ON POINTED HOPF ALGEBRAS OF DIMENSION p^5

MATÍAS GRAÑA

ABSTRACT. We describe all possible coradically graded pointed Hopf algebras of dimension p^5 (p an odd prime number) over an algebraically closed field of characteristic 0.

1. Introduction

The *lifting procedure* described in [2] is a powerful tool for classifying pointed Hopf algebras. It has been applied successfully to the classification of pointed Hopf algebras of dimension p^3 in [2] and dimension p^4 in [4]. It has been used also in the classification of pointed Hopf algebras of dimension 32 in [10]. We describe here all pointed coradically graded Hopf algebras of dimension p^5 (we assume p is odd since the case $p = 2$ is treated in [10]). Some of these algebras are known and can be found in the referred articles as well as in [3], [8]. Classification problems of pointed Hopf algebras have been also treated in [6], [9] and [7].

Our main references for Hopf algebras are [13] and [11]. For Nichols algebras we refer to [12] and [1].

The article is organized as follows: in section 2 we give the notation and definitions we use and the first results we need. In section 3 we describe all possible Nichols algebras of dimension p^{5-j} over groups of order p^j, $j = 1, \dots, 4$. In section 4 we prove necessary auxiliary results; some of them have interest on their own, e.g. 4.4.3. In section 5 we prove that any pointed Hopf algebra of dimension p^5 over $\mathbf{k}$ is generated by group-like and skew-primitive elements. In other words, any coradically graded pointed Hopf algebra of dimension p^5 can be recovered by bosonization (or biproduct) from one of the Nichols algebras appearing in 3.3.2. Furthermore, this proves also that any pointed Hopf algebra of dimension p^5 can be recovered by lifting (in the sense of [2]) of one of these bosonizations. The classification of the pointed Hopf algebras of dimension p^5 could be done in principle using the *lifting procedure* of [2]. This article contains the first steps in this direction. In section 6 we address the remaining steps and consider some illustrating examples.

This is part of the doctoral thesis of the author, who thanks N. Andruskiewitsch for his guidance.

2. Notation and preliminary results

The letter $\mathbf{k}$ shall stand for an algebraically closed field of characteristic 0. All Hopf algebras are $\mathbf{k}$-algebras. For Γ a group and $g \in \Gamma$ we denote by Γ_g the isotropy subgroup $\Gamma_g = \{h \in \Gamma \mid hg = gh\}$. Let $q \in \mathbf{k}$. For $n \geq m \in \mathbb{N}$, we use the standard notation

$$(n)_q = \sum_{i=0}^{n-1} q^i, \ (0)_q = 1; \quad (n)_q^! = \prod_{i=1}^{n} (i)_q; \quad \binom{n}{m}_q = \frac{(n)_q^!}{(m)_q^!(n-m)_q^!}.$$

Date: December, 1998.
This work was partially supported by CONICET, CONICOR and SeCyT-UNC.
1991 Mathematics Subject Classification: 16W30.

For A a Hopf algebra, we say that A is *pointed* iff the simple subcoalgebras of A are 1-dimensional (iff the irreducible representations of A^* are 1-dimensional).

Let $A = \oplus_{i\geq 0} A(i)$ be a graded Hopf algebra. We say that A is *coradically graded* if the graduation corresponds to the coradical filtration of A, i.e. if

$$A_r = \oplus_{i=0}^{r} A(i) \quad \forall r \geq 0, \text{ where } A_0 \subseteq A_1 \subseteq \dots \text{ stands for the coradical filtration of } A.$$

In particular, A being coradically graded and pointed implies that $A(0) \simeq \mathbf{k}\Gamma$ where Γ is the group of group-likes of A.

Let H be a Hopf algebra. We denote by ${}_H^H\mathcal{YD}$ the category of (left-left) Yetter–Drinfeld modules over H (see [11]) and by c its braiding. Let A be a coradically graded pointed Hopf algebra and $A(0) = \mathbf{k}\Gamma$; then

$$R = A^{\operatorname{co} A(0)} = \{x \in A \mid (\operatorname{id} \otimes \pi)\Delta(x) = x \otimes 1\} = \oplus_i R(i), \tag{2.2.1}$$

(where $\pi : A \to A(0)$ is the canonical projection) is a braided Hopf algebra in the category ${}_{\mathbf{k}\Gamma}^{\mathbf{k}\Gamma}\mathcal{YD}$. The Hopf algebra A can be recovered by bosonization: $A = R\#\mathbf{k}\Gamma$. Furthermore, R is coradically graded and $R(0) = \mathbf{k}1$. If moreover R is generated as an algebra by $R(1)$ then we say that R is a *Nichols algebra.*

If R is a Nichols algebra then R is uniquely determined (up to isomorphism) by $V = R(1)$, which coincides with the space of primitive elements $\mathcal{P}(R)$. We denote $R = \mathfrak{B}(V)$.

We refer to the survey [1] for details on these constructions (Nichols algebras are called TOBAs in that article).

Proposition 2.2.2. *Let* $\mathbf{f}$ *be any field, and let* H *be a Hopf algebra over* $\mathbf{f}$*. Let* V *be an object in* ${}_H^H\mathcal{YD}$*. Suppose* V *has a basis* $\{x_1, \dots, x_\theta\}$ *such that* $c(x_i \otimes x_j) = b_{ij} x_j \otimes x_i$ *for certain* $b_{ij} \in \mathbf{f}$ *(since* c *is an automorphism,* $b_{ij} \in \mathbf{f}^\times$*). We take for each* $i = 1, \dots, \theta$

$$N_i = \begin{cases} \text{order of } b_{ii} & \text{if } b_{ii} \neq 1 \text{ and is a root of unity,} \\ \infty & \text{if } b_{ii} \text{ is not a root of unity,} \\ \infty & \text{if } b_{ii} = 1 \text{ and } \operatorname{char} \mathbf{f} = 0, \\ \operatorname{char} \mathbf{f} & \text{if } b_{ii} = 1 \text{ and } \operatorname{char} \mathbf{f} > 0. \end{cases}$$

Then $\dim \mathfrak{B}(V) \geq \prod_i N_i$*. Moreover, if* $\mathfrak{B}(V)$ *is finite dimensional, then the equality holds if and only if* $b_{ij} b_{ji} = 1$ $\forall i \neq j$.

Proof. See [2, §3]. □

We recall (see for instance [1]) that if Γ is a finite group, the category ${}_{\mathbf{k}\Gamma}^{\mathbf{k}\Gamma}\mathcal{YD}$ is semisimple. The simple objects are the modules $M(g, \rho)$ defined as follows: let $g \in \Gamma$, ρ an irreducible representation of the isotropy group Γ_g. Let W be the space affording ρ, and take

$$M(g, \rho) = \operatorname{Ind}_{\Gamma_g}^{\Gamma} W = \mathbf{k}\Gamma \otimes_{\mathbf{k}\Gamma_g} W,$$

with the usual module structure and the comodule structure given by

$$\delta(h \otimes w) = hgh^{-1} \otimes (h \otimes w) \in \mathbf{k}\Gamma \otimes M(g, \rho).$$

Remark 2.2.3. Since g is central in Γ_g, if ρ is an irreducible representation of Γ_g then the Schur lemma says that $\rho(g) = q \operatorname{id}$ for some $q \in \mathbf{k}^\times$.

