Cocycle Knot Invariants from Quandle Modules and Generalized Quandle Cohomology

J. Scott Carter*
University of South Alabama
Mobile, AL 36688
carter@mathstat.usouthal.edu

Mohamed Elhamdadi
University of South Florida
Tampa, FL 33620
emohamed@math.usf.edu

Matias Graña†
Universidad de Buenos Aires
1428 Buenos Aires, Argentina
matiasg@dm.uba.ar

Masahico Saito‡
University of South Florida
Tampa, FL 33620
saito@math.usf.edu

July 17, 2003

Abstract

Three new knot invariants are defined using cocycles of the generalized quandle homology theory that was proposed by Andruskiewitsch and Graña. We specialize that theory to the case when there is a group action on the coefficients.

First, quandle modules are used to generalize Burau representations and Alexander modules for classical knots. Second, 2-cocycles valued in non-abelian groups are used in a way similar to Hopf algebra invariants of classical knots. These invariants are shown to be of quantum type. Third, cocycles with group actions on coefficient groups are used to define quandle cocycle invariants for both classical knots and knotted surfaces. Concrete computational methods are provided and used to prove non-invertibility for a large family of knotted surfaces. In the classical case, the invariant can detect the chirality of 3-colorable knots in a number of cases.

1 Introduction

Quandle cohomology theory was developed [10] to define invariants of classical knots and knotted surfaces in state-sum form, called quandle cocycle (knot) invariants. The quandle cohomology theory is a modification of rack cohomology theory which was defined in [17], and studied from different perspectives. The cocycle knot invariants are analogous in their definitions to the Dijkgraaf-Witten invariants [13] of triangulated 3-manifolds with finite gauge groups, but they use quandle knot colorings as spins and cocycles as Boltzmann weights.

Two types of topological applications of cocycle knot invariants have been established and are being investigated actively: non-invertibility [10, 38] and the minimal triple point numbers [39] of

*Supported in part by NSF Grant DMS #9988107.
†Supported in part by CONICET and UBACyT X-066.
‡Supported in part by NSF Grant DMS #9988101.

knotted surfaces. A knot is *non-invertible* if it is not equivalent to itself with the orientation reversed while the the orientation of the space preserved. In both applications, quandle cocycle invariants produced results that are not obtained by other known methods; specifically, all known methods (e.g., [15, 19, 29, 37]) of proving non-invertibility of knotted surfaces do not apply directly to non-spherical knots (except Kawauchi's [27, 28] generalization of the Farber-Levine pairing, and it is not clear how this can be applied or how computations can be implemented for given examples), while quandle cocycle invariants can be applied to orientable surfaces of any genus. Thus the cocycle invariants are the only method available in detecting non-invertibility that can be applied regardless of genus. Furthermore concrete methods to compute these have been implemented via computers. The triple point numbers (the minimal number of triple points in projections) have been determined for some knotted surfaces for the first time by using the quandle cocycle invariants.

In this paper, we generalize the quandle cocycle invariants in three different directions, using generalizations of quandle homology theory provided by Andruskiewitsch and Graña [1]. The original and the generalized quandle homology theories can be compared to group cohomology theories, with trivial and non-trivial group actions on coefficient groups, respectively. Thus the generalization of the homology theory and knot invariants are substantial and essential; the original case is only the very special case when the action is trivial. Examples in wreath product of groups are given in Section 3 after preliminaries are provided in Section 2. Algebraic aspects of these examples are also studied. The three directions of generalizations are as follows. First, the actions of quandle modules (defined below) are regarded as generalizations of the Burau representation of braid groups. Thus we define invariants for classical knots, in Section 4, using such quandle modules, in a similar manner as Alexander modules are defined from Burau representations. Second, quandle 2-cocycles with non-abelian coefficients are used to define invariants for classical knots, defined in a similar manner as invariants defined from Hopf algebras [26], by sliding beads on knot diagrams, in Section 5. Generalizations of quandle cocycle invariants are given for classical knots in Section 6 and for knotted surfaces in Section 7, respectively. Computational methods, examples, and applications are provided. The invariant for the classical knots detect chirality of the 3-colorable knots through 9-crossings. As a main application of the invariant for knotted surfaces, we show that majority of $2k$-twist spun of 3-colorable knots in the classical knot table up to 9 crossings, as well as surfaces obtained from them by attaching trivial 1-handles, are non-invertible.

2 Preliminary

Quandles and knot colorings

A *quandle*, X, is a set with a binary operation $(a, b) \mapsto a * b$ such that

(I) For any $a \in X$, $a * a = a$.

(II) For any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$.

(III) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

A *rack* is a set with a binary operation that satisfies (II) and (III).

Racks and quandles have been studied in, for example, [4, 16, 24, 25, 33]. The axioms for a quandle correspond respectively to the Reidemeister moves of type I, II, and III (see [16, 25], for example). Quandle structures have been found in areas other than knot theory, see [1] and [4].

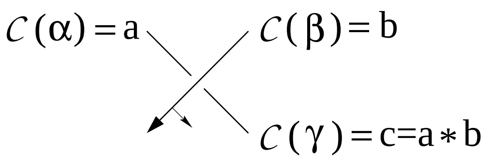

Figure 1: Quandle relation at a crossing

In Axiom (II), the element c that is uniquely determined from given $a, b \in X$ such that $a = c*b$, is denoted by $c = a\bar{*}b$. A function $f : X \to Y$ between quandles or racks is a *homomorphism* if $f(a*b) = f(a)*f(b)$ for any $a, b \in X$.

The following are typical examples of quandles. A group $X = G$ with n-fold conjugation as the quandle operation: $a*b = b^n a b^{-n}$ or $a*b = b^{-n} a b^n$. We denote by $\mathrm{Conj}(G)$ the quandle defined for a group G by $a*b = bab^{-1}$. Any subset of G that is closed under such conjugation is also a quandle.

Any $\Lambda(= \mathbb{Z}[t, t^{-1}])$-module M is a quandle with $a*b = ta + (1-t)b$, $a, b \in M$, that is called an *Alexander quandle*. For a positive integer n, $\mathbb{Z}_n[t, t^{-1}]/(h(t))$ is a quandle for a Laurent polynomial $h(t)$. It is finite if the coefficients of the highest and lowest degree terms of h are units in $\mathbb{Z}_n$.

Let n be a positive integer, and for elements $i, j \in \{0, 1, \ldots, n-1\}$, define $i*j \equiv 2j - i \pmod{n}$. Then $*$ defines a quandle structure called the *dihedral quandle*, R_n. This set can be identified with the set of reflections of a regular n-gon with conjugation as the quandle operation, but also is isomorphic to an Alexander quandle $\mathbb{Z}_n[t, t^{-1}]/(t+1)$. As a set of reflections of the regular n-gon, R_n can be considered as a subquandle of $\mathrm{Conj}(\Sigma_n)$.

Let X be a fixed quandle. Let K be a given oriented classical knot or link diagram, and let $\mathcal{R}$ be the set of (over-)arcs. The normals are given in such a way that (tangent, normal) agrees with the orientation of the plane, see Fig. 1. A (quandle) *coloring* $\mathcal{C}$ is a map $\mathcal{C} : \mathcal{R} \to X$ such that at every crossing, the relation depicted in Fig. 1 holds. More specifically, let β be the over-arc at a crossing, and α, γ be under-arcs such that the normal of the over-arc points from α to γ. (In this case, α is called the *source arc* and γ is called the *target arc*.) Then it is required that $\mathcal{C}(\gamma) = \mathcal{C}(\alpha) * \mathcal{C}(\beta)$. The color $\mathcal{C}(\gamma)$ depends only on the choice of orientation of the over-arc; therefore this rule defines the coloring at both positive and negative crossings. The colors $\mathcal{C}(\alpha)$, $\mathcal{C}(\beta)$ are called *source* colors.

Quandle Modules

We recall some information from [1], but with notation changed to match our conventions.

Let X be a quandle. Let $\Omega(X)$ be the free $\mathbb{Z}$-algebra generated by $\eta_{x,y}$, $\tau_{x,y}$ for $x, y \in X$ such that $\eta_{x,y}$ is invertible for every $x, y \in X$. Define $\mathbb{Z}(X)$ to be the quotient $\mathbb{Z}(X) = \Omega(X)/R$ where R is the ideal generated by

1. $$\eta_{x*y,z}\eta_{x,y} - \eta_{x*z,y*z}\eta_{x,z}$$
2. $$\eta_{x*y,z}\tau_{x,y} - \tau_{x*z,y*z}\eta_{y,z}$$
3. $$\tau_{x*y,z} - \eta_{x*z,y*z}\tau_{x,z} - \tau_{x*z,y*z}\tau_{y,z}$$
4. $$\tau_{x,x} + \eta_{x,x} - 1$$

The algebra $\mathbb{Z}(X)$ thus defined is called the *quandle algebra* over X. In $\mathbb{Z}(X)$, we define elements $\overline{\eta_{z,y}} = \eta^{-1}_{z\bar{*}y,y}$ and $\overline{\tau_{z,y}} = -\overline{\eta_{z,y}}\tau_{z\bar{*}y,y}$. The convenience of such quantities will become apparent by examining type II moves.

A *representation* of $\mathbb{Z}(X)$ is an abelian group G together with (1) a collection of automorphisms $\eta_{x,y} \in \mathrm{Aut}(G)$, and (2) a collection of endomorphisms $\tau_{x,y} \in \mathrm{End}(G)$ such that the relations above hold. More precisely, there is an algebra homomorphism $\mathbb{Z}(X) \to \mathrm{End}(G)$, and we denote the image of the generators by the same symbols. Given a representation of $\mathbb{Z}(X)$ we say that G is a $\mathbb{Z}(X)$-module, or *a quandle module*. The action of $\mathbb{Z}(X)$ on G is written by the left action, and denoted by $(\rho, g) \mapsto \rho g (= \rho \cdot g = \rho(g))$, for $g \in G$ and $\rho \in End(G)$.

Example 2.1 **[1]** Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ denote the ring of Laurent polynomials. Then any Λ-module M is a $\mathbb{Z}(X)$-module for any quandle X, by $\eta_{x,y}(a) = ta$ and $\tau_{x,y}(b) = (1-t)(b)$ for any $x, y \in X$.

The group $G_X = \langle x \in X \mid x * y = yxy^{-1} \rangle$ is called the *enveloping group* [1] (and the *associated group* in [16]). For any quandle X, any G_X-module M is a $\mathbb{Z}(X)$-module by $\eta_{x,y}(a) = ya$ and $\tau_{x,y}(b) = (1 - x * y)(b)$, where $x, y \in X$, $a, b \in M$.

We invite the reader to examine Figs. 2 and 3 to see the geometric motivation for the quandle module axioms. For the time being, ignore the terms $\kappa_{x,y}$ in the figures. Detailed explanations of these figures will be given in Section 4.

