Pointed Hopf algebras of dimension 32

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Abstract

We give a complete classification of the 32-dimensional pointed Hopf algebras over an algebraically closed field $\mathbf{k}$ with $\operatorname{char} \mathbf{k} \neq 2$. It turns out that there are infinite families of isomorphism classes of pointed Hopf algebras of dimension 32. In [AS1], [BDG] and [Ge] are given families of counterexamples for the tenth Kaplansky conjecture. Up to now, 32 is the lowest dimension where Kaplansky conjecture fails.

1. Introduction

In recent years there has been certain activity in classification problems of pointed Hopf algebras. In [AS1], [BDG], [Ge] independently the authors find infinite families of pointed Hopf algebras of dimension p^4 (p an odd prime number), solving thus by the negative the tenth Kaplansky conjecture. In [D] there is a classification of pointed Hopf algebras of dimension p^n with coradical of dimension p^{n-1}.

In [AS1], [AS2], [AS3] there are classifications of pointed Hopf algebras of dimension p^3, p^4 and a large number of strong results towards the classification of pointed Hopf algebras with commutative coradical. Using the techniques of these articles, namely the "lifting procedure" described below, we classify here the pointed Hopf algebras of dimension 32 over an algebraically closed field of

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characteristic $\neq 2$. Related results can be found in [CDR] (the classification of the pointed Hopf algebras of dimension 16), in [Be] (classification of Ore extension Hopf algebras with coradical $\mathbf{k}(C_2 \times C_2)$ and $\mathbf{k}C_4$), and in [G] (the description of the coradically graded pointed Hopf algebras of dimension p^5).

One of the key results is Corollary 2.14, where we compute the Nichols algebras $\mathfrak{B}(V)$ (see 1.2 for the definition of Nichols algebras), V the Yetter–Drinfeld modules of dimension 2 over the groups of order 2^j, $1 \leq j \leq 4$.

Our main result is

Theorem 1.0.1. *Let A be a pointed Hopf algebra of dimension 32 over the algebraically closed field $\mathbf{k}$, $\operatorname{char}\mathbf{k} \neq 2$. Let $\Gamma = G(A)$ be the group of group-likes of A.*
If $\Gamma = C_2$ then A is the algebra of section 3.
If $\Gamma = C_2 \times C_2$ then A is one of the 6 algebras of 4.1.
If $\Gamma = C_4$ then A is one of the 12 algebras of 4.2.
If $\Gamma = C_2 \times C_2 \times C_2$ then A is one of the 6 algebras of 5.1.
If $\Gamma = C_2 \times C_4$ then A is in the infinite family of algebras of 5.2.
If $\Gamma = C_8$ then A is in the infinite family of algebras of 5.3.
If $\Gamma = \mathbb{D}_4$ the dihedral group, then A is one of the 7 algebras of 5.4.
If $\Gamma = \mathbb{H}$ the quaternionic group, then A is one of the 3 algebras of 5.5.
If $\Gamma = C_2 \times C_2 \times C_2 \times C_2$ then A is the algebra of 6.1.
If $\Gamma = C_2 \times C_2 \times C_4$ then A is one of the 7 algebras of 6.2.
If $\Gamma = C_4 \times C_4$ then A is one of the 4 algebras of 6.3.
If $\Gamma = C_2 \times C_8$ then A is one of the 14 algebras of 6.4.
If $\Gamma = C_{16}$ then A is one of the 7 algebras of 6.5.
If Γ is one of the non abelian groups of order 16 then A is one of the 13 algebras of 6.6.
If Γ has order 32 then A is a group algebra.

1.1. **Some basic definitions and conventions.** Throughout the article $\mathbf{k}$ shall be an algebraically closed field of characteristic $\neq 2$. The degrees for graded algebras shall be denoted by parentheses, i.e. $H(i)$ is the homogeneous component of degree i of the graded algebra H. The symbols $\mathbf{i}, \xi$ shall denote fixed roots of unity of orders $4, 8$ respectively, such that $\mathbf{i} = \xi^2$.

Let H be a Hopf algebra. We denote by ${}^H_H\mathcal{YD}$ the category of (left-left) Yetter–Drinfeld modules over H, i.e. M is an object of ${}^H_H\mathcal{YD}$ if it is a left H-module (with structure $h \otimes m \mapsto h \rightharpoonup m$), a left H-comodule (with structure

$\delta(m) = m_{(-1)} \otimes m_{(0)})$ and such that

$$\delta(h \rightharpoonup m) = h_{(1)} m_{(-1)} \mathcal{S}(h_{(3)}) \otimes h_{(2)} m_{(0)}.$$

We denote by c the braiding in ${}^H_H\mathcal{YD}$, which is given by

$$c = c_{M,N} : M \otimes N \to N \otimes M, \quad m \otimes n \mapsto m_{(-1)} \rightharpoonup n \otimes m_{(0)}.$$

When H is clear from the context, we shall call a module in ${}^H_H\mathcal{YD}$ a YD-module. Unless explicitly stated, the letter V shall denote a YD-module.

For Γ a group and $g \in \Gamma$ we denote by Γ_g the commutator subgroup

$$\Gamma_g = \{h \in \Gamma \mid hg = gh\}.$$

It is not difficult to prove (see for instance [AG]) that if $H = \mathbf{k}\Gamma$ and $|\Gamma| \neq 0$ in $\mathbf{k}$ then ${}^H_H\mathcal{YD}$ is a semisimple category and the irreducible objects are given as follows: let $g \in \Gamma$ and $\rho : \Gamma_g \to \operatorname{Aut}(W) \in \widehat{\Gamma_g}$ an irreducible representation. Then the induced representation

$$M(g, \rho) = \operatorname{Ind}_{\Gamma_g}^{\Gamma} W = \mathbf{k}\Gamma \otimes_{\mathbf{k}\Gamma_g} W$$

with the usual module structure and the comodule structure given by

$$\delta(h \otimes w) = hgh^{-1} \otimes (h \otimes w) \in \mathbf{k}\Gamma \otimes M(g, \rho)$$

is an irreducible YD-module. Furthermore, let $\{g_1, \dots, g_n\}$ be a subset of Γ consisting of one element for each conjugacy class, and let $\{\rho_i^1, \dots, \rho_i^{m_i}\}$ be the irreducible representations of Γ_{g_i}. Then the objects $M(g_i, \rho_i^j)$ ($1 \le i \le n$, $1 \le j \le m_i$) gives a full collection of non-isomorphic irreducible objects of ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$.

We extend the definition of $M(g, \rho)$ to the case in which ρ is not irreducible. Observe that $\dim M(g, \rho) = \deg \rho \times [\Gamma : \Gamma_g]$. Hence, if Γ is abelian, the irreducible modules of ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ are 1-dimensional. In this case, let $V = \oplus_i M(h_i, \chi_i)$ where $h_i \in \Gamma$ and $\chi_i \in \hat{\Gamma}$, and denote by x_i a generator of $M(h_i, \chi_i)$. By the definition of the braiding,

$$c(x_i \otimes x_j) = \chi_j(h_i) x_j \otimes x_i. \tag{1.1.1}$$

Let $q \in \mathbf{k}$. For $n \ge m \in \mathbb{N}$, we use the standard notation

$$(n)_q = \sum_{i=0}^{n-1} q^i, \quad (n)_q^! = \prod_{i=1}^{n} (i)_q, \quad \binom{n}{m}_q = \frac{(n)_q^!}{(m)_q^! (n-m)_q^!}.$$

Let H be a graded Hopf algebra, $H = \oplus_i H(i)$. Let $\pi : H \to H(0)$ and $\iota : H(0) \to H$ be respectively the canonical projection and the canonical inclusion. Then

$$R = H^{\operatorname{co}\pi} = H^{\operatorname{co} H(0)} = \{x \in H \mid (\operatorname{id} \otimes \pi)\Delta(x) = x \otimes 1\}$$

is a braided Hopf algebra in ${}^{H(0)}_{H(0)}\mathcal{YD}$. Furthermore, it is graded: $R = \oplus_i R(i)$, with $R(i) = R \cap H(i)$. The algebra H can be reconstructed as a bosonization, or biproduct (see e.g. [AG])

$$H = R \# H(0).$$

A braided Hopf algebra R in ${}^{K}_{K}\mathcal{YD}$ (K a Hopf algebra) is called *Nichols algebra* if

(1.1.2) $R = \oplus_i R(i)$ is a graded braided Hopf algebra,

(1.1.3) $R(0) = \mathbf{k}1$,

(1.1.4) $\mathcal{P}(R) = R(1)$,

(1.1.5) $R(1)$ generates R,

where $\mathcal{P}(R)$ stands for the space of primitive elements of R. It is not known whether in characteristic 0 and for R finite dimensional the condition (1.1.5), resp. (1.1.4), is superfluous, though there are some results in this direction. For instance, in [AS1, Thm. 3.2] is proved that if $\dim R(1) = 1$ (R finite dimensional and $\operatorname{char} \mathbf{k} = 0$) then ((1.1.2)–(1.1.4))$\Rightarrow$(1.1.5). A Nichols algebra R is uniquely determined by its primitive elements $\mathcal{P}(R)$. We use hence the notation $R = \mathfrak{B}(V)$ for the Nichols algebra R with $R(1) = V$, whenever $V \in {}^{K}_{K}\mathcal{YD}$. We specify sometimes the graduation of $\mathfrak{B}(V)$ by

$$\mathfrak{B}(V) = \oplus_i \mathfrak{B}^i(V).$$

The construction $V \mapsto \mathfrak{B}(V)$ can be made with the so called *quantum-shuffles* (see for instance [AG]), though this construction does not depend on the braided category ${}^{K}_{K}\mathcal{YD}$, but only on the pair (V, c). The quantum-shuffles construction applied to any pair (V, c) ($c \in \operatorname{End}(V \otimes V)$ a solution of the braid equation), the result has been called *Quantum Shuffle Algebra* in [Ros]. Nichols algebras have some particularly useful properties which distinguish them from this general concept of Quantum Shuffle Algebras, e.g. the kind of Poincaré duality stated in 2.7.

We shall use the following notation for finite abelian groups: let $\Gamma = C_{n_1} \times \cdots \times C_{n_t}$. We denote by g_j a generator of the j-th factor, i.e.

Γ can be presented by generators g_j and relations $g_j^{n_j} = [g_j, g_k] = 1$. Furthermore, let $\alpha_j = -1, \mathbf{i}, \xi, \sqrt{\xi}$ depending whether $n_j = 2, 4, 8, 16$ (recall that $\xi^2 = \mathbf{i}$, $\mathbf{i}^2 = -1$). We denote by $\hat{g}_i$ the character of Γ given by $\hat{g}_i(g_j) = \alpha_i^{\delta_{ij}}$. Then $\hat{\Gamma} \simeq \Gamma$ with generators $\hat{g}_i$, $i = 1, \dots, t$.

If $x_1, \dots, x_r$ are vectors in a space V, we denote by $(x_1, \dots, x_r)$ the linear span of them. If $g_1, \dots, g_r$ are elements in Γ, we denote by $(g_1, \dots, g_r)$ the subgroup generated by them.

If $V \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ and Γ is abelian then $V = \oplus_{i=1}^{\theta} M(h_i, \chi_i)$. We denote by x_i a generator of $M(h_i, \chi_i)$, whence $V = (x_1, \dots, x_\theta)$ and c is given by the matrix (b_{ij}), with $b_{ij} = \chi_j(h_i)$.

1.2. **Lifting Procedure.** Let A be a finite dimensional pointed Hopf algebra. Let $A_0 \subset A_1 \subset \dots$ be its coradical filtration, and consider

$$H = \oplus_i H(i) = \operatorname{gr} A = \oplus_i (A_i / A_{i-1}) \quad (A_{-1} = 0).$$

It is a coradically graded Hopf algebra, i.e. $H_n = \oplus_{i=0}^{n} H(i)$. Take $R = H^{\operatorname{co} H(0)}$. This is a coradically graded Hopf algebra in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$, where $\Gamma = G(A)$ is the group of group-likes of A (and hence $H(0) = \mathbf{k}\Gamma$). Notice that the order of Γ divides the dimension of A, by virtue of the Nichols–Zoeller theorem. Furthermore, $R(0) = \mathbf{k}1$, from where R satisfies (1.1.2), (1.1.3) and (1.1.4). If moreover R satisfies (1.1.5) then it is a Nichols algebra, and hence $R = \mathfrak{B}(R(1))$. Since R is an invariant of A, we shall consider its "infinitesimal part" defined by

$$V = V(A) = R(1), \tag{1.2.1}$$

and call $\dim V$ *the rank of* A. It is clear that the rank of A is equal to $\dim A_1 / \dim A_0 - 1$.

The lifting procedure goes the other way: let $n \in \mathbb{N}$ be fixed. We want to classify the n-dimensional pointed Hopf algebras over $\mathbf{k}$. We can proceed as follows: for each group Γ such that $|\Gamma|$ divides n,

1. Determine the modules $V \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ such that $\dim \mathfrak{B}(V) \le \dfrac{n}{|\Gamma|}$.
2. Classify the isomorphism classes of the bosonizations $\mathfrak{B}(V)\#\mathbf{k}\Gamma$, where V runs over the modules in step 1. Note that two bosonizations $\mathfrak{B}(V)\#\mathbf{k}\Gamma$ and $\mathfrak{B}(V)'\#\mathbf{k}\Gamma$ may be isomorphic even though $\mathfrak{B}(V)$ and $\mathfrak{B}(V)'$ (thus, V and V') are not isomorphic (see 2.16).

3. For each isomorphism class $\mathfrak{B}(V)\#\mathbf{k}\Gamma$ in step 2, classify all the liftings, i.e. all the pointed Hopf algebras (up to isomorphism) such that $\operatorname{gr} A = \mathfrak{B}(V)\#\mathbf{k}\Gamma$.
4. Prove, if possible, that these are all the n-dimensional pointed Hopf algebras. Equivalently, show that a graded braided Hopf algebra R in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ such that $\dim R = n/|\Gamma|$ and satisfying (1.1.2)–(1.1.4), also satisfies (1.1.5) (for this purpose it is useful to classify those V in step 1 such that $\dim \mathfrak{B}(V) < n/|\Gamma|$).

1.3. **Contents.** The article is organized as follows: in section 2 we compute the Nichols algebras that appear applying the procedure above for 32-dimensional pointed Hopf algebras.

From 3 to 6.6 we follow the steps 1, 2 and 3 of the procedure for each of the groups of order 2^m, $m \le 4$. In section 7 we prove the statement in step 4 of the procedure.

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Our main references on Hopf algebras are [S] and [M]. We refer to the survey article [AG] for definitions and properties of braided Hopf algebras, Nichols algebras (called there TOBAs) and for further references on the subject.

2. The Nichols algebras we shall need

For future reference, we give two families of examples of Nichols algebras. The first one corresponds to Taft algebras, and the second one to the more general case of Quantum Linear Spaces (QLS for brevity).

Example 2.1. Let V be 1-dimensional and let x be a generator of V. Suppose $c(x \otimes x) = qx \otimes x$, where $q \in \mathbf{k}^{\times}$. If q is a root of unity of order $n > 1$, or if $q = 1$ and $\operatorname{char} \mathbf{k} = n > 0$, then $\mathfrak{B}(V) = \mathbf{k}[x]/(x^n)$ as algebra (and hence $\dim \mathfrak{B}(V) = n$). If q is not a root of unity (which can happen for instance for $V \in {}^{\mathbf{k}\mathbb{Z}}_{\mathbf{k}\mathbb{Z}}\mathcal{YD}$), or if $q = 1$ and $\operatorname{char} \mathbf{k} = 0$, then $\mathfrak{B}(V) = \mathbf{k}[x]$ and hence it is infinite dimensional. The comultiplication in both cases is given by $\Delta(x^d) = \sum_{i=0}^{d} \binom{d}{i}_q x^i \otimes x^{d-i}$.

Remark 2.2. The above example shows the following useful result: let H be a finite dimensional Hopf algebra and $R \in {}^{H}_{H}\mathcal{YD}$ a braided Hopf algebra of dimension N. Suppose $\operatorname{char}\mathbf{k}$ does not divide N. If $x \in \mathcal{P}(R)$ is such that $\mathbf{k}x$ is a sub-YD-module of R and $c(x \otimes x) = qx \otimes x$, where $q \in \mathbf{k}^\times$, then $q \neq 1$.

Proof. Suppose $q = 1$. If $\operatorname{char}\mathbf{k} = 0$ then R would be infinite dimensional, if $\operatorname{char}\mathbf{k} = p > 0$ then x would generate an algebra of dimension a power of p, but then, taking the bosonization of both algebras, p would divide N. $\square$

Definition 2.3. Let $q \in \mathbf{k}^\times$. We define $N(q)$ by

$$N(q) = \begin{cases} \text{order of } q & \text{if } q \neq 1 \text{ and is a root of unity,} \\ \infty & \text{if } q \text{ is not a root of unity,} \\ \infty & \text{if } q = 1 \text{ and } \operatorname{char}\mathbf{k} = 0, \\ \operatorname{char}\mathbf{k} & \text{if } q = 1 \text{ and } \operatorname{char}\mathbf{k} > 0. \end{cases}$$

Definition 2.4. Let $V \in {}^H_H\mathcal{YD}$ be of dimension θ. We shall say that V (or c) *has a matrix* if there exists a basis $\{x_1, \dots, x_\theta\}$ of V and a matrix $(b_{ij}) \in M_\theta(\mathbf{k}^\times)$ such that the braiding c is given by

$$c(x_i \otimes x_j) = b_{ij} x_j \otimes x_i.$$

When we want to specify the basis, we say that V has a matrix (b_{ij}) *in the basis* $\{x_1, \dots, x_\theta\}$. If $H = \mathbf{k}\Gamma$ and Γ is abelian, then every finite dimensional module has a matrix (see (1.1.1)).