Definition 2.2.4. We say that $V \in {}^H_H\mathcal{YD}$ *has a matrix* (b_{ij}) if it has a basis $\{x_1, \dots, x_\theta\}$ such that $c(x_i \otimes x_j) = b_{ij} x_j \otimes x_i$. This happens for instance if Γ is abelian. This happens also under a weaker condition: let $V = \oplus_i M(g_i, \rho_i)$ and suppose that the subgroup Γ' of Γ generated by the conjugacy classes of all the g_i is abelian. Then V *comes from the abelian case* in the sense of [1, Def. 3.1.8] and consequently has a matrix. In this case V can be considered as a Yetter–Drinfeld module over Γ' and $\mathfrak{B}(V)\#\mathbf{k}\Gamma$ can be reconstructed as an extension of Γ/Γ' by $\mathfrak{B}(V)\#\mathbf{k}\Gamma'$. A sufficient condition for Γ' to be abelian in the case $V = M(g, \rho)$ is that the isotropy subgroup Γ_g be invariant in Γ (see [1, Lemma 3.1.9]). Since we are working in characteristic 0, if V has a matrix (b_{ij}) and $\mathfrak{B}(V)$ is finite dimensional then by 2.2.2 $b_{ii} \neq 1 \ \forall i$.

If V has a matrix (b_{ij}) with $b_{ij} b_{ji} = 1 \ \forall i \neq j$ then it can be shown that $\mathfrak{B}(V)$ has a PBW basis of the form

$$\{x_1^{n_1} \cdots x_\theta^{n_\theta} \mid 0 \leq n_i < N_i\}$$

(N_i defined as in 2.2.2) and the relations are given by

$$x_i^{N_i} = 0, \quad x_i x_j = b_{ij} x_j x_i \ \forall i > j.$$

Thus $\mathfrak{B}(V)$ is a quantum linear space as an algebra. We notice that the lines $\mathbf{k}x_i$ $(i = 1, \dots, \theta)$ are not Yetter–Drinfeld submodules in general. In order to agree with the terminology of [2], we shall denote such an algebra by QLS only when the lines $\mathbf{k}x_i$ are Yetter–Drinfeld modules $\forall i$. Thus, a QLS in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ is given by a module $V = \oplus_{i=1}^\theta M(g_i, \chi_i)$ where

$$(2.2.5) \qquad \begin{cases} g_1, \dots, g_\theta \in \Gamma \text{ are central elements,} \\ \chi_1, \dots, \chi_\theta \in \hat{\Gamma} \text{ are characters such that} \\ \chi_i(g_j)\chi_j(g_i) = 1 \ \forall i \neq j. \end{cases}$$

For $V \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$, we shall say that $\mathfrak{B}(V)$ is a QLS over $\Gamma' \subset \Gamma$ if V is a Yetter–Drinfeld module in ${}^{\mathbf{k}\Gamma'}_{\mathbf{k}\Gamma'}\mathcal{YD}$ and conditions 2.2.5 hold for Γ'. A 1-dimensional QLS will be called also *Quantum Line* (or QL), and a 2-dimensional QLS will be called also *Quantum Plane* (or QP).

According to [3], if V has a matrix (b_{ij}) we say that V *is of Cartan type* if there exists a (generalized) Cartan matrix (a_{ij}) such that

$$b_{ij} b_{ji} = b_{ii}^{a_{ij}} \quad \forall i, j = 1, \dots, \theta.$$

We transfer to V the terminology over the Cartan matrix (a_{ij}).

Lemma 2.2.6. *Let g be central in Γ and ρ an irreducible representation of Γ. Let $V = M(g, \rho)$. By 2.2.3, g acts by a scalar on V, say q. Let N be the order of q. Then* $\dim \mathfrak{B}(V) \geq N^{\deg \rho}$.

Proof. Since g is central, V comes from the abelian case, and consequently c has a matrix (b_{ij}). It is straightforward to see that $b_{ij} = q \ \forall i, j$. Then 2.2.2 applies and the result follows. □

Lemma 2.2.7. *Let $g \in \Gamma$ and ρ an irreducible representation of Γ_g. Let $V = M(g, \rho)$. Suppose* $\dim V < p$, *where p is the smallest prime dividing $|\Gamma|$. Then g is central,* $\deg \rho = 1$ *and thus* $\mathfrak{B}(V)$ *is a QL over Γ with* $\dim \mathfrak{B}(V) = N$, *where N is the order of $\rho(g)$.*

Proof. We have $\dim V = [\Gamma : \Gamma_g] \deg(\rho) < p$. Since $[\Gamma : \Gamma_g]$ and $\deg(\rho)$ both divide $|\Gamma|$, necessarily $[\Gamma : \Gamma_g] = 1$, whence g is central, and $\deg(\rho) = 1$. The result follows by 2.2.2. □

Remark 2.2.8. Since $\dim \mathfrak{B}(V) \geq 1 + \dim V$, the hypothesis of the preceding lemma is verified if $\dim \mathfrak{B}(V) \leq p$.

Remark 2.2.9. By 2.2.7, we have that if $V = \oplus_i M(g_i, \rho_i)$ is such that $\dim V < p$, then g_i is central $\forall i$ and ρ_i is a character $\forall i$, and furthermore $\dim \mathfrak{B}(V) \geq \prod_i N_i$, where N_i is the order of $\rho_i(g_i)$.

3. Main results

Lemma 3.3.1. *Let A be a coradically graded pointed Hopf algebra of dimension p^5. Let $\Gamma = G(A)$ be the group of group-likes of A. Let $R = A^{\mathrm{co}\,\mathbf{k}\Gamma} \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ be the coinvariants (thus $A = R\#\mathbf{k}\Gamma$) and let $V = R(1)$ be the primitive elements of R. Assume that V generates R as an algebra (i.e. $R = \mathfrak{B}(V)$). Then the following possibilities arise:*

1. *If $|\Gamma| = p^5$ then $V = 0$ and $A = \mathbf{k}\Gamma$.*
2. *If $|\Gamma| = p^4$ then V is 1-dimensional and R is a QLS.*
3. *If $|\Gamma| = p^3$ then V may be 1 or 2-dimensional and R is a QLS over some subgroup Γ' of Γ.*
4. *If $\Gamma = C_p \times C_p$ then V is 2-dimensional (and then R is a twisting of a Nichols algebra of type A_2) or V is 3-dimensional (and R is a QLS).*
5. *If $\Gamma = C_{p^2}$ then V is 2-dimensional (and then R is a QLS or a twisting of a Nichols algebra of type A_2) or V is 3-dimensional (and R is a QLS).*
6. *If $\Gamma = C_p$ then either V is 2-dimensional, R is of type B_2 and necessarily $p \equiv 1 \mod 4$, or V is 3-dimensional, R is of type $A_2 \times A_1$ and $p = 3$.*

Proof. We prove that $\mathfrak{B}(V)$ is of the claimed form:

