mathbf{S}^n)$. We have that

$$\Delta(x) = \sum_{k+l=n} \mathbf{S}_{k,l} x \in \sum_{k+l=n} A^k \otimes A^l,$$

but $\mathbf{S}^n = (\mathbf{S}^k \otimes \mathbf{S}^l)\mathbf{S}_{k,l}$, and hence

$$\mathbf{S}_{k,l}(x) \in \ker(\mathbf{S}^k \otimes \mathbf{S}^l) = \ker \mathbf{S}^k \otimes V^{\otimes l} + V^{\otimes k} \otimes \ker \mathbf{S}^l,$$

whence $\mathbf{S}_{k,l}(x) \in I \otimes A + A \otimes I$ if $k, l < n$. Then $(p \otimes p)(\mathbf{S}_{k,l}(x)) = 0$ if $k, l < n$, which implies that

$$\Delta(p(x)) = \sum_{k+l=n} (p \otimes p)\mathbf{S}_{k,l}(x) = (p \otimes p)(\mathbf{S}_{n,0}(x) + \mathbf{S}_{0,n}(x)) = p(x) \otimes 1 + 1 \otimes p(x).$$

Thus $p(x) \in \mathcal{P}(R)$, but $p(x) \in R(n)$ and $n > 1$, whence $p(x) = 0$ and then $x \in I$. This proves the first inclusion.

We have now the quotient morphism of coalgebras (of braided Hopf algebras, in fact)

$$A(V)/\ker \mathbf{S} \to R,$$

which is injective on $(A(V)/\ker(\mathbf{S}))_1$ (the second term of the coradical filtration), since

$$(A(V)/\ker(\mathbf{S}))_1 = \mathbf{k} \oplus V = R(0) \oplus R(1).$$

By [Mon93, 5.3.1], the quotient morphism is injective, which says that $\ker(\mathbf{S}) = I$. □

Remark 3.2.13. We note that $R = \mathfrak{t}(V)$ depends as a braided Hopf algebra only on the braiding $c_{V,V} \in \mathrm{End}(V \otimes V)$. This allows to consider R in different categories, as long as $c_{V,V}$ remains unchanged.

We give now a second construction of a TOBA. See [Nic78] for details.

Definition 3.2.14 (Nichols)**.** A graded bialgebra $B = \bigoplus_{i\geq 0} B(i)$ is called *a bialgebra of type one* if it verifies the following conditions:

$$(\oplus_{i\geq 1} B(i))^2 = \oplus_{i\geq 2} B(i), \tag{3.2.15}$$

$$B(0) \wedge B(0) = B(0) \oplus B(1). \tag{3.2.16}$$

We define similarly the notion of *Hopf algebra of type one.*

Remark 3.2.17. Let $B = \bigoplus_{i\geq 0} B(i)$ be a bialgebra. As in the braided case, it follows from [Mon93, Lemma 5.2.10] that B has an antipode if and only if $B(0)$ does.

Nichols constructs bialgebras of type one starting out from Hopf bimodules. We relate his construction to that of Schauenburg. In order to do this we need the following

Lemma 3.2.18. *Let H be a Hopf algebra, $R = \bigoplus_{n\geq 0} R(n)$ be a graded Hopf algebra in ${}^H_H\mathcal{YD}$, and $A = R\#H$. This is a graded Hopf algebra with respect to the grading*

$$A = \bigoplus_{n\geq 0} A(n), \ \ A(n) = (R(n) \otimes H) \subseteq R\#H.$$

If R is a TOBA then A is a bialgebra of type one such that $A(0) \simeq H$. Conversely, let $B = \bigoplus_{n\geq 0} B(n)$ be a graded Hopf algebra. We have the canonical morphisms of Hopf algebras $B(0) \hookrightarrow B$ and $B \twoheadrightarrow B(0)$. Let $R = B^{\mathrm{co}\pi}$; it is a graded Hopf algebra in ${}^{B(0)}_{B(0)}\mathcal{YD}$. Hence, if B is a Hopf algebra of type one then R is a TOBA.

Proof. Condition (3.2.4) is easily seen to be equivalent to the condition $A(0) = H$. Then the equivalence between (3.2.5) and (3.2.15) is a consequence of the following: let M and N be subspaces of R. We claim that if N is an H-submodule then $(MN)\#H = (M\#H)(N\#H)$. For this, let $m \in M$, $n \in N$, $h \in H$. Then $(mn\#h) = (m\#1)(n\#h) \in (M\#H)(N\#H)$, which implies one inclusion. The other is immediate under the hypothesis of N being a submodule.

As we remarked after the definition 3.2.3, it is easy to see that (3.2.6) is equivalent to the condition $\mathbf{k} \wedge \mathbf{k} = R(0) \oplus R(1)$. The equivalence between conditions (3.2.6) and (3.2.16) is a consequence of the following: let M and N be subspaces of R. We claim that if N is an H-subcomodule then $(M\#H) \wedge (N\#H) = (M \wedge N)\#H$. To see this, we consider both subspaces as kernels of certain morphism: let us denote by τ the usual flip $x \otimes y \mapsto y \otimes x$. If X is a subspace of Y, we denote by $p_X : Y \to Y/X$ the canonical projection. Since N is a subcomodule, R/N has an H-comodule structure, which we denote also by δ. Thus,

