FINITE-DIMENSIONAL POINTED HOPF ALGEBRAS WITH ALTERNATING GROUPS ARE TRIVIAL

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Abstract. Any finite-dimensional complex pointed Hopf algebra with group of group-likes isomorphic to the alternating group $\mathbb{A}_m$, $m > 4$, is a group algebra.

1. Introduction

This paper contributes to the classification of finite-dimensional pointed Hopf algebras over the field $\mathbb{C}$ of complex numbers. Our basic reference is [AS]; see *loc. cit.* for unexplained terminology and notation. If G denotes a finite group, we would like to know all pointed Hopf algebras H with $G(H) \simeq G$ and $\dim H < \infty$. For this, we need to solve the following problem. Let $\mathcal{O}$ be a conjugacy class of G, $\sigma \in \mathcal{O}$ fixed, ρ an irreducible representation of the centralizer $C_G(\sigma)$, $M(\mathcal{O}, \rho)$ the corresponding irreducible Yetter-Drinfeld module and $\mathfrak{B}(\mathcal{O}, \rho)$ the associated Nichols algebra. If (V, c) is a braided vector space, that is $c \in \mathbf{GL}(V \otimes V)$ is a solution of the braid equation, then $\mathfrak{B}(V)$ denotes its Nichols algebra; for shortness, we write $\mathfrak{B}(\mathcal{O}, \rho)$ instead of $\mathfrak{B}(M(\mathcal{O}, \rho))$. The problem is:

For which pairs $(\mathcal{O}, \rho)$ *is the dimension of* $\mathfrak{B}(\mathcal{O}, \rho)$ *finite?*

We denote by $\widehat{G}$ the set of isomorphism classes of irreducible representations of a group G. We use the rack notation $x \triangleright y = xyx^{-1}$, x, $y \in G$. See [AG] for information on racks. If $\sigma \in G$ and $\rho \in \widehat{C_G(\sigma)}$, then $\rho(\sigma)$ is a scalar denoted $q_{\sigma\sigma}$.

This article is continuation of [AF1], where we began the study of finite-dimensional pointed Hopf algebras with group of group-like elements isomorphic to $\mathbb{A}_m$.

Theorem 1.1. *Let* $G = \mathbb{A}_m$, $m \geq 5$. *If* $\mathcal{O}$ *is any conjugacy class of* G, $\sigma \in \mathcal{O}$ *is fixed and* $\rho \in \widehat{C_G(\sigma)}$, *then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Date: April 21, 2009.

2000 *Mathematics Subject Classification.* 16W30; 17B37.

This work was partially supported by ANPCyT-Foncyt, CONICET, Ministerio de Ciencia y Tecnología (Córdoba) and Secyt (UNC).

By the Lifting Method [AS], we conclude:

Theorem 1.2. *Let $G = \mathbb{A}_m$, $m \geq 5$. Any finite-dimensional pointed Hopf algebra $\mathcal{H}$ with $G(\mathcal{H}) \simeq \mathbb{A}_m$, is isomorphic to $\mathbb{C}\mathbb{A}_m$.* □

This result was known for the particular cases $m = 5$ and $m = 7$ [AF1, F]. We prove it for $m \geq 6$. Since $\mathbb{A}_3$ is abelian, finite dimensional Nichols algebras over it are classified, there are 25 of them. Nichols algebras over $\mathbb{A}_4$ are infinite-dimensional except for four pairs corresponding to the classes of $(1\,2\,3)$ and $(1\,3\,2)$ and the non-trivial characters of $\mathbb{Z}/3$. Actually, these four algebras are connected to each other either by an outer automorphism of $\mathbb{A}_4$ or by the Galois group of $\mathbb{Q}(\zeta_3)|\mathbb{Q}$ (the cyclotomic extension by third roots of unity). Therefore, there is only one pair to study for $\mathbb{A}_4$.

In a previous version of this paper, we proved Theorem 1.2 for $m \geq 7$, but were unable to decide the dimension of the Nichols algebra attached to the pair formed by the class of $(1\,2\,3\,4)(5\,6)$ and the character $\rho = \chi_{(-1)}$, corresponding to $\mathbb{A}_6$. It was observed that this class contains a subrack with 18 elements, a union of 2 subracks of order 9, identified as a union of two conjugacy classes in $\mathbb{F}_9 \rtimes \mathbb{Z}/4$. This pair can be discarded now too, and consequently we also finish $\mathbb{A}_6$, by means of [HS1, Th. 8.6].

The paper is organized as follows. In Section 2, we spell out some techniques, based on the analysis of braided vector subspaces of diagonal type, that are needed for our arguments but are also useful elsewhere. In Section 3 we prove Theorem 1.1.

2. Some techniques

2.1. **An abelian subrack with 3 elements.** We begin by recording a result that is needed in Lemma 2.4 and will be also useful elsewhere.

Lemma 2.1. *Let G be a finite group, $\mathcal{O}$ be the conjugacy class of σ_1 in G and $(\rho, V) \in \widehat{C_G(\sigma_1)}$. Let $\sigma_2 \neq \sigma_3 \in \mathcal{O} - \{\sigma_1\}$; let $g_1 = e$, g_2, $g_3 \in G$ such that $\sigma_i = g_i \sigma_1 g_i^{-1}$, for all i. Assume that*

- *$\sigma_1^h = \sigma_2\sigma_3$ for an odd integer h,*
- *g_3g_2 and g_2g_3 belong to $C_G(\sigma_1)$, and*
- *$\sigma_i\sigma_j = \sigma_j\sigma_i$, $1 \leq i, j \leq 3$.*

Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$*, for any $\rho \in \widehat{C_G(\sigma_1)}$.*

Proof. Since $\sigma_i\sigma_j = \sigma_j\sigma_i$, there exists $w \in V - 0$ and $\lambda_i \in \mathbb{C}$ such that $\rho(\sigma_i)(w) = \lambda_i w$ for $i = 1, 2, 3$. For any $1 \leq i, j \leq 3$, we call $\gamma_{ij} = g_j^{-1}\sigma_i g_j$. It is easy to see that $\gamma_{ij} \in C_G(\sigma_1)$ and that

$$\gamma = (\gamma_{ij}) = \begin{pmatrix} \sigma_1 & \sigma_3 & \sigma_2 \\ \sigma_2 & \sigma_1 & \sigma_2^h \sigma_1^{-1} \\ \sigma_3 & \sigma_3^h \sigma_1^{-1} & \sigma_1 \end{pmatrix}.$$