1. Immediate.
2. By 2.2.9 we have $\dim V = 1$, $V = (x) = M(g, \chi)$, $g \in Z(\Gamma)$. Furthermore, by the same remark, $\chi(g) = q$ is such that $q^p = 1$ (and $q \neq 1$ since A is finite dimensional), from where by 2.2.4 the structure of R is given by
$$x^p = 0, \quad \varepsilon(x) = 0,$$
$$\Delta(x^r) = \sum_{i=0}^{r} \binom{r}{i}_q x^i \otimes x^{r-i},$$
$$\delta(x) = g \otimes x, \quad h \rightharpoonup x = \chi(h)x.$$
Let $a = x\#1 \in A$. Then A is generated by Γ and a, with the structure given by
$$a^p = 0, \quad \varepsilon(a) = 0, \quad hah^{-1} = \chi(h)a \ \ \forall h \in \Gamma,$$
$$\Delta(a^r) = \sum_{i=0}^{r} \binom{r}{i}_q (a^i g^{r-i}) \otimes a^{r-i}.$$
3. The bound $\dim V \leq 2$ is a consequence of 4.4.3 below. If V is 1-dimensional then $V = M(g, \chi)$ and A is given exactly as in the case $|\Gamma| = p^4$ with the only exception that q has order p^2 and the relation on a is $a^{p^2} = 0$. If V is 2-dimensional, [1, Prop 3.1.11] applies and V comes from the abelian case, i.e. V has a basis $\{x_1, x_2\}$ with $(x_i) = M(g_i, \chi_i)$ (g_i and χ_i are respectively central elements and characters of a certain subgroup Γ' of Γ). Let N_i be the order of $\chi_i(g_i)$. Then by 2.2.2 we have $p^2 \geq N_1N_2$, from where $N_1 = N_2 = p$ ($\chi_i(g_i) \neq 1$ since A is finite dimensional), and again by 2.2.2 we have that $\mathfrak{B}(V)$ is a QLS over Γ'. Let $b_{12} = \chi_2(g_1)$, and let for each $h \in \Gamma$ the matrix $\rho(h)_{ij}$ be defined by $h \rightharpoonup x_j = \sum_{i=1}^{2} \rho(h)_{ij} x_i$. Then A is generated by Γ, a_1, a_2 with structure and relations given by
$$a_i^p = 0, \quad \varepsilon(a_i) = 0, \quad ha_jh^{-1} = \sum_{i=1}^{2} \rho(h)_{ij} a_i \ \ \forall h \in \Gamma,$$
$$\Delta(a_i) = g_i \otimes a_i + 1 \otimes a_i,$$
$$a_1a_2 = b_{12}a_2a_1.$$

(4) The bounds $2 \leq \dim V \leq 3$ are immediate consequences of 2.2.2. If $\dim V = 2$ then by 4.4.10 it is a twisting of an algebra of type A_2. If $\dim V = 3$ then by 2.2.2 it is a QLS.
(5) As in the case $\Gamma = C_p \times C_p$, the bounds $2 \leq \dim V \leq 3$ are consequences of 2.2.2. Suppose $\dim V = 2$, V have basis $\{x_1, x_2\}$ and c be given in this basis by the matrix (b_{ij}). If b_{11} (resp. b_{22}) has order p^2 then by, 2.2.2, b_{22} (resp. b_{11}) has order p and $\mathfrak{B}(V)$ is a QLS. If both b_{11} and b_{22} have order p then by 4.4.9 and 4.4.10 $\mathfrak{B}(V)$ is a twisting of an algebra of type A_2. If $\dim V = 3$ then by 2.2.2 it is a QLS.
(6) This is proved in [3, Thm. 1.3].

□

In section 5 we prove that if Γ is a group of order p^j and $R = \oplus_i R(i) \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ is a coradically graded braided Hopf algebra of dimension p^{5-j} with $R(1) \simeq \mathbf{k}$, then R is generated by $R(1)$. With this and the previous lemma we can prove

Theorem 3.3.2. *Let $A = \oplus_i A(i)$ be a coradically graded pointed Hopf algebra of dimension p^5. Let $\Gamma = G(A)$ be the group of group-likes of A. Let $R = \oplus_i R(i) = A^{\mathrm{co}A(0)} \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ and let $V = R(1)$. Then R is generated by V (i.e. $A = \mathfrak{B}(V)\#\mathbf{k}\Gamma$) and $\mathfrak{B}(V)$ is one in the list below. By $B(\cdot)$ we denote the group of order p^4 in* [5, Page 145].

Γ	$\dim \mathfrak{B}(V)$	*Type*	*Conditions*
$(C_p)^4$	1	*QLS*	
$(C_p)^2 \times C_{p^2}$	1	*QLS*	
$C_{p^2} \times C_{p^2}$	1	*QLS*	
$C_p \times C_{p^3}$	1	*QLS*	
C_{p^4}	1	*QLS*	
$B(vi)$	1	*QLS*	
$B(vii)$	1	*QLS*	
$B(viii)$	1	*QLS*	
$B(ix)$	1	*QLS*	
$B(x)$	1	*QLS*	
$B(xiv)$	1	*QLS*	
$(C_p)^3$	2	*QLS*	
$C_{p^2} \times C_p$	1	*QLS*	
	2	*QLS*	
C_{p^3}	1	*QLS*	
	2	*QLS*	
$(C_p)^2$	2	A_2	
	3	*QLS*	
C_{p^2}	2	A_2	$p = 3$ *or* $p \equiv 1 \mod 3$
C_p	2	B_2	$p \equiv 1 \mod 4$
	3	$A_2 \times A_1$	$p = 3$

Proof. For the groups of order p^4, the only condition for the existence of a QLS is the existence of a central element $g \in \Gamma$ and a character $\chi \in \hat{\Gamma}$ such that $\chi(g)$ has order p. This is possible if and only if $g \notin [\Gamma, \Gamma]$ ($[\Gamma, \Gamma]$ the commutator subgroup of Γ). It follows by inspection of each case that the groups in the table are those Γ such that $Z(\Gamma) \not\subset [\Gamma, \Gamma]$.

We go now to $|\Gamma| = p^3$. It is clear that QLS of rank one exist for $\Gamma = C_{p^2} \times C_p$ and $\Gamma = C_{p^3}$, and not for $\Gamma = (C_p)^3$. The two non abelian groups of order p^3 verify that their centers are included in their commutators subgroups, whence the 1-dimensional Yetter–Drinfeld modules give infinite dimensional Nichols algebras. We prove now that for the three abelian groups there exist QLS of rank 2: let q_1, q_2, q_3 denote respectively (fixed) roots of unity of orders p, p^2, p^3. We denote the generators of $(C_p)^3$ by $\{g_1, g_2, g_3\}$ and the generators of $\widehat{(C_p)^3}$ by $\{\hat{g}_1, \hat{g}_2, \hat{g}_3\}$, where $\hat{g}_i(g_j) = q_1^{\delta_{ij}}$. We denote the generators of $C_{p^2} \times C_p$ by $\{g_1, g_2\}$ and the generators of $\widehat{C_{p^2} \times C_p}$ by $\{\hat{g}_1, \hat{g}_2\}$, where $\hat{g}_i(g_j) = q_{3-i}^{\delta_{ij}}$. We denote the generator of C_{p^3} by $\{g\}$ and the generator of $\widehat{C_{p^3}}$ by $\{\hat{g}\}$, where $\hat{g}(g) = q_3$. It is straightforward that the following Yetter–Drinfeld modules give QLS of dimension p^2:

$$\begin{aligned} \Gamma = (C_p)^3, &\quad V = M(g_1, \hat{g}_1) \oplus M(g_2, \hat{g}_2), \\ \Gamma = C_{p^2} \times C_p, &\quad V = M(g_1^p, \hat{g}_1) \oplus M(g_1^{-p}, \hat{g}_1), \\ \Gamma = C_{p^3}, &\quad V = M(g^{p^2}, \hat{g}) \oplus M(g^{-p^2}, \hat{g}). \end{aligned}$$