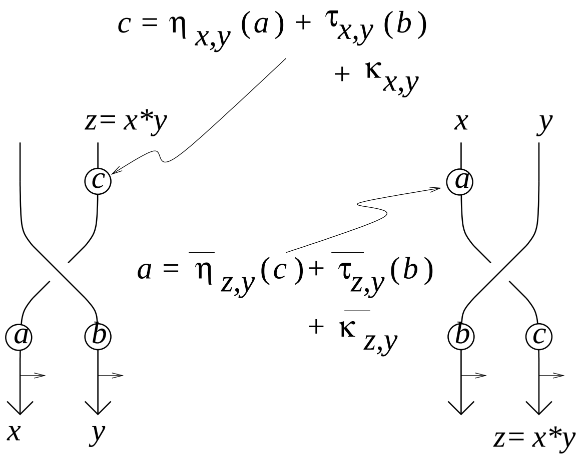

Figure 2: The geometric notation at a crossing

Generalized quandle homology theory

Consider the free right $\mathbb{Z}(X)$-module $C_n(X) = \mathbb{Z}(X)X^n$ with basis X^n (for $n = 0$, X^0 is a singleton $\{x_0\}$, for a fixed element $x_0 \in X$). In [1], boundary operators $\partial = \partial_n : C_{n+1}(X) \to C_n(X)$ are defined by

$$\partial(x_1, \ldots, x_{n+1}) \quad = \quad (-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i \eta_{[x_1,\ldots,\widehat{x_i},\ldots,x_{n+1}],[x_i,\ldots,x_{n+1}]}(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1})$$

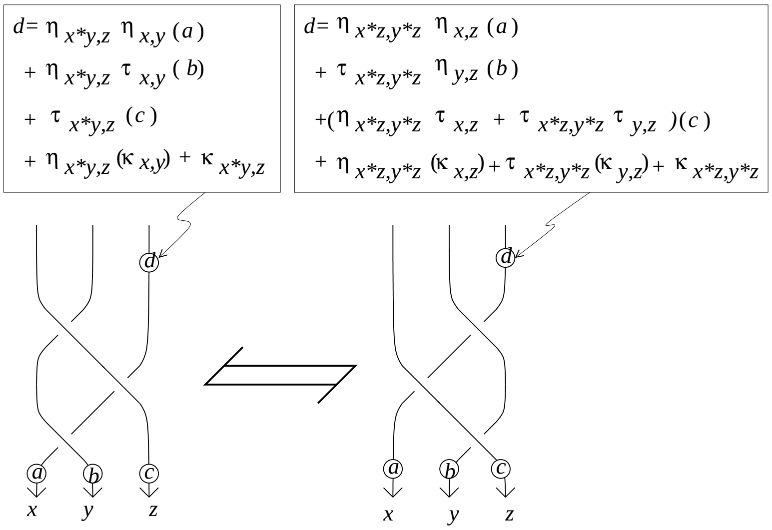

Figure 3: Reidemeister moves and the quandle algebra definition

$$-(-1)^{n+1}\sum_{i=2}^{n+1}(-1)^i(x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_{n+1})$$

$$+(-1)^{n+1}\tau_{[x_1,x_3,\ldots,x_{n+1}],[x_2,x_3,\ldots,x_{n+1}]}(x_2, \ldots, x_{n+1}),$$

where $\qquad [x_1, x_2, \ldots, x_n] = ((\cdots(x_1 * x_2) * x_3) * \cdots) * x_n$

for $n > 0$, and $\partial_1(x) = -\tau_{x \bar{*} x_0, x_0}$ for $n = 0$. The notational conventions are slightly different from [1].In particular, the 2-cocycle condition for a 2-cochain $\kappa_{x,y}$ in this homology theory is written as

$$\eta_{x*y,z}(\kappa_{x,y}) + \kappa_{x*y,z} = \eta_{x*z,y*z}(\kappa_{x,z}) + \tau_{x*z,y*z}(\kappa_{y,z}) + \kappa_{x*z,y*z},$$

for any $x, y, z \in X$. We call this a *generalized (rack) 2-cocycle condition.* When κ further satisfies $\kappa_{x,x} = 0$ for any $x \in X$, we call it a *generalized quandle 2-cocycle.*

Dynamical cocycles

Let X be a quandle and S be a non-empty set. Let $\alpha : X \times X \to \mathrm{Fun}(S \times S, S) = S^{S\times S}$ be a function, so that for $x, y \in X$ and $a, b \in S$ we have $\alpha_{x,y}(a, b) \in S$.

Then it is checked by computations that $S \times X$ is a quandle by the operation $(a, x) * (b, y) = (\alpha_{x,y}(a, b), x*y)$, where $x*y$ denotes the quandle product in X, if and only if α satisfies the following conditions:

1. $\alpha_{x,x}(a, a) = a$ for all $x \in X$ and $a \in S$;
2. $\alpha_{x,y}(-, b) : S \to S$ is a bijection for all $x, y \in X$ and for all $b \in S$;
3. $\alpha_{x*y,z}(\alpha_{x,y}(a, b), c) = \alpha_{x*z,y*z}(\alpha_{x,z}(a, c), \alpha_{y,z}(b, c))$ for all $x, y, z \in X$ and $a, b, c \in S$.

Such a function α is called a *dynamical quandle cocycle* [1]. The quandle constructed above is denoted by $S \times_\alpha X$, and is called the *extension* of X by a dynamical cocycle α. The construction is general, as Andruskiewitsch and Graña show:

Lemma 2.2 [1] *Let $p : Y \to X$ be a surjective quandle homomorphism between finite quandles such that the cardinality of $p^{-1}(x)$ is a constant for all $x \in X$. Then Y is isomorphic to an extension $S \times_\alpha X$ of X by some dynamical cocycle on the set S such that $|S| = |p^{-1}(x)|$.*

Example 2.3 [1] Let G be a $\mathbb{Z}(X)$-module for the quandle X, and κ be a generalized 2-cocycle. For $a, b \in G$, let

$$\alpha_{x,y}(a, b) = \eta_{x,y}(a) + \tau_{x,y}(b) + \kappa_{x,y}.$$

Then it can be verified directly that α is a dynamical cocycle. In particular, even with $\kappa = 0$, a $\mathbb{Z}(X)$-module structure on the abelian group G defines a quandle structure $G \times_\alpha X$.

3 Group extensions and quandle modules

The purpose of this section is to provide examples of quandle modules and cocycles from group extensions and group cocycles. Let $0 \to N \xrightarrow{i} E \xrightarrow{\pi} H \to 1$ be a short exact sequence of groups that expresses the group E as a twisted semi-direct product $E = N \rtimes_\theta H$ by a group 2-cocycle θ, where N is an abelian group (see page 91 of [5]). Thus we have a set-theoretic section $s : H \to E$ that is normalized, in the sense that $s(1_H) = 1_E$, and the elements of E can be written as pairs (a, x) where $a \in N$ and $x \in H$, by the bijection $(a, x) \mapsto i(a)s(x)$. We have

$$s(x)s(y) = i(\theta(x, y))s(xy), \quad x, y \in H,$$

and the multiplication rule in E is given by $(a, x)\cdot(b, y) = (a+x(b)+\theta(x, y), xy)$, where $x(b)$ denotes the action of H on N that gives E the structure of a twisted semi-direct product, $E = N \rtimes_\theta H$. Recall here that the group 2-cocycle condition is

$$\theta(x, y) + \theta(xy, z) = x\theta(y, z) + \theta(x, yz), \quad x, y, z \in H.$$

Let $0 \to N \xrightarrow{i} E \xrightarrow{\pi} H \to 1$ be an exact sequence, and $E = N \rtimes_\theta H$ be the corresponding twisted semi-direct product. Consider E and G as quandles, $\mathrm{Conj}(E)$ and $\mathrm{Conj}(H)$, respectively, where $a * b = bab^{-1}$. Lemma 2.2 implies that E is an extension of H by a dynamical cocycle $\alpha : H \times H \to N^{N \times N}$.

Proposition 3.1 *The dynamical cocycle α, in this case, is written by $\alpha_{x,y}(a, b) = \eta_{x,y}(a)+\tau_{x,y}(b)+\kappa_{x,y}$ for any $x, y \in H$ and $a, b \in N$, where $\eta_{x,y}(a) = ya$, $\tau_{x,y}(b) = (1 - x * y)(b)$, and*

$$\kappa_{x,y} = \theta(y, x) - yx\theta(y^{-1}, y) + \theta(yx, y^{-1}).$$

Similarly, if the quandle structure is defined by $r * s = s^{-1}rs$, then we obtain $\eta_{x,y}(a) = y^{-1}a$, $\tau_{x,y}(b) = (y^{-1}x - y^{-1})(b)$ and

$$\kappa_{x,y} = -\theta(y^{-1}, y) + y^{-1}\theta(x, y) + \theta(y^{-1}, xy).$$

Proof. For $(b, y) \in E$, one has

$$(b, y)^{-1} = (-y^{-1}(b) - \theta(y^{-1}, y), y^{-1}) = (-y^{-1}(b) - y^{-1}\theta(y, y^{-1}), y^{-1})$$

(see page 92 of [5]), and compute

$$\begin{aligned}(a,x)*(b,y) &= (b,y)(a,x)(b,y)^{-1}\\ &= (b+y(a)-(yxy^{-1})(b)+\theta(y,x)-yx\theta(y^{-1},y)+\theta(yx,y^{-1}),yxy^{-1})\end{aligned}$$

so that

$$\begin{aligned}\alpha_{x,y}(a,b) &= \eta_{x,y}(a)+\tau_{x,y}(b)+\kappa_{x,y}\\ &= y(a)+(1-x*y)(b)+[\theta(y,x)-yx\theta(y^{-1},y)+\theta(yx,y^{-1})],\end{aligned}$$

and we obtain the formulas. Note that by expanding the terms $(a,x)(b,y)^{-1}$ first, we obtain an equivalent formula

$$\kappa_{x,y}=-yx\theta(y^{-1},y)+y\theta(x,y^{-1})+\theta(y,xy^{-1}),$$

which follows from the group 2-cocycle condition from the first formula, as well. The second case is similar. ■

Thus the second item in Example 2.1 occurs in semi-direct product of groups, when $\theta=0$ and hence $\kappa=0$. Note also that the second case of Lemma 3.1 agrees with the quandle action considered by Ohtsuki [35].

Example 3.2 The wreath product of groups gives specific examples as follows. Let $N=(\mathbb{Z}_q)^n$ for some $q\in\{0,1,\ldots\}$. (In case $q=0$, then N is the direct product of the integers, and when $q=1$, then N is trivial.) The symmetric group $H=\Sigma_n$ acts on N by permutation of the factors $\sigma(x_1,\ldots,x_n)=\sigma(\vec{x})=(x_{\sigma^{-1}(1)},\ldots,x_{\sigma^{-1}(n)})$, for $\sigma\in\Sigma_n$ and $\vec{x}=(x_j)_{j=1}^n\in N$. In this situation, E is called a wreath product and denoted by $E=(\mathbb{Z}_q)\wr\Sigma_n$. In this case, $\kappa=0$, and the quandle module structure can be computed explicitly by matrices over $\mathbb{Z}_q$. In [8], such computations were used to obtain non-trivial colorings of some twist-spun knots by dynamical extensions of R_n.