Then, $W = \operatorname{span}\{g_1 w, g_2 w, g_3 w\}$ is a braided vector subspace of $M(\mathcal{O}, \rho)$ of abelian type with Dynkin diagram given by Figure 1. Assume that $\dim \mathfrak{B}(\mathcal{O}, \rho)$ is finite. Then $\lambda_1 \neq 1$; also $\lambda_1^h \neq 1$, for otherwise $g_2 w, g_3 w$ span a braided vector subspace of Cartan type with Dynkin diagram $A_1^{(1)}$. Thus, we should have $\lambda_1 = -1$ and h even, by [H2, Table 3], but this is a contradiction to the hypothesis on h. □

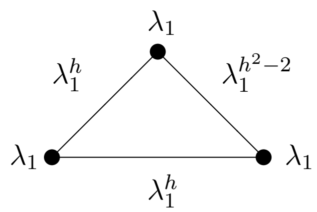

FIGURE 1

2.2. **The technique of a suitable subgroup.** Notice that if (V, c) is a braided vector space and $W \subseteq V$ is a subspace such that $c(W \otimes W) = W \otimes W$, then $\mathfrak{B}(W) \subseteq \mathfrak{B}(V)$. In particular, if $\mathfrak{B}(W)$ is infinite dimensional, also is $\mathfrak{B}(V)$.

Let G be a finite group, $\sigma \in G$, $\mathcal{O}_\sigma^G = \mathcal{O}$ its conjugacy class, $C_G(\sigma)$ its centralizer and $\rho \in \widehat{C_G(\sigma)}$. If H is a subgroup of G and $\sigma \in H$, then $\mathcal{O}_\sigma^H = \mathcal{O}^H$ denotes the conjugacy class of σ in H.

Lemma 2.2. *If* $\dim \mathfrak{B}(\mathcal{O}^H, \tau) = \infty$ *for all* $\tau \in \widehat{C_H(\sigma)}$, *then* $\dim \mathfrak{B}(\mathcal{O}^G, \rho) = \infty$ *for all* $\rho \in \widehat{C_G(\sigma)}$.

Proof. Since $M = \operatorname{Ind}_{C_G(\sigma)}^G \rho = \oplus_{s \in \mathcal{O}^G} V_s$, where $V_s = \{v \in V \mid \delta(v) = s \otimes v\}$, we have that $M^H := \oplus_{s \in \mathcal{O}^H} V_s \subseteq M$ is a Yetter-Drinfeld module over H. □

Now assume that $\sigma_1, \sigma_2 \in H$. Let $\mathcal{O}_i$ be the conjugacy class in H of σ_i. Assume that $\mathcal{O}_1 \neq \mathcal{O}_2$.

Lemma 2.3. *If* $\mathfrak{B}(M(\mathcal{O}_1, \tau_1) \oplus M(\mathcal{O}_2, \tau_2))$ *has infinite dimension for all pairs* $\tau_1 \in \widehat{C_H(\sigma_1)}$, $\tau_2 \in \widehat{C_H(\sigma_2)}$, *then* $\mathfrak{B}(\mathcal{O}^G, \rho)$ *is infinite dimensional.*

Proof. As before, $M = \oplus_{s \in \mathcal{O}^G} V_s$ and then $\oplus_{s \in \mathcal{O}_1 \cup \mathcal{O}_2} V_s \subseteq M$ is a Yetter-Drinfeld module over H. □

2.3. **The group $\mathbb{A}_4 \times \mathbb{Z}/r$ for r odd.** Let r be an odd integer and let $G = \mathbb{A}_4 \times \mathbb{Z}/r$. Assume that $\mathbb{Z}/r$ is generated by τ.

Lemma 2.4. *Let $\mathcal{O}$ be the conjugacy class of $\sigma = ((1\,2)(3\,4), \tau)$ in G. Then, $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ for every $\rho \in \widehat{C_G(\sigma)}$.*

Proof. Apply Lemma 2.1 with $\sigma_1 = ((1\,2)(3\,4), \tau)$, $\sigma_2 = ((1\,3)(2\,4), \tau)$, $\sigma_3 = ((1\,4)(2\,3), \tau)$, $g_1 = e$, $g_2 = (1\,3\,2) \times 1$, $g_3 = g_2^{-1}$ and $h = r + 2$. □

Remark 2.5. The case $r = 1$ of this Lemma is known (see for example [AF1, Prop. 2.4]) and it is used to kill the conjugacy class of involutions in $\mathbb{A}_4$.

2.4. **The group $\mathbf{SL}(2, 3)$.** Let $G = \mathbf{SL}(2, 3)$; recall $|G| = 24$. Here is one presentation of G by generators and relations:

$$\mathbf{SL}(2,3) \simeq \langle x, y, z \,|\, x^4 = y^4 = z^3 = 1, x^2 = y^2, y^{-1}xy = x^{-1}, \\ z^{-1}xz = y^{-1}, z^{-1}yz = yx^{-1} \rangle.$$

This presentation can be realized by choosing

$$x = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{1}$$

Let us consider the conjugacy class of $\sigma = x \in G$, explicitly

$$\mathcal{O}_\sigma = \left\{ \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \right\}.$$

We numerate these elements as $\sigma_1, \ldots, \sigma_6$, in this order. The centralizer $C_G(\sigma)$ is the cyclic group of order 4 generated by σ.

Definition 2.6. The rack underlying the orbit of σ in $\mathbf{SL}(2, 3)$ will be denoted $\mathcal{D}_2^3$.

Lemma 2.7. [FGV, Subs. 3.2] *If $(\rho, V) \in \widehat{C_G(\sigma)}$, then* $\dim \mathfrak{B}(\mathcal{O}_\sigma, \rho) = \infty$.

For completeness, we include a proof of this result.