Let $V \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ be finite dimensional. We shall say that V (or c) *comes from the abelian case* if there exists an invariant abelian subgroup $\Gamma' \subset \Gamma$ such that the image of $\delta : V \to \mathbf{k}\Gamma \otimes V$ is included in $\mathbf{k}\Gamma' \otimes V$. In this case, V can be considered as a module in ${}^{\mathbf{k}\Gamma'}_{\mathbf{k}\Gamma'}\mathcal{YD}$ (thus, it has a matrix) and the bosonization $\mathfrak{B}(V)\#\mathbf{k}\Gamma$ can be reconstructed as an extension of Γ/Γ' by $\mathfrak{B}(V)\#\mathbf{k}\Gamma'$. We note that the condition on Γ' to be invariant can be dropped: it is not difficult to see that if there exists Γ' abelian such that the image of δ lies in $\mathbf{k}\Gamma' \otimes V$, then there exists $\Gamma'' \subset \Gamma'$ invariant (and clearly abelian) such that the image of δ lies in $\mathbf{k}\Gamma'' \otimes V$.

Example 2.5. Let $V \in {}^H_H\mathcal{YD}$ be of dimension θ, and suppose V has a matrix (b_{ij}) in the basis $\{x_1, \dots, x_\theta\}$. Let $N_i = N(b_{ii})$, see 2.3. Then it can be proved (see [AS1, Prop. 3.5]) that

$$\dim \mathfrak{B}(V) \geq \prod_{i=1}^{n} N_i.$$

Furthermore, if $N_i < \infty$ for all i, then the equality holds if and only if

$$b_{ij} b_{ji} = 1 \text{ for all } i \neq j. \tag{2.6}$$

In this case, a PBW basis of $\mathfrak{B}(V)$ is given by

$$\{x_1^{n_1}\cdots x_\theta^{n_\theta} \mid 0 \le n_i < N_i\},$$

and the multiplication is given by the following relations:

$$x_i^{N_i} = 0, \quad x_jx_i = b_{ji}x_ix_j \text{ for } i<j.$$

Hence, as an algebra, $\mathfrak{B}(V)$ is a quantum linear space. Let now $H = \mathbf{k}\Gamma$. In order to agree with the notation of [AS1], we shall say that $\mathfrak{B}(V)$ is a *Quantum Linear Space* (or QLS) if V satisfies (2.6) and the lines $\mathbf{k}x_i$ are submodules of V in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$. If V can be considered as a module in ${}^{\mathbf{k}\Gamma'}_{\mathbf{k}\Gamma'}\mathcal{YD}$ for certain subgroup $\Gamma' \subset \Gamma$ (that is, if the image of the coaction δ lies in $\Gamma' \otimes V$) and it is a QLS in ${}^{\mathbf{k}\Gamma'}_{\mathbf{k}\Gamma'}\mathcal{YD}$ then we say that it is a QLS *over* Γ'.

We shall use an important result about Nichols algebras due to Nichols:

Proposition 2.7. *Let $R = \oplus_{i=0}^N R(i)$ be a Nichols algebra (in some braided rigid abelian category). Suppose that $R(N) \neq 0$. Then*

$$\dim R(i) = \dim R(N-i) \quad \forall i = 0,\dots,N.$$

Proof. See for instance [AG, Pr. 3.2.2]. □

Lemma 2.8. *Let A be a 2^n-dimensional pointed Hopf algebra with $A_0 = \mathbf{k}\Gamma$. Suppose $|\Gamma| \ge 2^{n-2}$ and let $V = V(A) \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ be the module defined by* (1.2.1). *Then* $\dim V \le 2$.

Proof. Since $2^n = |\Gamma| \times \dim \mathfrak{B}(V)$ we have $\dim \mathfrak{B}(V) \le 4$; but if $\dim V \ge 3$, then $\dim \mathfrak{B}(V) \ge 5$ by 2.7. □

In [N] the author introduces derivations for Nichols algebras, which turns out to be a useful tool for computing bases (and hence their dimensions).

Definition 2.9. Let $V \in {}^H_H\mathcal{YD}$. For $i + j = n$, we denote by

$$\Delta_{i,j} : \mathfrak{B}^n(V) \to \mathfrak{B}^i(V) \otimes \mathfrak{B}^j(V)$$

the (i,j)-component of the comultiplication of $\mathfrak{B}(V)$. It is proved in [Sch] (or see [AG, Def. 3.2.10]) that $\Delta_{i,j}$ is injective $\forall i,j$. Let $\{x_1,\dots,x_\theta\}$ be a basis of V and let $\{x_1^*,\dots,x_\theta^*\}$ be its dual basis. We denote by ∂_{x_i} the differential operator on $\mathfrak{B}(V)$ given by

$$\partial_{x_i}(z) = (\mathrm{id}\otimes x_i^*)\Delta_{n-1,1}(z), \quad \text{if } z \in \mathfrak{B}^n(V),\ n>0, \quad \text{and } \partial_{x_i}(1) = 0.$$

By the injectivity of $\Delta_{i,j}$ it is clear that for $z \in \mathfrak{B}^n(V)$ $(n > 0)$ we have $z = 0$ if and only if $\partial_{x_i}(z) = 0$ for all $i = 1, \dots, \theta$. Suppose now that $V \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ and some x_i^* vanishes on the homogeneous components of degree $\neq g_i$ for some $g_i \in \Gamma$ (that is, $x_i^*(y) = 0$ if $\delta(y) = h \otimes y$ and $h \neq g_i$). Then it is easy to see that ∂_{x_i} verifies the Leibniz rule

$$\partial_{x_i}(z_1 z_2) = \partial_{x_i}(z_1)(g_i \rightharpoonup z_2) + z_1 \partial_{x_i}(z_2).$$

Since usually we shall take bases $\{x_1, \dots, x_\theta\}$ of homogeneous elements (i.e. $\delta(x_i) = g_i \otimes x_i$ for some $g_i \in \Gamma$), the previous condition will hold for all x_i^*.

Definition 2.10. Let A be a Hopf algebra. We shall denote by Ad the adjoint action

$$\mathrm{Ad}_x(y) = x_{(1)} y \mathcal{S}(x_{(2)}).$$

Let B be a braided Hopf algebra with multiplication μ, let $c \in \mathrm{End}(B \otimes B)$ be the braiding. We shall denote by Ad the braided adjoint action, given by

$$\mathrm{Ad}_x(y) = \mu(\mu \otimes \mathcal{S})(\mathrm{id} \otimes c)(\Delta \otimes \mathrm{id})(x \otimes y).$$

Note that if $x \in \mathcal{P}(B)$ then the adjoint is nothing but $\mathrm{Ad}_x(y) = \mu(\mathrm{id} - c)(x \otimes y)$. Since in the cases we are interested in the endomorphism c is given by the context, we do not make a reference to c in the notation.

We need the following relation between both adjoint actions: let H be a graded Hopf algebra, $\pi : H \to H(0)$ the canonical projection and $B \in {}^{H(0)}_{H(0)}\mathcal{YD}$ the braided Hopf algebra of coinvariants (then $H \simeq B \# H(0)$). Let $x \in B(1)$ (it is a primitive element). Then it is straightforward to see that

$$\mathrm{Ad}_{(b\#1)}(c\#1) = (\mathrm{Ad}_b(c))\#1 \quad \forall c \in B,$$

where the left Ad is the classical one, while the right Ad is the braided one.

We shall compute now the Nichols algebras appearing in next sections.

Proposition 2.11. *Let $V \in {}^H_H\mathcal{YD}$ with basis $\{x, y\}$. Suppose that c has a matrix (b_{ij}) in this basis, and suppose that $b_{12}b_{21} \neq 1$, $b_{11} \neq 1$, $b_{22} \neq 1$. Let $z = \mathrm{Ad}_x(y) = xy - b_{12}yx$. Let $N_i = N(b_{ii})$ (see 2.3) and suppose that $N_i < \infty$ (i.e., that b_{ii} is a root of unity, or $b_{ii} = 1$ and $\mathrm{char}\,\mathbf{k} > 0$). Let $q = b_{11}b_{12}b_{21}b_{22}$.*

1. *There exists n_{12} (respectively n_{21}) such that $(\mathrm{Ad}_x)^{n_{12}}(y) = 0$ (respectively $(\mathrm{Ad}_y)^{n_{21}}(x) = 0$). Let n_{12} (resp. n_{21}) be the least positive integer with such property. Then $n_{12} \leq N_1$ and $n_{21} \leq N_2$. Suppose that there exists m_{12} (resp. m_{21}) such that $b_{12}b_{21}b_{11}^{m_{12}} = 1$ (resp. $b_{12}b_{21}b_{22}^{m_{21}} = 1$) and*

let m_{12} (resp. m_{21}) be the least non negative integer with such property. Then $n_{12} = m_{12} + 1$ (resp. $n_{21} = m_{21} + 1$).

2. *If $q = 1$ and $\operatorname{char} \mathbf{k} = 0$ then $z^n \neq 0$ $\forall n$ and then $\mathfrak{B}(V)$ is infinite dimensional. If $q = 1$ and $\operatorname{char} \mathbf{k} \neq 0$ then $\dim \mathfrak{B}(V) \geq N_1 N_2 \operatorname{char} \mathbf{k}$.*
3. *Suppose $b_{11} = b_{22} = -1$ and let $N = N(q)$. Then $\mathfrak{B}(V)$ has dimension $4N$, with a PBW basis $\{x^a y^b z^c \mid 0 \leq a, b < 2,\ 0 \leq c < N\}$.*

Proof. 1. We compute inductively the derivatives of $(\mathrm{Ad}_x)^n(y)$ (see 2.9):

$$\partial_x((\mathrm{Ad}_x)^1(y)) = \partial_x(xy - b_{12}yx) = 0.$$

Suppose that $\partial_x((\mathrm{Ad}_x)^n(y)) = 0$, then

$$\begin{aligned}
\partial_x((\mathrm{Ad}_x)^{n+1}(y)) &= \partial_x\left(x(\mathrm{Ad}_x)^n(y) - b_{11}^n b_{12}(\mathrm{Ad}_x)^n(y)x\right) \\
&= b_{11}^n b_{12}(\mathrm{Ad}_x)^n(y) - b_{11}^n b_{12}(\mathrm{Ad}_x)^n(y) = 0, \\
\partial_y((\mathrm{Ad}_x)^1(y)) &= \partial_y(xy - b_{12}yx) = (1 - b_{12}b_{21})x.
\end{aligned}$$

Let $\sigma_n = (1 - b_{12}b_{21}b_{11}^n)$. Suppose that $\partial_y((\mathrm{Ad}_x)^n(y)) = \sigma_0\sigma_1 \cdots \sigma_{n-1}x^n$. Then

$$\begin{aligned}
\partial_y((\mathrm{Ad}_x)^{n+1}(y)) &= \partial_y(x(\mathrm{Ad}_x)^n(y) - b_{11}^n b_{12}(\mathrm{Ad}_x)^n(y)x) \\
&= \sigma_0 \cdots \sigma_{n-1}x^{n+1} - b_{11}^n b_{12}\sigma_0 \cdots \sigma_{n-1}b_{21}x^{n+1} \\
&= \sigma_0 \cdots \sigma_n x^{n+1}.
\end{aligned}$$

Thus, since $x^{N_1} = 0$, we have

$$\partial_x((\mathrm{Ad}_x)^{N_1}(y)) = \partial_y((\mathrm{Ad}_x)^{N_1}(y)) = 0,$$

and hence $(\mathrm{Ad}_x)^{N_1}(y) = 0$, which proves that n_{12} exists and $n_{12} \leq N_1$. Furthermore, if $b_{12}b_{21}b_{11}^{m_{12}} = 1$, we see that

$$\partial_x((\mathrm{Ad}_x)^{m_{12}+1}(y)) = \partial_y((\mathrm{Ad}_x)^{m_{12}+1}(y)) = 0,$$

whence $n_{12} \leq m_{12} + 1$ (notice that if m_{12} exists, then $m_{12} < N_1$). Moreover, the elements x^i do not vanish for $i < N_1$, whence if m_{12} exists then $\partial_y((\mathrm{Ad}_x)^{m_{12}}(y)) \neq 0$, from where $n_{12} > m_{12}$ and thus n_{12} is exactly $m_{12} + 1$.

2. We compute the derivatives of z^n. We do it for a general $q = b_{11}b_{12}b_{21}b_{22}$ (keep the notation $\sigma_n = (1 - b_{12}b_{21}b_{11}^n)$).

$$\begin{aligned}
\partial_x(z) &= \partial_x(\mathrm{Ad}_x\, y) = 0, \\
\partial_y(z) &= \partial_y(\mathrm{Ad}_x\, y) = \sigma_0 x, \\
\partial_x\partial_y(z) &= \partial_x(\sigma_0 x) = \sigma_0,
\end{aligned}$$

$$
\begin{aligned}
\partial_y(z^n) &= \sum_{i=0}^{n-1} (b_{21}b_{22})^i \sigma_0 z^{n-1-i} x z^i, \\
\partial_x\partial_y(z^n) &= \sum_{i=0}^{n-1} (b_{21}b_{22})^i (b_{11}b_{12})^i \sigma_0 z^{n-1} \\
&= (n)_q \sigma_0 z^{n-1}.
\end{aligned}
$$

Let now $q = 1$. If $\operatorname{char} \mathbf{k} = 0$, we see inductively that $z^n \neq 0\ \forall n$. Furthermore, since $z^n \in \mathfrak{B}^{2n}(V)$, the set $\{z^n \mid n \geq 0\}$ is linearly independent, and thus $\mathfrak{B}(V)$ is infinite dimensional. If $\operatorname{char} \mathbf{k} > 0$, we see inductively that $z^n \neq 0\ \forall n < \operatorname{char} \mathbf{k}$. Furthermore, with the same computations as in 3 below, we see that the set

$$
\{x^a y^b z^c \mid 0 \leq a < N_1,\ 0 \leq b < N_2,\ 0 \leq c < \operatorname{char} \mathbf{k}\}
$$

is linearly independent, whence $\dim \mathfrak{B}(V) \geq N_1 N_2 \operatorname{char} \mathbf{k}$.

3. By the same computations than in 2, we have that the set $\{z^i \mid 0 \leq i < N\}$ is linearly independent. We first prove that $z^N = 0$:

$$
\begin{aligned}
\partial_x(z^N) &= 0 \quad \text{(as in 2)} \\
\partial_y(z^N) &= \sum_{i=0}^{N-1} (b_{21}b_{22})^i \sigma_0 z^{N-1-i} x z^i,
\end{aligned}
$$

but this element vanishes in $\mathfrak{B}(V)$, since

$$
\begin{aligned}
\partial_x(\partial_y z^N) &= (N)_q \sigma_0 z^{N-1} = 0 \quad \text{(as in 2)} \\
\partial_y(\partial_y z^N) &= \sum_{i=0}^{N-1} (b_{21}b_{22})^i \sigma_0 [\sum_{j=0}^{N-2-i} (b_{21}b_{22})^{j+i} b_{21} z^{N-2-i-j} x z^j x z^i \\
&\qquad + \sum_{j=0}^{i-1} (b_{21}b_{22})^j z^{N-1-i} x z^{i-j-1} x z^j] \\
&= \sum_{k+l \leq N-2} (b_{21}b_{22})^{k+2l} \sigma_0 b_{21}(1+b_{22}) z^{N-2-k-l} x z^k x z^l \\
&= 0 \text{ (since } b_{22} = -1).
\end{aligned}
$$

Now, since $b_{11} = b_{22} = -1$, we have $x^2 = y^2 = 0$. Furthermore,

$$
\begin{aligned}
yx &= b_{12}^{-1} xy - b_{12}^{-1} z, \\
zx &= (xy - b_{12}yx)x = xyx = -b_{12}^{-1} x(xy - b_{12}yx) = -b_{12}^{-1} xz, \\
zy &= (xy - b_{12}yx)y = -b_{12}yxy = -b_{12}y(xy - b_{12}yx) = -b_{12}yz.
\end{aligned}
$$

Thus the set $\{x^a y^b z^c \mid 0 \leq a, b < 2,\ 0 \leq c < N\}$ generates $\mathfrak{B}(V)$ as a vector space. Let $\alpha_{b,c} \in \mathbf{k}$ such that

$$\sum_{\substack{0 \leq b < 2 \\ 0 \leq c < N}} \alpha_{b,c} y^b z^c = 0.$$

We compute

$$\begin{aligned} \partial_y(yz^c) &= (b_{21}b_{22})^c z^c + y\partial_y z^c, \\ \partial_x\partial_y(y) &= 0, \\ \partial_x\partial_y(yz^c) &= (c)_q \sigma_0 y z^{c-1} \quad (c > 0). \end{aligned}$$

Applying $(\partial_x\partial_y)^{N-1}$, we see that $\alpha_{0,N-1} = \alpha_{1,N-1} = 0$. Inductively, we see that $\alpha_{b,c} = 0\ \forall b, c$. In the same vein, let $\alpha_{a,b,c} \in \mathbf{k}$ such that

$$\sum_{\substack{0 \leq a, b < 2 \\ 0 \leq c < N}} \alpha_{a,b,c} x^a y^b z^c = 0.$$

We compute

$$\partial_x(xy^b z^c) = b_{11}^c b_{22}^{b+c} y^b z^c,$$

from where we see that $\alpha_{1,b,c} = 0\ \forall b, c$, and thus $\alpha_{a,b,c} = 0\ \forall a, b, c$, which proves the statement.