For the two non abelian groups we should have V a Yetter–Drinfeld module of dimension 2. There are three possibilities:

(1) $V = M(h_1, \chi_1) \oplus M(h_2, \chi_2)$ where h_i are central and χ_i are characters, but by the same reason as in the rank one case, this would give infinite dimensional Nichols algebras.
(2) $V = M(g, \chi)$ where χ is a character and $[\Gamma : \Gamma_g] = 2$, but this is impossible since $p \neq 2$ (this case arises when $p = 2$, see [10]).
(3) $V = M(g, \rho)$ where g is central and ρ is an irreducible representation of Γ with $\deg \rho = 2$. Since $p \neq 2$, the Frobenius theorem applies to say that this is impossible (this case arises when $p = 2$, see [10]).

Let now $\Gamma = (C_p)^2$. It is immediate that there are no QLS of rank 1 neither 2, since there should be a character with a p^2-th root of unity in the image. The existence of a QLS of rank 3 is a consequence of [2, Lemma 4.1]. An explicit construction is as follows: let Γ have generators $\{g_1, g_2\}$ and $\hat{\Gamma}$ have generators $\{\hat{g}_1, \hat{g}_2\}$ where $\hat{g}_i(g_j) = q_1^{\delta_{ij}}$ (as before q_1 is a fixed p-th root of unity). Take then $V = M(g_1, \hat{g}_1) \oplus M(g_1, \hat{g}_1^{-1}) \oplus M(g_2, \hat{g}_2)$. It is straightforward to see that V generates a QLS. For a construction of a Nichols algebra of type A_2, let $r = \frac{1}{2} \in \mathbb{Z}/p$ (the construction for $p = 2$ is slightly different, see [10]). Take then $V = M(g_1, \hat{g}_1 \hat{g}_2^{-r}) \oplus M(g_2, \hat{g}_1^{-r} \hat{g}_2)$. It is clear then that V has the matrix

$$(b_{ij}) = \begin{pmatrix} q_1 & q_1^{-r} \\ q_1^{-r} & q_1 \end{pmatrix}, \text{ whence } b_{ij} b_{ji} = b_{ii}^{a_{ij}} \text{ with } (a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Let $\Gamma = C_{p^2}$. The non-existence of a p^3-dimensional QLS is a consequence of 4.4.1. Let $g, \hat{g}$ be respectively generators of Γ and $\hat{\Gamma}$, and let $q = \hat{g}(g)$. Suppose $V \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ generates an algebra of type A_2. Let $V = M(g^{e_1}, \hat{g}^{f_1}) \oplus M(g^{e_2}, \hat{g}^{f_2})$. Since V has a matrix $b_{ij} = q^{e_i f_j}$ and b_{11}, b_{22} must have order p, then p divides e_1 and e_2, or p divides f_1 and f_2. Then the same arguments as in [3, Thm 1.3] give the condition $p = 3$ or $p \equiv 1 \mod 3$. Furthermore, let b be such that $b^2 + b + 1 \equiv 0 \mod p$ (the condition on p is equivalent to the existence of such a b) and take $e_1 = p$, $f_1 = 1$, $e_2 = -p(b+1)$, $f_2 = b$. It is straightforward to see that this gives a Cartan matrix of type A_2.

For $\Gamma = C_p$ it is proved in [3, Thm 1.3] that there exist an algebra of type B_2 if and only if $p \equiv 1 \mod 4$, and of type $A_2 \times A_1$ if and only if $p = 3$. $\square$

4. Subsidiary results

The following lemma may be considered as an addendum to [2, Lemma 4.2].

Lemma 4.4.1. *Let $\Gamma = C_{p^n}$ and $V \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ generate a finite dimensional QLS. Then V may be 1-dimensional (and hence $\dim \mathfrak{B}(V) = p^v$ with $1 \leq v \leq n$) or it may be 2-dimensional (and hence $\dim \mathfrak{B}(V) = p^{2v}$ with $1 \leq v \leq n$).*

Proof. The bound $\dim V \leq 2$ is the content of [2, Lemma 4.2]. Let Γ have a generator g and $\hat{\Gamma}$ have a generator $\hat{g}$. Let $q = \hat{g}(g)$, which is a primitive p^n-th root of unity. If V is 1-dimensional, the result is an easy consequence of 2.2.2.

Suppose $V = M(g^{e_1'}, \hat{g}^{f_1'}) \oplus M(g^{e_2'}, \hat{g}^{f_2'})$, and let $e_i' = p^r e_i$ such that e_1, e_2 are not both divisible by p, $f_i' = p^s f_i$ such that f_1, f_2 are not both divisible by p. Then V has a matrix given by $b_{ij} = q^{e_i' f_j'} = q^{e_i f_j p^{r+s}}$. Since $\mathfrak{B}(V)$ is finite dimensional, $r + s < n$ (for if not $b_{11} = b_{22} = 1$). Let $u = n - r - s$. Suppose $p \not| e_1$ (if $p \not| e_2$ it is analogous). Suppose first that $p \not| f_2$ then b_{12} has order p^u. Since V generates a QLS, $b_{21} b_{12}^{-1}$ also has order p^u and thus $p \not| e_2$, $p \not| f_1$. This implies the result with $v = u$.

Suppose then $p | f_2$. Then $p \not| f_1$, and let $f_2 = p^t b$, $e_2 = p^y a$ with $p \not| b$, $p \not| a$. We prove that $t = y$: we have $t < u$ since if not $b_{22} = 1$. Now, b_{12} has order p^{u-t}, whence b_{21} has order p^{u-t}. Since $p \not| f_1$ then $p^t | e_2$, whence $y \geq t$. By similar considerations, $y \leq t$. This implies the result with $v = u - t$. $\square$

We shall make use of the following important tool for Nichols algebras.