Proof. The class $\mathcal{O}_\sigma$ is real because it contains all elements of order 4 in G; hence, we only need to consider $\rho = \chi \in \widehat{C_G(\sigma)}$ such that $\chi(\sigma) = -1$, cf. [AZ, 2.2]. Let

$$g_1 = \mathrm{id}, \quad g_2 = \sigma_3, \quad g_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad g_4 = \sigma g_3, \quad g_5 = \sigma g_6, \quad g_6 = g_3^{-1}. \tag{2}$$

Then $g_i \triangleright \sigma_1 = \sigma_j$. For any i, j, there is an index denoted $i \triangleright j$ and $\gamma_{ij} \in C_G(\sigma)$ such that $\sigma_i g_j = g_{i\triangleright j}\gamma_{ij}$. A straightforward computation shows that

$$\gamma = (\gamma_{ij}) = \begin{pmatrix} \sigma & \sigma^{-1} & 1 & \sigma^2 & \sigma^2 & 1 \\ \sigma^{-1} & \sigma & \sigma^2 & 1 & 1 & \sigma^2 \\ 1 & \sigma^2 & \sigma & \sigma^{-1} & \sigma & \sigma \\ \sigma^2 & 1 & \sigma^{-1} & \sigma & \sigma^{-1} & \sigma^{-1} \\ \sigma^{-1} & \sigma^{-1} & \sigma & \sigma & \sigma & \sigma^{-1} \\ \sigma & \sigma & \sigma^{-1} & \sigma^{-1} & \sigma^{-1} & \sigma \end{pmatrix}. \tag{3}$$

Let $v \in V - 0$. We define

$$\begin{aligned} u_1 &:= g_1 v + g_2 v, & u_3 &:= g_3 v + g_4 v, & u_5 &:= g_5 v + g_6 v, \\ u_2 &:= g_1 v - g_2 v, & u_4 &:= g_3 v - g_4 v, & u_6 &:= g_5 v - g_6 v. \end{aligned} \tag{4}$$

By straightforward computations, we can see that, in this basis, $M(\mathcal{O}, \rho)$ is a braided vector space of diagonal type with matrix given by

$$\mathcal{Q} = \begin{pmatrix} -1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 & -1 \end{pmatrix}.$$

Hence $M(\mathcal{O}, \rho)$ is of Cartan type with Dynkin diagram given by Figure 2; this is not of finite type, and $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, by [H1]. □

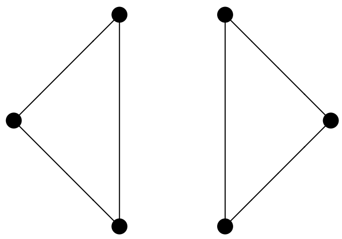

FIGURE 2

2.5. **The group $\mathbf{SL}(2,3) \times \mathbb{Z}/r$ for r a prime number.** In this subsection, we present a useful variant of the criterium given in 2.4 that will be used in [AFGV]. Let $G = \mathbf{SL}(2,3) \times \mathbb{Z}/r$. Let us consider the conjugacy class $\mathcal{O}$ of $\sigma = (x, \tau)$, where $x = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and τ has order r. The centralizer of σ is $C_G(\sigma) = \langle x \rangle \times \langle \tau \rangle \simeq \mathbb{Z}/4 \times \mathbb{Z}/r$. We consider Nichols algebras associated to pairs $(\mathcal{O}, \rho)$, where $\rho = \rho_1 \otimes \rho_2 \in \widehat{C_G(\sigma)}$, $\rho_1 \in \widehat{\mathbb{Z}/4}$ and $\rho_2 \in \widehat{\mathbb{Z}/r}$.

Lemma 2.8. *If* $(\rho, V) \in \widehat{C_G(\sigma)}$, *then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. The braided vector space $M(\mathcal{O}, \rho)$ is $(V \otimes W, c_{V\otimes W})$, where (V, c_V) is the braided vector space associated to $(\mathcal{O}_x, \rho_1)$, (W, c_W) is the braided vector space associated to $(\mathcal{O}_\tau, \rho_2)$, and

$$c_{V\otimes W} = (\mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id})(c_V \otimes c_W)(\mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id}).$$

First notice that if $r \geq 3$ the conjugacy class $\mathcal{O}$ is quasi-real of type j, with j an odd number, and $g = g^{j^2}$. Then, by [FGV, Corollary 2.2] we only need to consider $\rho(g) = \zeta$, where $\zeta = -1$ or ζ is a cubic root of 1. Moreover, if $r > 3$, since r is a prime number, we only need to consider the case $\rho(g) = -1$. Also, if $r = 2$, then $\mathcal{O}$ is a real conjugacy class and by [AZ, Lemma 2.2] we need to consider $\rho(g) = -1$. Therefore, we only need to consider $\rho = \chi_{(-1)} \otimes \epsilon$ or $\rho = \epsilon \otimes \chi$, where $\chi(g) = \zeta$ is a primitive r-root of 1. If $\rho = \chi_{(-1)} \otimes \epsilon$, the result follows in a analogous way to the proof of Lemma 2.7. Assume then that $\rho = \varepsilon \otimes \chi$ for $r = 2$ or 3. We call $\nu_i = (\sigma_i, \tau)$ and $h_i = (g_i, \mathrm{id})$, where σ_i and g_i are as in the proof of Lemma 2.7. Then $h_i \triangleright \nu_1 = \nu_i$, $1 \leq i \leq 6$, and $\nu_i h_j = h_{i\triangleright j}\delta_{ij}$, where $\delta_{ij} = (\gamma_{ij}, \tau)$, with γ_{ij} given by (3), $1 \leq i, j \leq 6$. Let $v \in V - 0$. We define W as the $\mathbb{C}$-span of $\{u_l \,|\, 1 \leq l \leq 6\}$, where

$$\begin{aligned} &u_1 := h_1 v + h_2 v, && u_3 := h_3 v + h_4 v, && u_5 := h_5 v + h_6 v, \\ &u_2 := h_1 v - h_2 v, && u_4 := h_3 v - h_4 v, && u_6 := h_5 v - h_6 v. \end{aligned} \tag{5}$$

By straightforward computations, we can see that, in this basis, $M(\mathcal{O}, \rho)$ is a braided vector space of diagonal type with matrix given by

$$\mathcal{Q} = \begin{pmatrix} \zeta & \zeta & \zeta & -\zeta & \zeta & -\zeta \\ \zeta & \zeta & \zeta & -\zeta & \zeta & -\zeta \\ \zeta & -\zeta & \zeta & \zeta & \zeta & -\zeta \\ \zeta & -\zeta & \zeta & \zeta & \zeta & -\zeta \\ \zeta & -\zeta & \zeta & -\zeta & \zeta & \zeta \\ \zeta & -\zeta & \zeta & -\zeta & \zeta & \zeta \end{pmatrix}.$$