$$\begin{aligned}
(M\wedge N)\#H &= \ker((p_M\otimes p_N)\Delta_R)\#H \\
&= \ker((p_M\otimes p_N\otimes \mathrm{id})(\Delta_R\otimes \mathrm{id})) \\
&= \ker((p_M\otimes \mathrm{id}\otimes p_N\otimes \mathrm{id})(\mathrm{id}\otimes\tau\otimes \mathrm{id})(\Delta_R\otimes\Delta_H)) \\
&= \ker((\mathrm{id}\otimes m_H\otimes \mathrm{id}\otimes \mathrm{id})(\mathrm{id}\otimes\tau\otimes \mathrm{id}\otimes \mathrm{id})(\mathrm{id}\otimes \mathrm{id}\otimes\delta\otimes \mathrm{id})(p_M\otimes \mathrm{id}\otimes p_N\otimes \mathrm{id})(\mathrm{id}\otimes\tau\otimes \mathrm{id})(\Delta_R\otimes\Delta_H)) \\
&= \ker((p_M\otimes m_H\otimes p_N\otimes \mathrm{id})(\mathrm{id}\otimes \mathrm{id}\otimes\tau\otimes \mathrm{id})(\mathrm{id}\otimes\delta\otimes \mathrm{id}\otimes \mathrm{id})(\Delta_R\otimes\Delta_H)) \\
&= \ker((p_M\otimes \mathrm{id}\otimes p_N\otimes \mathrm{id})(\mathrm{id}\otimes c\otimes \mathrm{id})(\Delta_R\otimes\Delta_H)) \\
&= \ker((p_{(M\#H)}\otimes p_{(N\#H)})\Delta_{R\#H}) \\
&= (M\#H)\wedge(N\#H).
\end{aligned}$$

□

Let $(P,\delta_P: P\to P\otimes H)$ be a right H-comodule, and $(Q,\delta_Q: Q\to H\otimes Q)$ be a left H-comodule. We denote as usual by $\Box_H$ the cotensor product, i.e.

$$P\Box_H Q = \mathrm{Eq}(P\otimes Q \underset{\mathrm{id}\otimes\delta_Q}{\overset{\delta_P\otimes\mathrm{id}}{\rightrightarrows}} P\otimes H\otimes Q) = \{\sum p_i\otimes q_i | \sum (p_i)_{(0)}\otimes(p_i)_{(1)}\otimes q_i = p_i\otimes (q_i)_{(-1)}\otimes(q_i)_{(0)}\}.$$

Let H be a Hopf algebra and $M\in {}^H_H\mathcal{M}^H_H$ (see 1.1.17). We denote by

$$\begin{aligned}
A_H(M) &= T_H(M) = H\oplus M\oplus (M\otimes_H M)\oplus(M\otimes_H M\otimes_H M)\oplus\cdots = \bigoplus_{i\geq 0} A^i_H(M), \\
C_H(M) &= T_H(M) = H\oplus M\oplus (M\Box_H M)\oplus(M\Box_H M\Box_H M)\oplus\cdots = \bigoplus_{i\geq 0} C^i_H(M).
\end{aligned}$$

As before, $A_H(M)$ (resp. $C_H(M)$) has a canonical graded multiplication (resp. comultiplication) given by projection

$$A^i_H(M)\otimes A^j_H(M)\to A^{i+j}_H(M) = A^i_H(M)\otimes_H A^j_H(M)$$

(resp. inclusion $C^{i+j}_H = C^i_H\Box_H C^j_H \to C^i_H\otimes C^j_H$). Moreover, $A_H(M)$ can be endowed with a (unique) comultiplication which makes it into a bialgebra such that in degree 1 it is given by

$$A^1_H(M) = M \xrightarrow{\delta_l+\delta_r} (H\otimes M)\oplus(M\otimes H) = [A_H(M)\otimes A_H(M)](1),$$

and $C_H(M)$ can be endowed with a (unique) multiplication which makes it into a bialgebra such that in degree 1 it is given by

$$[C_H(M)\otimes C_H(M)](1) = (H\otimes M)\oplus(M\otimes H)\xrightarrow{m_l+m_r} M = C^1_H(M).$$

There exists a unique bialgebra map $A_H(M)\to C_H(M)$ which is the identity on degrees 0 and 1. We denote by $B_H(M)$ its image. This is a bialgebra of type one. Moreover, since H is a Hopf algebra, $B_H(M)$ is a Hopf algebra. This construction is related to that of Schauenburg by the following diagram. For $\mathcal{C}$ a braided category, we denote by $\mathcal{H}f(\mathcal{C})$ the subcategory of the Hopf algebras in $\mathcal{C}$, by $\#$ the bosonization functor and by S the functor ${}^H_H\mathcal{YD}\to {}^H_H\mathcal{M}^H_H$ of proposition 1.1.17. In this diagram we denote also by B (instead of $\mathfrak{t}$) the functor giving the TOBA in ${}^H_H\mathcal{YD}$.

Then we have a commutative diagram

$$\begin{array}{ccc} {}^H_H\mathcal{YD} & \xrightarrow{B} & \mathcal{H}f({}^H_H\mathcal{YD}) \\ {\scriptstyle S}\downarrow & & {\scriptstyle \#}\downarrow \\ {}^H_H\mathcal{M}^H_H & \xrightarrow{B} & \mathcal{H}f({}_{\mathbf{k}}\mathcal{M}). \end{array}$$

The proof that this diagram commutes is straightforward but tedious. One can verify that the diagram commutes replacing B with A and C, the tensor and cotensor bialgebras, and then note that the following diagram commutes $\forall V \in {}^H_H\mathcal{YD}$:

$$\begin{array}{ccc} (AV)\#H & \longrightarrow & A(SV) \\ \downarrow & & \downarrow \\ (CV)\#H & \longrightarrow & C(SV), \end{array}$$

where the left and right sides are the (universal) morphisms $A \to C$, and the top and bottom sides are the natural equivalences given by the commutativity of the first diagram with B replaced by A and C respectively.

Remark 3.2.19. Let V be a $\mathbf{k}$-vector space and $c \in \operatorname{Aut}(V \otimes V)$, satisfying the braid equation, namely

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

We remark that we can define $\mathfrak{t}(V) = AV/\ker(\mathbf{S})$ in the same vein as before, where $\mathbf{S}(v) = \sum_{x\in\mathbb{S}_n} s(x)(v)$ for $v \in A^n(V)$, and the group $\mathbb{B}_n$ acting on $V^{\otimes n}$ via c.