Next we consider the 2-cocycle κ in terms of sections. In the group case, recall that the equality $s(x)s(y)=i(\theta(x,y))s(xy)$ expresses the group 2-cocycle θ as an obstruction to the section being a homomorphism. There is a similar interpretation for quandle 2-cocycles.

Lemma 3.3 *The 2-cocycle κ in Proposition 3.1 satisfies $s(x)*s(y)=i(\kappa_{x,y})s(x*y)$.*

Proof. One computes

$$s(x)*s(y)=(0,x)*(0,y)=(\alpha_{x,y}(0,0),x*y)=(\kappa_{x,y},x*y)=i(\kappa_{x,y})s(x*y)$$

as desired. ■

Lemma 3.4 *Let κ be as above. Then we have $\kappa_{x,y}=\theta(y,x)-\theta(yxy^{-1},y)$.*

Proof. From Lemma 3.3 we have

$$\begin{aligned}s(y)s(x)s(y)^{-1} &= i\kappa_{x,y}s(yxy^{-1}),\\ i\theta(y,x)s(yx) &= i\kappa_{x,y}s(yxy^{-1})s(y)\\ &= i\kappa_{x,y}i\theta(yxy^{-1},y)s(yxy^{-1}y),\end{aligned}$$

and we obtain the formula, which is simpler than that of Proposition 3.1. ■

Lemma 3.5 *Let κ and θ be as above. If θ is a coboundary, then so is κ.*

Proof. For a certain group 1-cochain γ,

$$\theta(x,y) = \delta_G\gamma(x,y) = \gamma(xy) - \gamma(x) - x\gamma(y),$$

where δ_G (resp. δ_Q) denotes the group (resp. quandle) coboundary homomorphism. By Lemma 3.4,

$$\kappa_{x,y} = \gamma(yxy^{-1}) - y\gamma(x) - (1 - yxy^{-1})\gamma(y) = \delta_Q\gamma(x,y)$$

as desired. ■

Thus functors from group homology theories to quandle homology theories are expected.

Let $E = N \rtimes_\theta H$ be as above, a twisted semi-direct product. Let X be a subquandle of $\mathrm{Conj}(H)$, and $\tilde{X} = \pi^{-1}(X)$, where $\pi : E \to H$ is the projection. Then $\tilde{X}$ is a subquandle of $\mathrm{Conj}(E)$, and π induces the quandle homomorphism $\pi : \tilde{X} \to X$, which is a dynamical extension.

Example 3.6 Let $X = R_n$ be the dihedral quandle of order n (a positive integer), which is a subquandle of $\mathrm{Conj}(\Sigma_n)$. Let $E = (\mathbb{Z}_q) \wr \Sigma_n$ be the wreath product as in Example 3.2,

$$0 \to N = (Z_q)^n \to E = (\mathbb{Z}_q) \wr \Sigma_n \xrightarrow{\pi} \Sigma_n \to 1.$$

Then $\tilde{X} = \pi^{-1}(X)$ is a subquandle of $\mathrm{Conj}(E)$, and $\tilde{X} = (\mathbb{Z}_q)^n \times_\alpha X$ is a dynamical extension of X.

As for the first cocycle groups, we have the following interpretation. Let X be a quandle, A a $\mathbb{Z}(X)$-module. Consider the dynamical extension $A \times_\alpha X$ of X by A with the dynamical cocycle $\alpha_{x,y} = \eta_{x,y} + \tau_{x,y}$ as before. For a given 1-cochain $f \in C^1(X;A) = \mathrm{Hom}_{\mathbb{Z}(X)}(\mathbb{Z}(X)X, A)$, define a section $\hat{f} : X \to A \times_\alpha X$ by $\hat{f}(x) = (f(x), x)$, which is indeed a section: $\pi \circ \hat{f} = \mathrm{id}_X$ for the projection $\pi : A \times_\alpha X \to X$.

Lemma 3.7 *The section $\hat{f}$ is a quandle homomorphism if and only if $f \in Z^1(X;A)$.*

Proof. The 1-cocycle condition is written as $f(x * y) = \eta_{x,y}f(x) + \tau_{x,y}f(y)$ for any $x, y \in X$. Thus we compute

$$\begin{aligned} \hat{f}(x*y) &= (f(x*y), x*y) \\ \hat{f}(x) * \hat{f}(y) &= (\alpha_{x,y}(f(x), f(y)), x*y) = (\eta_{x,y}f(x) + \tau_{x,y}f(y), x*y) \quad \blacksquare. \end{aligned}$$

Alternatively, it can be stated that there is a one-to-one correspondence between $Z^1(X;A)$ and the set of sections that are quandle homomorphisms.

4 Knot invariants from quandle modules

Quandle modules and braids

A braid word w (of k-strings), or a k-braid word, is a product of standard generators $\sigma_1, \ldots, \sigma_{k-1}$ of the braid group $\mathcal{B}_k$ of k-strings and their inverses. A braid word w represents an element $[w]$ of the braid group $\mathcal{B}_k$. Geometrically, w is represented by a diagram in a rectangular box with k end points at the top, and k end points at the bottom, where the strings go down monotonically. Each generator or its inverse is represented by a crossing in a diagram. We use the same letter w for a choice of such a diagram. Let $\hat{w}$ denote the closure of the diagram w. Quandle colorings of w are defined in exactly the same manner as in the case of knots. However, the quandle elements at the top and the bottom of a diagram of w do not necessarily coincide. For the closure $\hat{w}$, the quandle elements at the top and the bottom of a diagram of w coincide, when we consider a coloring of a link $\hat{w}$.

Let X be a quandle. Let $\gamma_1, \ldots, \gamma_k$ be the bottom arcs of w. For a given vector $\vec{x} = (x_1, \ldots, x_k) \in X^k$, assign these elements $x_1, \ldots, x_k$ on $\gamma_1, \ldots, \gamma_k$ as their colors, respectively. Then from the definition, a coloring $\mathcal{C}$ of w by X is uniquely determined such that $\mathcal{C}(\gamma_i) = x_i$, $i = 1, \ldots, k$. We call such a coloring $\mathcal{C}$ the *coloring induced from* $\vec{x}$. Let $\delta_1, \ldots, \delta_k$ be the arcs at the top. Let $\vec{y} = (y_1, \ldots, y_k) = (\mathcal{C}(\delta_1), \ldots, \mathcal{C}(\delta_k)) \in X^k$ be the colors assigned to the top arcs, that are uniquely determined from $\vec{x}$. Denote this situation by a left action, $\vec{y} = w \cdot \vec{x}$. The colors $\vec{x}$ and $\vec{y}$ are called *bottom* and *top colors* or *color vectors*, respectively. See Fig. 4.

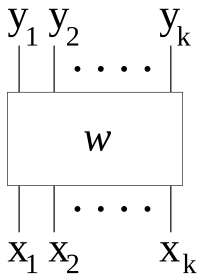

Figure 4: A quandle coloring of a braid word w

Let X be a quandle and G be a quandle module. For a dynamical cocycle $\alpha = \eta + \tau + \kappa$ — which acts on $(a, b) \in G^2$ by $\alpha_{x,y}(a, b) = \eta_{x,y}(a) + \tau_{x,y}(b) + \kappa_{x,y}$ for any $(x, y) \in X^2$ — let $\tilde{X} = G \times_\alpha X$ be the dynamical extension. If $\vec{r} = ((a_1, x_1), \ldots, (a_k, x_k))$ and $\vec{s} = ((b_1, y_1), \ldots, (b_k, y_k)) \in \tilde{X}^k$ are bottom and top colors of $w \in \mathcal{B}_k$ by $\tilde{X}$, respectively, then we write this situation by $\vec{b} = M(w, \vec{x}) \cdot \vec{a}$, where $\vec{a} = (a_1, \ldots, a_k)$, $\vec{b} = (b_1, \ldots, b_k) \in G^k$. Thus $M(w, \vec{x})$ represents a map $M(w, \vec{x}) : G^k \to G^k$.

Lemma 4.1 *If $[w] = [w'] \in \mathcal{B}_k$, then $M(w, \vec{x}) = M(w', \vec{x}) : G^k \to G^k$.*

Proof. The invariance under the braid relations are checked from the definitions. In particular,

$$\begin{aligned}
&M(\sigma_1\sigma_2\sigma_1, (x, y, z))(a, b, c) \\
&\quad = \ (c,\ \eta_{y,z}(b) + \tau_{y,z}(c) + \kappa_{y,z},\ \eta_{x*y,z}\eta_{x,y}(a) + \eta_{x*y,z}\tau_{x,y}(b) + \tau_{x*y,z}(c) + \eta_{x*y,z}(\kappa_{x,y}) + \kappa_{x*y,z}), \\
&M(\sigma_2\sigma_1\sigma_2, (x, y, z))(a, b, c) \\
&\quad = \ (c,\ \eta_{y,z}(b) + \tau_{y,z}(c) + \kappa_{y,z}, \\
&\qquad\quad \eta_{x*z,y*z}\eta_{x,z}(a) + \tau_{x*z,y*z}\eta_{y,z}(b) + (\eta_{x*z,y*z}\tau_{x,z} + \tau_{x*z,y*z}\tau_{y,z})(c) \\
&\qquad\quad + \eta_{x*z,y*z}(\kappa_{x,z}) + \tau_{x*z,y*z}(\kappa_{y,z}) + \kappa_{x*z,y*z})
\end{aligned}$$

and the equality follows from the quandle module conditions and the generalized 2-cocycle condition. ■

This is not a braid group representation on G^k, as it depends on the color of w by X. However, in the case in which the coloring by X is trivial so $x_1 = x_2 = \cdots = x_k$, and $\kappa = 0$, then it is a braid group representation. We call the map $M(-, \vec{x}) : \mathcal{B}_k \to \mathrm{Map}(G^k, G^k)$ a *colored representation.*

For a standard braid generator, this situation is diagrammatically represented as depicted in Fig. 2, the left figure for a negative crossing, and the right one for positive. In the calculations given in this section, the left figure represents the braid generator σ_j, and the right represents the inverse. In the figure, the colors by X are assigned to arcs. Elements of G are put in small circles on arcs. We imagine these circles sliding up through a crossing, at which the dynamical cocycle α acts and changes the elements when a circled elements goes under a crossing. Going over a crossing does not change the element in a circle. From type II Reidemeister moves, the definition of $\overline{\eta}$ and $\overline{\tau}$ is recovered. Figure 3 shows that the quandle module conditions correspond to the type III move with this diagrammatic convention.