Hence, $M(\mathcal{O}, \rho)$ is of Cartan type with Dynkin diagram of infinite type (if $r = 2$ the Dynkin diagram is $A_5^{(1)}$). Thus $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$. □

3. Nichols algebras over $\mathbb{A}_m$

3.1. **Notations on symmetric groups.** Let $\sigma \in \mathbb{S}_m$. We say that σ is of type $(1^{n_1}, 2^{n_2}, \ldots, m^{n_m})$ if the decomposition of σ as product of disjoint cycles contains n_j cycles of length j, for every j, $1 \leq j \leq m$. Let $A_j =$

$A_{1,j} \cdots A_{n_j,j}$ be the product of the $n_j \geq 0$ disjoint j-cycles $A_{1,j}, \ldots, A_{n_j,j}$ of σ. Then

$$\sigma = A_1 \cdots A_m; \tag{6}$$

we shall omit A_j when $n_j = 0$. The *even* and the *odd parts* of σ are

$$\sigma_e := \prod_{j \text{ even}} A_j, \qquad \sigma_o := \prod_{1<j \text{ odd}} A_j. \tag{7}$$

Thus, $\sigma = A_1\sigma_e\sigma_o = \sigma_e\sigma_o$; we need to define σ_o in this way for simplicity of some statements and proofs. We say also that σ has type $(1^{n_1}, 2^{n_2}, \ldots, \sigma_o)$, for brevity.

The centralizer $\mathbb{S}_m^\sigma = T_1 \times \cdots \times T_m$, where

$$T_j = \langle A_{1,j}, \ldots, A_{n_j,j}\rangle \rtimes \langle B_{1,j}, \ldots, B_{n_j-1,j}\rangle \simeq (\mathbb{Z}/j)^{n_j} \rtimes \mathbb{S}_{n_j}, \tag{8}$$

$1 \leq j \leq m$. We describe the irreducible representations of the centralizers. If $\rho = (\rho, V) \in \widehat{C_{\mathbb{S}_m}(\sigma)}$, then $\rho = \rho_1 \otimes \cdots \otimes \rho_m$, where $\rho_j \in \widehat{T_j}$ has the form

$$\rho_j = \operatorname{Ind}_{(\mathbb{Z}/j)^{n_j} \rtimes \mathbb{S}_{n_j}^{\chi_j}}^{(\mathbb{Z}/j)^{n_j} \rtimes \mathbb{S}_{n_j}} (\chi_j \otimes \mu_j), \tag{9}$$

with $\chi_j \in \widehat{(\mathbb{Z}/j)^{n_j}}$ and $\mu_j \in \widehat{\mathbb{S}_{n_j}^{\chi_j}}$ – see [S, Section 8.2].

Remark 3.1. Let $\sigma \in \mathbb{A}_m$. Then $\sigma = A_1\sigma_e\sigma_o$, see (7); clearly, $\sigma_e, \sigma_o \in \mathbb{A}_m$. Since σ_e, $\sigma_o \in Z(C_{\mathbb{A}_m}(\sigma))$, the center of $C_{\mathbb{A}_m}(\sigma)$, ρ acts by a scalar on σ_e and σ_o, i. e. $\rho(\sigma_e) = \lambda \operatorname{Id}$ and $\rho(\sigma_o) = \widetilde{\lambda} \operatorname{Id}$. Hence, $q_{\sigma\sigma} = \lambda\widetilde{\lambda}$. Notice that if the orders of σ_e and σ_o are relatively prime and $q_{\sigma\sigma} = -1$, then $\lambda = -1$ and $\widetilde{\lambda} = 1$.

We introduce some elements of $\mathbb{S}_m$ attached to a cycle α that will be used later. Let $\alpha = (i_1\ i_2\ i_3\ \cdots\ i_{4n})$ be a $4n$-cycle in $\mathbb{A}_m$. We define

$$g_\alpha := \prod_{l=1}^{2n} (l \quad 4n - l + 1) \tag{10}$$

Thus, $g_\alpha \in \mathbb{A}_m$ is an involution and $g_\alpha \triangleright \alpha = \alpha^{-1}$.

3.2. **Scheme of the proof of Theorem 1.1.** We proceed to the strategy of the proof of Theorem 1.1, postponing to later subsections the consideration of some particular cases. Let $G = \mathbb{A}_m$, with $m \geq 6$, $\sigma \in G$ of type $(1^{n_1}, 2^{n_2}, \ldots, m^{n_m})$, $\mathcal{O}$ its conjugacy class and $\rho \in \widehat{C_G(\sigma)}$. *Assume that* $\dim \mathfrak{B}(\mathcal{O}, \rho) < \infty$. Then σ is real with even order and $q_{\sigma\sigma} = -1$ by [AF1, 2.3]; $\mathcal{O}_\sigma^{\mathbb{A}_m} = \mathcal{O}_\sigma^{\mathbb{S}_m}$ and $[C_{\mathbb{S}_m}(\sigma) : C_{\mathbb{A}_m}(\sigma)] = 2$ (see for instance [JL, Proposition 12.17]). Hence, any subrack of $\mathcal{O}_\sigma^{\mathbb{S}_m}$ is obviously a subrack of $\mathcal{O}_\sigma^{\mathbb{A}_m}$ and we may apply the techniques from [AF2].

(a) If $j \geq 6$ is even and has an odd divisor, then $n_j = 0$. Otherwise, $\mathcal{O}_\sigma^{\mathbb{A}_m}$ contains a subrack of type $\mathcal{D}_p^{(2)}$, with p odd prime, by [AF2, 2.11] and $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ by [AF2, 2.9].

(b) $n_{2^k} \leq 2$, for all $k \geq 2$. Otherwise, $\mathcal{O}_\sigma^{\mathbb{A}_m}$ contains a subrack of type $\mathcal{D}_3$ by the proof of [AF2, 3.10] and $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ by [AF2, 3.8].

(c) The type of σ_e is $(2^{n_2}, 4^{n_4})$, by Proposition 3.5.

So far, we have that

$$\sigma = A_1 \sigma_e \sigma_o$$

where A_1 is of type (1^{n_1}), σ_e is of type $(2^{n_2}, 4^{n_4})$, with $n_2 + n_4$ even and $n_4 \leq 2$, and σ_o is of type $(3^{n_3}, 5^{n_5}, \dots)$.