The last way we give to construct a TOBA is by means of a bilinear form on $A(M)$. The idea is the same Lusztig uses to construct the algebra $\mathbf{f}$, and is in fact the motivation for Schauenburg to construct the morphism $\mathbf{S}$ (see [Lus93], [Sch96]). Müller uses this presentation to prove that the nilpotent part $\mathfrak{n}_+$ of the Frobenius-Lusztig kernel $\mathfrak{u}$ is a TOBA over $\mathfrak{u}_0$ (see [Mül98]). In our context the form is not a pairing between $A(V)$ and itself, but between $A(V)$ and $A(W)$, W being another (possibly the same) vector space. We begin with a useful result.

Lemma 3.2.20. *Let U, Z be $\mathbf{k}$-vector spaces with an action of $\mathbb{B}_n$ (in the usual cases $U = V^{\otimes n}$, $Z = W^{\otimes n}$). We denote for $u \in U$*

$$\mathbf{S}^n u = \sum_{x\in\mathbb{S}_n} s(x)(u),$$

and analogously for $z \in Z$. Suppose we have a bilinear form $(|) : U \otimes Z \to \mathbf{k}$ such that either
(a). $(\sigma_i(u)|z) = (u|\sigma_i(z))$ *or*
(b). $(\sigma_i(u)|z) = (u|\sigma_{n-i}(z))$.
Then we have $(\mathbf{S}^n u|z) = (u|\mathbf{S}^n z)$ for $u \in U$, $z \in Z$.

Proof. Let $x \in \mathbb{S}_n$, $x = \tau_{i_1} \cdots \tau_{i_d}$ with $\ell(x) = d$. Then $s(x) = \sigma_{i_1} \cdots \sigma_{i_d}$ and $s(x^{-1}) = \sigma_{i_d} \cdots \sigma_{i_1}$, since $x^{-1} = \tau_{i_d} \cdots \tau_{i_1}$ and $\ell(x^{-1}) = \ell(x) = d$. Furthermore, let $T \in \mathbb{S}_n$ be defined by $T : \{1, \dots, n\} \to \{1, \dots, n\}$, $T(i) = n + 1 - i$. Let D be the inner automorphism defined by T, that is, $D : \mathbb{S}_n \to \mathbb{S}_n$, $D(x) = TxT^{-1}$. We observe that $D(\tau_i) = \tau_{n-i}$ for $1 \le i \le n - 1$. Moreover, since $T^2 = \mathrm{id}$, we have $D^2 = \mathrm{id}$. Thus, if $x \in \mathbb{S}_n$, $x = \tau_{i_1} \cdots \tau_{i_d}$ and $\ell(x) = d$, we have $D(x^{-1}) = D(\tau_{i_d} \cdots \tau_{i_1}) = \tau_{n-i_d} \cdots \tau_{n-i_1}$, whence $\ell(D(x^{-1})) \leq \ell(x)$. Since $D((D(x^{-1}))^{-1}) = (D^2(x^{-1}))^{-1} = x$ and the previous inequality holds true $\forall x \in \mathbb{S}_n$, we have $\ell(D(x^{-1})) = \ell(x)$. Thus,

$$s(D(x^{-1})) = s(\tau_{n-i_d} \cdots \tau_{n-i_1}) = \sigma_{n-i_d} \cdots \sigma_{n-i_1}.$$

We have now in case (a) that

$$(s(x)(u)|z) = (\sigma_{i_1} \cdots \sigma_{i_d}(u)|z) = (u|\sigma_{i_d} \cdots \sigma_{i_1}(z)) = (u|s(x^{-1})(z)),$$

and hence

$$(\mathbf{S}^n u|z) = \sum_{x \in \mathbb{S}_n} (s(x)u|z) = \sum_{x \in \mathbb{S}_n} (u|s(x^{-1})z) = (u|\mathbf{S}^n z).$$

In case (b), we have

$$(s(x)(u)|z) = (\sigma_{i_1} \cdots \sigma_{i_d}(u)|z) = (u|\sigma_{n-i_d} \cdots \sigma_{n-i_1}(z)) = (u|s(D(x^{-1}))(z)),$$

and hence

$$(\mathbf{S}^n u|z) = \sum_{x \in \mathbb{S}_n} (s(x)u|z) = \sum_{x \in \mathbb{S}_n} (u|s(D(x^{-1}))z) = (u|\mathbf{S}^n z).$$

□

Remark 3.2.21. Bilinear forms as in 3.2.20 happen to exist very often. Suppose for instance that we have a bilinear form $(|) : V \otimes W \to \mathbf{k}$ such that one of the following cases arises:

(a). $(c_V(v_1 \otimes v_2)|w_1 \otimes w_2) = (v_1 \otimes v_2|c_W(w_1 \otimes w_2))$ for the form $(v_1 \otimes v_2|w_1 \otimes w_2) = (v_1|w_1)(v_2|w_2)$.
(b). $(c_V(v_1 \otimes v_2)|w_1 \otimes w_2) = (v_1 \otimes v_2|c_W(w_1 \otimes w_2))$ for the form $(v_1 \otimes v_2|w_1 \otimes w_2) = (v_1|w_2)(v_2|w_1)$.

In case (a) we define

$$(v_1 \otimes \ldots \otimes v_n|w_1 \otimes \ldots \otimes w_n)_> = \prod_i (v_i|w_i),$$

and this fits into case (a) of 3.2.20.

In case (b) we define

$$(v_1 \otimes \ldots \otimes v_n|w_1 \otimes \ldots \otimes w_n)_< = \prod_i (v_i|w_{n+1-i}),$$

and this fits into case (b) of 3.2.20.