Module invariants

Let w be a k-braid word, and denote by $\hat{w}$ the closure of w. Let X be a quandle and G be a quandle module. Let $\alpha = \eta + \tau$ be a dynamical cocycle, which acts on $(a, b) \in G^2$ by $\alpha_{x,y}(a, b) = \eta_{x,y}(a) + \tau_{x,y}(b)$ for any $(x, y) \in X^2$.

Theorem 4.2 *Let L be a link represented as a closed braid $\hat{w}$, where w is a k-braid word, and $\mathrm{Col}_X(L)$ be the set of colorings of L by a quandle X. For $\mathcal{C} \in \mathrm{Col}_X(L)$, let $\vec{x}$ be the color vector of bottom strings of w that is the restriction of $\mathcal{C}$. Then the family*

$$\tilde{\Phi}(X, \alpha\ ; L) = \{G^k / \mathrm{Im}(M(w, \vec{x}) - I)\}_{\mathcal{C} \in \mathrm{Col}_X(L)}$$

of isomorphism classes of modules presented by the maps $(M(w, \vec{x}) - I)$, where I denotes the identity, is independent of choice of w that represents L as its closed braid, and thus defines a link invariant.

Proof. By the Lemma 4.1, $M(w) = M(w, \vec{x})$ does not depend on the choice of a braid word. We use Markov's theorem to prove the statement. First note that the set of colorings remains unchanged by a stabilization, in the sense that there is a natural bijection (the colorings on bottom strings $(x_1, \ldots, x_k)$ of a braid word w extend uniquely to the coloring $(x_1, \ldots, x_k, x_k)$ of $w\sigma_k^{\pm 1}$, a stabilization of w). There is a bijection of colorings between conjugates, as well. Hence it is sufficient to prove that, for a given coloring, the isomorphism class of the module defined in the statement remains unchanged by conjugation and stabilization for a natural induced coloring. The invariance under conjugation is seen from the fact that conjugation by a braid word induces a conjugation by a matrix, and the module is isomorphic under conjugate presentation matrices. So we investigate the stabilization.

We represent maps of G^k by k by k matrices whose entries represent maps of G. The braid generator σ_k in the stabilization $w\sigma_k^{\pm 1}$ is represented by the matrix $M(\sigma_k) = I_{k-1} \oplus \begin{bmatrix} O & I \\ W & I - W \end{bmatrix}$, where I denotes the identity map of G, and I_k denotes the identity map on G^k. This is because the kth and $(k+1)$st strings receive the same color. The block matrix $\begin{bmatrix} O & I \\ W & I - W \end{bmatrix}$, in particular, represents the map

$$(a, b) \mapsto (b, \alpha_{x,x}(a, b)) = (b, \eta_{x,x}(a) + \tau_{x,x}(b))$$

where $x = x_k$, so that W corresponds to the action by η, and hence W is an isomorphism of G. The fact that $\tau_{x,x}$ corresponds to the matrix $I - W$ follows from the condition $\eta_{x,x} + \tau_{x,x} = 1$ in a quandle algebra.

Express the matrix $M(w)$, where w is regarded as a $(k+1)$-braid word after stabilization, though originally a k-braid word, by a matrix $\begin{bmatrix} M_{11} & \cdots & M_{1k} & O \\ \vdots & & \vdots & \vdots \\ M_{k1} & \cdots & M_{kk} & O \\ O & \cdots & O & I \end{bmatrix}$ where M_{ij}, $1 \leq i, j \leq k$, are maps of G, and $M(w)$ was written as a $(k+1) \times (k+1)$ matrix of these maps, I denotes the identity map on G. Then $M(w\sigma_k)$ is represented by the matrix

$$M(w)M(\sigma_k) = \begin{bmatrix} M_{11} & \cdots & M_{1\ (k-1)} & O & M_{1k} \\ \vdots & & \vdots & \vdots & \vdots \\ M_{k1} & \cdots & M_{k\ (k-1)} & O & M_{kk} \\ O & \cdots & O & W & I - W \end{bmatrix}.$$

Hence

$$M(w)M(\sigma_k) - I_k = \begin{bmatrix} M_{11} - I & \cdots & M_{1\ (k-1)} & O & M_{1k} \\ \vdots & & \vdots & \vdots & \vdots \\ M_{(k-1)\ 1} & \cdots & M_{(k-1)\ (k-1)} - I & O & M_{(k-1)\ k} \\ M_{k1} & \cdots & M_{k\ (k-1)} & -I & M_{kk} \\ O & \cdots & O & W & -W \end{bmatrix},$$

which is column reduced to $\begin{bmatrix} M_{11} - I & \cdots & M_{1\ (k-1)} & O & M_{1k} \\ \vdots & & \vdots & \vdots & \vdots \\ M_{(k-1)\ 1} & \cdots & M_{(k-1)\ (k-1)} - I & O & M_{(k-1)\ k} \\ M_{k1} & \cdots & M_{k\ (k-1)} & -I & M_{kk} - I \\ O & \cdots & O & W & O \end{bmatrix}$ and is further row reduced to $\begin{bmatrix} M_{11} - I & \cdots & M_{1\ (k-1)} & O & M_{1k} \\ \vdots & & \vdots & \vdots & \vdots \\ M_{(k-1)\ 1} & \cdots & M_{(k-1)\ (k-1)} - I & O & M_{(k-1)\ k} \\ M_{k1} & \cdots & M_{k\ (k-1)} & O & M_{kk} - I \\ O & \cdots & O & W & O \end{bmatrix}$ that represents the isomorphic module as $M(w)$ does, since W is an isomorphism. The invariance under stabilization by σ_k^{-1} follows similarly. ■

This theorem implies that the following is well-defined.

Definition 4.3 The family of modules $\tilde{\Phi}(X, \alpha\ ; L) = \{G^k / \mathrm{Im}(M(w, \vec{x}) - I)\}_{\mathcal{C} \in \mathrm{Col}_X(L)}$ is called the *quandle module invariant.*

Specific examples can be constructed from wreath products. Let X be a subquandle of $\mathrm{Conj}(\Sigma_n)$, and E be a wreath product extension by $N = (\mathbb{Z}_q)^n$ where q, n are positive integers, so that

$$0 \to N = (\mathbb{Z}_q)^n \xrightarrow{i} E \xrightarrow{\pi} \Sigma_n \to 1,$$

and $E = (\mathbb{Z}_q)^n \rtimes \Sigma_n$. Let $\tilde{X} = \pi^{-1}(X)$, as before, and let α be the corresponding dynamical cocycle, so that $\tilde{X} = (\mathbb{Z}_q)^n \times_\alpha X$.

Example 4.4 Let $X = R_n$ denote the n-element dihedral quandle, regarded as a subset of Σ_n. The action on $\mathbb{Z}^n$ is by permutations. Then the quandle module invariant is defined as above, with $q = 0$. We ran programs in *Maple*, *Mathematica*, and/or *C*, independently, to compute the quandle module invariants, for $n = 3, 5, 7, 11$, and 13, through the nine-crossing knots. There are 84 such prime knots in the table. We used Jones's [23] table in which knots are given in braid form. For $n = 3$, 5, 7, 11, and 13, there are 33, 17, 10, and 7 knots that are non-trivially colored by R_n, respectively. We summarize our data in Table 1 and 2 for R_3 and R_5, respectively, below. For each knot we list the torsion subgroups, the rank of the free part, and whether the coloring is trivial or not. Thus the entry in Table A-1 for knot 8_{15} indicates that there are three trivial colorings of 8_{15} which give the invariant $\mathbb{Z}_{33} \oplus \mathbb{Z}^3$, and six colorings give the invariant $\mathbb{Z}_2 \oplus \mathbb{Z}^4$. For colorings by R_7, R_{11} and R_{13}, the results are summarized as follows, where D denotes the determinant of a given knot.

- All 7-colorable knots up to 9-crossings, except 9_{41}, have the module invariant $(\mathbb{Z}_D)^3 \oplus \mathbb{Z}^7$, for 7 trivial colorings, and $\mathbb{Z}^{10}$ for 42 non-trivial colorings. For 9_{41}, it is $(\mathbb{Z}_D)^6 \oplus \mathbb{Z}^7$ for 7 trivial colorings, and $\mathbb{Z}^{10}$ for 336 non-trivial colorings.
- All 11-colorable knots up to 9-crossings have the module invariant $(\mathbb{Z}_D)^5 \oplus \mathbb{Z}^{11}$, for 11 trivial colorings, and $\mathbb{Z}^{16}$ for 110 non-trivial colorings.
- All 13-colorable knots up to 9-crossings have the module invariant $(\mathbb{Z}_D)^6 \oplus \mathbb{Z}^{13}$, for 13 trivial colorings, and $\mathbb{Z}^{19}$ for 156 non-trivial colorings.

We expect that the monotony of values for 7, 11, and 13 is due to the fact that the knots considered are all of relatively small crossing numbers and/or bridge numbers.

Remark 4.5 The construction of the quandle module invariant is similar to the construction of Alexander modules from Burau representation. Also, the cokernel appears in the definition of the Bowen-Franks [3] groups for symbolic dynamical systems. Thus relations to covering spaces, as well as dynamical systems related to braid groups, are expected.

In [30, 40] group representations of knot groups are used to define twisted Alexander polynomials. In that situation, the representation can be viewed as a matrix (which depends on the image of the meridian) assigned to a crossing. The quandle module invariant is related when the quandle colorings are given by knot group representations. The general relation can be understood via Fox calculus.

The quandle module invariants might give lower bounds to the number of strands needed in a braid representation of a knot. Future studies will include more detailed investigations of this invariant, the non-abelian cocycle invariant (Section 5), and the generalized cocycle invariant of classical knots (Section 6). Our main purpose in the current paper is to indicate the strength of the generalized cocycle invariants for knotted surfaces (Section 7).