(d) $n_4 > 0$. Otherwise, σ is of type $(1^{n_1}, 2^{n_2}, \sigma_o)$; here n_2 is even, because $(1^{n_1}, 2^{n_2}, \sigma_o) \notin \mathbb{A}_m$ if n_2 is odd. Then we conclude by Prop. 3.6.

(e) $n_2 \leq 2$. Otherwise, $\mathcal{O}_\sigma^{\mathbb{A}_m}$ contains a subrack of type $\mathcal{D}_3$ by the proof of [AF2, 3.12] and $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ by [AF2, 3.8] – note that $\sigma \neq \sigma^{-1}$ because $n_4 > 0$.

(f) If $n_2 > 0$, then $n_1 = 0$. Otherwise, $\mathcal{O}_\sigma^{\mathbb{A}_m}$ contains a subrack of type $\mathcal{D}_3$ by the proof of [AF2, 3.9] and $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ by [AF2, 3.8] – note that $\sigma \neq \sigma^{-1}$ because $n_4 > 0$.

(g) σ_o is trivial by Prop. 3.7.

(h) The remaining types are: $(2, 4)$, excluded by Prop. 3.9; $(1^{n_1}, 4^2)$, excluded by Prop. 3.2; and $(2^2, 4^2)$, excluded by Prop. 3.3.

3.3. **The classes** $(1^{n_1}, 4^2)$ **and** $(2^2, 4^2)$**.** We now apply the technique of the subgroup with $H = \mathbf{SL}(2, 3)$.

Proposition 3.2. *Let $G = \mathbb{A}_m$ or $\mathbb{S}_m$, $\sigma \in G$, $\mathcal{O}$ the conjugacy class of σ and $\rho \in \widehat{C_G(\sigma)}$. If the type of σ is $(1^{n_1}, 4^2)$, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.*

Proof. The group $\mathbf{SL}(2, 3)$ acts faithfully on $\mathbb{F}_3 \times \mathbb{F}_3$, and also on $\mathbb{F}_3 \times \mathbb{F}_3 \setminus \{(0, 0)\}$, which consists of 8 elements. Therefore, we get an injective morphism $\psi : \mathbf{SL}(2, 3) \to \mathbb{S}_8 \subseteq \mathbb{S}_m$. Using a particular labelling of the elements, this map is given by $x \mapsto (1\ 3\ 2\ 6)(4\ 5\ 8\ 7)$, $y \mapsto (1\ 4\ 2\ 8)(3\ 7\ 6\ 5)$, $z \mapsto (1\ 4\ 7)(2\ 8\ 5)$, whence the image lies in $\mathbb{A}_8 \subseteq \mathbb{A}_m$. By Lemma 2.7, the claims follows. □

Proposition 3.3. *Let $\sigma \in \mathbb{A}_{12}$, $\mathcal{O}$ the conjugacy class of σ and $\rho \in \widehat{C_{\mathbb{A}_{12}}(\sigma)}$. If the type of σ is $(2^2, 4^2)$, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.*

Proof. As before, we have a faithful permutation action of $\mathbf{SL}(2, 3)$, which is the product $\psi \times \varphi$, where ψ is the morphism in the proof of Proposition 3.2, and $\varphi : \mathbf{SL}(2, 3) \to \mathbb{A}_4$ is given by

$$x \mapsto (9\ 10)(11\ 12), \quad y \mapsto (9\ 11)(10\ 12), \quad z \mapsto (9\ 11\ 12)$$

(notice that the group generated by (9 10)(11 12), (9 11)(10 12), (9 11 12) is isomorphic to $\mathbb{A}_4$). The image of $\psi \times \varphi$ lies in $\mathbb{A}_8 \times \mathbb{A}_4 \subseteq \mathbb{A}_{12}$ and the type of $(\psi \times \varphi)(x)$ is $(2^2, 4^2)$. By Lemma 2.7, the claims follows. $\square$

Remark 3.4. This argument applies also to the class of $\sigma \in \mathbb{S}_m$ with type $(2^2, 4^2)$. This was dealt with transversal subracks in [AF2].

3.4. **Remaining cases.**

Proposition 3.5. *Let $\sigma \in \mathbb{A}_m$ be of type $(1^{n_1}, 2^{n_2}, 4^{n_4}, \dots, (2^k)^{n_{2^k}}, \sigma_o)$, with $k \geq 3$ and $n_{2^k} > 0$, $\mathcal{O}$ the conjugacy class of σ in $\mathbb{A}_m$ and $\rho = (\rho, V) \in \widehat{C_{\mathbb{A}_m}(\sigma)}$. Then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. As explained in Subsection 3.2 (b), we may assume $n_{2^k} = 1$ or 2.

(I) Assume that $n_{2^k} = 1$. Let $\alpha = (i_1\, i_2\, \cdots\, i_{2^k})$ be the 2^k-cycle appearing in the decomposition of σ as product of disjoint cycles, and we call

$$\mathbf{I} := (i_1\, i_3\, i_5\, \cdots\, i_{2^k-1}) \quad \text{and} \quad \mathbf{P} := (i_2\, i_4\, i_6\, \cdots\, i_{2^k}).$$

In the proof of [AF2, Lemma 2.11], it was shown that

(a) $\mathbf{I}$ and $\mathbf{P}$ are disjoint 2^{k-1}-cycles,
(b) $\alpha^2 = \mathbf{IP}$,
(c) $\alpha \mathbf{I} \alpha^{-1} = \mathbf{P}$, (hence $\sigma \mathbf{I} \sigma^{-1} = \mathbf{P}$),
(d) $\mathbf{P}^t \alpha \mathbf{P}^t = \alpha^{2t+1}$, for all integer t.

For notational convenience, we set

$$r := 2^{k-3}, \qquad \widetilde{g}_l := \mathbf{P}^{2^r l}, \quad 1 \leq l \leq 4.$$

Notice that

(i) if $k \geq 4$, then $\widetilde{g}_l = (\mathbf{P}^{2^{r-1} l})^2 \in \mathbb{A}_m$.
(ii) if $k = 3$, then $\widetilde{g}_4 = \mathrm{id}$ and $\widetilde{g}_2 = \mathbf{P}^2$ are in $\mathbb{A}_m$, whereas $\widetilde{g}_1 = \mathbf{P}$ and $\widetilde{g}_3 = \mathbf{P}^3$ are not in $\mathbb{A}_m$.