It is clear that if $(|)$ is non degenerate, then $(|)_>$ (resp. $(|)_<$) is non degenerate. These cases are satisfied in the following examples.

Example 3.2.22. Let $(i \cdot j)_{1 \leq i,j \leq n}$ be a Cartan datum (see [Lus93] for the definition), and let q be an indeterminate over $\mathbf{k}$. We take $V = \mathbf{k}(q)\theta_1 \oplus \ldots \oplus \mathbf{k}(q)\theta_n$, and define $c_V(\theta_i \otimes \theta_j) = q^{i \cdot j}\theta_j \otimes \theta_i$. Furthermore, we take $(|) : V \otimes V \to \mathbf{k}(q)$ given by

$$(\theta_i|\theta_j) = (1 - q^{-2i \cdot i})^{-1}\delta_{i,j}.$$

It is easy to see that this is a non degenerate bilinear form such that

$$(c(\theta_{i_1} \otimes \theta_{i_2})|\theta_{j_1} \otimes \theta_{j_2}) = (\theta_{i_1} \otimes \theta_{i_2}|c(\theta_{j_1} \otimes \theta_{j_2})),$$

whence we are in case (a) of the above remark.

Example 3.2.23. Let $\mathcal{C}$ be a braided abelian rigid category which can be embedded in $\mathcal{C}'$, a braided abelian category in which countable direct sums exist. This is the case for instance of ${}^H_H\mathcal{YD} \hookrightarrow {}^H_H\mathcal{YD}_\infty$ for H any Hopf algebra. Let ${}^*V$ be the left dual of V in $\mathcal{C}$, and $(|) : V \otimes {}^*V \to \mathbf{k}$ be the evaluation map. Lemma 2.1.5 tells that this fits into case (b) of the remark.

Definition 3.2.24. Let U, Z be $\mathbf{k}\mathbb{B}_n$-modules with a bilinear form $(|) : U \otimes Z \to \mathbf{k}$. We denote

$$[,] : U \otimes Z \to \mathbf{k}, \quad [u, z] = (\mathbf{S}^n u|z).$$

According to this, for V, W $\mathbf{k}$-vector spaces with braidings c_V, c_W and a bilinear form $(|) : V \otimes W \to \mathbf{k}$ satisfying (a) of remark 3.2.21 (resp. (b)), we define $[,] : AV \otimes AW \to \mathbf{k}$ by

1. $[1, 1] = 1$.
2. $[u, z] = 0$ if $u \in A^i V$, $z \in A^j W$ and $i \neq j$.
3. $\begin{cases} [u, z] = [u, z]_> = (\mathbf{S}^n u|z)_> & \text{if } u \in A^n V,\ z \in A^n W \text{ for the case (a)} \\ [v, w] = [v, w]_< = (\mathbf{S}^n v|w)_< & \text{if } u \in A^n V,\ z \in A^n W \text{ for the case (b)} \end{cases}$

Lemma 3.2.25. *Let V, W be as above. Let us suppose that we are in case (a) (resp. (b)) of remark 3.2.21. Then we have respectively*

$$\text{(a) } [u, z \cdot z']_> = [u_{(1)}, z]_>[u_{(2)}, z']_>, \quad [u \cdot u', z]_> = [u, z_{(1)}]_>[u', z_{(2)}]_>.$$
$$\text{(b) } [u, z \cdot z']_< = [u_{(1)}, z']_<[u_{(2)}, z]_<, \quad [u \cdot u', z]_< = [u, z_{(2)}]_<[u', z_{(1)}]_<.$$

Proof. For $u \in A^n V$ and $i + j = n$, we denote

$$(\mathbf{S}_{i,j}(u))_i \otimes (\mathbf{S}_{i,j}(u))_j = \mathbf{S}_{i,j}(u) \in A^i V \otimes A^j V.$$

In case (a) we have, for $z \in A^i W$, $z' \in A^j W$ and $u \in A^n V$,

$$\begin{aligned}[u, z \cdot z'] &= (\mathbf{S}^n u|z \cdot z') = ((\mathbf{S}^i \otimes \mathbf{S}^j)(\mathbf{S}_{i,j}u)|z \cdot z') \\ &= (\mathbf{S}^i(\mathbf{S}_{i,j}u)_i|z)(\mathbf{S}^j(\mathbf{S}_{i,j}u)_j|z').\end{aligned}$$

From the other hand, we have

$$\begin{aligned}[u_{(1)}, z][u_{(2)}, z'] &= \sum_{k+l=n} [(\mathbf{S}_{k,l}u)_k, z][(\mathbf{S}_{k,l}u)_l, z'] \\ &= [(\mathbf{S}_{i,j}u)_i, z][(\mathbf{S}_{i,j}u)_j, z'] \\ &= (\mathbf{S}^i(\mathbf{S}_{i,j}u)_i|z)(\mathbf{S}^j(\mathbf{S}_{i,j}u)_j|z').\end{aligned}$$

The other equality for case (a) is analogous, using lemma 3.2.20. The same proof aplies to the case (b), but replacing $(\mathbf{S}_{i,j}(u))_i \otimes (\mathbf{S}_{i,j}(u))_j$ by $(\mathbf{S}_{i,j}(u))_j \otimes (\mathbf{S}_{i,j}(u))_i$. □

We are in position now to give the last construction of a TOBA.