□

Remark 2.12. It can be proved in general that if V has a matrix (b_{ij}) in a basis $\{x_1, \dots, x_\theta\}$, then the operator $(\partial_{x_i})^{N(b_{ii})}$ vanishes. Furthermore, the duality of [AG, Prop. 3.2.30] translates products into compositions.

Definition 2.13. Let V be a θ-dimensional YD module, and suppose V has a matrix (b_{ij}). According to [AS2], we say that V *is of Cartan type* (or CT) if there exists a matrix $(a_{ij}) \in M_\theta(\mathbb{Z})$ such that

$$b_{ij}b_{ji} = b_{ii}^{a_{ij}} \quad \forall i, j.$$

If we choose $a_{ii} = 2$ and $-$order of $b_{ii} < a_{ij} \leq 0$ for $i \neq j$, then (a_{ij}) is a generalized Cartan matrix. We transfer to V the terminology on (a_{ij}).

Corollary 2.14.

1. *Let $V \in {}^H_H\mathcal{YD}$, and suppose that V has one of the following matrices in the basis $\{x, y\}$ (recall from the beginning of section 1.1 that $\mathbf{i}, \xi$ denote fixed roots of unity of orders 4 and 8 respectively).*

$$b_q^1 = \begin{pmatrix} -1 & q^{-1} \\ q & -1 \end{pmatrix} \ (q \in \mathbf{k}^\times), \qquad b^2 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix},$$

$$b^3_{\pm+} = \begin{pmatrix} -1 & 1 \\ \pm\mathbf{i} & -1 \end{pmatrix}, \qquad b^3_{\pm-} = \begin{pmatrix} -1 & -1 \\ \pm\mathbf{i} & -1 \end{pmatrix},$$

$$b^4_{\pm} = \begin{pmatrix} \pm\mathbf{i} & \pm\mathbf{i} \\ -1 & -1 \end{pmatrix}, \qquad b^5_{\pm} = \begin{pmatrix} \pm\mathbf{i} & \mp\mathbf{i} \\ -1 & -1 \end{pmatrix},$$

$$b^6_k = \begin{pmatrix} -1 & 1 \\ \xi^k & -1 \end{pmatrix} \qquad (k = 1, 3, 5, 7).$$

The information about $\mathfrak{B}(V)$ that we need is given in the following table. We put $z_1 = \mathrm{Ad}_x(y)$ or $z_1 = \mathrm{Ad}_y(x)$ depending on the case.

TABLE 1: Matrices and algebras

b	$\dim \mathfrak{B}(V)$	*CT?*	z_1	$\lvert x\rvert$	$\lvert y\rvert$	$\lvert z_1\rvert$
b^1_q	4	*Yes*	0	2	2	
b^2	8	*Yes*	$\mathrm{Ad}_x(y) = xy + yx$	2	2	2
$b^3_{\pm+}$	16	*No*	$\mathrm{Ad}_x(y) = xy + yx$	2	2	4
$b^3_{\pm-}$	16	*No*	$\mathrm{Ad}_x(y) = xy + yx$	2	2	4
$b^4_{\pm}$	16	*No*	$\mathrm{Ad}_y(x) = xy + yx$	4	2	2
$b^5_{\pm}$	$\geq 8N(1) \geq 24$	*No*	$\mathrm{Ad}_y(x) = xy + yx$	4	2	$N(1)$
b^6_k	32	*No*	$\mathrm{Ad}_y(x) = xy - yx$	2	2	8

We denote by $|w|$ the nilpotency order of w, i.e. $w^{|w|} = 0,\ w^{|w|-1} \neq 0$.

2. *Suppose V is 3-dimensional and has a matrix $b^1_{q_1,q_2,q_3}$ in the basis $\{x, y, z\}$, where*

$$b^1_{q_1,q_2,q_3} = \begin{pmatrix} -1 & q_1^{-1} & q_3^{-1} \\ q_1 & -1 & q_2^{-1} \\ q_3 & q_2 & -1 \end{pmatrix} \qquad (q_1, q_2, q_3 \in \mathbf{k}^\times).$$

Hence $\mathfrak{B}(V)$ is an 8-dimensional QLS, with relations

$$x^2 = y^2 = z^2 = 0, \qquad yx = q_1xy,\ zx = q_3xz,\ zy = q_2yz.$$

Proof. The cases b^1_q, $b^1_{q_1,q_2,q_3}$ are particular cases of example 2.5.

The cases b^2, $b^3_{\pm+}$, $b^3_{\pm-}$, $b^5_{\pm}$ and b^6_k are particular cases of 2.11.

For $b^4_{\pm}$, it is straightforward to see that

$$zy = yz, \qquad zx = -\mathbf{i}xz,$$

and, using the same techniques than in 2.11, that there is a PBW basis $\{x^ay^bz^c \mid 0 \leq a < 4,\ 0 \leq b, c < 2\}$. ∎

Remark 2.15. Let V be an object in $\mathcal{C}$, a braided abelian category. Suppose V has a matrix

$$b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

in the basis $\{x, y\}$. Then it is clear that V has the matrix

$$[b]^{\rho} := \begin{pmatrix} b_{22} & b_{21} \\ b_{12} & b_{11} \end{pmatrix}$$

in the basis $\{y, x\}$.

Consider now $\overline{\mathcal{C}}$ the category with same objects and morphisms as $\mathcal{C}$ but with the inverse braiding $\bar{c} = c^{-1}$. Consider $\bar{V} = V$ in $\overline{\mathcal{C}}$. Then $\bar{V}$ has a matrix

$$[b]^{\tau} := \begin{pmatrix} b_{11}^{-1} & b_{21}^{-1} \\ b_{12}^{-1} & b_{22}^{-1} \end{pmatrix}$$

in the basis $\{x, y\}$. On the other hand, it can be shown that $\dim \mathfrak{B}(\bar{V}) = \dim \mathfrak{B}(V)$. Notice, for instance, that $[b^3_{\pm +}]^{\rho\tau} = b^3_{\mp +}$.

Let $M \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ and $f \in \operatorname{Aut}(\Gamma)$. We write $(M)^f$ for the YD-module with the same underlying space as M but with the structure given by

$$h \rightharpoonup^f m = f^{-1}(h) \rightharpoonup m, \quad \delta^f(m) = (f \otimes \mathrm{id})\delta(m).$$

Note that $M(g, \rho)^f = M(f(g), \rho \circ f^{-1})$. In order to classify non isomorphic algebras, the following criterium will be applied:

Lemma 2.16. *Let $V = \oplus_{i=1}^n M(g_i, \rho_i)$ and $V' = \oplus_{i=1}^{n'} M(g_i', \rho_i')$, where ρ_i and ρ_i' are irreducible. Then $\mathfrak{B}(V)\#\mathbf{k}\Gamma \simeq \mathfrak{B}(V')\#\mathbf{k}\Gamma$ if and only if $n = n'$ and there exist $f \in \operatorname{Aut}(\Gamma)$, $\sigma \in \mathbb{S}_n$ such that $M(g_i', \rho_i') \simeq (M(g_{\sigma(i)}, \rho_{\sigma(i)}))^f$.*

Proof. Mimic [AS2, Prop. 6.3], where it is proved for Γ abelian. □

For the last step of the lifting procedure we shall use the following Proposition, which gives an invariant way to lift the elements of the bosonized algebras.

Proposition 2.17. *Let A be a pointed Hopf algebra, $H = \operatorname{gr} A$ be its associated graded Hopf algebra and Γ the group of group-likes of A. Let $V = V(A) \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ be the YD-module associated to A as in* (1.2.1)*, that is, $V = (H(1))^{\mathrm{co} H(0)}$. We have $H(1) = V\#\mathbf{k}\Gamma$. Take on $H(1)$ the Γ action given by*

$$g \cdot (v\#h) = (g \rightharpoonup v)\#ghg^{-1}$$

Consider on A the adjoint action of Γ. It restricts to the coradical filtration, and then A_1 becomes a $\mathbf{k}\Gamma$-module. Let $p_0 : A_1 \to H(1) = A_1/A_0$ be the canonical projection. Then

1. *p_0 is a morphism of $\mathbf{k}\Gamma$-modules.*
2. *Let $\lambda \in \mathbf{k}^\Gamma$, $\lambda(g) = \delta_{g,1}$. Then $p = (\mathrm{id}\,\#\lambda) \circ p_0 : A_1 \to V$ is a morphism of $\mathbf{k}\Gamma$-modules.*
3. *There exists a section $s : V \to A_1$ of $\mathbf{k}\Gamma$-modules to p such that if $\delta(x) = g \otimes x$ then $s(x) \in \mathcal{P}_{g,1}(A)$.*

Proof. The items 1 and 2 are straightforward looking at the definition of the action $\rightharpoonup$. We prove 3. For $g, h \in \Gamma$, let $\mathcal{P}'_{g,h} \subset A_1$ be subspaces such that $\mathcal{P}_{g,h}(A) = \mathbf{k}(g-h) \oplus \mathcal{P}'_{g,h}$. Then, by the Taft–Wilson theorem,

$$A_1 = \mathbf{k}\Gamma \oplus (\bigoplus_{g,h\in\Gamma} \mathcal{P}'_{g,h}).$$

We have the exact sequence of vector spaces

$$0 \to \mathbf{k}\Gamma = A_0 \to A_1 \to V\#\mathbf{k}\Gamma \to 0.$$

Consider now the inclusion

$$i : \oplus_{g\in\Gamma}\mathcal{P}'_{g,1} \hookrightarrow A_1.$$

It is easy to verify that the composition $p \circ i$ is an isomorphism of vector spaces (a word of warning is needed here: $\oplus_{g\in\Gamma}\mathcal{P}'_{g,1}$ need not to be a $\mathbf{k}\Gamma$-submodule of A_1). Let $s_0 : V \to \oplus_{g\in\Gamma}\mathcal{P}'_{g,1}$, $s_0 = (p \circ i)^{-1}$, and consider as usual

$$s : V \to A_1, \quad s = \frac{1}{|\Gamma|}\sum_{h\in\Gamma}(h \cdot s_0), \text{ i.e. } s(x) = \frac{1}{|\Gamma|}\sum_{h\in\Gamma} h \cdot (s_0(h^{-1} \rightharpoonup x)).$$

It is clear that s is a $\mathbf{k}\Gamma$ section to p.

Let $V^g = \{x \in V \mid \delta(x) = g \otimes x\}$; then $V = \oplus_{g\in\Gamma}V^g$. Moreover, let $a \in \mathcal{P}'_{g,1}$. Then $\Delta a = g \otimes a + a \otimes 1$, and if $b = a + \mathbf{k}\Gamma \in A_1/A_0$ is the image of a under $A_1 \to A_1/A_0$, we have $\Delta b = g \otimes b + b \otimes 1$. Then $b \in (V\#\mathbf{k}\Gamma)^{\mathrm{co}H(0)}$, whence $b = v\#1$ for some $v \in V$ such that $\Delta(v\#1) = g \otimes (v\#1) + (v\#1) \otimes 1$. This implies that $\delta(v) = g \otimes v$, and hence $v = (\mathrm{id}\,\#\lambda)(v\#1) \in V^g$. Thus $p(\mathcal{P}'_{g,1}) \subseteq V^g$ for all $g \in \Gamma$. Now, since $p \circ i : \oplus_{g\in\Gamma}\mathcal{P}'_{g,1} \to \oplus_{g\in\Gamma}V^g$ is an isomorphism, we must have that $p \circ i|_{\mathcal{P}'_{g,1}} : \mathcal{P}'_{g,1} \to V^g$ is an isomorphism, whence $s_0(V^g) = \mathcal{P}'_{g,1}$ for all $g \in \Gamma$.

Let now $x \in V^g$. We claim that $s(x) \in \mathcal{P}_{g,1}(A)$. This is true for, by the definition of s, we have

$$\begin{aligned} s(x) &= \frac{1}{|\Gamma|}\sum_{h\in\Gamma} h\cdot s_0(h^{-1}\rightharpoonup x) \in \sum_{h\in\Gamma} h\cdot s_0(V^{h^{-1}gh}) = \sum_{h\in\Gamma} h\cdot \mathcal{P}'_{h^{-1}gh,1} \\ &\subseteq \sum_{h\in\Gamma} h\cdot \mathcal{P}_{h^{-1}gh,1}(A) = \sum_{h\in\Gamma}\mathcal{P}_{g,1}(A) = \mathcal{P}_{g,1}(A). \end{aligned}$$

□

We will use the above proposition as follows:

Tactic 2.18. If V has a basis $\{x_1,\dots,x_\theta\}$ such that $x_i \in V^{g_i}$, the following proposition enables us to consider $a_1,\dots,a_\theta$ such that $a_i \in \mathcal{P}_{g_i,1}(A)$, the projection $A_1 \to A_1/A_0$ sends a_i to $(x_i\#1) \in V\#\mathbf{k}\Gamma$, and if $g \in \Gamma$ acts on V by some matrix T in the basis $\{x_i\}$, then g acts on the linear span $(a_1,\dots,a_\theta)$ by the same matrix (in the basis $\{a_i\}$). Furthermore, since V generates $\mathfrak{B}(V)$, the a_i's together with Γ generate A. In general, suppose $\sum_i(\alpha_i x_{i,1}\cdots x_{i,n}) = 0 \in \mathfrak{B}^n(V)$. Then $b = \sum_i(\alpha_i a_{i,1}\cdots a_{i,n}) \in A$ lies in the kernel of the projection, i.e. $b \in A_{n-1}$. Usually, we shall have $b \in \mathcal{P}_{g,1}(A)$ for some $g \in \Gamma$, from where b shall lie in A_1.

The following is a very easy but useful remark:

Remark 2.19. Let A be a finite dimensional Hopf algebra such that $\operatorname{char}\mathbf{k}$ does not divide $\dim A$. Let $g \in A$ be a group-like and $a \in \mathcal{P}_{g,1}(A)$ such that $ga = \zeta ag$. Let n be the order of ζ. Then, by the same computations as in 2.1, $a^n \in \mathcal{P}_{g^n,1}(A)$. If $g^n = 1$ we must have $a^n = 0$, for, if not, a would generate a sub Hopf algebra of dimension a power of $\operatorname{char}\mathbf{k}$ (if $\operatorname{char}\mathbf{k} > 0$) or ∞ (if $\operatorname{char}\mathbf{k} = 0$), leading to a contradiction.

If we are lifting a QLS, we have $a_i \in \mathcal{P}_{g_i,1}(A)$ and $g_i a_j g_i^{-1} = b_{ij}a_j$. If $i \neq j$, $b_{ij}b_{ji} = 1$ and $g_ig_j = g_jg_i$. Then

$$\begin{aligned} \Delta(a_ia_j - b_{ij}a_ja_i) &= g_ig_j\otimes a_ia_j + g_ia_j\otimes a_i + a_ig_j\otimes a_j + a_ia_j\otimes 1 \\ &\quad - b_{ij}g_jg_i\otimes a_ja_i - b_{ij}a_jg_i\otimes a_i - b_{ij}g_ja_i\otimes a_j - b_{ij}a_ja_i\otimes 1 \\ &= g_ig_j\otimes(a_ia_j - b_{ij}a_ja_i) + (a_ia_j - b_{ij}a_ja_i)\otimes 1, \end{aligned}$$

from where $a_ia_j - b_{ij}a_ja_i \in \mathcal{P}_{g_ig_j,1}(A)$. As above, if $\operatorname{char}\mathbf{k} \nmid \dim A$ and $g_ig_j = 1$, then $a_ia_j - b_{ij}a_ja_i = 0$.

3. $\Gamma = C_2$

Let $V = \oplus_i M(h_i, \chi_i)$. For $\mathfrak{B}(V)$ to have dimension 16, we must have $b_{ii} = -1\ \forall i$, where $(b_{ij}) = (\chi_j(h_i))$ is the matrix of V. Thus, $V = \oplus_i M(g, \hat{g})$, and hence $(b_{ij}) = \hat{g}(g) = -1\ \forall ij$, from where $\mathfrak{B}(V)$ is a QLS with generators $x_1, \dots, x_\theta$ and relations $x_i^2 = 0$, $x_i x_j = -x_j x_i$. Thus, $\dim \mathfrak{B}(V) = 2^\theta$, and $\theta = 4$. Let $a_1, \dots, a_4$ be as in 2.18. Then $g a_i g = -a_i$ and $a_i \in \mathcal{P}_{g,1}(A)$. We have by 2.19 that $a_i^2 \in \mathcal{P}(A)$, and the same is true for $a_i a_j + a_j a_i$. Hence $a_i^2 = a_i a_j + a_j a_i = 0\ \forall i, j$. There exist thus only one Hopf algebra of dimension 32 with coradical $\mathbf{k}C_2$. It is clear, using the same reasoning, that there exists a Hopf algebra with coradical $\mathbf{k}C_2$ and dimension n if and only if $n = 2^m$ for some $m \in \mathbb{N}$. Furthermore, if $n = 2^m$, there exists only one such Hopf algebra. This is the content of [N, Thm. 4.2.1], but we included here a proof for completeness.