Definition 4.4.2. Let $V \in {}^H_H\mathcal{YD}$ and $c = c_{V,V}$. For $i + j = n$, we denote by $\Delta_{i,j} : \mathfrak{B}^n(V) \to \mathfrak{B}^i(V) \otimes \mathfrak{B}^j(V)$ the (i,j)-component of the comultiplication of $\mathfrak{B}(V)$. It is proved in [14] (or see [1, Def. 3.2.10]) that $\Delta_{i,j}$ is injective $\forall i,j$. Let $\{x_1, \dots, x_\theta\}$ be a basis of V and let $\{x_1^*, \dots, x_\theta^*\}$ be its dual basis. We denote by ∂_{x_i} the differential operator on $\mathfrak{B}(V)$ given by

$$\partial_{x_i}(z) = (\mathrm{id} \otimes x_i^*)\Delta_{n-1,1}(z), \quad \text{if } z \in \mathfrak{B}^n(V),\ n > 0, \quad \text{and } \partial_{x_i}(1) = 0.$$

By the injectivity of $\Delta_{i,j}$ it is immediate that for $z \in \mathfrak{B}^n(V)$ $(n > 0)$ we have $z = 0$ if and only if $\partial_{x_i}(z) = 0$ for all $i = 1, \dots, \theta$. Suppose now that $V \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ and ∂_{x_i} verifies that there exists $g \in \Gamma$ such that $\partial_{x_i}(v) = 0$ if $\delta(v) = h \otimes v$ and $h \neq g$ (this happens for instance if $\delta(x_j) = g_j \otimes x_j$, $j = 1, \dots, \theta$ and $g = g_i$). Then it is easy to see that ∂_{x_i} verifies the Leibniz rule

$$\partial_{x_i}(z_1 z_2) = \partial_{x_i}(z_1)(g \rightharpoonup z_2) + z_1 \partial_{x_i}(z_2).$$

The following theorem is proved in [2, Thm 0.2] when Γ is an abelian group.

Theorem 4.4.3. *Let Γ be a finite group. Let $V \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ be such that $\dim \mathfrak{B}(V) = p^2$, where p is the smallest prime number dividing $|\Gamma|$. Then $\dim V \leq 2$ and $\mathfrak{B}(V)$ is a QLS over some subgroup $\Gamma' \subset \Gamma$. Furthermore, if $p > 2$ then $V = M(g, \chi)$ with g central, χ is a character such that $\chi(g)$ has order p^2 and hence $\mathfrak{B}(V)$ is a QL over Γ, or $V = M(g_1, \chi_1) \oplus M(g_2, \chi_2)$ where g_i is central, χ_i is a character $(i = 1, 2)$ such that $\chi_i(g_i)$ has order p and hence $\mathfrak{B}(V)$ is a QP over Γ.*

Proof. Let $V = \oplus_{i=1}^{\theta} M(g_i, \rho_i)$. It can be shown that $\dim \mathfrak{B}(V) \geq \dim \mathfrak{B}(M(g_i, \rho_i))$ $\forall i$. Let $I = [\Gamma : \Gamma_{g_1}]$ and $d = \deg(\rho_1)$. We have $\dim M(g_1, \rho_1) = dI$. The numbers d, I can be equal to 1 or $\geq p$. Since $\dim \mathfrak{B}(V) \geq 1 + \dim V$ we have $\dim V < p^2$.

Suppose first that $d \geq p$. This implies that $I = 1$, from where g_1 is central in Γ. By 2.2.6 we have $p^2 = \dim \mathfrak{B}(V) \geq \dim \mathfrak{B}(M(g_1, \rho_1)) \geq N^d$ with N the order of q, where $q\,\mathrm{id} = \rho_1(g_1)$. Since $\mathfrak{B}(V)$ is finite dimensional, we have $q \neq 1$ and hence $N \geq p$. If $p > 2$ we have a contradiction. If

$p = 2$ we must have $\theta = 1$, $d = 2$, $N = 2$. The condition $d = 2$ implies that V comes from the abelian case, as explained in 2.2.4. The condition on N tells that $q = -1$. Furthermore, 2.2.2 says that $\mathfrak{B}(V)$ is a QLS, and it is shown in [1] that the matrix of c is $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$.

Suppose then that $I \geq p$. This implies that $d = 1$, from where ρ_1 is a character of Γ_{g_1}. Let $q = \rho_1(g_1)$ and let N be the order of q. Let x be a generator of the space affording ρ_1, and let $\{h_1 = 1, h_2, \dots, h_I\}$ be a set of representatives of the cosets of Γ/Γ_{g_1}. Then $M(g_1, \rho_1)$ has as basis the elements $\{h_1 \rightharpoonup x, \dots, h_I \rightharpoonup x\}$ and we have

(4.4.4)
$$c(h_i \rightharpoonup x \otimes h_i \rightharpoonup x) = h_i g_1 h_i^{-1} h_i \rightharpoonup x \otimes h_i \rightharpoonup x = h_i g_1 \rightharpoonup x \otimes h_i \rightharpoonup x = q(h_i \rightharpoonup x \otimes h_i \rightharpoonup x).$$

It is straightforward to see using derivations that the elements
$$\{1, (h_i \rightharpoonup x)^r \mid 1 < r < N,\ i = 1, \dots, I\}$$
are linearly independent, from where

$$p^2 = \dim \mathfrak{B}(V) \geq 1 + I(N-1). \tag{4.4.5}$$

Thus, $N \leq p$. On the other hand, if $q \neq 1$ for if not it would be easy to see using derivations that the elements $\{x^r \mid r \geq 0\}$ would be linearly independent and $\mathfrak{B}(V)$ would be infinite dimensional (note that we had not proved at the moment that $\mathbf{k}x$ is a sub-YD-module neither that $M(g_1, \rho_1)$ comes from the abelian case, and hence 2.2.2 can not be used.) We have then proved that $N = p$. Suppose for a moment that $I > p$. It is clear that if $p > 2$ then $I \geq p+2$, but then (4.4.5) tells that this is a contradiction. If $p = 2$ then $I = 3$, but then [1, Prop. 3.2.2] tells that $\dim \mathfrak{B}(M(g_1, \rho_1)) \geq 5$, also a contradiction. Hence, we have that $I = p$ and then Γ_g, having index the smallest prime dividing $|\Gamma|$, is invariant in Γ. As said in 2.2.4, this implies that $\mathfrak{B}(M(g_1, \rho_1))$ comes from the abelian case, but then 2.2.2 applies and 4.4.4 tells that $\dim \mathfrak{B}(M(g_1, \rho_1)) \geq p^p$. This is a contradiction if $p > 2$. If $p = 2$ then $\theta = 1$, $q = -1$ and it is proved in [1] that the matrix of c is
$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

Suppose finally that $I = d = 1$. Then g_1 is central and ρ_1 is a character. Let $q = \rho_1(g_1)$ and let N be its order. Then $\dim \mathfrak{B}(V) \geq \dim \mathfrak{B}(M(g_1, \rho_1)) = N$ implies that $N \leq p^2$. If $N = p^2$ then $\theta = 1$ and the result follows at once. If $N < p^2$ then N is a prime number $\geq p$. Since $\dim \mathfrak{B}(M(g_1, \rho_1)) = N$, we have $\theta > 1$. Since $\dim \mathfrak{B}(M(g_2, \rho_2)) \leq p^2 - 1$ (because if x is a generator of $M(g_1, \rho_1)$ then x does not belong to $\mathfrak{B}(M(g_2, \rho_2))$) by the same arguments as above applied to $M(g_2, \rho_2)$ we have necessarily that g_2 is central and ρ_2 is a character. Let N_2 be the order of $\rho_2(g_2)$. Thus $N_2 < p^2$, and since g_1, g_2 are both central then $M(g_1, \rho_1) \oplus M(g_2, \rho_2)$ comes from the abelian case, whence by 2.2.2, $\mathfrak{B}(M(g_1, \rho_1) \oplus M(g_2, \rho_2))$ has dimension $\geq NN_2$. This implies $N = N_2 = p$, $\theta = 2$ and $\mathfrak{B}(V)$ is a QLS over Γ. □

Remark 4.4.6. It is proved in [8] in a different way that if $\dim \mathfrak{B}(V) = p$, p the smallest prime number dividing Γ, then $\dim V = 1$ and $\mathfrak{B}(V)$ is a QLS. It is proved also, with the same ideas as here, in [3, Prop 7.5].