Definition 3.2.26. Let V, W be $\mathbf{k}$-vector spaces with braidings c_V, c_W and let $(|) : V \otimes W \to \mathbf{k}$ be a non degenerate bilinear form satisfying (a) (resp. (b)) of remark 3.2.21. We take $[,] : A(V) \otimes A(W) \to \mathbf{k}$ as in definition 3.2.24. Let $\mathcal{I} = \{v \in AV \mid [v, w] = 0\ \forall w \in AW\}$, $\mathcal{I}' = \{w \in AW \mid [v, w] = 0\ \forall v \in AV\}$ be the radicals of the form $[,]$. Since $(|)$ is non degenerate, it is clear that $\mathcal{I} = \oplus_{n \geq 0} \mathcal{I}_n = \oplus_{n \geq 0} \ker(\mathbf{S}^n : A^n V \to A^n V)$, and $\mathcal{I}' = \oplus_{n \geq 0} \mathcal{I}'_n = \oplus_{n \geq 0} \ker(\mathbf{S}^n : A^n W \to A^n W)$. Hence, another way to define $\mathfrak{t}(V)$ is to take $A(V)$ and divide out by the left radical of the form $[,]$ (resp. for $\mathfrak{t}(W)$, we take $A(W)$ and divide out by the right radical).

Remark 3.2.27. For the definition of the TOBA it is necessary for $(|)$ to be non degenerate, though it is not necessary for the definition of the form $[,]$ in definition 3.2.24.

Remark 3.2.28. In the case of example 3.2.22 we get the algebra $\mathbf{f} = \mathfrak{t}(\mathbf{k}(q)\theta_1 \oplus \ldots \oplus \mathbf{k}(q)\theta_n)$. For $V \in {}^H_H\mathcal{YD}$, $W = {}^*V \in {}^H_H\mathcal{YD}$, we get the same object $\mathfrak{t}(V)$ as before, and hence this is really an alternative form for the construction.

Lemma 3.2.25 says that the left and right radicals of $[,]$ are Hopf ideals, and hence the relations between AV and AW give similar relations between $\mathfrak{t}(V)$ and $\mathfrak{t}(W)$.

Theorem 3.2.29. *Let V, W be as in lemma 3.2.25. There is a unique non degenerate bilinear form $\mathfrak{t}(V) \otimes \mathfrak{t}(W) \to \mathbf{k}$ such that*

1. $[1,1] = 1$,
2. $[\mathfrak{t}^i(V), \mathfrak{t}^j(W)] = 0$ *if* $i \neq j$,
3. $[v,w] = (v|w)$ *if* $v \in \mathfrak{t}^1(V)$, $w \in \mathfrak{t}^1(W)$,
4. $[u, z\cdot z'] = \begin{cases} [u_{(1)}, z][u_{(2)}, z'] & \text{in case (a)}, \\ [u_{(1)}, z'][u_{(2)}, z] & \text{in case (b)}, \end{cases}$
5. $[u\cdot u', z] = \begin{cases} [u, z_{(1)}][u', z_{(2)}] & \text{in case (a)}, \\ [u, z_{(2)}][u', z_{(1)}] & \text{in case (b)}, \end{cases}$

where we denote $\mathfrak{t}^n(V) = A^nV/\mathcal{I}_n$ the component in degree n.

Proof. As in definition 3.2.24, we define the form $[,] = [,]_>$ in case (a), and $[,] = [,]_<$ in (b). Lemma 3.2.25 allows to consider $[,] : \mathfrak{t}(V) \otimes \mathfrak{t}(W) \to \mathbf{k}$ induced from $[,] : AV \otimes AW \to \mathbf{k}$, which turns out to be a non degenerate bilinear form satisfying 1–5. The uniqueness follows easily by induction. □

When V, W are objects in $\mathcal{C}$, a braided abelian category, and the pairing $[,] : AV \otimes AW \to \mathbf{k}$ is a morphism in $\mathcal{C}$, we have a relation between $\mathfrak{t}(W)$ and ${}^*\mathfrak{t}(V)$, provided this latter object exists in $\mathcal{C}$. We close the section giving the explicit relation for V, ${}^*V$ in ${}^H_H\mathcal{YD}$:

Proposition 3.2.30. *Let V be an object in ${}^H_H\mathcal{YD}$. If $\mathfrak{t}(V)$ is finite dimensional, then $\mathfrak{t}({}^*V) \simeq ({}^*\mathfrak{t}(V))^{\mathrm{bop}}$.*

Proof. First, $\mathfrak{t}({}^*V)$ can be identified, via $[,]$, to ${}^*\mathfrak{t}(V)$ as a vector space. This identification is furthermore an isomorphism in ${}^H_H\mathcal{YD}$, since $[,]$ is a morphism in ${}^H_H\mathcal{YD}$. We have to check the relation between $[,]$ and the products and coproducts in $({}^*\mathfrak{t}(V))^{\mathrm{bop}}$, $\mathfrak{t}(V)$ and $\mathfrak{t}({}^*V)$. We do it for the multiplication in $\mathfrak{t}({}^*V)$, the other one being analogous.

Let $\{{}_\alpha r\}_\alpha$, $\{{}^\alpha r\}_\alpha$ be dual bases of $\mathfrak{t}(V)$. We have for $u \in \mathfrak{t}(V)$, $f, g \in ({}^*\mathfrak{t}(V))^{\mathrm{bop}}$, we have

$$\begin{aligned}
\langle u, & m_{\mathrm{bop}}(f \otimes g)\rangle \\
&= \langle u, (f_{(-1)}g)f_{(0)}\rangle \\
&= \langle u_{(1)}, u_{(2)(-1)}f_{(-1)}g\rangle\langle u_{(2)(0)}, f_{(0)}\rangle \\
&= \langle u_{(1)}, u_{(2)(-1)}\mathcal{S}({}_\alpha r_{(-1)})g\rangle \\
&\qquad \langle u_{(2)(0)}, {}^\alpha r\rangle\langle {}_\alpha r_{(0)}, f\rangle \\
&= \langle u_{(1)}, u_{(2)(-2)}\mathcal{S}(u_{(2)(-1)})g\rangle\langle u_{(2)(0)}, f\rangle \\
&= \langle u_{(1)}, g\rangle\langle u_{(2)}, f\rangle,
\end{aligned}$$

but this is the same equality for $\mathfrak{t}({}^*V)$. □

3.3. Concrete examples.

We present now two families of braided Hopf algebras discovered by Milinski and Schneider. Both families are particular cases of Hopf algebras in Yetter–Drinfeld categories over group algebras of Coxeter groups (see [MS97]) and have the form $A(V)/I$ for certain V and $I \subset \ker(\mathbf{S})$. It is not known in general whether or not $I = \ker(\mathbf{S})$ (that is, whether or not they are TOBAs). Most of the results in this section are taken from to [MS97], an exception is Proposition 3.3.9.