4. Γ with order 4

4.1. $\Gamma = C_2 \times C_2$. Necessarily the rank is 2 or 3: for rank 1 the root b_{11} has order 2, whence $\dim(\mathfrak{B}(V)\#\mathbf{k}\Gamma)$ would be 8, and rank ≥ 4 is impossible by 2.5. The following is a list of the irreducible YD-modules giving Nichols algebras of dimension 2, with the action of $\mathrm{Aut}(\Gamma)$ as in the lemma 2.16. In this case, $\mathrm{Aut}(\Gamma)$ is isomorphic to $\mathbb{S}_3$, generated by f_1 and f_2, where, using the notation at the end of 1.1,

$$f_1 = (g_1 \mapsto g_1 g_2,\ g_2 \mapsto g_2), \quad f_2 = (g_1 \mapsto g_1,\ g_2 \mapsto g_1 g_2).$$

Thus, we have

TABLE 2: modules / $\mathrm{Aut}(\Gamma)$, $\Gamma = C_2 \times C_2$

$M(h,\chi)$	h	χ	$(M(h,\chi))^{f_1}$	$(M(h,\chi))^{f_2}$
Y_2^1	g_1	$\hat{g}_1$	Y_2^5	Y_2^2
Y_2^2	g_1	$\hat{g}_1\hat{g}_2$	Y_2^6	Y_2^1
Y_2^3	g_2	$\hat{g}_2$	Y_2^4	Y_2^6
Y_2^4	g_2	$\hat{g}_1\hat{g}_2$	Y_2^3	Y_2^5
Y_2^5	$g_1 g_2$	$\hat{g}_1$	Y_2^1	Y_2^4
Y_2^6	$g_1 g_2$	$\hat{g}_2$	Y_2^2	Y_2^3

Consider then $V = M_1 \oplus M_2 = M(h_1, \chi_1) \oplus M(h_2, \chi_2)$. Since $\mathrm{Aut}(\Gamma)$ acts transitively, we may suppose that $M_1 = Y_2^1$. For rank 2 this gives 6 possibilities, but the cases $Y_2^1 \oplus Y_2^4$ and $Y_2^1 \oplus Y_2^6$ are equivalent by means of $f_1 \circ f_2$. Hence, the possibilities are

TABLE 3: Rank 2, $\Gamma = C_2 \times C_2$						
V	h_1	h_2	χ_1	χ_2	(b_{ij})	$\dim \mathfrak{B}(V)$
V_2^1	g_1	g_1	$\hat{g}_1$	$\hat{g}_1$	b^1_{-1}	4
V_2^2	g_1	g_1	$\hat{g}_1$	$\hat{g}_1\hat{g}_2$	b^1_{-1}	4
V_2^3	g_1	g_2	$\hat{g}_1$	$\hat{g}_2$	b^1_1	4
V_2^4	g_1	g_2	$\hat{g}_1$	$\hat{g}_1\hat{g}_2$	b^2	8
V_2^5	g_1	g_1g_2	$\hat{g}_1$	$\hat{g}_1$	b^1_{-1}	4

For rank 3, $V = \oplus_{i=1}^3 M_i = \oplus_{i=1}^3 M(h_i, \chi_i)$. Since the module V_2^4 must not appear as a submodule, we have the following possibilities:

TABLE 4: Rank 3, $\Gamma = C_2 \times C_2$								
V	h_1	h_2	h_3	χ_1	χ_2	χ_3	(b_{ij})	$\dim \mathfrak{B}(V)$
W_2^1	g_1	g_1	g_1	$\hat{g}_1$	$\hat{g}_1$	$\hat{g}_1$	$b^1_{-1,-1,-1}$	8
W_2^2	g_1	g_1	g_1	$\hat{g}_1$	$\hat{g}_1$	$\hat{g}_1\hat{g}_2$	$b^1_{-1,-1,-1}$	8
W_2^3	g_1	g_1	g_1g_2	$\hat{g}_1$	$\hat{g}_1$	$\hat{g}_1$	$b^1_{-1,-1,-1}$	8
W_2^4	g_1	g_1	g_2	$\hat{g}_1$	$\hat{g}_1$	$\hat{g}_2$	$b^1_{1,-1,-1}$	8

We compute now the liftings.

4.1.1. $V = V_2^4$. We take a_1 and a_2 as in 2.18. Hence we have

$$\Delta(a_1) = g_1 \otimes a_1 + a_1 \otimes 1, \quad g_1 a_1 = -a_1 g_1, \quad g_2 a_1 = a_1 g_2;$$
$$\Delta(a_2) = g_2 \otimes a_2 + a_2 \otimes 1, \quad g_1 a_2 = -a_2 g_1, \quad g_2 a_2 = -a_2 g_2.$$

Since $a_1^2 \in \mathcal{P}(A)$, we must have $a_1^2 = 0$ by 2.19. For the same reason, $a_2^2 = 0$. Take now

$$a_3 = \mathrm{Ad}_{a_1}(a_2) = a_1 a_2 + g_1 a_2 \mathcal{S}(a_1) = a_1 a_2 + a_2 a_1,$$

then

$$\begin{aligned}\Delta(a_3) &= g_1 g_2 \otimes a_1 a_2 + g_1 a_2 \otimes a_1 + a_1 g_2 \otimes a_2 + a_1 a_2 \otimes 1\\ &= g_1 g_2 \otimes a_1 a_2 - a_2 g_1 \otimes a_1 + a_1 g_2 \otimes a_2 + a_1 a_2 \otimes 1.\end{aligned}$$

It is straightforward to see that $a_3^2 \in \mathcal{P}(A)$, and thus $a_3^2 = 0$. Furthermore, it is clear from the definition of a_3 that

$$a_1 a_3 = a_3 a_1, \ a_2 a_3 = a_3 a_2,$$

and hence the unique lifting is the bosonization $\mathfrak{B}(V_2^2)\#\mathbf{k}\Gamma$.

4.1.2. $V = W_2^1$. As in 2.18, we can take a_1, a_2 and a_3 such that

$$\Delta(a_i) = g_1 \otimes a_i + a_i \otimes 1, \quad g_1 a_i = -a_i g_1, \quad g_2 a_i = a_i g_2.$$

We have $a_i^2 \in \mathcal{P}(A)$ and $a_i a_j + a_j a_i \in \mathcal{P}(A)$, whence $a_i^2 = a_i a_j + a_j a_i = 0\ \forall i,j$ by 2.19. Hence, there is only one lifting: the bosonization $\mathfrak{B}(W_2^1)\#\mathbf{k}\Gamma$.

4.1.3. $V = W_2^2$. This case is the same as the previous one, with the only difference that $g_2 a_3 = -a_3 g_2$, while $g_2 a_i = a_i g_2$ for $i = 1, 2$. The same arguments say that there is no other lifting than the bosonization $\mathfrak{B}(W_2^2)\#\mathbf{k}\Gamma$.

4.1.4. $V = W_2^3$. We have now $g_1 a_i = -a_i g_1$ and $g_2 a_i = a_i g_2$. The difference with the case $V = W_2^1$ is now that $\Delta(a_3) = g_1 g_2 \otimes a_3 + a_3 \otimes 1$. As before, $a_i^2 \in \mathcal{P}(A)$ and hence $a_i^2 = 0$ by 2.19. Furthermore, $a_1 a_2 + a_2 a_1 \in \mathcal{P}(A)$, from where $a_1 a_2 = -a_2 a_1$. However $a_i a_3 + a_3 a_i \in \mathcal{P}_{g_2,1}(A)$ for $i = 1, 2$ and hence, there exist λ_1, $\lambda_2 \in \mathbf{k}$ such that $a_i a_3 + a_3 a_i = \lambda_i(g_2 - 1)$. Now, let $W' = (a_1, a_2)$ and let $T \in \mathrm{GL}_2(\mathbf{k})$, $T = (T_{ij})$. If we replace the basis $\{a_1, a_2\}$ by $\{Ta_1, Ta_2\}$ (we denote also by T the automorphism of W' with matrix T in the basis $\{a_1, a_2\}$), then the constants $\binom{\lambda_1}{\lambda_2}$ are replaced by $T\binom{\lambda_1}{\lambda_2}$. Hence, there are 2 non isomorphic liftings: they can be characterized by the rank of the morphism $W' \to \mathbf{k}(g_2 - 1)$, $a \mapsto aa_3 + a_3 a$, which can be 0 or 1. Thus we have

1. The bosonization $\mathfrak{B}(W_2^3)\#\mathbf{k}\Gamma$.
2. The algebra with the above relations plus $a_1 a_3 + a_3 a_1 = g_2 - 1$, $a_2 a_3 + a_3 a_2 = 0$.

4.1.5. $V = W_2^4$. We have here $\Delta(a_i) = g_1 \otimes a_i + a_i \otimes 1$ for $i = 1, 2$, and $\Delta(a_3) = g_2 \otimes a_3 + a_3 \otimes 1$, plus $g_1 a_i = -a_i g_1$, $g_2 a_i = a_i g_2$ for $i = 1, 2$, $g_1 a_3 = a_3 g_1$, $g_2 a_3 = -a_3 g_2$. As before, $a_i^2 = 0$ for $i = 1, 2, 3$, and also $a_1 a_2 + a_2 a_1 = 0$. Furthermore, $a_i a_3 - a_3 a_i \in \mathcal{P}_{g_1 g_2, 1}(A)$, from where $a_i a_3 - a_3 a_1 = \lambda_i(g_1 g_2 - 1)$ for $i = 1, 2$. These last conditions may add some unwanted relation which would decrease the dimension of A. To see which λ_i's give a 32-dimensional algebra, we use the diamond lemma:

$$g_1 a_3 a_1 = g_1 a_1 a_3 + \lambda_1 g_1(g_1 g_2 - 1) = -a_1 a_3 g_1 + \lambda_1 g_2 - \lambda_1 g_1$$
$$g_1 a_3 a_1 = -a_3 a_1 g_1 = -a_1 a_3 g_1 - \lambda_1(g_1 g_2 - 1) g_1 = -a_1 a_3 g_1 - \lambda_1 g_2 + \lambda_1 g_1,$$

from where $\lambda_1 = 0$. An analogous work with the monomial $g_1 a_3 a_2$ tells that $\lambda_2 = 0$, and there is only one lifting: the bosonization $\mathfrak{B}(W_2^4)\#\mathbf{k}\Gamma$.

4.2. $\Gamma = C_4$. By similar considerations as for $\Gamma = C_2 \times C_2$, the rank must be 2 or 3. Here $\mathrm{Aut}(\Gamma) \simeq C_2$, generated by $f(g) = g^3$. The list of YD-modules giving Nichols algebras of dimension a power of 2 is

TABLE 5: modules / $\mathrm{Aut}(\Gamma)$, $\Gamma = C_4$				
$M(h,\chi)$	h	χ	$(M(h,\chi))^f$	$\dim \mathfrak{B}(M(h,\chi))$
Y_3^1	g	$\hat{g}$	Y_3^8	4
Y_3^2	g	$\hat{g}^2$	Y_3^7	2
Y_3^3	g	$\hat{g}^3$	Y_3^6	4
Y_3^4	g^2	$\hat{g}$	Y_3^5	2
Y_3^5	g^2	$\hat{g}^3$	Y_3^4	2
Y_3^6	g^3	$\hat{g}$	Y_3^3	4
Y_3^7	g^3	$\hat{g}^2$	Y_3^2	2
Y_3^8	g^3	$\hat{g}^3$	Y_3^1	4

Hence, for rank 2 we have the following non isomorphic possibilities (we exclude the cases $Y_3^i \oplus Y_3^j$ with $i, j \in \{1, 3, 6, 8\}$ by virtue of 2.5).

TABLE 6: Rank 2, $\Gamma = C_4$										
V	h_1	h_2	χ_1	χ_2	b_{11}	b_{12}	b_{21}	b_{22}	b_{ij}	$\dim \mathfrak{B}(V)$
V_3^1	g	g	$\hat{g}$	$\hat{g}^2$	$\mathbf{i}$	-1	$\mathbf{i}$	-1	$[b^4_-]^\tau$	64
V_3^2	g	g	$\hat{g}^3$	$\hat{g}^2$	$-\mathbf{i}$	-1	$-\mathbf{i}$	-1	$[b^4_+]^\tau$	64
V_3^3	g	g	$\hat{g}^2$	$\hat{g}^2$	-1	-1	-1	-1	b^1_{-1}	4
V_3^4	g	g^3	$\hat{g}^2$	$\hat{g}^3$	-1	$-\mathbf{i}$	-1	$\mathbf{i}$	$[b^5_-]^{\rho\tau}$	≥ 24
V_3^5	g	g^3	$\hat{g}^2$	$\hat{g}$	-1	$\mathbf{i}$	-1	$-\mathbf{i}$	$[b^5_+]^{\rho\tau}$	≥ 24
V_3^6	g	g^3	$\hat{g}^2$	$\hat{g}^2$	-1	-1	-1	-1	b^1_{-1}	4
V_3^7	g	g^2	$\hat{g}$	$\hat{g}$	$\mathbf{i}$	$\mathbf{i}$	-1	-1	b^4_+	64
V_3^8	g	g^2	$\hat{g}$	$\hat{g}^3$	$\mathbf{i}$	$-\mathbf{i}$	-1	-1	b^5_+	≥ 24
V_3^9	g	g^2	$\hat{g}^3$	$\hat{g}$	$-\mathbf{i}$	$\mathbf{i}$	-1	-1	b^5_-	≥ 24
V_3^{10}	g	g^2	$\hat{g}^3$	$\hat{g}^3$	$-\mathbf{i}$	$-\mathbf{i}$	-1	-1	b^4_-	64
V_3^{11}	g	g^2	$\hat{g}^2$	$\hat{g}$	-1	$\mathbf{i}$	1	-1	$[b^3_{++}]^\rho$	16
V_3^{12}	g	g^2	$\hat{g}^2$	$\hat{g}^3$	-1	$-\mathbf{i}$	1	-1	$[b^3_{-+}]^\rho$	16
V_3^{13}	g^2	g^2	$\hat{g}$	$\hat{g}$	-1	-1	-1	-1	b^1_{-1}	4
V_3^{14}	g^2	g^2	$\hat{g}$	$\hat{g}^3$	-1	-1	-1	-1	b^1_{-1}	4

Thus, we have no rank 2 modules over C_4 giving 8-dimensional Nichols algebras. For rank 3 the possible subcases are V_3^3, V_3^6, V_3^{13} and V_3^{14}. We have hence the following cases:

TABLE 7: Rank 3, $\Gamma = C_4$								
V	h_1	h_2	h_3	χ_1	χ_2	χ_3	b_{ij}	$\dim \mathfrak{B}(V)$
W_3^1	g	g	g	$\hat{g}^2$	$\hat{g}^2$	$\hat{g}^2$	$b^1_{-1,-1,-1}$	8
W_3^2	g	g	g^3	$\hat{g}^2$	$\hat{g}^2$	$\hat{g}^2$	$b^1_{-1,-1,-1}$	8
W_3^3	g^2	g^2	g^2	$\hat{g}$	$\hat{g}$	$\hat{g}$	$b^1_{-1,-1,-1}$	8
W_3^4	g^2	g^2	g^2	$\hat{g}$	$\hat{g}$	$\hat{g}^3$	$b^1_{-1,-1,-1}$	8

We compute now the liftings:

4.2.1. $V = W_3^1$. We take a_i, $i = 1, 2, 3$ as in 2.18, i.e. $\Delta(a_i) = g \otimes a_i + a_i \otimes 1$, $ga_i = -a_i g$. Hence, by 2.19, $a_i^2 \in \mathcal{P}_{g^2,1}(A)$, and $a_i a_j + a_j a_i \in \mathcal{P}_{g^2,1}(A)$, from where there exist $\lambda_i, 1 \le i \le 6$ such that

$$\begin{aligned}
&a_1^2 = \lambda_1(g^2-1),\ a_2^2 = \lambda_2(g^2-1),\ a_3^2 = \lambda_3(g^2-1),\\
&a_1a_2 + a_2a_1 = \lambda_4(g^2-1),\\
&a_1a_3 + a_3a_1 = \lambda_5(g^2-1),\\
&a_2a_3 + a_3a_2 = \lambda_6(g^2-1).
\end{aligned}$$

Thus, we have a quadratic form $f : W = (a_1, a_2, a_3) \to \mathbf{k}(g^2-1)$, $f(a) = a^2$. Furthermore, the basis $\{a_1, a_2, a_3\}$ of W can be replaced by any other basis of W, as a result of what we must consider the equivalence class of the quadratic form f in order to have non isomorphic liftings. Since $\operatorname{char} \mathbf{k} \neq 2$ and $\mathbf{k}$ is algebraically closed, f is characterized by its rank, and hence there are 4 liftings (up to isomorphism):

1. $a_ia_j + a_ja_i = 0$ for $i \neq j$, $a_i^2 = 0$ for $i = 1, 2, 3$. This is the bosonization $\mathfrak{B}(W_3^1)\#\mathbf{k}\Gamma$.
2. $a_ia_j + a_ja_i = 0$ for $i \neq j$, $a_i^2 = 0$ for $i = 1, 2$, $a_3^2 = (g^2-1)$.
3. $a_ia_j + a_ja_i = 0$ for $i \neq j$, $a_1^2 = 0$, $a_i^2 = (g^2-1)$ for $i = 2, 3$.
4. $a_ia_j + a_ja_i = 0$ for $i \neq j$, $a_i^2 = (g^2-1)$ for $i = 1, 2, 3$.

