POINTED HOPF ALGEBRAS OVER SOME SPORADIC SIMPLE GROUPS

N. ANDRUSKIEWITSCH, F. FANTINO, M. GRAÑA, L. VENDRAMIN

ABSTRACT. Any finite-dimensional complex pointed Hopf algebra with group of group-likes isomorphic to a sporadic group G, where G is either the Mathieu groups M_{22} or M_{24}, the Janko groups J_1, J_2 or J_3, the Suzuki or the Held group, is a group algebra.
RÉSUMÉ. Soit G un des groupes de Mathieu M_{22} ou M_{24}, ou un des groupes de Janko J_1, J_2 ou J_3, ou le groupe de Suzuki ou le groupe de Held. Soit H une algèbre de Hopf pointée de dimension finie dont le group des *grouplikes* est isomorphe a G. Alors H est isomorphe a l'algèbre de groupe de G.

1. INTRODUCTION

Let $\Bbbk$ be a field of characteristic 0. In this Note, we announce a new contribution to the classification of finite-dimensional Hopf algebras over $\Bbbk$. As is known, different classes of finite-dimensional Hopf algebras have to be studied separately because the pertaining methods are radically different. There is a method for pointed Hopf algebras (those whose coradical is a group algebra $\Bbbk G$) that has been applied with satisfactory results when G is abelian [8]; an exposition of the method can be found in [7]. Recently, it appeared that many finite simple (or almost simple) groups G admit very few finite-dimensional, pointed non-semisimple Hopf algebras with coradical isomorphic to $\Bbbk G$:

- Any finite-dimensional complex pointed Hopf algebra with group of group-likes isomorphic to $\mathbb{A}_m$, $m \geq 5$, $m \neq 6$, is a group algebra [2].
- Same for the groups $SL(2, 2^n)$, $n > 1$ [10] and M_{20}, $M_{21} = PSL(3, 4)$ [11].
- Most of the pointed Hopf algebras over the symmetric groups have infinite dimension, with the exception of a short list of open possibilities, see [4] and references therein (see 2.1 below).

We are presently studying finite-dimensional pointed Hopf algebras over sporadic simple groups. As part of our results, we have the following.

Theorem 1. *Let G be any of the Mathieu groups M_{22}, M_{24}, the Janko groups J_1, J_2, J_3, the Suzuki group Suz, or the Held group He. If H is a finite-dimensional pointed Hopf algebra with $G(H) \simeq G$, then $H \simeq \Bbbk G$.*

2. OUTLINE OF THE PROOF

A complete proof of Theorem 1 for the groups M_{22} and M_{24} is contained in [9]; the proof for the other groups will be included in [3].

We sketch now the proof in two main reductions. The first one has been explained in several places, with detail in [7], but we include a brief summary for completeness. We assume the reader familiar with the important notion of the Nichols algebra of

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a braided vector space, discussed at length in [7]. We remind that if U is a braided vector subspace of V, then $\mathfrak{B}(U) \hookrightarrow \mathfrak{B}(V)$.

2.1. **A general reduction.** Let G be a finite group, H a pointed Hopf algebra with $G(H) \simeq G$. Then there are two basic invariants of H, a Yetter-Drinfeld module V over $\Bbbk G$ (called the infinitesimal braiding of H) and its Nichols algebra $\mathfrak{B}(V)$. We have $|G| \dim \mathfrak{B}(V) \leq \dim H$. Therefore, the following statements are equivalent:

(1) If H is a finite-dimensional pointed Hopf algebra with $G(H) \simeq G$, then $H \simeq \Bbbk G$.

(2) If $V \neq 0$ is a Yetter-Drinfeld module over $\Bbbk G$, then $\dim \mathfrak{B}(V) = \infty$.

(3) If V is an *irreducible* Yetter-Drinfeld module over $\Bbbk G$, then $\dim \mathfrak{B}(V) = \infty$.