Remark 4.4.7. We note that the proof of 4.4.3 above says that there are no V in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ such that $\dim \mathfrak{B}(V) = \pi^2$ if π is a prime number smaller than every prime dividing $|\Gamma|$.

The previous theorem implies the following result:

Corollary 4.4.8. *Let A be a pointed Hopf algebra of dimension m whose coradical has dimension m/p^2, p the smallest prime number dividing m. Then p^3 divides m and* $\dim A_1 = (r+1)m/p^2$, *where $r = 1$ or 2.*

Proof. Consider the coradical filtration of A and let $H = \oplus_i H(i)$ be the associated graded algebra. Then H is pointed and $H(0) \simeq \mathbf{k}\Gamma$, Γ the group of group-likes of A, which has order m/p^2. Let $R = H^{\mathrm{co}H(0)} \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ and let $R' \subset R$ be the algebra generated by $R(1)$ ($R' = R$ iff R is a Nichols algebra). Thus $\dim R = p^2$ and, thanks to the Nichols–Zoeller theorem, $\dim R' = p^j$ with $0 \le j \le 2$. The case $j = 0$ would imply $\dim R(1) = 0$, which is impossible. The case $j = 1$ is also impossible, for in that case 4.4.6 says that $\dim R(1) = \dim R'(1) = 1$, and [2, Thm 3.2] says that R is a Nichols algebra. Then $R' = R$. Remark 4.4.7 says that p divides $|\Gamma|$ (whence p^3 divides m) and 4.4.3 says that $r = \dim R(1)$ may be 1 or 2, whence $\dim H(1) = r|\Gamma|$ and $\dim A_1 = \dim H(0) + \dim H(1) = (r+1)|\Gamma| = (r+1)m/p^2$. □

Lemma 4.4.9. *Let Γ be a p-group and V a 2-dimensional module in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ such that* $\dim \mathfrak{B}(V) = p^3$. *Recall that under this assumption c has a matrix (b_{ij}) in some basis $\{x, y\}$. Let q be a primitive p^2-th root of unity, and suppose $b_{ij} = q^{c_{ij}}$. If p divides c_{11} and c_{22} then p divides $c_{12} + c_{21}$.*

Proof. We have $x^p = y^p = 0$. Let $z = \mathrm{Ad}_x(y) = xy - b_{12}yx$. Let $\sigma = 1 - b_{12}b_{21}$. We compute

$$\partial_x(z) = b_{12}y - b_{12}y = 0, \qquad \partial_y(z) = x - b_{12}b_{21}x = \sigma x \Rightarrow \partial_x\partial_y(z) = \sigma.$$

Furthermore, we have

$$\begin{aligned}
\partial_x(x^r y^s z^t) &= (r)_{b_{11}} b_{11}^t b_{12}^{s+t} x^{r-1} y^s z^t,\\
\partial_y(y^s z^t) &= (s)_{b_{22}} b_{21}^t b_{22}^t y^{s-1} z^t + \sum_{i=0}^{t-1} \sigma b_{21}^i b_{22}^i y^s z^{t-1-i} x z^i,\\
\partial_x\partial_y(y^s z^t) &= \sigma (t)_{(b_{11}b_{12}b_{21}b_{22})} y^s z^{t-1}.
\end{aligned}$$

Thus, if $p \not| c_{12} + c_{21}$, the order of $(b_{11}b_{12}b_{21}b_{22})$ is p^2, from where the set $\{z^t \mid 0 \le t < p^2\}$ is linearly independent. This implies inductively that the set $\{y^s z^t \mid 0 \le s < p,\ 0 \le t < p^2\}$ is linearly independent, and then that the set $\{x^r y^s z^t \mid 0 \le r, s < p,\ 0 \le t < p^2\}$ is linearly independent, whence $\dim \mathfrak{B}(V) \ge p^4$. □

The following result is a consequence of [3, Cor. 1.2]. We give a direct proof here.

Lemma 4.4.10. *Let $V = (x, y)$ be a 2-dimensional module in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ such that* $\dim \mathfrak{B}(V) = p^3$. *Let V have a matrix (b_{ij}) and suppose that $b_{ij}^p = 1$ for all i, j. Then b_{ij} is a Cartan matrix of type A_2.*

Proof. Let $q = b_{11}$ and c_{ij} be given by $b_{ij} = q^{c_{ij}}$. We may suppose as said above that $b_{12} = b_{21}$. Let $b_{12} = q^a$, $b_{22} = q^c$. Take $z = \mathrm{Ad}_x(y) = xy - q^a yx$, and let $\sigma = 1 - q^{2a}$; thus $\sigma \ne 0$ since otherwise $\mathfrak{B}(V)$ would be a QLS and $\dim \mathfrak{B}(V)$ would be p^2. We compute as before

$$\partial_x(z) = 0, \qquad \partial_y(z) = \sigma x \Rightarrow \partial_x\partial_y(z) = \sigma,$$

from where

$$\begin{aligned}
\partial_x(x^r y^s z^t) &= (r)_q q^{t+a(s+t)} x^{r-1} y^s z^t,\\
\partial_y(y^s z^t) &= (s)_{q^c} q^{as+ct} y^{s-1} z^t + \sum_{i=0}^{t-1} \sigma q^{(a+c)i} y^s z^{t-1-i} x z^i,\\
\partial_x\partial_y(y^s z^t) &= \sigma (t)_{q^{1+2a+c}} y^s z^{t-1}.
\end{aligned}$$

As before, the set $\{x^r y^s z^t \mid 0 \leq r,s,t < p\}$ is linearly independent (as a remark, note that we must have $1+2a+c \not\equiv 0 \pmod p$ since if not $\mathfrak{B}(V)$ would be infinite dimensional). Let now $w = \mathrm{Ad}_x(z) = xz - q^{1+a}zx$, and we have

$$\partial_x(w) = 0, \qquad \partial_y(w) = \sigma x^2 - \sigma q^{1+a}q^a x^2 = \sigma(1-q^{2a+1})x^2.$$

The (x,y)-bidegree of w is $(2,1)$, from where the set $\{x^2y, xz, w\}$ must be linearly dependent in order for $\mathfrak{B}(V)$ to be p^3-dimensional, but this implies $2a+1=0$, which means that $b_{12}b_{21} = b_{11}^{-1}$.