4.2.2. $V = W_3^2$. Similarly, we have $a_1, a_2 \in \mathcal{P}_{g,1}(A)$, $a_3 \in \mathcal{P}_{g^3,1}(A)$ and such that $ga_i = -a_i g$ for $i = 1, 2, 3$. Now, $a_ia_3 + a_3a_i \in \mathcal{P}(A)$ $(i = 1, 2)$, whence $a_ia_3 + a_3a_i = 0$ for $i = 1, 2$. However, there exist λ_i, $1 \le i \le 4$ such that

$$a_i^2 = \lambda_i(g^2-1)\ (i = 1, 2, 3), \qquad a_1a_2 + a_2a_1 = \lambda_4(g^2-1).$$

Hence we have two quadratic forms:

$$\begin{aligned}
&f_1 : W_1 = (a_1, a_2) \to \mathbf{k}(g^2-1),\ f_1(a) = a^2,\\
&f_2 : W_2 = (a_3) \to \mathbf{k}(g^2-1),\ f_2(a) = a^2.
\end{aligned}$$

As before, we can replace $\{a_1, a_2\}$ by another basis of W_1, and the same for $\{a_3\}$. As with W_3^1, f_1 and f_2 are characterized by their ranks, whence there are 3 possibilities for f_1 and 2 for f_2, and finally there are 6 liftings:

1. $a_ia_j + a_ja_i = 0$ for $i \neq j$, $a_1^2 = 0$, $a_2^2 = 0$, $a_3^2 = 0$. This is the bosonization $\mathfrak{B}(W_3^2)\#\mathbf{k}\Gamma$.
2. $a_ia_j + a_ja_i = 0$ for $i \neq j$, $a_1^2 = 0$, $a_2^2 = (g^2 - 1)$, $a_3^2 = 0$.
3. $a_ia_j + a_ja_i = 0$ for $i \neq j$, $a_1^2 = (g^2 - 1)$, $a_2^2 = (g^2 - 1)$, $a_3^2 = 0$.
4. $a_ia_j + a_ja_i = 0$ for $i \neq j$, $a_1^2 = 0$, $a_2^2 = 0$, $a_3^2 = (g^2 - 1)$.
5. $a_ia_j + a_ja_i = 0$ for $i \neq j$, $a_1^2 = 0$, $a_2^2 = (g^2 - 1)$, $a_3^2 = (g^2 - 1)$.
6. $a_ia_j + a_ja_i = 0$ for $i \neq j$, $a_1^2 = (g^2 - 1)$, $a_2^2 = (g^2 - 1)$, $a_3^2 = (g^2 - 1)$.

4.2.3. $V = W_3^3$. The situation here is easier: we have $a_i \in \mathcal{P}_{g^2,1}(A)$ and $ga_i = \mathbf{i}a_ig$ for $i = 1, 2, 3$. Since $a_i^2 \in \mathcal{P}(A)$ and $a_ia_j + a_ja_i \in \mathcal{P}(A)$ for all i, j, by 2.19 we have $a_i^2 = a_ia_j + a_ja_i = 0$, and there is only one lifting: the bosonization $\mathfrak{B}(W_3^3)\#\mathbf{k}\Gamma$.

4.2.4. $V = W_3^4$. Similar to the previous case, the only difference being that $ga_i = \mathbf{i}a_ig$ for $i = 1, 2$ and $ga_3 = -\mathbf{i}a_3g$. There is only one lifting: the bosonization $\mathfrak{B}(W_3^3)\#\mathbf{k}\Gamma$.

5. Γ of order 8

By 2.8 we have $\dim V \leq 2$. If $\dim V = 1$, we have $V = M(h, \chi)$, $h \in Z(\Gamma)$ and χ a character such that $\chi(h)$ has order 4. If Γ is non abelian, we have $Z(\Gamma) \subseteq [\Gamma; \Gamma]$, whence any irreducible representation of degree 1 vanishes on the center, and $\chi(h) = 1$. This tells that there are no 32-dimensional pointed Hopf algebras with non abelian coradical of dimension 8.

Remark 5.0.1. In turn this explains why there are no 16-dimensional pointed Hopf algebras with non abelian coradical (see [CDR]).

If $\dim V = 2$. Then, by [AG, Prop. 3.1.11], V comes from the abelian case, and hence it has a matrix (b_{ij}). By 2.5, V must be a QLS over some subgroup $\Gamma' \subseteq \Gamma$, with $b_{11} = b_{22} = -1$. If Γ is non abelian, there are in principle three possibilities for V to be 2-dimensional:

5.0.2. Decomposition of V, Γ non abelian.

1. $V = M(h_1, \chi_1) \oplus M(h_2, \chi_2)$, where the h_i's are central in Γ and the χ_i's are characters,

2. $V = M(h, \chi)$, where $[\Gamma : \Gamma_h] = 2$ and χ is a character of Γ_h,
3. $V = M(h, \rho)$, where h is central in Γ and $\dim \rho = 2$.

Case 1 does not arise for the same reasons as above: any irreducible representation of degree 1 of a non abelian group of order 8 vanishes on the center, and we would have $b_{11} = b_{22} = 1$, contradicting 2.2. We shall proceed as follows:

Tactic 5.0.3. In case 2, let x be a generator of the space affording χ. Note that $\chi(h) = b_{11} = -1$. Let $t \notin \Gamma_h$. The conjugacy class of h is $\{h, {}^th = tht^{-1}\}$, and since $[\Gamma : \Gamma_h] = 2$ it is easy to prove that ${}^th \in \Gamma_h$. Notice that $t^{-1} \notin \Gamma_h$, whence $t^{-1}ht = tht^{-1} = {}^th$. Let $y = t \rightharpoonup x$ and let $\zeta = \chi({}^th)$. Then $\delta(y) = {}^th \otimes y$, and c can be computed as follows:

$$\begin{aligned}
c(x \otimes x) &= (h \rightharpoonup x) \otimes x = -x \otimes x, \\
c(x \otimes y) &= (h \rightharpoonup y) \otimes x = (ht \rightharpoonup x) \otimes x = (tt^{-1}ht \rightharpoonup x) \otimes x \\
&= (\zeta t \rightharpoonup x) \otimes x = \zeta y \otimes x, \\
c(y \otimes x) &= ({}^th \rightharpoonup x) \otimes y = \zeta x \otimes y, \\
c(y \otimes y) &= ({}^tht \rightharpoonup x) \otimes y = (th \rightharpoonup x) \otimes y = -y \otimes y.
\end{aligned}$$

Therefore, the matrix of c is $\begin{pmatrix} -1 & \zeta \\ \zeta & -1 \end{pmatrix}$, whence $1 = b_{12}b_{21} = \zeta^2$, i.e. $\zeta = \pm 1$. To compute the liftings, let a, b be liftings of x, y as in 2.18. Then $hah^{-1} = -a$ and $b = tat^{-1}$. The action of Γ on A_1 may be computed from the action of Γ on V, as explained in 2.18.

Tactic 5.0.4. In case 3, let x be a generator of the space affording ρ, and let $t \in \Gamma$ such that $y = \rho(t)(x)$ is linearly independent with x. Since ρ is irreducible and h is central, $\rho(h) = \zeta\,\mathrm{id}$ for some $\zeta \in \mathbf{k}^\times$. Then the matrix of c is $\begin{pmatrix} \zeta & \zeta \\ \zeta & \zeta \end{pmatrix}$, from where $\zeta = b_{11} = -1$. Let a, b be liftings of x, y as in 2.18. Then $b = tat^{-1}$ and $hbh^{-1} = -b$.

Remark 5.0.5. These cases do not arise when classifying pointed Hopf algebras of dimension p^5, p an odd prime number (see [G]), and there are no pointed Hopf algebras of dimension p^5 with non commutative coradical of dimension p^3.

We consider now the case Γ abelian.

5.0.6. Use of diamond lemma

If Γ is abelian and $\dim V = 1$, then $V = M(h,\chi)$ and, as in 2.18, we have $a = a_1 \in \mathcal{P}_{h,1}(A)$. Furthermore, for $\mathfrak{B}(V)$ to be 4-dimensional, we must have $\chi(h)$ of order 4. By 2.19, $a^4 \in \mathcal{P}_{h^4,1}(A) = \mathbf{k}(h^4-1)$ ($\chi(h)$ having order 4, this implies that h^4 does not coincide neither with h, h^2 nor h^3). We have hence A generated by Γ and a, with relations

$$tat^{-1} = \chi(t)a \ \forall t \in \Gamma, \qquad a^4 = \lambda(h^4-1).$$

If $h^4 \neq 1$, taking a suitable scalar multiple of a we may suppose that $\lambda \in \{0,1\}$. We use the diamond lemma to prove that A is 32-dimensional, but this adds some condition on λ. Let $t \in \Gamma$. We have

$$\begin{aligned} ta^4 &= \chi^4(t)a^4t = \chi^4(t)\lambda(h^4-1)t, \\ ta^4 &= t\lambda(h^4-1) = \lambda(h^4-1)t, \end{aligned}$$

whence

$$\lambda(\chi^4(t)-1)(h^4-1) = 0 \ \forall t \in \Gamma. \tag{5.0.7}$$

If $h^4 = 1$ we can take $\lambda = 0$, and hence condition (5.0.7) can be written in the more compact form

$$\lambda(\chi^4-1) = 0. \tag{5.0.8}$$

If Γ is abelian and $\dim V = 2$, then $V = M(h_1,\chi_1) \oplus M(h_2,\chi_2)$. We have as in 2.18 $a_i \in \mathcal{P}_{h_i,1}(A)$, $ta_it^{-1} = \chi_i(t)a_i$. As in 2.19, $a_i^2 \in \mathcal{P}_{h_i^2,1}(A)$ and $\mathrm{Ad}_{a_1}(a_2) = a_1a_2 - \chi_2(h_1)a_2a_1 \in \mathcal{P}_{h_1h_2,1}(A)$. Since $\mathfrak{B}(V)$ is 4-dimensional, by 2.5 it is a QP, and we have $\chi_2(h_1)\chi_1(h_2) = 1$, $\chi_i(h_i) = -1$. Now, $h_1h_2 \neq h_1$ and $h_1h_2 \neq h_2$, for if not $h_2 = 1$ or $h_1 = 1$, contradicting $\chi_i(h_i) = -1$. Moreover, if $h_1^2 = h_2$, we would have

$$-1 = \chi_2(h_2) = \chi_2(h_1^2) = (\chi_2(h_1))^2 = (\chi_1(h_2))^{-2} = (\chi_1(h_1))^{-4} = (-1)^{-4} = 1,$$

which is absurd. Thus $h_1^2 \neq h_2$, and analogously $h_2^2 \neq h_1$.
Hence, $\mathcal{P}_{h_i^2,1}(A) = \mathbf{k}(h_i^2-1)$ and $\mathcal{P}_{h_1h_2,1}(A) = \mathbf{k}(h_1h_2-1)$, and we have

(5.0.9)
$$a_i^2 = \lambda_i(h_i^2-1) \ (i=1,2), \qquad a_1a_2 - \chi_2(h_1)a_2a_1 = \lambda_3(h_1h_2-1).$$

Similarly to (5.0.8), we get

$$\lambda_1(\chi_1^2-1) = 0, \quad \lambda_2(\chi_2^2-1) = 0. \tag{5.0.10}$$

Doing the same with the monomials ta_2a_1, $t \in \Gamma$, we arrive to the condition

$$\lambda_3(\chi_1\chi_2 - 1) = 0. \tag{5.0.11}$$

We consider now group by group.

5.1. $\Gamma = C_2 \times C_2 \times C_2$**.** There are no 1-dimensional modules $M(h,\chi)$ in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ s.t. $\chi(h)$ has order 4, whence the rank must be 2. Now, the YD-modules giving 2-dimensional Nichols algebras are those $M(h,\chi)$ such that $\chi(h) = -1$, but $\mathrm{Aut}(\Gamma) \simeq \mathrm{GL}_3(\mathbb{Z}/2)$, and this group acts transitively on these modules. Hence we can suppose that $M_1 = M(g_1, \hat{g}_1)$ (we are using the notation at the end of 1.1). The subgroup of $\mathrm{Aut}(\Gamma)$ which fixes M_1 is isomorphic to $\mathbb{S}_3$, and is generated by

$$f_1 = (g_1 \mapsto g_1, g_2 \mapsto g_2g_3, g_3 \mapsto g_3), \quad f_2 = (g_1 \mapsto g_1, g_2 \mapsto g_2, g_3 \mapsto g_2g_3).$$

Thus we consider the action of this subgroup on the modules and choose one element in each orbit:

$$Y_4^1 = M(g_1, \hat{g}_1),\ Y_4^2 = M(g_1, \hat{g}_1\hat{g}_2),\ Y_4^3 = M(g_1g_2, \hat{g}_1),\ Y_4^4 = M(g_1g_2, \hat{g}_1\hat{g}_3),$$
$$Y_4^5 = M(g_1g_2, \hat{g}_2),\ Y_4^6 = M(g_2, \hat{g}_2),\ Y_4^7 = M(g_2, \hat{g}_1\hat{g}_2).$$

In order to give non isomorphic modules after taking the sums $Y_4^1 \oplus Y_4^i$, we must take into account the possibility of switching the order, as in lemma 2.16. The modules $Y_4^1 \oplus Y_4^5$ and $Y_4^1 \oplus Y_4^7$ give isomorphic algebras when its Nichols algebras are bosonized by means of the automorphism

$$(g_1 \mapsto g_2,\ g_2 \mapsto g_1g_2,\ g_3 \mapsto g_3),$$

and we consider only $Y_4^1 \oplus Y_4^5$. Hence, the list is

TABLE 8: Rank 2, $\Gamma = C_2 \times C_2 \times C_2$

V	h_1	h_2	χ_1	χ_2	b_{11}	b_{12}	b_{21}	b_{22}	b_{ij}	$\dim \mathfrak{B}(V)$
V_4^1	g_1	g_1	$\hat{g}_1$	$\hat{g}_1$	-1	-1	-1	-1	b^1_{-1}	4
V_4^2	g_1	g_1	$\hat{g}_1$	$\hat{g}_1\hat{g}_2$	-1	-1	-1	-1	b^1_{-1}	4
V_4^3	g_1	g_1g_2	$\hat{g}_1$	$\hat{g}_1$	-1	-1	-1	-1	b^1_{-1}	4
V_4^4	g_1	g_1g_2	$\hat{g}_1$	$\hat{g}_1\hat{g}_3$	-1	-1	-1	-1	b^1_{-1}	4
V_4^5	g_1	g_1g_2	$\hat{g}_1$	$\hat{g}_2$	-1	1	-1	-1	$[b^2]^\rho$	8
V_4^6	g_1	g_2	$\hat{g}_1$	$\hat{g}_2$	-1	1	1	-1	b^1_1	4

We compute the liftings:

5.1.1. $V = V_4^1$. We have, as in 2.18, $a_1, a_2 \in \mathcal{P}_{g_1,1}(A)$, and $g_1a_i = -a_ig_1$, $g_2a_i = a_ig_2$, $g_3a_i = a_ig_3$ for $i = 1, 2$. By 2.19, we have $a_i^2 \in \mathcal{P}(A)$ and also $a_1a_2 + a_2a_1 \in \mathcal{P}(A)$, whence $a_i^2 = a_1a_2 + a_2a_1 = 0$, and there is only one lifting: the bosonization $\mathfrak{B}(V_4^1)\#\mathbf{k}\Gamma$.

5.1.2. $V = V_4^2$. It is similar to the previous one, the only difference being that $g_2a_2 = -a_2g_2$. The unique lifting is the bosonization $\mathfrak{B}(V_4^2)\#\mathbf{k}\Gamma$

5.1.3. $V = V_4^3$. Here $a_1 \in \mathcal{P}_{g_1,1}(A)$, $a_2 \in \mathcal{P}_{g_1g_2,1}(A)$ and Γ acts by $g_1a_i = -a_ig_1$, $g_2a_i = a_ig_2$, $g_3a_i = a_ig_3$. As before, $a_i^2 \in \mathcal{P}(A)$, whence $a_i^2 = 0$ for $i = 1, 2$. However, $a_1a_2 + a_2a_1 \in \mathcal{P}_{g_2,1}(A)$, from where as in (5.0.9) we have $a_1a_2 + a_2a_1 = \lambda_3(g_2 - 1)$. Taking a suitable scalar multiple of a_1, we may suppose that $\lambda_3 = 0$ or $\lambda_3 = 1$. There are hence two liftings:

1. $a_i^2 = 0$, $a_1a_2 + a_2a_1 = 0$. This is the bosonization $\mathfrak{B}(V_4^3)\#\mathbf{k}\Gamma$.
2. $a_i^2 = 0$, $a_1a_2 + a_2a_1 = (g_2 - 1)$.

5.1.4. $V = V_4^4$. It is similar to V_4^3. We have $a_1 \in \mathcal{P}_{g_1,1}(A)$, $a_2 \in \mathcal{P}_{g_1g_2,1}(A)$ and Γ acts now by $g_1a_i = -a_ig_1$, $g_2a_i = a_ig_2$, $g_3a_1 = a_1g_3$, $g_3a_2 = -a_2g_3$. As before, $a_i^2 = 0$, $a_1a_2 + a_2a_1 = \lambda_3(g_2 - 1)$. Now, (5.0.11) tells that $\lambda_3 = 0$. There is only one lifting: the bosonization $\mathfrak{B}(V_4^4)\#\mathbf{k}\Gamma$.