Knot	Tor	Rank	Col type	Knot	Tor	Rank	Col type
3_1	3	3	3 Trivial	9_{11}	33	3	3 Trivial
	0	4	6 Non-trivial		0	4	6 Non-trivial
6_1	9	3	3 Trivial	9_{15}	39	3	3 Trivial
	0	4	6 Non-trivial		0	4	6 Non-trivial
7_4	15	3	3 Trivial	9_{16}	39	3	3 Trivial
	0	4	6 Non-trivial		2	4	6 Non-trivial
7_7	21	3	3 Trivial	9_{17}	39	3	3 Trivial
	0	4	6 Non-trivial		0	4	6 Non-trivial
8_5	21	3	3 Trivial	9_{23}	45	3	3 Trivial
	2	4	6 Non-Trivial		0	4	6 Non-trivial
8_{10}	27	3	3 Trivial	9_{24}	45	3	3 Trivial
	2	4	6 Non-trivial		2	4	6 Non-trivial
8_{11}	27	3	3 Trivial	9_{28}	51	3	3 Trivial
	0	4	6 Non-trivial		2	4	6 Non-trivial
8_{15}	33	3	3 Trivial	9_{29}	51	3	3 Trivial
	2	4	6 Non-trivial		0	4	6 Non-trivial
8_{18}	3, 15	3	3 Trivial	9_{34}	69	3	3 Trivial
	3	4	24 Non-trivial		0	4	6 Non-trivial
8_{19}	3	3	3 Trivial	9_{35}	3, 9	3	3 Trivial
	2	4	6 Non-Trivial		0	4	18 Non-trivial
8_{20}	9	3	3 Trivial		2	4	6 Non-trivial
	2	4	6 Non-trivial	9_{37}	3, 15	3	3 Trivial
8_{21}	15	3	3 Trivial		2	4	6 Non-trivial
	2	4	6 Non-trivial		0	4	18 Non-trivial
9_1	9	3	3 Trivial	9_{38}	57	3	3 Trivial
	0	4	6 Non-trivial		0	4	6 Non-trivial
9_2	15	3	3 Trivial	9_{40}	5, 15	3	3 Trivial
	0	4	6 Non-trivial		4	4	6 Non-trivial
9_4	21	3	3 Trivial	9_{46}	3, 3	3	3 Trivial
	0	4	6 Non-trivial		0	4	18 Non-trivial
9_6	27	3	3 Trivial		2	4	6 Non-trivial
	0	4	6 Non-trivial	9_{47}	3, 9	3	3 Trivial
9_{10}	33	3	3 Trivial		0	4	24 Non-trivial
	0	4	6 Non-trivial	9_{48}	3, 9	3	3 Trivial
					0	4	18 Non-trivial
					2	4	6 Non-trivial

Table 1: A table of module invariants for 3-colorable knots

Knot	Tor	Rank	Col type	Knot	Tor	Rank	Col type
4_1	5, 5	5	5 Trivial	9_2	15, 15	5	5 Trivial
	0	7	20 Non-trivial		0	7	20 Non-trivial
5_1	5, 5	5	5 Trivial	9_{12}	35, 35	5	5 Trivial
	0	7	20 Non-trivial		0	7	20 Non-trivial
7_4	15, 15	5	5 Trivial	9_{23}	45, 45	5	5 Trivial
	0	7	20 Non-trivial		0	7	20 Non-trivial
8_7	25, 25	5	5 Non-trivial	9_{24}	45, 45	5	5 Trivial
	0	7	20 Trivial		0	7	20 Non-trivial
8_8	25, 25	5	5 Trivial	9_{31}	55, 55	5	5 Trivial
	0	7	20 Non-trivial		0	7	20 Non-trivial
8_{16}	35, 35	5	5 Trivial	9_{37}	3, 3, 15, 15	5	5 Trivial
	0	7	20 Non-trivial		0	7	20 Non-Trivial
8_{18}	3, 3, 15, 15	5	5 Trivial	9_{39}	55, 55	5	5 Trivial
	2, 2	7	20 Non-trivial		0	7	20 Non-trivial
8_{21}	15, 15	5	5 Trivial	9_{40}	5, 5, 15, 15	5	5 Trivial
	0	7	20 Non-trivial		5	7	120 Non-Trivial
				9_{48}	5, 5, 5, 5	5	5 Trivial
					0	7	120 Non-trivial

Table 2: A table of module invariants for 5-colorable knots

5 Knot invariants from non-abelian 2-cocycles

Non-abelian 2-cocycles

Let X be a quandle and H a (not necessarily abelian) group. A function $\beta : X \times X \to H$ is a *rack 2-cocycle* [1] if

$$\beta(x_1, x_2)\beta(x_1 * x_2, x_3) = \beta(x_1, x_3)\beta(x_1 * x_3, x_2 * x_3)$$

is satisfied for any x_1, x_2, $x_3 \in X$. If a rack 2-cocycle further satisfies $\beta(x, x) = 1$ for any $x \in X$, then it is called a *quandle 2-cocycle* [1]. The set of quandle 2-cocycles is denoted by $Z^2_{\mathrm{CQ}}(X; H)$. Two cocycles β, β' are *cohomologous* if there is a function $\gamma : X \to H$ such that

$$\beta'(x_1, x_2) = \gamma(x_1)^{-1}\beta(x_1, x_2)\gamma(x_1 * x_2)$$

for any $x_1, x_2 \in X$. An equivalence class is called a *cohomology class*. The set of cohomology classes is denoted by $H^2_{\mathrm{CQ}}(X; H)$. These definitions agree with those in [10] if H is an abelian group. When H is not necessarily abelian, the 2-cocycles β are called (constant) *2-cocycles* in [1]. We call such a 2-cocycle *non-abelian* when H is not an abelian group.

Let S be a set and $\mathbb{S}_S$ denotes the permutation group on S. Let $E = S \times X$ and $\beta \in Z^2_{\mathrm{CQ}}(X; \mathbb{S}_S)$. Then the binary operation on E

$$(a_1, x_1) * (a_2, x_2) = (a_1 \cdot \beta(x_1, x_2), x_1 * x_2)$$

defines a quandle structure on E. We call this E the non-abelian extension of X by β, and denote it by $E = E(X, S, \beta)$.

A quandle X is *decomposable* [1] if it is a disjoint union $X = Y \sqcup Z$ such that $Y * X = Y$ and $Z * X = Z$, where $Y * X = \{y * x \mid y \in Y,\ x \in X\}$. A quandle is *indecomposable* if it is not decomposable. For a rack X, let $\phi_y : X \to X$ be the quandle isomorphism defined by $\phi_y(x) = x * y$, $x \in X$. The subgroup $\mathrm{Inn}(X)$ of the group $\mathrm{Aut}(X)$ of automorphisms of X generated by ϕ_y, $y \in X$, is called the inner automorphism group. The same groups are defined for quandle automorphisms for a quandle X.

Lemma 5.1 **[1]** *Suppose Y is indecomposable and $f : Y \to X$ is a quandle surjective homomorphism such that $\phi_x = \phi_y$ if $f(x) = f(y)$ for $x, y \in Y$. Then Y is a non-abelian extension $S \times_\beta X$ for some $\beta \in Z^2_{\mathrm{CQ}}(X; \mathbb{S}_S)$.*

The following construction was given in [1] (Example 2.13). Let X be an indecomposable finite rack, $x_0 \in X$ a fixed element, $G = \mathrm{Inn}(X)$, and $H = G_{x_0}$ be the subgroup of G whose elements fix x_0. Let S be a finite set and $\rho : H \to \mathbb{S}_S$ be a group homomorphism. There is a bijection $G/H \to X$ given by $g \mapsto g(x_0)$. Fix a set-theoretic section $s : X \to G$. Thus $s(x) \cdot x_0 = x$ for all $x \in X$.

Lemma 5.2 **[1]** *The element $t(x, y) = s(x)\phi_y s(x * y)^{-1}$ is in H for any $x, y \in X$, and $\beta(x, y) = \rho(t(x, y))$ is a rack 2-cocycle.*

The 2-cocycle β is a quandle 2-cocycle if and only if $\rho(\phi_{x_0}) = 1 \in \mathbb{S}_S$.

Definitions

We define a new cocycle invariant using the non-abelian cocycles. Let $L = K_1 \cup \cdots \cup K_r$ be a classical oriented link diagram on the plane, where $K_1, \ldots, K_r$ are connected components, for some positive integer r. Let $\mathcal{T}_i$, for, $i = 1, \ldots, r$, be the set of crossings such that the under-arc is from the component i.

Let X be a quandle, H a group, $\beta \in Z^2_{\mathrm{CQ}}(X; H)$. Let $\mathcal{C} \in \mathrm{Col}_X(L)$ be a coloring of L by X. Let $(b_1, \ldots, b_r)$ be the set of base points on the components $(K_1, \ldots, K_r)$, respectively. Let $(\tau_1^{(j)}, \ldots, \tau_{k(j)}^{(j)})$ be the crossings in $\mathcal{T}_j$, $j = 1, \ldots, r$, that appear in this order when one starts from b_j and travels K_j in the given orientation.

At a crossing τ, let x_τ be the color on the under-arc from which the normal of the over-arc points; let y_τ be the color on the over-arc. The *Boltzmann weight* at τ is $B(\tau, \mathcal{C}) = \beta(x_\tau, y_\tau)^{\epsilon(\tau)}$, where $\epsilon(\tau)$ is ± 1 depending on whether τ is positive or negative, respectively. For a group element $h \in H$, denote by $[h]$ the conjugacy class to which h belongs.

Definition 5.3 The family of vectors of conjugacy classes

$$\vec{\Psi}(L) = \vec{\Psi}_{(X,H,\beta)}(L) = ([\Psi_1(L, \mathcal{C})], \ldots, [\Psi_r(L, \mathcal{C})])_{\mathcal{C} \in \mathrm{Col}_X(L)}$$

where

$$\Psi_i(L, \mathcal{C}) = \prod_{j=1}^{k(i)} \beta(x_{\tau_j^{(i)}}, y_{\tau_j^{(i)}})^{\epsilon(\tau_j^{(i)})},$$

is called the *conjugacy quandle cocycle invariant* of a link.

These cocycle invariants include abelian cocycle invariants defined in [10] as a special case (when H is abelian).

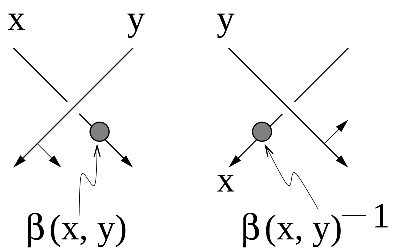

Figure 5: The beads interpretation of the invariant

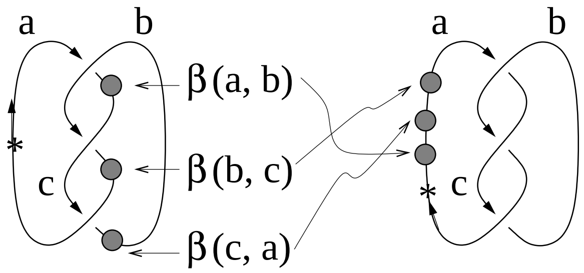

Figure 6: Making a beads necklace with trefoil

Remark 5.4 The invariant has the following interpretation of sliding beads along knots, similar to Hopf algebra invariants defined in [26]. Let a knot diagram K and its coloring $\mathcal{C}$ by a finite quandle X be given, and let $\Phi(K)$ be the cocycle invariant with a 2-cocycle $\beta \in Z^2_{\rm CQ}(X;H)$ for a (non-abelian) coefficient group H.

Put a bead on the underarc just below each crossing as shown in Fig. 5. Each bead is assigned the weight at the crossing, as in the left of the figure for a positive crossing and as in the right for a negative crossing, respectively. This process is depicted in Fig. 6 in the left for the trefoil.

Pick a base point (which is depicted by $*$ in the figure) on the diagram, and push it in the given orientation of the knot. With it the beads are pushed along in the order the base point encounters them. When the base point comes back to near the original position, it has collected all the beads. This situation is depicted in Fig. 6 in the right. The beads, read from the base point, are aligned in the order $\beta(a,b)$, $\beta(b,c)$, and $\beta(c,a)$ in this order in the figure, and the conjugacy class $[\beta(a,b)\beta(c,a)\beta(b,c)]$ is the contribution to the invariant for this coloring.