Example 3.3.1. Let $n \in \mathbb{N}$, and $H = \mathbf{k}\mathbb{S}_n$. We take V the $\mathbf{k}$-vector space with basis consisting of elements y_τ where τ runs over all (not only elementary) transpositions $\tau = (i,j),\ i \neq j$. We make V an object of ${}_H^H\mathcal{YD}$ taking

$$\begin{aligned}\delta(y_\tau) &= \tau \otimes y_\tau,\\ \sigma \rightharpoonup y_\tau &= \mathrm{sg}(\sigma) y_{\sigma\tau\sigma^{-1}}.\end{aligned}$$

The module V is nothing but $M(g,\rho)$ with g any transposition and ρ the restriction of the sign representation to the isotropy subgroup of g.

Let now J be the subspace of $A^2(V)$ generated by the elements

$$y_\tau^2 \qquad \forall \tau, \tag{3.3.2}$$

$$y_\tau y_{\tau'} + y_{\tau'} y_\tau \qquad \text{if } \tau\tau' = \tau'\tau, \tag{3.3.3}$$

$$y_\tau y_{\tau'} + y_{\tau'} y_{\tau''} + y_{\tau''} y_\tau \qquad \text{if } \tau\tau' = \tau''\tau. \tag{3.3.4}$$

Then $J = \ker(\mathbf{S}^2)$. Let I be the ideal generated by J. Since J is an H-submodule and an H-subcomodule, the same is true for I. Since J is a coideal, the same is true for I. Then $R_n^1 := A(V)/I$ is a Hopf algebra in ${}_H^H\mathcal{YD}$.

Example 3.3.5. Let p be an odd prime number. We take now $H = \mathbf{k}\mathbb{D}_p$, where $\mathbb{D}_p$ is the dihedral group, i.e. the group generated by ρ and σ, with relations

$$\rho^p = \sigma^2 = 1, \quad \sigma\rho = \rho^{p-1}\sigma.$$

The conjugacy class of σ is $\mathcal{O}_\sigma = \{\sigma, \rho\sigma, \dots, \rho^{p-1}\sigma\}$, and the isotropy subgroup is $(\mathbb{D}_p)_\sigma = \{1, \sigma\}$. We take now $\chi : (\mathbb{D}_p)_\sigma \to \mathbf{k}^\times$, $\chi(\sigma) = -1$, and then define $V = M(\sigma, \chi)$ as in 3.1. Let V_0 be the space affording χ. We denote y_0 a generator of V_0. We put $y_i = \rho^i \rightharpoonup y_0 \in M(\sigma,\chi)$. We take the subindices of the y_i to be on $\mathbb{Z}/p$, thus $y_{i+p} = y_i$. We then have

$$\begin{aligned}&\delta(y_i) = {}^{\rho^i}\sigma \otimes y_i = \rho^{2i}\sigma \otimes y_i,\\ &\rho^j \rightharpoonup y_i = y_{i+j}, \qquad \sigma \rightharpoonup y_i = -y_{-i},\\ &c_{V,V}(y_i \otimes y_j) = -y_{2i-j} \otimes y_i.\end{aligned}$$

To compute $\ker \mathbf{S}^2$ it is convenient to take a different basis in $V \otimes V$. Let ξ be a primitive p-root of unit (we may suppose $\mathbf{k}$ has a primitive p-root of unit, for, if not, we can take a suitable extension of $\mathbf{k}$). Let

$$w_k^r = \sum_{i=0}^{p-1} \xi^{ri} y_i \otimes y_{i+k}, \qquad 0 \le r, k < p.$$

Then the w_k^r form a basis of $V^{\otimes 2}$, and $c_{V,V}(w_k^r) = -\xi^{rk} w_k^r$, whence

$$\ker(\mathbf{S}^2) = \ker(\mathrm{id} + c) = \langle w_k^r,\ rk = 0\rangle \qquad (\text{and then } \dim\ker(\mathbf{S}^2) = 2p-1). \tag{3.3.6}$$

It is easy to see that $\langle w_0^0, w_0^1, \dots , w_0^{p-1}\rangle = \langle (y_0\otimes y_0),\ (y_1\otimes y_1), \dots ,\ (y_{p-1}\otimes y_{p-1})\rangle$. We define then R_p^2 to be $A(V)/I$, where I is generated by

$$\begin{array}{rl} & y_i\otimes y_i, \qquad 0 \le i < p, \\ w_1^0 = & y_0\otimes y_1 + y_1\otimes y_2 + \cdots + y_{p-1}\otimes y_0, \\ w_2^0 = & y_0\otimes y_2 + y_1\otimes y_3 + \cdots + y_{p-1}\otimes y_1, \\ \dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots \\ w_{p-1}^0 = & y_0\otimes y_{p-1} + y_1\otimes y_0 + \cdots + y_{p-1}\otimes y_{p-2}. \end{array}$$

Remark 3.3.7. The hypothesis p being an odd prime number is not necessary. It is used because it makes the relations simpler.