For every $1 \leq l \leq 4$, we define $g_l = \widetilde{g}_l$ in the case (i) or in the case (ii) with $l = 2$ or 4, and $g_l = \widetilde{g}_l \alpha$ in the case (ii) with $l = 1$ or 3. Then, $g_l \in \mathbb{A}_m$, $1 \leq l \leq 4$. We define $\alpha_l := g_l \triangleright \alpha$ and

$$\sigma_l := g_l \triangleright \sigma. \tag{11}$$

Notice that $\sigma_l = (g_l \triangleright \sigma_e)\, \sigma_o$, for all l. Then $(\sigma_l)_{1 \leq l \leq 4}$ is a subrack of $\mathcal{O}$ of type $\mathcal{D}_4$ in the sense of [AF2, Def. 2.2]. Notice that $\alpha_4 = \alpha$, $\alpha_2 = \alpha_4^{2^{k-1}+1}$ and $\alpha_3 = \alpha_1^{2^{k-1}+1}$. Thus, $\sigma_2 = \sigma_e^{2^{k-1}+1} \sigma_o$ because $\sigma_e^{2^{k-1}} = \alpha^{2^{k-1}}$. If we define $\tau_l := (g_l \triangleright \sigma_e)^{-1}\,\sigma_o$, for all l, then $(\sigma_l)_{1\le l\le 4} \cup (\tau_l)_{1\le l\le 4}$ is a subrack of $\mathcal{O}$ of type $\mathcal{D}_4^{(2)}$. Let

$$g := \prod_{t=2}^{k}\prod_{s=1}^{n_{2t}} g_{A_{s,2^t}} \in \mathbb{A}_m,$$

see (10). Then g is an involution in $\mathbb{A}_m$ such that $g \triangleright \sigma = \sigma_e^{-1}\sigma_o$. Let

$$h_l := g_l g, \qquad 1 \le l \le 4;$$

clearly, $h_l \triangleright \sigma = \tau_l$, $1 \le l \le 4$. By straightforward computations, we have the following relations:

$\cdot$	g_4	g_1	g_2	g_3
σ_4	$g_4\,\sigma$	$g_3\,\sigma\alpha^{2r}$	$g_2\,\sigma_2$	$g_1\,\sigma\alpha^{-2r}$
σ_1	$g_2\,\sigma\alpha^{-2r}$	$g_1\,\sigma$	$g_4\,\sigma\alpha^{2r}$	$g_3\,\sigma_2$
σ_2	$g_4\,\sigma_2$	$g_3\,\sigma\alpha^{-2r}$	$g_2\,\sigma$	$g_1\,\sigma\alpha^{2r}$
σ_3	$g_2\,\sigma\alpha^{2r}$	$g_1\,\sigma_2$	$g_4\,\sigma\alpha^{-2r}$	$g_3\,\sigma$
τ_4	$g_4\,\sigma_e^{-1}\sigma_o$	$g_3\,\sigma_e^{-1}\sigma_o\alpha^{2r}$	$g_2\,\sigma_e^{-2^{k-1}-1}\sigma_o$	$g_1\,\sigma_e^{-1}\sigma_o\alpha^{-2r}$
τ_1	$g_2\,\sigma_e^{-1}\sigma_o\alpha^{-2r}$	$g_1\,\sigma_e^{-1}\sigma_o$	$g_4\,\sigma_e^{-1}\sigma_o\alpha^{2r}$	$g_3\,\sigma_e^{-2^{k-1}-1}\sigma_o$
τ_2	$g_4\,\sigma_e^{-2^{k-1}-1}\sigma_o$	$g_3\,\sigma_e^{-1}\sigma_o\alpha^{-2r}$	$g_2\,\sigma_e^{-1}\sigma_o$	$g_1\,\sigma_e^{-1}\sigma_o\alpha^{2r}$
τ_3	$g_2\,\sigma_e^{-1}\sigma_o\alpha^{2r}$	$g_1\,\sigma_e^{-2^{k-1}-1}\sigma_o$	$g_4\,\sigma_e^{-1}\sigma_o\alpha^{-2r}$	$g_3\,\sigma^{-1}$

$\cdot$	h_4	h_1	h_2	h_3
σ_4	$h_4\,\sigma_e^{-1}\sigma_o$	$h_3\,\sigma_e^{-1}\sigma_o\alpha^{-2r}$	$h_2\,\sigma_e^{-2^{k-1}-1}\sigma_o$	$h_1\,\sigma_e^{-1}\sigma_o\alpha^{2r}$
σ_1	$h_2\,\sigma_e^{-1}\sigma_o\alpha^{2r}$	$h_1\,\sigma_e^{-1}\sigma_o$	$h_4\,\sigma_e^{-1}\sigma_o\alpha^{-2r}$	$h_3\,\sigma_e^{-2^{k-1}-1}\sigma_o$
σ_2	$h_4\,\sigma_e^{-2^{k-1}-1}\sigma_o$	$h_3\,\sigma_e^{-1}\sigma_o\alpha^{2r}$	$h_2\,\sigma_e^{-1}\sigma_o$	$h_1\,\sigma_e^{-1}\sigma_o\alpha^{-2r}$
σ_3	$h_2\,\sigma_e^{-1}\sigma_o\alpha^{-2r}$	$h_1\,\sigma_e^{-2^{k-1}-1}\sigma_o$	$h_4\,\sigma_e^{-1}\sigma_o\alpha^{2r}$	$h_3\,\sigma_e^{-1}\sigma_o$
τ_4	$h_4\,\sigma$	$h_3\,\sigma\,\alpha^{-2r}$	$h_2\,\sigma_2$	$h_1\,\sigma\,\alpha^{2r}$
τ_1	$h_2\,\sigma\alpha^{2r}$	$h_1\,\sigma$	$h_4\,\sigma\,\alpha^{-2r}$	$h_3\,\sigma_2$
τ_2	$h_4\,\sigma_2$	$h_3\,\sigma\,\alpha^{2r}$	$h_2\,\sigma$	$h_1\,\sigma\,\alpha^{-2r}$
τ_3	$h_2\,\sigma\,\alpha^{-2r}$	$h_1\,\sigma_2$	$h_4\,\sigma\,\alpha^{2r}$	$h_3\,\sigma$