Theorem 5.5 *The quandle cocycle invariant $\vec{\Psi}$ is well defined. Specifically, let L_1, L_2 be two link diagrams of ambient isotopic links, and $\vec{\Psi}(L_1), \vec{\Psi}(L_2)$ be their quandle cocycle invariants. Then there is a bijection $\eta : \vec{\Psi}(L_1) \to \vec{\Psi}(L_2)$.*

Proof. The fact that $\vec{\Psi}$ does not change by Reidemeister moves for fixed base points is similar to the proof in [10] for the knot case and that in [6] for the link case, proved for abelian cocycle invariants,

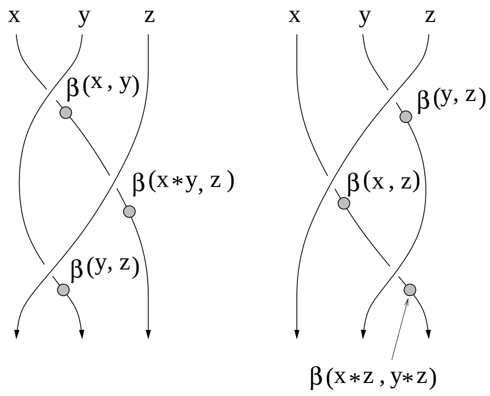

Figure 7: The beads and the type III move

except one just observes that the order of group elements is also preserved under Reidemeister moves.

Specifically, the changes under the type III Reidemeister move is depicted in Fig. 7. On the bottom string (that goes from top left to bottom right), two cocycles are assigned in both of the left and right of the figure, and they are equal with the given order (from top to bottom), by the 2-cocycle condition. The only cocycle assigned on the middle string has the same value for the left and right of the figure, and they also occupy the same position when the cocycles are read (as beads are slidden) along the component. Thus the ordered elements do not change by this move.

A change of base points causes cyclic permutations of Boltzmann weights, and hence the invariant is defined up to conjugacy. ■

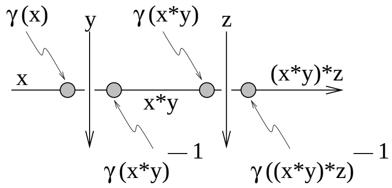

Figure 8: Canceling beads terms

Proposition 5.6 *The cocycle invariant $\vec{\Psi}$ is trivial (i.e., $\vec{\Psi}$ consists of vectors whose entries are the conjugacy class of the identity element) if the cocycle used is a coboundary.*

Proof. The proof is similar to the proof of an analogous theorem in [10]. If $\beta \in Z^2_{CQ}(X; H)$ is null-homologous, then $\beta(x, y) = \gamma(x)\gamma(x * y)^{-1}$ for some function $\gamma : X \to H$. Let $\beta(x, y)$, $\beta(x * y, z)$ be consecutive 2-cocycles in $\Psi_i(K, \mathcal{C})$, a contribution to the cocycle invariant for a coloring $\mathcal{C}$ for the ith component. Then in $\Psi_i(K, \mathcal{C})$, they form a product $\cdots \beta(x, y)\beta(x * y, z) \cdots$, which is equal to $\cdots (\gamma(x)\gamma(x * y)^{-1})(\gamma(x * y)\gamma((x * y) * z)^{-1}) \cdots$, and the middle terms cancel. The left and the right terms cancel with the next adjacent terms, and obtain $\Psi_i(K, \mathcal{C}) = 1$. This proves the proposition. The situation is easier to visualize diagrammatically. The bead representing $\beta(x, y)$ is represented by two separate ordered beads representing $\gamma(x)$ and $\gamma(x*y)^{-1}$ as depicted at the left crossing in Fig. 8. Thus all the beads cancel after going around each component once. ■

Recall that an action of a group H on a set X is said *free* if for all $h \in H$, $x \in X$ one has that $h \cdot x = x$ implies $h = 1$.

Proposition 5.7 *Let $H \subseteq \mathbb{S}_S$ and $\beta \in Z^2_{\rm CQ}(X;H)$. If the invariant $\vec{\Psi}(L)$ is trivial, then any coloring of L by X extends to a coloring by $E(X,S,\beta)$. Conversely, if any coloring of L by X extends to a coloring by $E(X,S,\beta)$, then $\vec{\Psi}(L)$ is trivial, provided that the action on S of the subgroup of H generated by the image of β is free.*

Proof. This proof is similar to the proof of the corresponding theorem in [6]. Pick an element $s_0 \in S$ and color the arc with the base point b_i on the ith component by $(s_0, x) \in E = E(X,S,\beta)$, where E is identified with $S \times X$. Go along the component, and after passing under the first crossing, the color of the second arc is required to be $(s_0\beta(x,y), x*y)$, if the over-arc is colored by $y \in X$. Continuing this process, when we come back to the base point, the color is $(s_0\Psi_i(L,\mathcal{C}), x)$, and the proposition follows. ■

The freeness requirement is satisfied, for example, if $S = H$ and H acts as multiplication on the right.

When a 2-cocycle constructed by the method of Lemma 5.2 is used to define the invariant, we have the following interpretation that is an analogue of Propositions 5.6 and 5.7. Let $L = K_1 \cup \cdots \cup K_r$ be an oriented link diagram, and $b_1, \ldots, b_r$ arbitrarily chosen and fixed base points on each component. Let β be a 2-cocycle constructed in Lemma 5.2, so that $\beta(x,y) = \rho(t(x,y))$ for any x, y in a quandle X, and $t(x,y) = s(y)\phi_y s(x*y)^{-1}$. From the proof of Proposition 5.7, the contribution to the cocycle invariant is written as

$$\begin{aligned}
\Psi_i(L,\mathcal{C}) &= \beta(x_{\tau_1^{(i)}}, y_{\tau_1^{(i)}})^{\epsilon(\tau_1^{(i)})} \cdot \beta(x_{\tau_2^{(i)}}, y_{\tau_2^{(i)}})^{\epsilon(\tau_2^{(i)})} \cdots \beta(x_{\tau_{k(i)}^{(i)}}, y_{\tau_{k(i)}^{(i)}})^{\epsilon(\tau_{k(i)}^{(i)})} \\
&= (\rho(s(x_{\tau_1^{(i)}})\phi_{y_{\tau_1^{(i)}}} s(x_{\tau_2^{(i)}})^{-1})) \cdot (\rho(s(x_{\tau_2^{(i)}})\phi_{y_{\tau_2^{(i)}}} s(x_{\tau_3^{(i)}})^{-1})) \cdots (\rho(s(x_{\tau_{k(i)}^{(i)}})\phi_{y_{\tau_{k(i)}^{(i)}}} s(x_{\tau_{k(i)}^{(i)}})^{-1})) \\
&= \rho(s(x_{\tau_1^{(i)}}) \cdot (\phi_{y_{\tau_1^{(i)}}}\phi_{y_{\tau_2^{(i)}}} \cdots \phi_{y_{\tau_{k(i)}^{(i)}}}) \cdot s(x_{\tau_{k(i)}^{(i)}})^{-1})
\end{aligned}$$

Thus we obtain the following.

Lemma 5.8 *If a 2-cocycle constructed in Lemma 5.2 is used to define the conjugacy cocycle invariant, then the contribution to the ith component $\Psi_i(L,\mathcal{C})$ is computed by*

$$\Psi_i(L,\mathcal{C}) = \rho(s(x_{\tau_1^{(i)}}) \cdot (\phi_{y_{\tau_1^{(i)}}}\phi_{y_{\tau_2^{(i)}}} \cdots \phi_{y_{\tau_{k(i)}^{(i)}}}) \cdot s(x_{\tau_1^{(i)}})^{-1}).$$

The expression $y_{\tau_1^{(i)}} \cdots y_{\tau_{k(i)}^{(i)}}$ is the sequence of colors that one encounters traveling along the component, picking up $y_{\tau_j^{(i)}}$ as one goes under the jth crossing. Thus this sequence corresponds to the longitudinal element in the fundamental group.

Constructions

We follow Lemma 5.2 to construct explicit examples of non-abelian cocycle invariants from conjugacy classes of groups. Let X be a conjugacy class in a finite group and let $G = \langle X \rangle$ be the subgroup generated by X. Let $x_0 \in G$ be a fixed element and $H = Z_{x_0} = \{x \in G \mid x_0 x = x x_0\}$. Let $\alpha : G \to \mathrm{Inn}(X)$ be the map induced from the conjugation $g \mapsto (x \mapsto gxg^{-1})$. Then the kernel of α is the center $Z(G)$ and $\mathrm{Inn}(X)$ is isomorphic to $G/Z(G)$.

To evaluate cocycle invariants for specific examples, we take $X = (2, 1, \ldots, 1)$ ($(n-2)$ copies of 1), the conjugacy class consisting of transpositions of $\mathbb{S}_n$, $n \geq 5$, then $G = \langle X \rangle = \mathbb{S}_n$, and $\mathrm{Inn}(X) = G$. Let $x_0 = (1\ 2)$, then

$$H = \langle (1\ 2), (i\ j) \mid 2 < i < j \leq n \rangle \cong \mathbb{Z}_2 \times \mathbb{S}_{n-2},$$

and $N(x_0)$, the normal closure of x_0 in H, is equal to $\langle (1\ 2) \rangle = \{1, (1\ 2)\}$, so that $H/N(x_0) \cong \mathbb{S}_{n-2}$, where $\mathbb{S}_{n-2}$ is identified with the symmetric group on $n-2$ letters $\{3, 4, \ldots, n\}$. Thus take $S = \{3, 4, \ldots, n\}$ and regard $H/N(x_0)$ as $\mathbb{S}_S$. Let the map $\rho : H \to H/N(x_0) = \mathbb{S}_S$ be the projection. Then $\rho(x_0) = 1$, so that the condition for a quandle cocycle is satisfied. Let $s : X \to G$ be defined by $s(i\ j) = (1\ i)(2\ j)$, then s defines a section. By Lemma 5.2, this set up defines a 2-cocycle $\beta \in Z^2_{\mathrm{CQ}}(X; \mathbb{S}_S)$.

Example 5.9 Let $n = 5$, then, for example, $\beta((1\ 4), (2\ 3))$ can be computed as

$$\begin{aligned}
\beta((1\ 4), (2\ 3)) &= \rho(t((1\ 4), (2\ 3))) \\
&= \rho(s((1\ 4))\phi_{(2\ 3)} s((1\ 4) * (2\ 3))^{-1}) \\
&= \rho((2\ 4)(2\ 3)[s((2\ 3)(1\ 4)(2\ 3))]^{-1}) \\
&= \rho((2\ 4)(2\ 3)(2\ 4)) \\
&= \rho((3\ 4)) \\
&= (3\ 4).
\end{aligned}$$

Example 5.10 We evaluate the cocycle invariant using the preceding 2-cocycle for a Hopf link $L = K_1 \cup K_2$ depicted in Fig. 9.