2.2. **Looking at subracks.** We focus on (3) above. The second reduction has been the basis of our recent papers. It starts from the well-known classification of irreducible Yetter-Drinfeld modules over $\Bbbk G$ by pairs $(\mathcal{O}, \rho)$, where $\mathcal{O}$ is a conjugacy class in G and ρ is an irreducible representation of the stabilizer G^s of a fixed point $s \in \mathcal{O}$. Now, the definition of the Nichols algebra $\mathfrak{B}(\mathcal{O}, \rho)$ of the corresponding Yetter-Drinfeld module $M(\mathcal{O}, \rho)$ just depends on the braiding. If $\dim \rho = 1$, then this braiding depends only on the *rack* $\mathcal{O}$ and a 2-cocycle $q : \mathcal{O} \times \mathcal{O} \to \Bbbk^\times$ [5]. Namely, $\mathcal{O}$ is a rack with the product $x \triangleright y := xyx^{-1}$, $M(\mathcal{O}, \rho)$ has a natural basis $(e_x)_{x \in \mathcal{O}}$ and the braiding is given by $c(e_x \otimes e_y) = q_{xy} e_{x \triangleright y} \otimes e_x$. If there exists a subrack X of $\mathcal{O}$ such that the Nichols algebra of the braided vector space defined by X and the restriction of q is infinite dimensional, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

We recall some examples of racks which are relevant in this work.

(i) Abelian racks: those racks X such that $x \triangleright y = y$ for all $x, y \in X$.

(ii) $\mathcal{D}_p$: the class of involutions in the dihedral group $\mathbb{D}_p$, p a prime.

(iii) $\mathfrak{O}$: the class of 4-cycles in $\mathbb{S}_4$.

(iv) Doubles of racks: if X is a rack, then $X^{(2)}$ denotes the disjoint union of two copies of X each acting on the other by left multiplication.

We are interested in finding subracks which are abelian, or isomorphic to $\mathcal{D}_p^{(2)}$ or to $\mathfrak{O}^{(2)}$, by the following reasons:

(A) If X is abelian, then the corresponding braided vector space is of diagonal type. Braided vector spaces of diagonal type with finite-dimensional Nichols algebra where classified in [13]; thus, we just need to check if the matrix (q_{xy}) belongs or not to the list in [13].

(B) If X is isomorphic either to $\mathcal{D}_p^{(2)}$ or to $\mathfrak{O}^{(2)}$, then for some specific cocycles, the related Nichols algebras have infinite dimension [6, Ths. 4.7, 4.8].

Variations.

(a) If $\dim \rho > 1$, similar arguments apply.

(b) Sometimes the rack X is not abelian, but the braided vector space produced by X and the 2-cocycle can be realized with an abelian rack, by a suitable change of basis.

(c) Let $F < G$ be a subgroup, $s \in F$, $\mathcal{O}^F$, resp. $\mathcal{O}^G$ the conjugacy class of s in F, resp. in G. If $\dim \mathfrak{B}(\mathcal{O}^F, \tau) = \infty$ for any irreducible representation τ of F^s, then $\dim \mathfrak{B}(\mathcal{O}^G, \rho) = \infty$ for any irreducible representation ρ of G^s.

(d) A conjugacy class $\mathcal{O}$ is real if $\mathcal{O} = \mathcal{O}^{-1}$. It is quasireal if $\mathcal{O} = \mathcal{O}^m$ for some integer m, $1 < m < N$, where N is the order of the elements in $\mathcal{O}$. The search of subracks isomorphic to $\mathcal{D}_p^{(2)}$ or to $\mathfrak{O}^{(2)}$, as well as the verification that the restriction of the cocycle q is as needed in (B), is greatly simplified in a real (quasireal) conjugacy class [1].

2.3. **Computations.** We now fix a sporadic group G.

- We extracted relevant information from the ATLAS [14] by using GAP [12] and the `AtlasRep` package [15].
- We checked with GAP [12] when a conjugacy class is real; the correspondence between conjugacy classes in a group G and in a subgroup H. We wrote GAP functions to find subracks of types (i),...,(iv).

These tools allow to apply the techniques sketched above to all pairs $(\mathcal{O}, \rho)$ and establish the validity of (3).

2.4. **Final remarks.** Some of the results presented here are part of the PhD theses of FF and LV.

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[a]Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba. CIEM – CONICET. Medina Allende s/n (5000) Ciudad Universitaria, Córdoba, Argentina

[b]Departamento de Matemática – FCEyN, Universidad de Buenos Aires, Pab. I – Ciudad Universitaria (1428) Buenos Aires – Argentina

[c]Instituto de Ciencias, Universidad de Gral. Sarmiento, J.M. Gutierrez 1150, Los Polvorines (1653), Buenos Aires – Argentina

E-mail address: `andrus@famaf.unc.edu.ar` [a]

E-mail address: `fantino@famaf.unc.edu.ar` [a]

E-mail address: `matiasg@dm.uba.ar` [b]

E-mail address: `lvendramin@dm.uba.ar` [b,c]