The algebras R_p^2 are infinite dimensional for $p > 7$. This is a consequence of the following

Theorem 3.3.8 (Golod–Shafarevich). *Let $V = \oplus_{n>0} V_n$ be a graded vector space, and $A = T(V)$ be the (graded) tensor algebra of V. Let I be a homogeneous ideal, and suppose I is generated (as an ideal) by the subspaces $\oplus_{n>0} I_n$. Let $R = T(V)/I$ be the quotient algebra. Let h_V and h_I be the Hilbert series of V and I, that is,*

$$h_V(t) = \sum_{n>0} \dim(V_n)t^n, \qquad h_I(t) = \sum_{n>0} \dim(I_n)t^n.$$

Let $g(t) = \sum g_n t^n = (1 - h_V(t) + h_I(t))^{-1}$ as formal power series. If $g_n \ge 0\ \ \forall n$, then R is infinite dimensional.

Proof. See [Ufn95]. □

Proposition 3.3.9. *The algebras R_p^2 are infinite dimensional for $p > 7$.*

Proof. We apply the theorem. We have $h_V(t) = pt$, and $h_I(t) = (2p - 1)t^2$. Then

$$g(t) = (1 - pt + (2p - 1)t^2)^{-1} = (1 - t/a)^{-1}(1 - t/b)^{-1} = (\sum_{n\ge 0}(t/a)^n)(\sum_{n\ge 0}(t/b)^n)$$

for a, b the roots of $(1-pt+(2p-1)t^2)$. If a and b are both real and positive then $g_n \ge 0\ \forall n$. This is true if $p^2 - 4(2p - 1) \ge 0$, which implies $p > 7$. □

For $p = 3$ we have $\mathbb{D}_3 \simeq \mathbb{S}_3$, and in fact $R_3^2 \simeq R_3^1$.

Proposition 3.3.10. *$R_3^2 \simeq R_3^1$ is a TOBA of dimension 12.*

Proof. It can be seen by direct computation using the relations that

$$\begin{aligned} y_0y_1y_0 &= -y_1y_2y_0 = y_1y_0y_1 = -y_0y_2y_1, \\ y_0y_1y_2 &= -y_0y_2y_0 = y_2y_1y_0 = -y_2y_0y_2, \\ y_1y_0y_2 &= -y_2y_1y_2 = y_2y_0y_1 = -y_1y_2y_1, \end{aligned}$$

and the other monomials in degree 3 vanish since in all of them appears y_i^2 for some i. This in turn implies

$$\begin{aligned} y_0y_1y_0y_2 &= -y_1y_2y_0y_2 = y_1y_0y_1y_2 = -y_0y_2y_1y_2 = y_0y_2y_0y_1 = -y_0y_1y_2y_1 \\ &= -y_2y_1y_0y_1 = y_2y_0y_2y_1 = -y_2y_0y_1y_0 = -y_1y_0y_2y_0 = y_2y_1y_2y_0 = y_1y_2y_1y_0, \end{aligned} \tag{3.3.11}$$

$$y_0y_1y_0y_1 = y_1y_2y_0y_1 = y_1y_0y_1y_0 = y_0y_2y_1y_0 = y_0y_1y_2y_0 = y_2y_0y_2y_0 = y_0y_2y_0y_2$$
$$= y_2y_1y_0y_2 = y_1y_0y_2y_1 = y_2y_1y_2y_1 = y_2y_0y_1y_2 = y_1y_2y_1y_2 = 0,$$

and the other monomials in degree 4 vanish since in all of them appears y_i^2 for some i. Moreover, the monomials in (3.3.11) are annihilated by multiplying them with any of the y_i, and then

$$R_3^2(n) = 0 \ \forall n \geq 5.$$

With this, we get the set of generators of R_3^2 consisting of

$$\{1,\ y_0,\ y_1,\ y_2,\ y_0y_1,\ y_1y_2,\ y_0y_2,\ y_1y_0,\ y_0y_1y_0,\ y_0y_1y_2,\ y_1y_0y_2,\ y_0y_1y_0y_2\}. \tag{3.3.12}$$

It can be proved that this set is indeed a basis taking the representation of rank 12 given by

$$y_0 \mapsto A_0 = E_{1,2} + E_{3,7} + E_{4,8} + E_{5,9} + E_{6,10} + E_{11,12};$$
$$y_1 \mapsto A_1 = E_{1,3} + E_{2,5} - E_{4,6} - E_{4,7} - E_{6,9} + E_{7,9} - E_{8,11} + E_{10,12};$$
$$y_2 \mapsto A_2 = E_{1,4} + E_{2,6} - E_{3,5} - E_{3,8} - E_{5,10} - E_{7,11} + E_{8,10} + E_{9,12};$$

where $E_{i,j}$ stands for the matrix with 1 in the entry (i,j) and 0 in the others. This is easily seen to be a representation (i.e. $A_0^2 = A_1^2 = A_2^2 = A_0A_1 + A_1A_2 + A_2A_0 = A_0A_2 + A_1A_0 + A_2A_1 = 0$) and to map the set in (3.3.12) to a linearly independent set, which says that $\dim R_3^2 = 12$. Alternatively one can use the Diamond Lemma.

We have to check now that R_3^2 is a TOBA. Let $V = R_3^2(1)$, and let $T = \mathfrak{t}(V)$. Since $I \subseteq \ker \mathbf{S}$, we know that there exists a surjective graded morphism $\pi : R_3^2 \twoheadrightarrow T$. Let N be such that $T(N) \neq 0$ and $T(i) = 0 \ \forall i > N$. By 3.2.2 we have that $\dim T(N) = 1$ and $\dim T(i) = \dim T(N-i)$. We have then the following possibilities:

1. $N = 4$, and then $\dim T(3) = \dim T(1) = \dim V = 3$, from where π is an isomorphism unless $\dim T(2) < 4$.
2. $N = 3$, and then $\dim T(2) = \dim T(1) = 3$.
3. $N = 2$, and then $\dim T(2) = \dim T(0) = 1$.