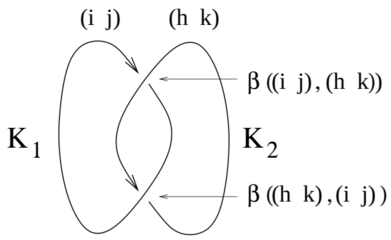

Figure 9: Hopf link

Let $n = 5$ as in Example 5.9. There are two arcs in a Hopf link, also denoted by K_1 and K_2. In the figure, the colors $(i\ j)$ and $(h\ k)$ are assigned. Consider the case $(i\ j) = (1\ 4)$ and $(h\ k) = (2\ 3)$, which certainly defines a coloring $\mathcal{C}$ of L. By the computation in Example 5.9, the contribution to the invariant $\vec{\Psi}(L)$ is

$$\Psi_1(L, \mathcal{C}) = \beta((1\ 4), (2\ 3)) = (3\ 4) = \Psi_2(L, \mathcal{C}),$$

which contributes the pair $([(3\ 4)], [(3\ 4)])$ of conjugacy classes of $\mathbb{S}_3 = \mathbb{S}_{\{3,4,5\}}$.

The non-abelian cocycle invariant is a quantum invariant

Let X be a finite quandle, G a finite group and $\beta : X \times X \to G$ a 2-cocycle. Let $\vec{\Psi}(L) = ([\Psi_1(L,\mathcal{C})], \dots, [\Psi_r(L,\mathcal{C})])$ be the above defined invariant.

Let $\rho : G \to \mathrm{Aut}(V)$ be a representation of G. Let $W = \mathbb{C}X \otimes V$ with the braiding given by $c((x \otimes v) \otimes (y \otimes w)) = (y \otimes w) \otimes (x * y \otimes v \cdot \beta(x,y))$. Since X and G are finite, W is a (right–right) Yetter-Drinfeld module over some finite group, (see [1]) and we can consider the quantum link invariants coloring links by W (the ribbon structure is the identity map, see [20]). Denote this invariant by $\Psi(L,W)$. Then we have the following, an analogue of [20].

Proposition 5.11 *$tr(\rho\vec{\Psi}(L)) = \Psi(L,W)$. (Here $tr(\rho\vec{\Psi}(L)) = \prod_{i=1}^r tr([\Psi_i(L,\mathcal{C})])$ is the trace in the tensor product via the inclusion $Aut(V) \times \cdots \times Aut(V) \hookrightarrow End(V \otimes \cdots \otimes V)$ (r components)).*

Proof. Consider L as the closure of a braid $z = \sigma_{i_1} \cdots \sigma_{i_l} \in \mathbb{B}_n$. Let $\bar{z} \in \mathbb{S}_n$ be the projection of z in the symmetric group. To compute $\Psi(L,W)$ we take the trace of the map defined by

$$z : W^{\epsilon_1} \otimes \cdots \otimes W^{\epsilon_n} \to W^{\epsilon_{\bar{z}(1)}} \otimes \cdots \otimes W^{\epsilon_{\bar{z}(n)}},$$

where $\epsilon_i = \pm 1$ and W^{-1} stands for W^*. To compute this trace we take the basis X of $\mathbb{C}X$, a basis $\{v_i \mid i \in I\}$ of V and the basis $\{x \otimes v_i \mid x \in X, i \in I\}$. Notice that the first components of the braiding in W are given by the quandle X. Thus, if we apply the map z to an element of the form $(x_1 \otimes v_{i_1}) \otimes \cdots \otimes (x_n \otimes v_{i_n})$, this element will not contribute to the invariant unless we receive $x_1, \dots, x_n$ as the first components; i.e. if $x_1, \dots, x_n$ defines a coloring of L. Consider therefore the colorings of L. Take a particular coloring given by $x_1, \dots, x_n$ at the bottom of the braid. For any n-tuple of elements $v_{i_1}, \dots, v_{i_n}$ in the basis of V, we apply successively $\sigma_{i_1}, \dots, \sigma_{i_l}$ and notice that the v_i's change by β exactly in the same way as defined in the invariant $\overline{\Psi}$. Now, $\mathrm{tr}(\rho\vec{\Psi})$ is the sum over the colorings of traces of products of $\rho(\beta_{x,y})$ for several x, y's, and Ψ is the sum over colorings of traces of maps made by $\beta_{x,y}$ for the same pairs x, y as for $\mathrm{tr}(\rho\vec{\Psi})$. The last step is to notice that the order of the β's used in the definition of $\vec{\Psi}$ is the same one that one gets by taking traces, thanks to the following remark: Let $f_1, \dots, f_n \in \mathrm{End}(V)$ be linear maps and consider the map $F \in \mathrm{End}(V^{\epsilon_1} \otimes \cdots \otimes V^{\epsilon_n})$ given by

$$F(x_1^{\epsilon_1} \otimes \cdots \otimes x_n^{\epsilon_n}) = \bar{z}(f^{\epsilon_1}(x_1^{\epsilon_1}) \otimes \cdots \otimes f^{\epsilon_n}(x_n^{\epsilon_n})).$$

Write $\bar{z} = (i_1, \dots, i_{a_1})(i_{a_1+1}, \dots, i_{a_1+a_2}) \cdots (i_{a_1+\cdots+a_{r-1}+1}, \dots, i_{a_1+\cdots+a_r})$ the decomposition of $\bar{z}$ in disjoint cycles. Then

$$\mathrm{tr}F = \prod_{j=1}^r \mathrm{tr}(f_{a_1+\cdots+a_{j-1}+1}^{\epsilon_{a_1+\cdots+a_{j-1}+1}} f_{a_1+\cdots+a_{j-1}+2}^{\epsilon_{a_1+\cdots+a_{j-1}+2}} \cdots f_{a_1+\cdots+a_j}^{\epsilon_{a_1+\cdots+a_j}}).$$

This is easy to see, though the notation is cumbersome. Let us illustrate it with the easy example $z = \sigma_1 \in B_2$; we must prove then that

$$\mathrm{tr}((v \otimes w) \mapsto (g(w) \otimes f(v)) = \mathrm{tr}(fg),$$

but the left hand side of the equation is $\sum_{i,j} g_{j,i} f_{i,j}$ (here $f_{i,j}$ and $g_{i,j}$ are the matrix coefficients of f and g in the basis $\{v_i\}$ of V) and this coincides with the right hand side. ■

6 Knot invariants from generalized 2-cocycles

Definitions

Let X be a finite quandle and $\mathbb{Z}(X)$ be its quandle algebra with generators $\{\eta_{x,y}^{\pm 1}\}_{x,y\in X}$ and $\{\tau_{x,y}\}_{x,y\in X}$. Recall (Example 2.1) that the enveloping group G_X of a quandle X is a quotient of the free group $F(X)$ on X defined by $G_X = \langle x \in X \mid x * y = yxy^{-1}\rangle$. Then $\mathbb{Z}(X)$ maps onto the group algebra $\mathbb{Z}G_X$ by $\eta_{x,y} \mapsto y$, $\tau_{x,y} \mapsto 1 - x * y$ so that any left $\mathbb{Z}G_X$-module has a structure of a $\mathbb{Z}(X)$-module [1]. Let G be an abelian group with this $\mathbb{Z}(X)$-module structure. Let $\kappa_{x,y}$ be a generalized quandle 2-cocycle of X with the coefficient group G. Thus the generalized 2-cocycle condition, in this setting, is written as

$$z\kappa_{x,y} + \kappa_{x*y,z} + ((x*y)*z)\kappa_{y,z} = \kappa_{y,z} + (y*z)\kappa_{x,z} + \kappa_{x*z,y*z}.$$

We define a cocycle invariant using this 2-cocycle.

A knot diagram K is given on the plane. Recall that it is oriented, and has orientation normals. There are four regions near each crossing, divided by the arcs of the diagram. The unique region into which both normals (to over- and under-arcs) point is called the *target* region. Let γ be an arc from the region at infinity of the plane to the target region of a given crossing r, that intersect K in finitely many points transversely, missing crossing points. Let a_i, $i = 1, \ldots, k$, in this order, be the arcs of K that intersect γ from the region at infinity to the crossing. Let $\mathcal{C}$ be a coloring of K by a fixed finite quandle X.

Definition 6.1 The *Boltzmann weight* $B(\mathcal{C}, r, \gamma)$ for the crossing r, for a coloring $\mathcal{C}$, with respect to γ, is defined by

$$B(\mathcal{C}, r, \gamma) = \pm(\mathcal{C}(a_1)^{\epsilon(a_1)}\mathcal{C}(a_2)^{\epsilon(a_2)} \cdots \mathcal{C}(a_k)^{\epsilon(a_k)})\kappa_{x,y} \in \mathbb{Z}G,$$

where x, y are the colors at the given crossing (x is assigned on the under-arc from which the normal of the over-arc points, and y is assigned to the over-arc), and the product

$$(\mathcal{C}(a_1)^{\epsilon(a_1)}\mathcal{C}(a_2)^{\epsilon(a_2)} \cdots \mathcal{C}(a_k)^{\epsilon(a_k)}) \in G_X$$

acts on G via the quandle module structure. The sign $\pm$ in front is determined by whether r is positive $(+)$ or negative $(-)$. The exponent $\epsilon(a_j)$ is 1 if the arc γ crosses the arc a_j against its normal, and is -1 otherwise, for $j = 1, \ldots, k$.

The situation is depicted in Fig. 10, when the arc intersects two arcs with colors u and v in this order.

Lemma 6.2 *The Boltzmann weight, does not depend on the choice of the arc γ, so that it will be denoted by $B(\mathcal{C}, r)$.*

Proof. It is sufficient to check the changes of the coefficient $(\mathcal{C}(a_1)^{\epsilon(a_1)}\mathcal{C}(a_2)^{\epsilon(a_2)} \cdots \mathcal{C}(a_k)^{\epsilon(a_k)}) \in G_X$ when the arc is homotoped. When a pair of intersection points with the knot diagram is canceled or introduced by a path that zig-zags, then their colors are inverses of each other, and are adjacent in the above sequence, so that the product does not change. It remains to check what happens when an arc is homotoped through each crossing, and the effect of the incident is depicted in Fig. 11.