Notice that $\alpha \in Z(C_{\mathbb{S}_m}(\sigma))$ and $\alpha^2 \in \mathbb{A}_m$; thus, $\alpha^{2r} \in Z(C_{\mathbb{A}_m}(\sigma))$, and $\rho(\alpha^{2r})$ acts by a scalar κ, with $\kappa^4 = 1$ because

$$\mathrm{Id} = \rho(\mathrm{id}) = \rho(\alpha^{2^k}) = \rho((\alpha^{2r})^4) = \kappa^4\,\mathrm{Id}\,.$$

We show that $\kappa = \pm 1$. If we call $\widetilde{\sigma} = \sigma_e\alpha^{-1}$, then $\sigma_e^{2r} = \widetilde{\sigma}^{2r}\alpha^{2r}$ and $\widetilde{\sigma}^{2r} \in Z(C_{\mathbb{A}_m}(\sigma))$; thus, $\rho(\widetilde{\sigma}^{2r})$ acts by a scalar $\widetilde{\kappa}$. Now

$$\mathrm{Id} = \rho(\sigma_e^{2r}) = \rho(\widetilde{\sigma}^{2r})\rho(\alpha^{2r}) = \widetilde{\kappa}\kappa\,\mathrm{Id}\,.$$

That is, $1 = \widetilde{\kappa}\kappa$. Now, $\widetilde{\sigma}$ is product of 2^t-cycles with $t \leq k-1$. Then, $\widetilde{\sigma}^{2r} = \widetilde{\sigma}^{2^{k-2}}$ and $(\widetilde{\sigma}^{2r})^2 = \widetilde{\sigma}^{2^{k-1}} = \mathrm{id}$. Hence, $\widetilde{\kappa}^2 = 1$, and $\kappa = \pm 1$.

Let $v \in V - 0$. We define $W :=$ $\mathbb{C}$-span of $\{u_l, w_l \mid 1 \leq l \leq 4\}$, where

$$\begin{aligned} u_1 &:= g_4 v + g_2 v, & w_1 &:= h_4 w + h_2 w, \\ u_2 &:= g_4 v - g_2 v, & w_2 &:= h_4 w - h_2 w, \\ u_3 &:= g_1 v + g_3 v, & w_3 &:= h_1 w + h_3 w, \\ u_4 &:= g_1 v - g_3 v, & w_4 &:= h_1 w - h_3 w. \end{aligned} \tag{12}$$

By straightforward computations, we can see that W is a braided vector subspace of $M(\mathcal{O}, \rho)$ of Cartan type with matrix of coefficients given by

$$\begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}, \text{ where } \quad Q = \begin{pmatrix} -1 & -1 & -\kappa & \kappa \\ -1 & -1 & -\kappa & \kappa \\ -\kappa & \kappa & -1 & -1 \\ -\kappa & \kappa & -1 & -1 \end{pmatrix},$$

and Dynkin diagram given by Figure 3 which is not of finite type. Therefore, $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, by [H1].

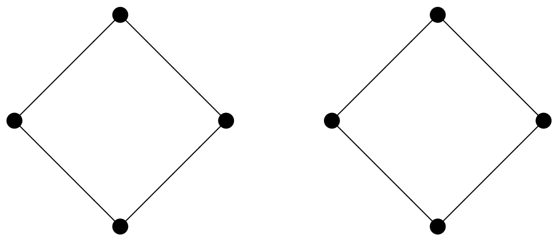

FIGURE 3

(II) Assume that $n_{2^k} = 2$. Let $A_{1,2^k} = (i_1\, i_2\, \cdots\, i_{2^k})$ and $A_{2,2^k} = (i_{2^k+1}\, i_{2^k+2}\, \cdots\, i_{2^{k+1}})$ the two 2^k-cycles appearing in σ, and let $\mathbf{I} = \mathbf{I}_1\mathbf{I}_2$ and $\mathbf{P} = \mathbf{P}_1\mathbf{P}_2$, with

$$\begin{aligned} \mathbf{I}_1 &:= (i_1\, i_3\, i_5\, \cdots\, i_{2^k-1}), & \mathbf{I}_2 &:= (i_{2^k+1}\, i_{2^k+3}\, i_{2^k+5}\, \cdots\, i_{2^{k+1}-1}), \\ \mathbf{P}_1 &:= (i_2\, i_4\, i_6\, \cdots\, i_{2^k}), & \mathbf{P}_2 &:= (i_{2^k+2}\, i_{2^k+4}\, i_{2^k+6}\, \cdots\, i_{2^{k+1}}). \end{aligned}$$

For every $1 \leq l \leq 4$, we define $g_l = \widetilde{g}_l$ in the case $k \geq 4$ or in the case $k = 3$ with $l = 0$ or 2, and we define $g_l = \widetilde{g}_l A_{1,2^k}$ in the case (ii) with $l = 1$ or 3. Then, $g_l \in \mathbb{A}_m$, $1 \leq l \leq 4$. Now, we take σ_l as in (11), τ_l, h_l, $1 \leq l \leq 4$, as in the case (I) above and we proceed in an analogous way. □

Proposition 3.6. *Let $\sigma \in \mathbb{A}_m$ be of type $(1^{n_1}, 2^{n_2}, \sigma_o)$, $\mathcal{O}$ the conjugacy class of σ in $\mathbb{A}_m$ and $\rho = (\rho, V) \in \widehat{C_{\mathbb{A}_m}(\sigma)}$. Then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. Notice that $n_2 = 2k$ is even. Assume first that $\sigma_o = e$. For every l, $1 \le l \le k$, we define

$$C_l = (4l-3 \quad 4l-2)(4l-1 \quad 4l),$$
$$D_l = (4l-3 \quad 4l-1)(4l-2 \quad 4l),$$
$$\alpha_l = (4l-2 \quad 4l-1)(4l-3 \quad 4l-2) = (4l-1 \quad 4l-2 \quad 4l-3).$$

It is easy to see that the group generated by C_l, D_l and α_l is isomorphic to $\mathbb{A}_4$. Moreover, the group generated by

$$C = C_1 \cdots C_k, \; D = D_1 \cdots D_k \text{ and } \alpha = \alpha_1 \cdots \alpha_k$$

is also isomorphic to $\mathbb{A}_4$ and C is an involution, conjugate to σ in $\mathbb{A}_m$. Then, the Nichols algebra $\mathfrak{B}(\mathcal{O}, \rho)$ is infinite dimensional. Now, if $\sigma_o \neq e$, as before, we have that σ belongs to a subgroup isomorphic to $\mathbb{A}_4 \times \langle \sigma_o \rangle$. Then, the result follows from Lemma 2.4. □

In our next Proposition, we apply the technique of the octahedral subrack $\mathfrak{O}$ introduced in [AF2, Sec. 4], and based in results of [AHS].