5.1.5. $V = V_4^6$. Now $a_1 \in \mathcal{P}_{g_1,1}(A)$, $a_2 \in \mathcal{P}_{g_2,1}(A)$ and Γ acts by

$$g_1a_1 = -a_1g_1, \quad g_1a_2 = a_2g_1, \quad g_2a_1 = a_1g_2, \quad g_2a_2 = -a_2g_2, \quad g_3a_i = a_ig_3.$$

As before, $a_i^2 \in \mathcal{P}(A)$, whence $a_i^2 = 0$. Furthermore, $a_1a_2 - a_2a_1 \in \mathcal{P}_{g_1g_2,1}(A)$, from where $a_1a_2 - a_2a_1 = \lambda_3(g_1g_2 - 1)$. Now, (5.0.11) tells that $\lambda_3 = 0$. There is only one lifting: the bosonization $\mathfrak{B}(V_4^6)\#\mathbf{k}\Gamma$.

5.2. $\Gamma = C_2 \times C_4$. We have, using the notation at the end of 1.1, $\mathrm{Aut}(\Gamma) \simeq \mathbb{D}_4$ with generators

$$f_1 = (g_1 \mapsto g_1g_2^2,\ g_2 \mapsto g_1g_2), \quad f_2 = (g_1 \mapsto g_1g_2^2,\ g_2 \mapsto g_2),$$

with relations $f_1^4 = f_2^2 = 1$, $f_1f_2 = f_2f_1^3$. Hence there are 3 orbits in Γ under the action of $\mathrm{Aut}(\Gamma)$: $\{g_2, g_1g_2, g_2^3, g_1g_2^3\}$, $\{g_1, g_1g_2^2\}$ and $\{g_2^2\}$. There are 7 orbits of irreducible YD-modules giving Nichols algebras of dimension a power of 2. The following is a list of one element per orbit and the subgroup of

$\mathrm{Aut}(\Gamma)$ fixing the chosen element.

TABLE 9: modules/ $\mathrm{Aut}(\Gamma)$, $\Gamma = C_2 \times C_4$				
$M(h,\chi)$	h	χ	$\dim \mathfrak{B}(M)$	$(\mathrm{Aut}(\Gamma))_M$
Y_5^1	g_2	$\hat{g}_2$	4	1
Y_5^2	g_2	$\hat{g}_2^2$	2	(f_2)
Y_5^3	g_2	$\hat{g}_2^3$	4	1
Y_5^4	g_2	$\hat{g}_1\hat{g}_2^2$	2	(f_2)
Y_5^5	g_1	$\hat{g}_1\hat{g}_2^3$	2	$(f_2 f_1)$
Y_5^6	g_1	$\hat{g}_1$	2	(f_1^2)
Y_5^7	g_2^2	$\hat{g}_2$	2	$(f_1 f_2)$

Hence there are two rank 1 Nichols algebras of dimension 4 over $C_2 \times C_4$. We go now to rank 2, i.e. $V = M_1 \oplus M_2$. Since $\dim \mathfrak{B}(M_i) = 2$, we can suppose that M_1 is one of Y_5^2, Y_5^4, Y_5^5, Y_5^6 or Y_5^7. We give the list, which is the hardest to compute (we take into account 2.16).

TABLE 10: Rank 2, $\Gamma = C_2 \times C_4$										
V	h_1	h_2	χ_1	χ_2	b_{11}	b_{12}	b_{21}	b_{22}	b_{ij}	$\dim \mathfrak{B}(V)$
V_5^1	g_2	g_2	$\hat{g}_2^2$	$\hat{g}_2^2$	-1	-1	-1	-1	b_{-1}^1	4
V_5^2	g_2	g_2	$\hat{g}_2^2$	$\hat{g}_1\hat{g}_2^2$	-1	-1	-1	-1	b_{-1}^1	4
V_5^3	g_2	g_2	$\hat{g}_1\hat{g}_2^2$	$\hat{g}_1\hat{g}_2^2$	-1	-1	-1	-1	b_{-1}^1	4
V_5^4	g_2	$g_1 g_2$	$\hat{g}_2^2$	$\hat{g}_2^2$	-1	-1	-1	-1	b_{-1}^1	4
V_5^5	g_2	$g_1 g_2$	$\hat{g}_2^2$	$\hat{g}_1$	-1	1	-1	-1	$[b^2]^\rho$	8
V_5^6	g_2	$g_1 g_2$	$\hat{g}_1\hat{g}_2^2$	$\hat{g}_1$	-1	1	1	-1	b_1^1	4
V_5^7	g_2	g_2^3	$\hat{g}_2^2$	$\hat{g}_2^2$	-1	-1	-1	-1	b_{-1}^1	4
V_5^8	g_2	g_2^3	$\hat{g}_1\hat{g}_2^2$	$\hat{g}_2^2$	-1	-1	-1	-1	b_{-1}^1	4
V_5^9	g_2	g_2^3	$\hat{g}_1\hat{g}_2^2$	$\hat{g}_1\hat{g}_2^2$	-1	-1	-1	-1	b_{-1}^1	4
V_5^{10}	g_2	g_1	$\hat{g}_2^2$	$\hat{g}_1$	-1	1	1	-1	b_1^1	4
V_5^{11}	g_2	g_1	$\hat{g}_2^2$	$\hat{g}_1\hat{g}_2^2$	-1	-1	1	-1	b^2	8
V_5^{12}	g_2	g_1	$\hat{g}_1\hat{g}_2^2$	$\hat{g}_1$	-1	1	-1	-1	$[b^2]^\rho$	8
V_5^{13}	g_2	g_1	$\hat{g}_1\hat{g}_2^2$	$\hat{g}_1\hat{g}_2^2$	-1	-1	-1	-1	b_{-1}^1	4
V_5^{14}	g_2	g_1	$\hat{g}_2^2$	$\hat{g}_1\hat{g}_2^3$	-1	$-\mathbf{i}$	1	-1	$[b_{-+}^3]^\rho$	16
V_5^{15}	g_2	g_1	$\hat{g}_2^2$	$\hat{g}_1\hat{g}_2$	-1	$\mathbf{i}$	1	-1	$[b_{++}^3]^\rho$	16
V_5^{16}	g_2	g_1	$\hat{g}_1\hat{g}_2^2$	$\hat{g}_1\hat{g}_2^3$	-1	$-\mathbf{i}$	-1	-1	$[b_{--}^3]^\rho$	16
V_5^{17}	g_2	g_1	$\hat{g}_1\hat{g}_2^2$	$\hat{g}_1\hat{g}_2$	-1	$\mathbf{i}$	-1	-1	$[b_{+-}^3]^\rho$	16
V_5^{18}	g_2	g_2^2	$\hat{g}_2^2$	$\hat{g}_1\hat{g}_2$	-1	$\mathbf{i}$	1	-1	$[b_{++}^3]^\rho$	16
Continues in next page										

V	h_1	h_2	χ_1	χ_2	b_{11}	b_{12}	b_{21}	b_{22}	b_{ij}	$\dim \mathfrak{B}(V)$
V_5^{19}	g_2	g_2^2	$\hat{g}_2^2$	$\hat{g}_1\hat{g}_2^3$	-1	$-\mathbf{i}$	1	-1	$[b^3_{-+}]^\rho$	16
V_5^{20}	g_2	g_2^2	$\hat{g}_1\hat{g}_2^2$	$\hat{g}_2$	-1	$\mathbf{i}$	1	-1	$[b^3_{++}]^\rho$	16
V_5^{21}	g_2	g_2^2	$\hat{g}_1\hat{g}_2^2$	$\hat{g}_2^3$	-1	$-\mathbf{i}$	1	-1	$[b^3_{-+}]^\rho$	16
V_5^{22}	g_2^2	g_2^2	$\hat{g}_2$	$\hat{g}_2$	-1	-1	-1	-1	b^1_{-1}	4
V_5^{23}	g_2^2	g_2^2	$\hat{g}_2$	$\hat{g}_1\hat{g}_2$	-1	-1	-1	-1	b^1_{-1}	4
V_5^{24}	g_2^2	g_2^2	$\hat{g}_2$	$\hat{g}_2^3$	-1	-1	-1	-1	b^1_{-1}	4
V_5^{25}	g_2^2	g_1	$\hat{g}_2$	$\hat{g}_1$	-1	1	1	-1	b^1_1	4
V_5^{26}	g_2^2	g_1	$\hat{g}_2$	$\hat{g}_1\hat{g}_2^3$	-1	-1	1	-1	b^2	8
V_5^{27}	g_2^2	g_1	$\hat{g}_1\hat{g}_2^3$	$\hat{g}_1\hat{g}_2^3$	-1	-1	-1	-1	b^1_{-1}	4
V_5^{28}	g_2^2	g_1	$\hat{g}_1\hat{g}_2^3$	$\hat{g}_1\hat{g}_2$	-1	-1	-1	-1	b^1_{-1}	4
V_5^{29}	g_2^2	g_1	$\hat{g}_1\hat{g}_2$	$\hat{g}_1$	-1	1	-1	-1	$[b^2]^\rho$	8
V_5^{30}	g_2^2	g_1	$\hat{g}_2^3$	$\hat{g}_1$	-1	1	1	-1	b^1_1	4
V_5^{31}	g_2^2	$g_1g_2^2$	$\hat{g}_2$	$\hat{g}_1$	-1	1	-1	-1	$[b^2]^\rho$	8
V_5^{32}	g_1	g_1	$\hat{g}_1\hat{g}_2^3$	$\hat{g}_1\hat{g}_2^3$	-1	-1	-1	-1	b^1_{-1}	4
V_5^{33}	g_1	g_1	$\hat{g}_1\hat{g}_2^3$	$\hat{g}_1\hat{g}_2$	-1	-1	-1	-1	b^1_{-1}	4
V_5^{34}	g_1	g_1	$\hat{g}_1\hat{g}_2^3$	$\hat{g}_1$	-1	-1	-1	-1	b^1_{-1}	4
V_5^{35}	g_1	g_1	$\hat{g}_1$	$\hat{g}_1$	-1	-1	-1	-1	b^1_{-1}	4
V_5^{36}	g_1	g_1	$\hat{g}_1$	$\hat{g}_1\hat{g}_2^2$	-1	-1	-1	-1	b^1_{-1}	4
V_5^{37}	g_1	$g_1g_2^2$	$\hat{g}_1\hat{g}_2^3$	$\hat{g}_2$	-1	1	1	-1	b^1_1	4
V_5^{38}	g_1	$g_1g_2^2$	$\hat{g}_1\hat{g}_2^3$	$\hat{g}_1$	-1	-1	1	-1	b^2	8
V_5^{39}	g_1	$g_1g_2^2$	$\hat{g}_1$	$\hat{g}_1$	-1	-1	-1	-1	b^1_{-1}	4
V_5^{40}	g_1	$g_1g_2^2$	$\hat{g}_1$	$\hat{g}_1\hat{g}_2^2$	-1	-1	-1	-1	b^1_{-1}	4

We compute the liftings. For rank 1 we have

5.2.1. $V = Y_5^1$. As in 2.18, we have $a \in \mathcal{P}_{g_2,1}(A)$, $g_1a = ag_1$, $g_2a = \mathbf{i}ag_2$. By 2.19, we have $a^4 = 0$ and there is only one lifting: the bosonization $\mathfrak{B}(Y_5^1)\#\mathbf{k}\Gamma$.

5.2.2. $V = Y_5^3$. Analogous to the previous one, the only difference being that $g_2a = -\mathbf{i}ag_2$. There is only one lifting: the bosonization $\mathfrak{B}(Y_5^3)\#\mathbf{k}\Gamma$.

We go now to rank 2. As explained in 5.0.6, we have a_1, a_2 with $a_i \in \mathcal{P}_{h_i,1}(A)$, $ta_it^{-1} = \chi_i(t)a_i$,

$$a_1^2 = \lambda_1(h_1^2 - 1), \quad a_2^2 = \lambda_2(h_2^2 - 1), \quad a_1a_2 - \chi_2(h_1)a_2a_1 = \lambda_3(h_1h_2 - 1).$$

Before computing the liftings, we notice some facts.

Remark 5.2.3. In certain cases, some element in $\mathrm{Aut}(\Gamma)$ interchanges M_1 and M_2. In this situation the lifting corresponding to $(\lambda_1, \lambda_2, \lambda_3)$ is isomorphic to that corresponding to $(\lambda_2, \lambda_1, \pm\lambda_3)$ (the $\pm$ sign would depend on $\chi_2(h_1)$, but in the cases in which this happens it will turn out that $\lambda_3 = 0$).

Remark 5.2.4. In some cases $M_1 = M_2$, $h_1^2 \neq 1$ and $\chi_1^2 = 1$. In those cases, let W be the linear span of a_1, a_2. We can change the basis $\{a_1, a_2\}$ by any other basis of W, and (similarly to the case W_3^1) we have a quadratic form $f : W \to \mathbf{k}(h_1^2 - 1)$, $f(a) = a^2$. As in that case, A is characterized by the rank of this form, whence there shall be 3 liftings.

Remark 5.2.5. In the case V_5^4 (and V_6^2, V_6^4 for $\Gamma = C_8$) there is no restriction to λ_1, λ_2 and λ_3. If we replace $\{a_1, a_2\}$ by $\{sa_1, ta_2\}$ $(s, t \in \mathbf{k}^\times)$, then we get an isomorphic algebra but with $(\lambda_1, \lambda_2, \lambda_3)$ replaced by $(s^2\lambda_1, t^2\lambda_2, st\lambda_3)$. There are thus infinitely many liftings, which are parametrized by the quotient variety $\mathbf{k}^3/\sim$, where

$$(\lambda_1, \lambda_2, \lambda_3) \sim (s^2\lambda_1, t^2\lambda_2, st\lambda_3), \qquad s, t \in \mathbf{k}^\times.$$

Remark 5.2.6. If for some reason (for instance 2.19 or (5.0.10)) $\lambda_1 = 0$ or $\lambda_2 = 0$ then taking suitable scalar multiples of a_1 or a_2 we can suppose also that $\lambda_3 \in \{0, 1\}$.

Here is a list of the liftings up to isomorphism. We put

$$a_3 = a_1a_2 - \chi_2(h_1)a_2a_1.$$

TABLE 11: Rank 2, $\Gamma = C_2 \times C_4$, liftings						
V	Lift.	$-b_{12}$	a_1^2	a_2^2	a_3	Remarks
V_5^1	3	$+$	0	0	0	See 5.2.4
			$g_2^2 - 1$	0	0	
			$g_2^2 - 1$	$g_2^2 - 1$	0	
V_5^2	4	$+$	0	0	0	See 5.0.11
			$g_2^2 - 1$	0	0	
			0	$g_2^2 - 1$	0	
			$g_2^2 - 1$	$g_2^2 - 1$	0	
V_5^3	3	$+$	0	0	0	See 5.2.4
			$g_2^2 - 1$	0	0	
			$g_2^2 - 1$	$g_2^2 - 1$	0	
Continues in next page						

V	Lift.	$-b_{12}$	a_1^2	a_2^2	a_3	Remarks
V_5^4	∞	$+$	$\lambda_1(g_2^2-1), \lambda_2(g_2^2-1), \lambda_3(g_1g_2^2-1)$ $(\lambda_1,\lambda_2,\lambda_3)/\sim$			See 5.2.5
V_5^6	3	$-$	0	0	0	See 5.0.11 and 5.2.3
			g_2^2-1	0	0	
			g_2^2-1	g_2^2-1	0	
V_5^7	3	$+$	0	0	0	See 2.19 and 5.2.3
			g_2^2-1	0	0	
			g_2^2-1	g_2^2-1	0	
V_5^8	4	$+$	0	0	0	See 2.19
			g_2^2-1	0	0	
			0	g_2^2-1	0	
			g_2^2-1	g_2^2-1	0	
V_5^9	3	$+$	0	0	0	See 5.2.3
			g_2^2-1	0	0	
			g_2^2-1	g_2^2-1	0	
V_5^{10}	2	$-$	0	0	0	See 2.19 and 5.0.11
			g_2^2-1	0	0	
V_5^{13}	4	$+$	0	0	0	See 2.19 and 5.2.6
			g_2^2-1	0	0	
			0	0	g_1g_2-1	
			g_2^2-1	0	g_1g_2-1	
V_5^{22}	1	$+$	0	0	0	See 2.19
V_5^{23}	1	$+$	0	0	0	See 2.19
V_5^{24}	1	$+$	0	0	0	See 2.19
V_5^{25}	1	$-$	0	0	0	See 2.19 and 5.0.11
V_5^{27}	1	$+$	0	0	0	See 2.19 and 5.0.11
V_5^{28}	2	$+$	0	0	0	See 2.19 and 5.2.6
			0	0	$g_1g_2^2-1$	
V_5^{30}	1	$-$	0	0	0	See 2.19 and 5.0.11
V_5^{32}	1	$+$	0	0	0	See 2.19
V_5^{33}	1	$+$	0	0	0	See 2.19
V_5^{34}	1	$+$	0	0	0	See 2.19
V_5^{35}	1	$+$	0	0	0	See 2.19
V_5^{36}	1	$+$	0	0	0	See 2.19

Continues in next page

V	Lift.	$-b_{12}$	a_1^2	a_2^2	a_3	Remarks
V_5^{37}	1	$-$	0	0	0	See 2.19 and 5.0.11
V_5^{39}	1	$+$	0	0	0	See 2.19 and 5.2.6
			0	0	g_2^2-1	
V_5^{40}	1	$+$	0	0	0	See 2.19 and 5.0.11

5.3. $\Gamma = C_8$. We have $\mathrm{Aut}(\Gamma) \simeq C_2 \times C_2$, generated by

$$f_1 = (g \mapsto g^3), \quad f_2 = (g \mapsto g^5).$$

The orbits of Γ are $\{g, g^3, g^5, g^7\}$, $\{g^2, g^6\}$ and $\{g^4\}$. There are 12 orbits of irreducible YD-modules giving Nichols algebras of dimension a power of 2. We choose an element in each:

TABLE 12: modules/ $\mathrm{Aut}(\Gamma)$, $\Gamma = C_8$			
$M(h,\chi)$	h	χ	$\dim \mathfrak{B}(M)$
Y_6^1	g	$\hat{g}$	8
Y_6^2	g	$\hat{g}^3$	8
Y_6^3	g	$\hat{g}^5$	8
Y_6^4	g	$\hat{g}^7$	8
Y_6^5	g	$\hat{g}^2$	4
Y_6^6	g	$\hat{g}^6$	4
Y_6^7	g	$\hat{g}^4$	2
Y_6^8	g^2	$\hat{g}$	4
Y_6^9	g^2	$\hat{g}^3$	4
Y_6^{10}	g^2	$\hat{g}^2$	2
Y_6^{11}	g^2	$\hat{g}^6$	2
Y_6^{12}	g^4	$\hat{g}$	2

Hence there are 4 different 4-dimensional Nichols algebras of rank 1. Now, for rank 2, we suppose M_1 is one of Y_6^7, Y_6^{10}, Y_6^{11} or Y_6^{12}. We have the following possibilities (we take into account 2.16).