With the same reasoning, we must have $b_{12}b_{21} = b_{22}^{-1}$, and thus $b_{ij}b_{ji} = b_{ii}^{c_{ij}}$ with

$$c_{ij} \equiv \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \mod p,$$

and b_{ij} is a Cartan matrix of type A_2. As a remark, note that $b_{11} = b_{22}$. □

5. The classification is complete

We have to prove that those of 3.3.2 are all the coradically graded pointed Hopf algebras of dimension p^5. This amounts to be the same as to prove that a coradically graded pointed Hopf algebra is generated by its homogeneous component of degree 1, which in turn is equivalent to prove that if $A = \oplus_i A(i)$ is a coradically graded pointed Hopf algebra and $R = A^{\mathrm{co}A_0}$ is its algebra of coinvariants then R is a Nichols algebra. As in [3, §8], let $S = R^*$ be its dual. Then S is a graded braided Hopf algebra in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$, $S = \oplus_i S(i)$ and is generated by $S(0) \oplus S(1)$. Furthermore, we have a surjection $S \twoheadrightarrow S'$, $S' = \mathfrak{B}(S(1))$. We have to prove that S is coradically graded, i.e. that $\mathcal{P}(S) = S(1)$. This is the same as to say that $S' = S$. Now, [2, Thm 3.2] plus 4.4.6 solve the problem for the cases in which Γ has order p^4 or p^3, and [3, Thm 8.2] and [3, Lemma 8.5] solve the problem for the case in which Γ has order p or when $\Gamma = C_p \times C_p$. The following theorem solves the pending case.

Theorem 5.5.1. *Let Γ be a finite group, p the smallest prime number dividing $|\Gamma|$. Let $S = \oplus_i S(i)$ in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ be a graded braided Hopf algebra of dimension p^3 such that $S(0) = \mathbf{k}$ and S is generated by $S(1)$. Suppose that $S(1)$ comes from the abelian case, i.e. there exist an abelian subgroup $\Gamma' \subset \Gamma$ such that $S(1)$ is a YD-module over Γ'. Then S is a Nichols algebra.*

Proof. We prove the statement for $p > 2$, the case $p = 2$ being treated in [10]. Let $S' = \mathfrak{B}(S(1))$, and consider the canonical projection $S \twoheadrightarrow S'$. We must prove that this is an isomorphism. If $\dim S(1) = 3$ then by 2.2.2 we have $\dim S' \geq p^3$, but this implies that $S' = S$ and S is a Nichols algebra. If $\dim S(1) = 1$ then [2, Thm 3.2] shows that S is a Nichols algebra. We are led to consider hence the case $\dim S(1) = 2$. We have $\dim S' \leq p^3$, and we suppose $\dim S' < p^3$. Then by 2.2.2 we have $\dim S' \geq p^2$, whence $\dim S' = p^2$. Now theorem 4.4.3 says that S' is a QLS over Γ', $S'(1)$ has a basis $\{x, y\}$ and the braiding c has a matrix (b_{ij}) in this basis, where b_{ii} are primitive p-th roots of unity and $b_{12}b_{21} = 1$. Furthermore, the linear spans $\mathbf{k}x$ and $\mathbf{k}y$ are sub-YD-modules over Γ'. Let $z = x_1x_2 - b_{12}x_2x_1 \in S$. If we prove that $z = 0$ in S then $\dim S = p^2$, but this would be a contradiction and we would be done.

Suppose $z \neq 0$. Now, it is immediate to see that z is primitive in S. Consider the coradical filtration of S and let $T = \oplus_i T(i)$ be the associated graded algebra. We have $x, y, z \in S_1$, and consider $\bar{x}, \bar{y}, \bar{z} \in T(1)$. It is easy to see that these elements are linearly independent. We compute the matrix of c for $\{\bar{x}, \bar{y}, \bar{z}\}$. It is given by

$$(b'_{ij}) = \begin{pmatrix} b_{11} & b_{12} & b_{11}b_{12} \\ b_{21} & b_{22} & b_{21}b_{22} \\ b_{11}b_{21} & b_{12}b_{22} & b_{11}b_{12}b_{21}b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{11}b_{12} \\ b_{21} & b_{22} & b_{21}b_{22} \\ b_{11}b_{21} & b_{12}b_{22} & b_{11}b_{22} \end{pmatrix}.$$

Consider now the canonical projection $T \twoheadrightarrow T' = \mathfrak{B}(T(1))$. Since $\mathbf{k}x$, $\mathbf{k}y$ and $\mathbf{k}z$ are sub-YD-modules of S over Γ', then $\mathbf{k}\bar{x}$, $\mathbf{k}\bar{y}$ and $\mathbf{k}\bar{z}$ are sub-YD-modules of T over Γ'. Thus, if $W = (\bar{x}, \bar{y}, \bar{z})$ we have $\dim \mathfrak{B}(W) \leq \dim T' \leq \dim T = p^3$. Now 2.2.2 applies (notice that b'_{ii} has order p $\forall i$, since $b_{11}b_{22} = 1$ would imply $\dim \mathfrak{B}(W) = \infty$), and $\mathfrak{B}(W)$ is a QLS, but this implies that $1 = b_{11}^2 b_{12} b_{21} = b_{11}^2$, which is impossible since $p \neq 2$. Hence, $z = 0$ in S and the theorem is proved. □

6. Final remarks

In order to give a complete classification of the pointed Hopf algebras of dimension p^5, the following steps should be accomplished:

(1) For each Nichols algebra R in 3.3.2, give all the modules M in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ such that $\mathfrak{B}(M) \simeq R$.
(2) Classify the isomorphism classes of the bosonizations of the Nichols algebras in the previous step (note that there exist non isomorphic Nichols algebras which give isomorphic algebras after bosonization).
(3) For each coradically graded p^5-dimensional Hopf algebra in the previous step, classify all the liftings.

These steps are highly non trivial. For instance, let $\Gamma = C_{p^n}$ where $n > 0$ and $p \neq 2$ and let $0 \leq s \leq n$. The number of QLS of rank 1 over Γ with dimension p^s is given by

$$\sum_{\substack{i+j-n=s \\ i,j \leq n}} \phi(i,j),$$

while the number of isomorphisms classes of these QLS after bosonization is given by

$$\sum_{\substack{i+j-n=s \\ i,j \leq n}} \frac{\phi(i,j)}{I(i,j)},$$

where

$$\phi(i) = p^{i-1}(p-1),$$
$$\phi(i,j) = \phi(i)\phi(j),$$
$$I(k_1, \dots, k_r) = \phi(\max\{k_1, \dots, k_r\}).$$

Furthermore, the number of QLS of rank 2 over Γ with dimension p^s (s even, by 4.4.1) is given by

$$\frac{1}{2} \sum_{i_1,j_1,i_2,j_2} \frac{\phi(i_1,j_1)\phi(i_2,j_2)}{L_n(i_1,j_2)},$$

while the number of isomorphism classes of these QLS after bosonization is given by

$$\frac{1}{2} \sum_{i_1,j_1,i_2,j_2} \frac{\phi(i_1,j_1)\phi(i_2,j_2)}{L_n(i_1,j_2) I(i_1,i_2,j_1,j_2)},$$

where

$$L_n(i,j) = \phi(i+j-n),$$

and the sum is over the tuples such that

$$i_1 + j_1 - n = s_1 \geq 1,\ i_2 + j_2 - n = s_2 \geq 1,\ s_1 + s_2 = s,\ i_1, j_1, i_2, j_2 \leq n,\ i_1 + j_2 = i_2 + j_1.$$

As a result of this, the number of coradically graded non isomorphic Hopf algebras of dimension p^5 with coradical C_{p^4}, C_{p^3} is respectively $2(p^2-1)$ and $p(p-1)[2+\frac{p(p-1)(p+2)}{2}]$.