We see that in any case π is an isomorphism unless $\dim T(2) < 4$, but $\dim T(2)$ is the codimension of $\ker \mathbf{S}^2$ in $V \otimes V$, and we know from 3.3.6 that it is equal to 4. $\square$

We give now the bosonised algebra. We denote also by y_i the element $(y_i\#1)$, and by $g_0,\ g_1$ the group-likes $(1\#\sigma)$, $(1\#\rho^2\sigma)$ (which generate $\mathbb{D}_3$). The bosonization is thus the algebra presented by generators $g_0,\ g_1,\ y_0,\ y_1,\ y_2$ with relations

$$g_i^2 = 1 \qquad i = 0,1; \tag{3.3.13}$$
$$g_1g_0g_1 = g_0g_1g_0; \tag{3.3.14}$$
$$y_j^2 = 0 \qquad j = 0,1,2; \tag{3.3.15}$$
$$y_0y_1 + y_1y_2 + y_2y_0 = 0; \tag{3.3.16}$$
$$y_1y_0 + y_0y_2 + y_2y_1 = 0; \tag{3.3.17}$$
$$g_0y_0g_0 = -y_0, \quad g_0y_1g_0 = -y_2, \quad g_0y_2g_0 = -y_1; \tag{3.3.18}$$
$$g_1y_0g_1 = -y_2, \quad g_1y_1g_1 = -y_1, \quad g_1y_2g_1 = -y_0. \tag{3.3.19}$$

The Hopf algebra structure is determined by

$$\Delta(g_i) = g_i \otimes g_i \quad (i = 0, 1), \qquad \Delta(y_i) = y_i \otimes 1 + g_i \otimes y_i \quad (i = 0, 1, 2),$$

where we denote $g_2 = g_0 g_1 g_0$. This Hopf algebra has dimension 72; it is pointed and its coradical is isomorphic to the group algebra of $\mathbb{S}_3$.

Remark 3.3.20. More Hopf algebras with coradical $\mathbf{k}\mathbb{S}_3$ appear replacing the relations (3.3.16) and (3.3.17) by

$$y_0 y_1 + y_1 y_2 + y_2 y_0 = \lambda_1 (g_0 g_1 - 1),$$
$$y_1 y_0 + y_0 y_2 + y_2 y_1 = \lambda_2 (g_1 g_0 - 1),$$

with $\lambda_1, \lambda_2 \in \mathbf{k}$. We will consider this and related problems in a separated article.

Remark 3.3.21. The relations (3.3.15) can be twisted as in the preceding remark, but in this case one must replace the group $\mathbb{S}_3$ by a covering of it, using the relations $g_i^{2N} = 1$ instead of $g_i^2 = 1$.

It is shown in [MS97] that R_4^1 is finite dimensional. It is not known whether R_n^1 is finite dimensional or not for $n > 4$.

It is not known whether R_5^2 and R_7^2 are finite dimensional or not. It is not known neither whether the algebras obtained dividing $A(V)$ by $\ker(\mathbf{S})$ (and not only by the ideal generated by $\ker(\mathbf{S}^2)$) are finite dimensional or not.

Example 3.3.22. As a last example, we take $\Gamma = \mathbb{D}_4$, $H = \mathbf{k}\Gamma$. The conjugacy class of σ is $\mathcal{O}_\sigma = \{\sigma, \rho^2\sigma\}$, and the conjugacy class of $\rho\sigma$ is $\mathcal{O}_{\rho\sigma} = \{\rho\sigma, \rho^3\sigma\}$. We take then, in a similar way to R_p^2, the module V in ${}^H_H\mathcal{YD}$ with basis $\{z_0, z_1, z_2, z_3\}$ with the structure given by $\delta(z_i) = \rho^i\sigma \otimes z_i$, $\rho^j \rightharpoonup z_i = z_{i+2j}$ and $\sigma \rightharpoonup z_i = -z_{-i}$ (where as before we take the subindices of the z_i to be on $\mathbb{Z}/4$). Then V can be decomposed as

$$V = \langle z_0, z_2 \rangle \oplus \langle z_1, z_3 \rangle = V_0 \oplus V_1,$$

which are irreducible. We have as before that the braiding is given by

$$c(z_i \otimes z_j) = -z_{2i-j} \otimes z_i.$$

Let $T_0 = \mathfrak{t}(V_0)$ and $T_1 = \mathfrak{t}(V_1)$. It is easy to see that T_i $(i = 0, 1)$ have dimension 4, and their respective ideals $\ker \mathbf{S}$ are generated by

$$z_0^2 = z_2^2 = 0, \quad z_0 z_2 + z_2 z_0 = 0,$$
$$z_1^2 = z_3^2 = 0, \quad z_1 z_3 + z_3 z_1 = 0.$$

The TOBA $T = \mathfrak{t}(V)$ is much more complicated to compute. Let $a = z_1 z_2 + z_0 z_1$ and $b = z_1 z_0 + z_2 z_1$. Then the elements a^2, b^2 and $ab + ba$ are primitive in the quotient algebra of $A(V)$ modulo the quadratic relations, which says that they belong to $\ker \mathbf{S}$. We have then a graded braided Hopf algebra dividing out $A(V)$ by the relations

$$\begin{aligned}
&z_0^2 = z_2^2 = 0, \quad z_0 z_2 + z_2 z_0 = 0,\\
&z_1^2 = z_3^2 = 0, \quad z_1 z_3 + z_3 z_1 = 0,\\
&z_0 z_1 + z_1 z_2 + z_2 z_3 + z_3 z_0 = 0,\\
&z_0 z_3 + z_1 z_0 + z_2 z_1 + z_3 z_2 = 0,\\
&a^2 = b^2 = 0, \quad ab + ba = 0.
\end{aligned}$$

Using the Diamond Lemma, it can be shown that the dimension of this algebra is 64. This is done in [MS97].

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