TABLE 13: Rank 2, $\Gamma = C_8$										
V	h_1	h_2	χ_1	χ_2	b_{11}	b_{12}	b_{21}	b_{22}	b_{ij}	$\dim \mathfrak{B}(V)$
V_6^1	g	g	$\hat{g}^4$	$\hat{g}^4$	-1	-1	-1	-1	b_{-1}^1	4
V_6^2	g	g^3	$\hat{g}^4$	$\hat{g}^4$	-1	-1	-1	-1	b_{-1}^1	4
V_6^3	g	g^7	$\hat{g}^4$	$\hat{g}^4$	-1	-1	-1	-1	b_{-1}^1	4
V_6^4	g	g^5	$\hat{g}^4$	$\hat{g}^4$	-1	-1	-1	-1	b_{-1}^1	4
Continues in next page										

V	h_1	h_2	χ_1	χ_2	b_{11}	b_{12}	b_{21}	b_{22}	b_{ij}	$\dim \mathfrak{B}(V)$
V_6^5	g	g^2	$\hat{g}^4$	$\hat{g}^2$	-1	$\mathbf{i}$	1	-1	$[b^3_{++}]^\rho$	16
V_6^6	g	g^2	$\hat{g}^4$	$\hat{g}^6$	-1	$-\mathbf{i}$	1	-1	$[b^3_{-+}]^\rho$	16
V_6^7	g	g^6	$\hat{g}^4$	$\hat{g}^2$	-1	$\mathbf{i}$	1	-1	$[b^3_{++}]^\rho$	16
V_6^8	g	g^6	$\hat{g}^4$	$\hat{g}^6$	-1	$-\mathbf{i}$	1	-1	$[b^3_{-+}]^\rho$	16
V_6^9	g	g^4	$\hat{g}^4$	$\hat{g}$	-1	ξ	1	-1	$[b^6_1]^\rho$	32
V_6^{10}	g	g^4	$\hat{g}^4$	$\hat{g}^3$	-1	ξ^3	1	-1	$[b^6_3]^\rho$	32
V_6^{11}	g	g^4	$\hat{g}^4$	$\hat{g}^7$	-1	ξ^7	1	-1	$[b^6_7]^\rho$	32
V_6^{12}	g	g^4	$\hat{g}^4$	$\hat{g}^5$	-1	ξ^5	1	-1	$[b^6_5]^\rho$	32
V_6^{13}	g^2	g^2	$\hat{g}^2$	$\hat{g}^2$	-1	-1	-1	-1	b^1_{-1}	4
V_6^{14}	g^2	g^2	$\hat{g}^2$	$\hat{g}^6$	-1	-1	-1	-1	b^1_{-1}	4
V_6^{15}	g^2	g^2	$\hat{g}^6$	$\hat{g}^6$	-1	-1	-1	-1	b^1_{-1}	4
V_6^{16}	g^2	g^6	$\hat{g}^2$	$\hat{g}^6$	-1	-1	-1	-1	b^1_{-1}	4
V_6^{17}	g^2	g^6	$\hat{g}^6$	$\hat{g}^6$	-1	-1	-1	-1	b^1_{-1}	4
V_6^{18}	g^2	g^6	$\hat{g}^6$	$\hat{g}^2$	-1	-1	-1	-1	b^1_{-1}	4
V_6^{19}	g^4	g^2	$\hat{g}$	$\hat{g}^2$	-1	1	$\mathbf{i}$	-1	b^3_{++}	16
V_6^{20}	g^4	g^2	$\hat{g}$	$\hat{g}^6$	-1	1	$\mathbf{i}$	-1	b^3_{++}	16
V_6^{21}	g^4	g^6	$\hat{g}$	$\hat{g}^6$	-1	1	$-\mathbf{i}$	-1	b^3_{-+}	16
V_6^{22}	g^4	g^6	$\hat{g}$	$\hat{g}^2$	-1	1	$-\mathbf{i}$	-1	b^3_{-+}	16
V_6^{23}	g^4	g^4	$\hat{g}$	$\hat{g}$	-1	-1	-1	-1	b^1_{-1}	4
V_6^{24}	g^4	g^4	$\hat{g}$	$\hat{g}^3$	-1	-1	-1	-1	b^1_{-1}	4
V_6^{25}	g^4	g^4	$\hat{g}$	$\hat{g}^7$	-1	-1	-1	-1	b^1_{-1}	4
V_6^{26}	g^4	g^4	$\hat{g}$	$\hat{g}^5$	-1	-1	-1	-1	b^1_{-1}	4

We compute now the liftings. For rank 1, we have the following algebras, where we take a as in 5.0.6.

TABLE 14: Rank 1, $\Gamma = C_8$				
V	h	χ	a^4	Remarks
Y_6^5	g	$\hat{g}^2$	0	
			$g^4 - 1$	
Y_6^6	g	$\hat{g}^6$	0	
			$g^4 - 1$	
Y_6^8	g^2	$\hat{g}$	0	See (2.19)
Y_6^9	g^2	$\hat{g}^3$	0	See (2.19)

For rank 2, as in 5.0.6, we have a_1, a_2 with $a_i \in \mathcal{P}_{h_i,1}(A)$, $ta_it^{-1} = \chi_i(t)a_i$,

$$a_i^2 = \lambda_i(h_i^2 - 1) \ (i = 1, 2), \quad a_1a_2 + a_2a_1 = \lambda_3(h_1h_2 - 1)$$

(notice that $b_{12} = \chi_2(h_1) = -1$ in all the 4-dimensional cases). Equations (5.0.10), (5.0.11) and remarks 5.2.3, 5.2.4, 5.2.5 and 5.2.6 apply here also. We put

$$a_3 = a_1 a_2 + a_2 a_1.$$

TABLE 15: Rank 2, $\Gamma = C_8$, liftings

V	Lift.	a_1^2	a_2^2	a_3	Remarks
V_6^1	3	0	0	0	See 5.2.4
		g^2-1	0	0	
		g^2-1	g^2-1	0	
V_6^2	∞	$\lambda_1(g^2-1), \lambda_2(g^6-1), \lambda_3(g^4-1)$			See 5.2.5
		$(\lambda_1, \lambda_2, \lambda_3)/\sim$			
V_6^3	3	0	0	0	See 5.2.3
		g^2-1	0	0	
		g^2-1	g^6-1	0	
V_6^4	∞	$\lambda_1(g^2-1), \lambda_2(g^2-1), \lambda_3(g^6-1)$			See 5.2.5
		$(\lambda_1, \lambda_2, \lambda_3)/\sim$			
V_6^{13}	1	0	0	0	See 5.0.10 and 5.0.11
V_6^{14}	2	0	0	0	See 5.0.10 and 5.2.6
		0	0	g^4-1	
V_6^{15}	1	0	0	0	See 5.0.10 and 5.0.11
V_6^{16}	1	0	0	0	See 2.19 and 5.0.10
V_6^{17}	1	0	0	0	See 2.19 and 5.0.10
V_6^{18}	1	0	0	0	See 5.0.10 and 5.0.11
V_6^{23}	1	0	0	0	See 2.19
V_6^{24}	1	0	0	0	See 2.19
V_6^{25}	1	0	0	0	See 2.19
V_6^{26}	1	0	0	0	See 2.19

5.4. $\Gamma = \mathbb{D}_4$. We take $\mathbb{D}_4$ generated by r and σ with relations $r^4 = \sigma^2 = 1$, $\sigma r = r^3\sigma$. The conjugacy classes of Γ are $\{1\}$, $\{r^2\}$, $\{\sigma, r^2\sigma\}$, $\{r\sigma, r^3\sigma\}$ and $\{r, r^3\}$. As explained in 5.0.2, there are two possibilities for V. For case 5.0.2(2), h must be in a conjugacy class with 2 elements, and hence we can take $h = \sigma$, $h = r\sigma$ or $h = r$. However, the cases $h = \sigma$ and $h = r\sigma$ give isomorphic algebras when bosonized by means of the morphism ($r \mapsto r$, $\sigma \mapsto r\sigma$), and hence we take only the cases $h = \sigma$ and $h = r$. The commutator subgroups are

$$\Gamma_\sigma = \{1, \sigma, r^2, r^2\sigma\} \simeq C_2 \times C_2, \qquad \Gamma_r = \{1, r, r^2, r^3\} \simeq C_4.$$

Case 5.0.2(3) arises taking $h = r^2$ and $\rho = \rho_0$ the irreducible representation given by

$$\rho_0(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho_0(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The following is the list of irreducible YD-modules giving Nichols algebras of dimension 4:

TABLE 16: $\Gamma = \mathbb{D}_4$				
$M(h,\rho)$	h	ρ	$b_{12} = b_{21}$	$\dim \mathfrak{B}(M)$
Y_7^1	σ	$\sigma \mapsto -1, \quad r^2 \mapsto 1$	-1	4
Y_7^2	σ	$\sigma \mapsto -1, \quad r^2 \mapsto -1$	1	4
Y_7^3	r	$r \mapsto -1$	-1	4
Y_7^4	r^2	ρ_0	-1	4

We compute the liftings. Though we can state some general conditions as in (5.0.11), it turns out to be simpler to use the diamond lemma in each case.

5.4.1. $V = Y_7^1$. As in 5.0.3, we have $a \in \mathcal{P}_{\sigma,1}(A)$ and $b = rar^3 \in \mathcal{P}_{r^2\sigma,1}(A)$. Since

$$\sigma \rightharpoonup x = -x, \quad r \rightharpoonup x = y, \quad r \rightharpoonup y = r^2 \rightharpoonup x = x,$$
$$\sigma \rightharpoonup y = \sigma r \rightharpoonup x = r^3\sigma \rightharpoonup x = r(r^2\sigma) \rightharpoonup x = -r \rightharpoonup x = -y,$$

we have

$$\sigma a \sigma = -a, \quad rar^3 = b, \quad \sigma b \sigma = -b, \quad rbr^3 = a.$$

By 2.19, $a^2 = b^2 = 0$, and $ab + ba \in \mathcal{P}_{r^2,1}(A)$, from where (taking a suitable scalar multiple of a or b), we have $ab + ba = \lambda(r^2 - 1)$, with $\lambda \in \{0, 1\}$. There are hence two liftings:

1. $a^2 = b^2 = 0$, $ab + ba = 0$. This is the bosonization $\mathfrak{B}(Y_7^1)\#\mathbf{k}\Gamma$.
2. $a^2 = b^2 = 0$, $ab + ba = (r^2 - 1)$.

5.4.2. $V = Y_7^2$. It is similar to the case Y_7^1: we have $a \in \mathcal{P}_{\sigma,1}(A)$ and $b = rar^3 \in \mathcal{P}_{r^2\sigma,1}(A)$ such that

$$\sigma a \sigma = -a, \quad rar^3 = b, \quad \sigma b \sigma = b, \quad rbr^3 = -a.$$

As in that case, $a^2 = b^2 = 0$ by 2.19, and $ab - ba = \lambda(r^2 - 1)$ with $\lambda \in \{0, 1\}$, but now the diamond lemma applies:

$$0 = bba = bab - \lambda b(r^2 - 1) = abb - \lambda(r^2 - 1)b - \lambda b(r^2 - 1) = 2\lambda b,$$

from where $\lambda = 0$ for A to be 32-dimensional.

There is hence only one lifting: the bosonization $\mathfrak{B}(Y_7^2)\#\mathbf{k}\Gamma$.

5.4.3. $V = Y_7^3$. By 5.0.3, we have here $a \in \mathcal{P}_{r,1}(A)$, $b = \sigma a \sigma \in \mathcal{P}_{r^3,1}(A)$, and

$$rar^3 = -a, \quad rbr^3 = -b, \quad \sigma a \sigma = b, \quad \sigma b \sigma = a.$$

We have now $a^2 = \lambda_1(r^2 - 1)$, $b^2 = \lambda_2(r^2 - 1)$. Since $ab + ba \in \mathcal{P}(A)$ then $ab + ba = 0$ by 2.19. We can take $\lambda_i \in \{0, 1\}$, from where we have four liftings. However, the automorphism $r \mapsto r^3$, $\sigma \mapsto \sigma$ gives an isomorphism between the algebra corresponding to $(\lambda_1 = 0, \lambda_2 = 1)$ and that of $(\lambda_1 = 1, \lambda_2 = 0)$, and thus there are three non isomorphic liftings:

1. $a^2 = 0$, $b^2 = 0$, $ab + ba = 0$. This is the bosonization $\mathfrak{B}(Y_7^3)\#\mathbf{k}\Gamma$.
2. $a^2 = (r^2 - 1)$, $b^2 = 0$, $ab + ba = 0$.
3. $a^2 = (r^2 - 1)$, $b^2 = (r^2 - 1)$, $ab + ba = 0$.

5.4.4. $V = Y_7^4$. As in 5.0.4 (with $t = r$), we have here $a, b \in \mathcal{P}_{r^2,1}(A)$, with

$$rar^3 = b, \quad rbr^3 = -a, \quad \sigma a \sigma = b, \quad \sigma b \sigma = a.$$

By 2.19, we have $a^2 = b^2 = ab + ba = 0$, and thus there is only one lifting: the bosonization $\mathfrak{B}(Y_7^4)\#\mathbf{k}\Gamma$.

5.5. $\Gamma = \mathbb{H}$. We take $\mathbb{H} = \{e, (-e), \underline{i}, -\underline{i}, \underline{j}, -\underline{j}, \underline{k}, -\underline{k}\}$ with the standard relations $\underline{i}\underline{j} = \underline{k}$, etc. The conjugacy classes are $\{e\}$, $\{(-e)\}$, $\{\underline{i}, -\underline{i}\}$, $\{\underline{j}, -\underline{j}\}$ and $\{\underline{k}, -\underline{k}\}$. Since the last three classes give isomorphic algebras via the morphisms

$$(\underline{i} \mapsto \underline{j} \mapsto \underline{k} \mapsto \underline{i}) \quad \text{and} \quad (\underline{i} \mapsto -\underline{i},\ \underline{j} \mapsto -\underline{j},\ \underline{k} \mapsto -\underline{k}),$$

there is only one case as in 5.0.2(2), which is given by $h = \underline{i}$, $\chi(\underline{i}) = -1$.

There is also only one case as in 5.0.2(3), which is given by $h = (-e)$, $\rho = \rho_0$, where

$$\rho_0(\underline{i}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \rho_0(\underline{j}) = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \ \rho_0(\underline{k}) = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}.$$

The list of 4-dimensional Nichols algebras giving non isomorphic algebras when bosonized is given hence by

TABLE 17: $\Gamma = \mathbb{H}$				
$M(h,\rho)$	h	ρ	$b_{12} = b_{21}$	$\dim \mathfrak{B}(M)$
Y_8^1	$\underline{i}$	$\underline{i} \mapsto -1$	-1	4
Y_8^2	$(-e)$	ρ_0	-1	4

The liftings are:

5.5.1. $V = Y_8^1$. As in 5.0.3 we have $a \in \mathcal{P}_{\underline{i},1}(A)$ and $b = \underline{j}a(-\underline{j}) \in \mathcal{P}_{-\underline{i},1}(A)$, with the action given by

$$\underline{i}a(-\underline{i}) = -a, \quad \underline{j}a(-\underline{j}) = b, \quad \underline{k}a(-\underline{k}) = -b,$$
$$\underline{i}b(-\underline{i}) = -b, \quad \underline{j}b(-\underline{j}) = a, \quad \underline{k}b(-\underline{k}) = -a.$$

We have $\lambda_1, \lambda_2 \in \{0,1\}$ such that $a^2 = \lambda_1((-e)-1)$, $b^2 = \lambda_2((-e)-1)$, $ab + ba \in \mathcal{P}(A) \Rightarrow ab + ba = 0$. The diamond lemma says, however, that $\lambda_1 = \lambda_2$, since

$$\underline{j}a^2 = \lambda_1 \underline{j}((-e)-1) = \lambda_1((-\underline{j}) - \underline{j}),$$
$$\underline{j}a^2 = b\underline{j}a = b^2\underline{j} = \lambda_2((-e)-1)\underline{j} = \lambda_2((-\underline{j}) - \underline{j}).$$

Hence, there are two liftings:

1. $a^2 = 0$, $b^2 = 0$, $ab + ba = 0$. This is the bosonization $\mathfrak{B}(Y_8^1)\#\mathbf{k}\Gamma$.
2. $a^2 = ((-e)-1)$, $b^2 = ((-e)-1)$, $ab + ba = 0$.