Proposition 3.7. *Let $\sigma \in \mathbb{A}_m$ be of type $(1^{n_1}, 2^{n_2}, 4^{n_4}, \sigma_o)$, with $n_4 > 0$ and $\sigma_o \neq \mathrm{id}$, $\mathcal{O}$ the conjugacy class of σ and $\rho \in \widehat{C_{\mathbb{A}_m}(\sigma)}$. Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.*

Proof. We can assume $0 < n_4 \le 2$ by Subsection 3.2 (b). We have two possibilities.

(i) Case $n_4 = 1$. We assume $\sigma = A_2(1\,2\,3\,4)\sigma_o$; so $\sigma_e = A_2(1\,2\,3\,4)$. The condition $q_{\sigma\sigma} = -1$, implies that ρ acts by $\lambda = -1$ on σ_e and by $\widetilde{\lambda} = 1$ on σ_o –see Remark 3.1. We define

$$\alpha_1 = (1\,2\,3\,4), \qquad \alpha_2 = (1\,2\,4\,3), \qquad \alpha_3 = (1\,3\,2\,4),$$
$$\alpha_4 = (1\,3\,4\,2), \qquad \alpha_5 = (1\,4\,2\,3), \qquad \alpha_6 = (1\,4\,3\,2),$$

$\sigma_l = A_2\alpha_l\sigma_o$ and $\tau_l = A_2\alpha_l\sigma_o^{-1}$, $1 \le l \le 6$. It is easy to see that the family $(\sigma_l, \tau_l)_{1\le l\le 6}$ is a subrack of $\mathcal{O}$ of type $\mathfrak{O}^{(2)}$. Let $g \in \mathbb{A}_m$ such that $g \triangleright \sigma_o = \sigma_o^{-1}$ and $g \triangleright \sigma_e = \sigma_e$; thus $g \triangleright \sigma = \tau_1$. Also $g^{-1} \triangleright \sigma_o = \sigma_o^{-1}$. We check the conditions (H4)-(H7) of [AF2, Th. 4.11]:

$$\rho(\sigma_6) = \rho(A_2\alpha_6\sigma_o) = \rho(\sigma_e^{-1}\sigma_o) = \lambda^{-1}\widetilde{\lambda} = -1,$$
$$\rho(\tau_1) = \rho(A_2\alpha_1\sigma_o^{-1}) = \rho(\sigma_e\sigma_o^{-1}) = \lambda\widetilde{\lambda}^{-1} = -1,$$
$$\rho(g^{-1}\sigma_1 g) = \rho(A_2\alpha_1\sigma_o^{-1}) = -1,$$
$$\rho(g^{-1}\sigma_6 g) = \rho(A_2\alpha_6\sigma_o) = \rho(\sigma_e^{-1}\sigma_o^{-1}) = q_{\sigma\sigma} = -1.$$

Now the result follows from [AF2, Th. 4.11].

(ii) Case $n_4 = 2$. We take $\sigma = A_2(1\,2\,3\,4)(5\,6\,7\,8)\sigma_o$ and we define

$$\alpha_1 = (1\,2\,3\,4)(5\,6\,7\,8), \quad \alpha_2 = (1\,2\,4\,3)(5\,6\,8\,7), \quad \alpha_3 = (1\,3\,2\,4)(5\,7\,6\,8),$$
$$\alpha_4 = (1\,3\,4\,2)(5\,7\,8\,6), \quad \alpha_5 = (1\,4\,2\,3)(5\,8\,6\,7), \quad \alpha_6 = (1\,4\,3\,2)(5\,8\,7\,6).$$

Now we proceed in an analogous way to the previous case. □

To deal with de conjugacy class of type $(2,4)$ of $\mathbb{A}_6$ we need to recall a very useful theorem.

Theorem 3.8. [HS1, Theorem 8.6] *Let $g, h \in G$ and $V = \bigoplus_{s\in\mathcal{O}_g} V_s$, $W = \bigoplus_{t\in\mathcal{O}_h} W_t$ be irreducible objects in ${}^G_G\mathcal{YD}$. If $\dim \mathfrak{B}(V \oplus W)$ is finite-dimensional, then for all $s \in \mathcal{O}_g$ and $t \in \mathcal{O}_h$, $(st)^2 = (ts)^2$.*

Proposition 3.9. *Let $\sigma \in \mathbb{A}_6$ be of type $(2,4)$, $\mathcal{O}$ the conjugacy class of σ in $\mathbb{A}_6$ and $\rho = (\rho, V) \in \widehat{C_{\mathbb{A}_6}(\sigma)}$. Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.*

Proof. Let H be the subgroup of $\mathbb{A}_6$ generated by $\sigma_1 = (1\,2)(3\,5\,4\,6)$, $\sigma_2 = (1\,2\,5\,3)(4\,6)$ and $\sigma_3 = (2\,3\,4)$. Notice that $H \simeq (\mathbb{Z}/3 \times \mathbb{Z}/3) \rtimes \mathbb{Z}/4$ has order 36. Also, it is easy to see that σ_1 and σ_2 are not conjugate in H. If $\mathcal{O}_i$ is the conjugacy class of σ_i in H, then $\mathcal{O}_i$ has 9 elements. Let V_i be an irreducible object in ${}^H_H\mathcal{YD}$ of the form $M(\mathcal{O}_i, \rho_i)$, for $\rho_i \in \widehat{C_H(\sigma_i)}$. Since $(\sigma_1\sigma_2)^2 \neq (\sigma_2\sigma_1)^2$ then, by Theorem 3.8, $\dim \mathfrak{B}(V_1 \oplus V_2) = \infty$. Now, the result follows from Lemma 2.3. □

Acknowledgements. We have used [GAP] to perform some computations. We thank István Heckenberger and Hans-Jürgen Schneider for interesting discussions.

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