See also the discussion in [3, §9] for the first step, [3, §6] for the second. In particular, a necessary and sufficient condition for two YD-modules to give isomorphic algebras after bosonization is given in [3, Prop. 6.3].

As an example on the last step, let A be a pointed Hopf algebra of dimension p^5 with coradical Γ of order p^4. Let H be the associated graded Hopf algebra and R its invariants as in 2.2.1. Then $H = R\#\mathbf{k}\Gamma$, where $R = \mathfrak{B}(V)$. We have then $V = M(h, \chi)$ where h is central and χ is a character such that $\chi(h)$ has order p. Let x be a generator of V. Then, by [10, Prop. 2.0.17], x can be lift to $a \in A$ such that $\Delta(a) = h \otimes a + a \otimes 1$ and $gag^{-1} = \chi(g)a\ \forall g \in \Gamma$. Since the elements $\{x^i\#g \mid 0 \leq i < p,\ g \in \Gamma\}$ are a basis of H, then the elements $\{a^i g \mid 0 \leq i < p,\ g \in \Gamma\}$ are a basis of A. The lifting A is then determined by the element a^p, the case $a^p = 0$ being the bosonization $A = R\#\mathbf{k}\Gamma$. It is easy to see that a^p is a skew-primitive, $\Delta(a^p) = h^p \otimes a^p + a^p \otimes 1$. Looking at the space of skew-primitives, this implies that

$$a^p = \lambda(h^p - 1), \quad \lambda \in \mathbf{k}.$$

Taking a suitable scalar multiple of a we may suppose that $\lambda \in \{0, 1\}$, from where there are no more that 2 liftings. In some of the cases we must have $a^p = 0$. These cases are given by the diamond lemma:

$$\begin{aligned} ga^p &= g\lambda(h^p - 1) = \lambda(h^p - 1)g \\ ga^p &= \chi(g)aga^{p-1} = \ldots = \chi^p(g)a^p g = \chi^p(g)\lambda(h^p - 1)g, \end{aligned}$$

from where $\lambda(\chi^p - 1) = 0$ for A to be p^5-dimensional. This tells that over the group $B(vi)$ any pointed Hopf algebra of dimension p^5 is coradically graded.

On the other hand, it is clear that if $h^p = 1$ then $a^p = 0$. This tells that over the groups $B(viii)$, $B(ix)$, $B(x)$ and $B(xiv)$ any pointed Hopf algebra of dimension p^5 is coradically graded.

As a corollary: a pointed Hopf algebra of dimension p^5 and non abelian coradical is coradically graded, unless its coradical is isomorphic to $\mathbf{k}B(vii)$. We classify all the liftings in this case: $B(vii)$ can be presented with generators X, Y, Z and relations

$$X^{p^2} = Y^p = Z^p = 1, \quad [Z, Y] = X^p, \quad [X, Y] = [X, Z] = 1.$$

Hence $Z(B(vii)) = (X)$ while $[B(vii), B(vii)] = (X^p)$. Let q be a (fixed) p-th root of unity. The Yetter–Drinfeld modules generating Nichols algebras of dimension p are then

$$V = M(X^i, \chi) \text{ such that } p \not| i,\ \chi(X) = q^a\ (p \not| a),\ \chi(Y) = q^b,\ \chi(Z) = q^c.$$

However, it can be shown that most of them give isomorphic algebras after bosonization. We are led to consider two modules:

$$V_i = M(X, \chi_i)\ (i = 1, 2), \quad \chi_1(X) = q,\ \chi_1(Y) = \chi_1(Z) = 1, \quad \chi_2(X) = \chi_2(Y) = q,\ \chi_2(Z) = 1.$$

We have hence 2 pointed Hopf algebras of dimension p^5 with non abelian coradical and which are not coradically graded:

$A^{(1)}$ generated by a, X, Y, Z and the relations of $B(vii)$ plus

$$Xa = qaX,\ Ya = aY,\ Za = aZ,\ a^p = X^p - 1,\ \Delta(a) = X \otimes a + a \otimes 1,$$

$A^{(2)}$ generated by a, X, Y, Z and the relations of $B(vii)$ plus

$$Xa = qaX,\ Ya = qaY,\ Za = aZ,\ a^p = X^p - 1,\ \Delta(a) = X \otimes a + a \otimes 1.$$

A description of the liftings of QLS (resp. of the algebras of type A_2 over groups of exponent p) is made in [2] (resp. [4]).

References

[1] N. Andruskiewitsch & M. Graña, *Braided Hopf algebras over non abelian finite groups*, `q-alg 9802074`, Boletín de la Acad. Nac. Cs. Córdoba, Argentina **63** (1999) 45–78.

[2] N. Andruskiewitsch & H.-J. Schneider, *Lifting of quantum linear spaces and pointed Hopf algebras of order* p^3, J.Algebra **209** (1998) 659–691.

[3] N. Andruskiewitsch & H.-J. Schneider, *Finite quantum groups and Cartan matrices*, Adv. Math. **154** (2000), no. 1, 1–45.

[4] N. Andruskiewitsch & H.-J. Schneider, *Lifting of Nichols algebras of type* A_2 *and Pointed Hopf algebras of order* p^4, in Hopf algebras and quantum groups, Proceedings of the colloquium in Brussels 1998, ed. S. Caenepeel, (1999) 1–18.

[5] W. Burnside, *Theory of groups of finite order*, 2nd edition, Dover Pub., New York, 1955.

[6] M. Beattie, S. Dăscălescu & L. Grünenfelder, *On the number of types of finite-dimensional Hopf algebras*, Inventiones Math. **136** (1999) 1–7.

[7] S. Caenepeel S. Dăscălescu & Ş. Raianu, *Classifying pointed Hopf algebras of dimension* 16, Comm. Algebra **28** (2000), no. 2, 541–568.

[8] S. Dăscălescu *Pointed Hopf algebras with large coradical*, Comm. Algebra **27** 10 (1999) 4827–4851.

[9] S. Gelaki, *On pointed Hopf algebras and Kaplansky's tenth conjecture*, J.Algebra **209** (1998) 635–657.

[10] M. Graña, *Pointed Hopf algebras of dimension 32*, Comm. Algebra **28** 6 (2000) 2935–2976.

[11] S. Montgomery, *Hopf Algebras and their Actions on Rings*, AMS (1993), CMBS **82**.

[12] W.D. Nichols, *Bialgebras of type one*, Comm. in Alg. **6** (1978), 1521–1552.

[13] M.E. Sweedler, *Hopf algebras*, Benjamin, New York, 1969.

[14] P. Schauenburg, *A Characterization of the Borel-like Subalgebras of Quantum Enveloping Algebras*, Comm. in Alg. **24** (1996), 2811–2823.

Matías Graña, FCEyN – Buenos Aires
E-mail address: `matiasg@dm.uba.ar`