5.5.2. $V = Y_8^2$. As in 5.0.4 we have $a, b \in \mathcal{P}_{-e,1}(A)$ with the action given by

$$\underline{i}a(-\underline{i}) = b, \qquad \underline{j}a(-\underline{j}) = \mathbf{i}a, \qquad \underline{k}a(-\underline{k}) = \mathbf{i}b,$$
$$\underline{i}b(-\underline{i}) = -a, \quad \underline{j}b(-\underline{j}) = -\mathbf{i}b, \quad \underline{k}b(-\underline{k}) = \mathbf{i}a.$$

By 2.19, we have $a^2 = b^2 = 0$, and $ab + ba = 0$, and hence there is only one lifting: the bosonization $\mathfrak{B}(Y_8^2)\#\mathbf{k}\Gamma$.

6. Γ of order 16

Now $\dim \mathfrak{B}(V) = 2$, from where we must have $V = M(h,\chi)$, $h \in Z(\Gamma)$ and χ a character such that $\chi(h) = -1$. Hence, for Γ to have YD-modules V with $\dim \mathfrak{B}(V) = 2$, it is necessary that $Z(\Gamma) \not\subset [\Gamma;\Gamma]$. There are 14 groups of order 16. Among them, 3 do not verify this condition.

As explained in 2.18, we take $a \in \mathcal{P}_{h,1}(A)$ such that $ga = \chi(g)ag$ for $g \in \Gamma$. Furthermore, there exists $\lambda \in \{0,1\}$ such that $a^2 = \lambda(h^2-1)$. Analogously to (5.0.8), for A to be 32-dimensional we have

$$\lambda g(h^2-1) = ga^2 = \chi^2(g)a^2g = \lambda\chi^2(g)(h^2-1)g \quad \forall g \in \Gamma$$

whence $\lambda(\chi^2(g)-1)(h^2-1) = 0$. Combining this with 2.19 we get

$$\lambda(\chi^2 - 1) = 0. \tag{6.0.1}$$

6.1. $\Gamma = C_2 \times C_2 \times C_2 \times C_2$**.** As in the case $\Gamma = C_2 \times C_2 \times C_2$, the group $\mathrm{Aut}(\Gamma)$ acts transitively on the YD-modules V such that $\dim \mathfrak{B}(V) = 2$, from where we get only one algebra after bosonization. We take hence $V = M(g_1, \hat{g}_1)$. By 2.19, there is only one lifting: the bosonization $\mathfrak{B}(V)\#\mathbf{k}\Gamma$.

6.2. $\Gamma = C_2 \times C_2 \times C_4$**.** There are 4 orbits with respect to the action of $\mathrm{Aut}(\Gamma)$: $\{1\}$, $\{g_3^2\}$, $\{g_1^i g_2^j g_3^k,\ (i,j) \neq (0,0),\ k = 0,2\}$, $\{g_1^i g_2^j g_3^k,\ k = 1,3\}$ (we are using the notation at the end of 1.1). The first orbit, as always, does not give a Nichols algebra of dimension 2. For the others, we have the following list of YD-modules giving non isomorphic algebras when bosonized (we indclude only the cases $\dim \mathfrak{B}(V) = 2$). We include also their corresponding liftings:

TABLE 18: $\Gamma = C_2 \times C_2 \times C_4$, liftings

$M(h,\chi)$	Liftings	h	χ	a^2	Remarks
Y_9^1	1	g_3^2	$\hat{g}_3$	0	See 2.19
Y_9^2	1	g_1	$\hat{g}_1$	0	See 2.19
Y_9^3	1	g_1	$\hat{g}_1\hat{g}_3$	0	See 2.19
Y_9^4	2	g_3	$\hat{g}_3^2$	0	
				(g_3^2-1)	
Y_9^5	2	g_3	$\hat{g}_1\hat{g}_3^2$	0	
				(g_3^2-1)	

6.3. $\Gamma = C_4 \times C_4$**.** The orbits are $\{1\}$, $\{g_1^2, g_1^2g_2^2, g_2^2\}$, $\{g_1^i g_2^j,\ 2 \nmid i \text{ or } 2 \nmid j\}$ (we are using the notation at the end of 1.1). The possibilities are

TABLE 19: $\Gamma = C_4 \times C_4$, liftings

$M(h,\chi)$	Liftings	h	χ	a^2	Remarks
Y_{10}^1	2	g_1	$\hat{g}_1^2$	0	
				(g_1^2-1)	
Y_{10}^2	1	g_1	$\hat{g}_1^2\hat{g}_2$	0	See 6.0.1
Y_{10}^3	1	g_1^2	$\hat{g}_1$	0	See 2.19

6.4. $\Gamma = C_2 \times C_8$. The orbits are $\{1\}$, $\{g_2^4\}$, $\{g_1g_2^2, g_1g_2^6\}$, $\{g_1, g_1g_2^4\}$, $\{g_2^2, g_2^6\}$, $\{g_1^ig_2^j,\ j = 1,3,5,7\}$. The list is

TABLE 20: $\Gamma = C_2 \times C_8$, liftings					
$M(h,\chi)$	Liftings	h	χ	a^2	Remarks
Y_{11}^1	1	g_2^4	$\hat{g}_2$	0	See 2.19
Y_{11}^2	2	$g_1g_2^2$	$\hat{g}_1$	0	
				$(g_2^4 - 1)$	
Y_{11}^3	1	$g_1g_2^2$	$\hat{g}_2^2$	0	See 6.0.1
Y_{11}^4	1	g_1	$\hat{g}_1$	0	See 2.19
Y_{11}^5	1	g_1	$\hat{g}_1\hat{g}_2^2$	0	See 2.19
Y_{11}^6	1	g_1	$\hat{g}_1\hat{g}_2$	0	See 2.19
Y_{11}^7	1	g_2^2	$\hat{g}_2^2$	0	See 6.0.1
Y_{11}^8	1	g_2^2	$\hat{g}_2^6$	0	See 6.0.1
Y_{11}^9	1	g_2^2	$\hat{g}_1\hat{g}_2^2$	0	See 6.0.1
Y_{11}^{10}	2	g_2	$\hat{g}_2^4$	0	
				$(g_2^2 - 1)$	
Y_{11}^{11}	2	g_2	$\hat{g}_1\hat{g}_2^4$	0	
				$(g_2^2 - 1)$	

6.5. $\Gamma = C_{16}$. The orbits are

$$\{1\},\ \{g^8\},\ \{g^4, g^{12}\},\ \{g^2, g^6, g^{10}, g^{14}\},\ \{g^i,\ \ 2 \not| i\}.$$

The list is

TABLE 21: $\Gamma = C_{16}$, liftings					
$M(h,\chi)$	Liftings	h	χ	a^2	Remarks
Y_{12}^1	1	g^8	$\hat{g}$	0	See 2.19
Y_{12}^2	1	g^4	$\hat{g}^2$	0	See 6.0.1
Y_{12}^3	1	g^4	$\hat{g}^6$	0	See 6.0.1
Y_{12}^4	1	g^2	$\hat{g}^4$	0	See 6.0.1
Y_{12}^5	1	g^2	$\hat{g}^{12}$	0	See 6.0.1
Y_{12}^6	2	g	$\hat{g}^8$	0	
				$(g^2 - 1)$	

6.6. **Non abelian groups.** As said above, the non abelian groups which verify $Z(\Gamma) \not\subset [\Gamma;\Gamma]$ are 6, and can be presented in the following list:

TABLE 22: Non abelian groups of order 16				
Group	Generators, relations	$Z(\Gamma)$	$[\Gamma;\Gamma]$	Γ_{ab}
B_1	$(g_1,g_2)/(g_1^8, g_2^2, g_2g_1g_2g_1^3)$	(g_1^2)	(g_1^4)	$C_4 \times C_2$
B_2	$(g_1,g_2,g_3)/R_2$	(g_1)	(g_1^2)	$C_2 \times C_2 \times C_2$
B_3	$(g_1,g_2)/(g_1^4, g_2^4, g_2g_1g_2^3g_1)$	(g_1^2, g_2^2)	(g_1^2)	$C_2 \times C_4$
$B_4 = \mathbb{D}_4 \times C_2$	(r,σ,t)	(t,r^2)	(r^2)	$C_2 \times C_2 \times C_2$
B_5	$(g_1,g_2,g_3)/R_5$	(g_1^2, g_2)	(g_2)	$C_4 \times C_2$
$B_6 = \mathbb{H} \times C_2$	$(\underline{i},\underline{j},t)$	$(t,-e)$	$(-e)$	$C_2 \times C_2 \times C_2$

where we put for B_2 and B_5 the subgroups R_2 and R_5 respectively, generated by the elements

$$R_2: \quad g_1^4,\ g_2^2,\ g_3^2,\ [g_1;g_2],\ [g_1;g_3],\ [g_3;g_2]g_1^2,$$
$$R_5: \quad g_1^4,\ g_2^2,\ g_3^2,\ [g_1;g_2],\ [g_3;g_2],\ [g_1;g_3]g_2.$$

The name B_i corresponds to the classification in [Bu, Sect. 118]. In these groups, we have $[\Gamma;\Gamma] \subset Z(\Gamma)$, $(Z(\Gamma) - [\Gamma;\Gamma]) = \{h_1, h_2\}$. Hence, there are only two possibilities for h in each group.

For $\Gamma = B_4$, for instance, $h_1 = t$ and $h_2 = r^2t$. Since the automorphism

$$(r \mapsto r,\ \sigma \mapsto \sigma,\ t \mapsto r^2t)$$

carries h_1 to h_2, we consider only $h = t$. There are 4 characters $\chi \in \hat{\Gamma}$ such that $\chi(t) = -1$. They are given by

$$\chi(t) = -1, \quad \chi(r) = \pm 1, \quad \chi(\sigma) = \pm 1.$$

However, the automorphisms

$$(t \mapsto t,\ r \mapsto t^i r,\ \sigma \mapsto t^j \sigma) \quad i,j = 0,1$$

act transitively on these representations, whence we have only one algebra.

Doing the same analysis with the other groups, we arrive to the following list (we take a as explained at the beginning of the section):

TABLE 23: Γ non abelian of order 16, liftings							
Group	$M(h,\chi)$	Lift.	h	χ		a^2	Remarks
B_1	Y_{13}^1	1	g_1^2	$\chi(g_1)=\mathbf{i}$	$\chi(g_2)=1$	0	See 6.0.1
B_1	Y_{13}^2	1	g_1^2	$\chi(g_1)=\mathbf{i}$	$\chi(g_2)=-1$	0	See 6.0.1
B_1	Y_{13}^3	1	g_1^2	$\chi(g_1)=-\mathbf{i}$	$\chi(g_2)=1$	0	See 6.0.1
B_1	Y_{13}^4	1	g_1^2	$\chi(g_1)=-\mathbf{i}$	$\chi(g_2)=-1$	0	See 6.0.1
B_2	Y_{13}^5	2	g_1	$\chi(g_1)=-1$	$\chi(g_2)=1$	0	
					$\chi(g_3)=1$	g_1^2-1	
B_2	Y_{13}^6	2	g_1	$\chi(g_1)=-1$	$\chi(g_2)=-1$	0	
					$\chi(g_3)=-1$	g_1^2-1	
B_3	Y_{13}^7	1	g_2^2	$\chi(g_1)=1$	$\chi(g_2)=\mathbf{i}$	0	See 2.19
B_3	Y_{13}^8	1	$g_1^2g_2^2$	$\chi(g_1)=1$	$\chi(g_2)=\mathbf{i}$	0	See 2.19
B_4	Y_{13}^9	1	t	$\chi(t)=-1$	$\chi(r)=1$	0	See 2.19
					$\chi(\sigma)=1$		
B_5	Y_{13}^{10}	1	g_1^2	$\chi(g_1)=\mathbf{i}$	$\chi(g_2)=1$	0	See 2.19
					$\chi(g_3)=1$		
B_6	Y_{13}^{11}	1	t	$\chi(t)=-1$	$\chi(\underline{i})=1$	0	See 2.19
					$\chi(\underline{i})=1$		

7. The classification is complete

We use the techniques of [AS2, Sect. 8]. If A is a 32-dimensional pointed Hopf algebra, we apply the procedures described in 1.2 and get R a braided Hopf algebra in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ satisfying (1.1.2), (1.1.3) and (1.1.4). We may consider the (braided) dual R^* as in [AG, Def. 2.2.2], which is a braided Hopf algebra in ${}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ satisfying (1.1.2), (1.1.3) and (1.1.5). Let $W = R^*(1)$. We get in this way the sequence of braided Hopf algebras

$$TW \to R^* \twoheadrightarrow \mathfrak{B}(W),$$

whence $\mathfrak{B}(W)\#\mathbf{k}\Gamma$ is an algebra of dimension 2^n, with $n \le 5$. If $n = 5$ then $R^* = \mathfrak{B}(W)$, from where R^* satisfies (1.1.4) and hence R satisfies (1.1.5), whence R is a Nichols algebra and A is in the list above. We must prove thence that the above surjection $R^* \twoheadrightarrow \mathfrak{B}(W)$ is in fact a bijection. This is the same as to prove that all the relations in $\mathfrak{B}(W)$ are relations in R^*, i.e., if $z \in TW$ vanishes in $\mathfrak{B}(W)$, then it vanishes in R^*. For this, notice first that for Γ non abelian there are no bosonizations of Nichols algebras over Γ with dimension 2^n, $n < 5$ (see 5.0.1).

Consider then Γ abelian. If W is 1-dimensional, let x be a generator of W. We have $c(x \otimes x) = qx \otimes x$ for some q a root of unity. Let r be the order of q ($q \neq 1$ because of 2.2). Thus, $x^r \in \mathcal{P}(R^*)$ and $c(x^r \otimes x^r) = (x^r \otimes x^r)$, whence by 2.2 we have $x^r = 0 \in R^*$. This implies that R^* verifies (1.1.4), and we are done.

Consider now $\dim W \geq 2$. By 2.5, $\dim \mathfrak{B}(W) \geq 2^{\dim W}$, and then

$$\dim \mathfrak{B}(W)\#\mathbf{k}\Gamma \geq 2^{\dim W}|\Gamma| \geq 4|\Gamma|.$$

Thus, if R were not a Nichols algebra, $|\Gamma|$ would be ≤ 4.

Let $\Gamma = C_2$. We have $W = \oplus_{i=1}^{\theta} M(g, \hat{g})$. Then for $y \in W$ we have $y^2 \in \mathcal{P}(R^*)$ and $c(y^2 \otimes y^2) = y^2 \otimes y^2$, from where by 2.2 we have $y^2 = 0\ \forall y \in W$. These are the relations for $\mathfrak{B}(W)$, and we are done.

If $|\Gamma| = 4$, the bijectivity of the projection above is a consequence of the following

Proposition 7.1. *Let Γ be a finite group, $S = \oplus_i S(i) \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}$ a graded braided Hopf algebra that verifies* (1.1.3) *and* (1.1.5), *and such that* $\dim S = 8$. *Then S is a Nichols algebra.*

Proof. Let $V = S(1)$ and $R' = \mathfrak{B}(V)$. We have $S \twoheadrightarrow R'$ the canonical projection. If $\dim R' = 8$, we are done. Consider then $\dim R' < 8$. Since the projection induces an epimorphism $S\#\mathbf{k}\Gamma \twoheadrightarrow R'\#\mathbf{k}\Gamma$, we have $\dim R' = 1,\ 2$ or 4. Since

$$\dim R \geq \dim R(0) + \dim R(1) = \dim S(0) + \dim S(1) = 1 + \dim S(1),$$

we have $\dim R' = 2$ or 4. If $\dim R' = 2$, then $\dim S(1) = 1$ and by the considerations for the rank-1 case, we would have $S = R'$. Thus $\dim R' = 4$. Now 2.7 implies that $\dim R'(1) = 2$, and then by [AG, Prop 3.1.11], $R'(1)$ comes from the abelian case. Furthermore, by the same computations as in 5.0.3 and 5.0.4, the matrix of R' is $\begin{pmatrix} -1 & \zeta \\ \zeta & -1 \end{pmatrix}$ (with $\zeta = \pm 1$) in some basis $\{x, y\}$. Now, as in 2.1, we have $x^2 \in \mathcal{P}(S)$ and $y^2 \in \mathcal{P}(S)$, and it is straightforward to see that $z = xy - \zeta yx \in \mathcal{P}(S)$. Moreover,

$$c(x^2 \otimes x^2) = x^2 \otimes x^2, \quad c(y^2 \otimes y^2) = y^2 \otimes y^2, \quad c(z \otimes z) = z \otimes z.$$

Thus, thanks to 2.2, we have $x^2 = y^2 = z = 0$. Since S is generated by x and y, this implies $\dim S = 4$, a contradiction. $\square$

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