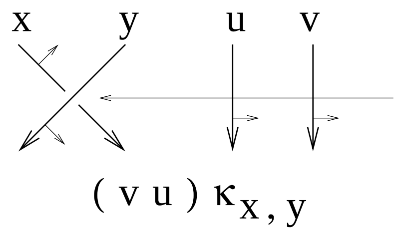

Figure 10: Weights on arcs

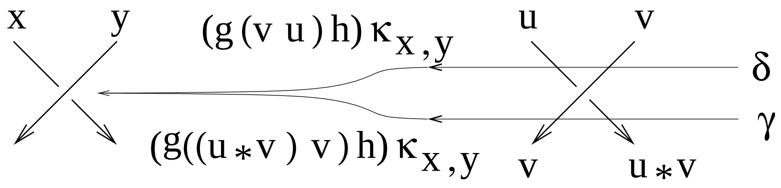

Figure 11: When an arc passes a crossing

There are two arcs, δ and γ, depicted in the figure. The given crossing r is the left crossing with colors x and y. The arc γ is obtained from δ by homotoping δ through a crossing with colors u and v as depicted. One sees that $B(\mathcal{C}, r, \delta) = (g(vu)h)\kappa_{x,y}$, where g and h are sequences of colors that δ intersects before and after, respectively, it intersects u and v. For the arc γ, we have $B(\mathcal{C}, r, \gamma) = (g((u * v)v))h)\kappa_{x,y}$, which agrees with $B(\mathcal{C}, r, \delta)$. Alternative orientations, and signs of crossings follow similarly. $\blacksquare$

Definition 6.3 The family $\Phi_\kappa(K) = \{\sum_r B(\mathcal{C}, r)\}_{\mathcal{C}}$ is called the quandle cocycle invariant with respect to the (generalized) 2-cocycle κ.

The invariant agrees with the quandle cocycle invariant $\Phi_\phi(K)$ defined in [10] when the quandle module structure of the coefficient group G is trivial, and with $\Phi_\phi(K)$ defined in [7], modulo the Alexander numbering convention in the Boltzmann weight in [7], when the coefficient group G is a $\mathbb{Z}[t, t^{-1}]$-module and the quandle module structure is given by $\eta_{x,y}(a) = ta$ and $\tau_{x,y}(b) = (1 - t)b$, for $x, y \in X$, $a, b \in G$. In the above papers, the state-sum form is used, instead of families. In this section we use families since it is easier to write for families of vectors.

We note that this definition contains the following data that were chosen and fixed: X, G, κ, and K.

Theorem 6.4 *The family $\Phi_\kappa(K)$ does not depend on the choice of a diagram of a given knot, so that it is a well-defined knot invariant.*

Proof. The proof is a routine check of Reidemeister moves, and it is straightforward for type I and II. The type III case is depicted, for one of the orientation choices, in Fig. 12, where g denotes the sequence of colors of arcs that appear before the arc γ intersects the crossings in consideration. It is seen from the figure that the contribution of the Boltzmann weights from these three crossings involved do not change before and after the move. The other orientation possibilities can be checked similarly. $\blacksquare$

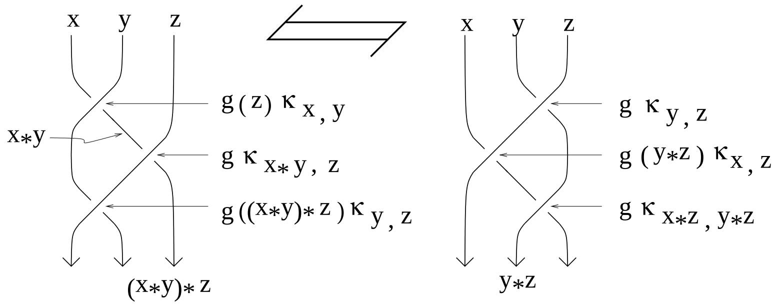

Figure 12: Reidemeister type III move and 2-cocycle condition

Lemma 6.5 *If $\kappa = \delta\lambda$ for some $\lambda \in C^1(X; A)$, then the cocycle invariant $\Phi_\kappa(L)$ is trivial for any link L. Moreover cohomologous cocycles yield the same invariants.*

Proof. The proof follows the same idea as in [10], and Proposition 5.6. We prove the case when $\delta\lambda = \kappa$, the case of cohomologous cocycles follows similarly. The coboundary of λ is given as

$$\delta\lambda(x, y) = -[y\lambda(x) + \lambda(y) - x * y\lambda(y) - \lambda(x * y)].$$

Near a crossing with source colors $x, y \in X$, assign $-y\lambda(x)$ to the source lower arc, $\lambda(x * y)$ to the target under-arc, $-\lambda(y)$ to the over-arc away from which the normal to the under-arc points, and $x * y\lambda(y)$ to the remaining over-arc, with the action of the sequence of group elements depending on the proximity of the arcs to the region at infinity. The situation is depicted in Fig. 13. In the figure, the arc from the region at infinity is depicted and named α, which crosses the arc colored by x_3. The arc α crosses other arcs, and the product of colors is denoted by g, so that the left crossing has the contribution $(gx_3)\kappa_{x_1,x_2}$ to the Boltzmann weight. The values $-(gx_3)x_2\lambda(x_1)$, $-(gx_3)\lambda(x_2)$, $(gx_3)x_1 * x_2\lambda(x_2)$, and $(gx_3)\lambda(x_1 * x_2)$ are assigned to the four arcs involved at the left crossing. The term $(gx_3)\lambda(x_1 * x_2)$ cancels with the same term assigned to the right of the horizontal arc coming from the contribution from the right crossing. Thus the assignments to arcs along consecutive crossings cancel. ■

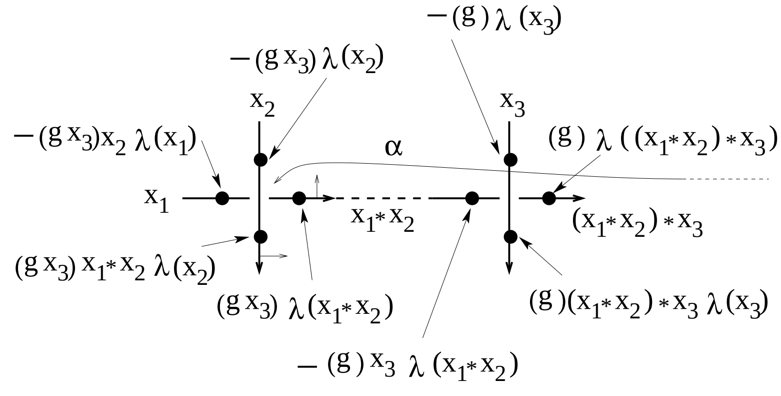

Figure 13: Coboundaries cancel

Proposition 6.6 *In the case when the quandle 2-cocycle κ is expressed by a group 2-cocycle as in Proposition 3.1, the invariant $\Phi_\kappa(K)$ is trivial for knots K (not necessarily for links).*

Proof. Let K denote a knot (not a link). Let $\pi = \pi_1(S^3 \setminus K)$ denote the fundamental group of the complement. The fundamental quandle $\pi_Q = \pi_Q(K)$ (see for example [16]) has π as its enveloping group $G_{\pi_Q} = \pi$. Let $E = A \rtimes_\theta G$ be an extension of a group G by an abelian group A twisted by a group 2-cocycle $\theta \in Z^2_{\text{Group}}(G; A)$. A quandle coloring of K by $\text{Conj}(G)$ is induced by a quandle homomorphism $f : \pi_Q \to \text{Conj}(G)$ which naturally lifts to a group homomorphism $f : \pi \to G$, denoted by the same letter. There is an action of π on A given via the group homomorphism f. Explicitly, if $x \in \pi$ and $a \in A$, then $x \cdot a = s(f(x))i(a)s(f(x))^{-1}$ where $s : G \to E$ is the section that gives rise to the description $E = A \rtimes_\theta G$, and $i : A \to E$ is the inclusion.

The sphere theorem gives that $S^3 \setminus K$ is a $K(\pi, 1)$-space, and so $H^2_{\text{Group}}(\pi; A)$ is trivial. Hence any group 2-cocycle is a coboundary for π. This implies that for generators α, β of π or π_Q corresponding to arcs of a given knot diagram, $\theta_{f(\alpha),f(\beta)} = f^\# \theta(\alpha, \beta) = \delta_G \gamma$, and by Lemma 3.5, $\kappa_{f(\alpha),f(\beta)} = f^\# \delta_Q \gamma$ Then Lemma 6.5 implies that the invariant is trivial. ■

Braids and the cocycle invariants

Let a knot or a link L be represented as a closed braid $\hat{w}$, where w is a k-braid word. Let X be a quandle and G be a quandle module, keeping the setting given at the beginning of this section.

Definition 6.7 Let $\vec{x} = (x_1, \ldots, x_k) \in X^k$ for some positive integer k. Define a *weighted sum* on G^k with respect to $\vec{x}$ by

$$\text{WS}_{\vec{x}}(\vec{a}) = \sum_{i=1}^{k} u_i a_i = (x_k \cdots x_2)a_1 + (x_k \cdots x_3)a_2 + \cdots + x_k a_{k-1} + a_k,$$

where $\vec{a} = (a_1, \ldots, a_k) \in G^k$ and $u_i = x_k \cdots x_{i+1}$ for $i = 1, \ldots, k-1$, and $u_k = 1$.

Let w be a k-braid word, $\vec{x} = (x_1, \ldots, x_k) \in X^k$ be a vector of colors assigned to the bottom strings of w, and $\mathcal{C}$ be the unique coloring of w by X determined by $\vec{x}$. Let the vector $\vec{y} = (y_1, \ldots y_k) \in X^k$ denote the color vector on the top strings that is induced by the vector $\vec{x} = (x_1, \ldots x_k) \in X^k$ which is assigned at the bottom. Fix this coloring $\mathcal{C}$. Let $\alpha^0_{x,y} = \eta_{x,y} + \tau_{x,y}$ be a quandle 2-cocycle with coefficient group G with $\kappa = 0$, where G is an X-module. Then any color vector $\vec{a} = (a_1, \ldots, a_k) \in G^k$ at the bottom strings uniquely determines a color vector $\vec{b} = (b_1, \ldots, b_k) \in G^k$ at the top strings of w with respect to α, that is, $\vec{b} = M(w, \vec{x}) \cdot \vec{a}$. Recall that in this section the quandle module structure is given by the G_X-module structure.

Lemma 6.8 *Let $\alpha^0_{x,y} = \eta_{x,y} + \tau_{x,y}$ be a quandle 2-cocycle with $\kappa = 0$, and $\vec{b} = M(w, \vec{x}) \cdot \vec{a}$, as above. Then we have $WS_{\vec{x}}(\vec{a}) = WS_{\vec{y}}(\vec{b})$.*

Proof. It is sufficient to prove the statement for a generator and its inverse. The inverse case is similar, so we compute the case when $w = \sigma_i$ for some i, $1 \leq i < k$. The only difference in the weighted sum, in this case, is the ith and $(i+1)$st terms. Let $u = x_k \cdots x_{i+2}$. For the bottom colors, the ith and $(i+1)$st terms are

$$\text{WS}_{\vec{x}}(\vec{a}) = \cdots + u x_{i+1} a_i + u a_{i+1} + \cdots.$$

On the other hand, one computes

$$\mathrm{WS}_{\vec{y}}(\vec{b}) = \cdots + u(x_i * x_{i+1})a_{i+1} + u[x_{i+1}a_i + (1 - x_i * x_{i+1})a_{i+1}] + \cdots$$

which agrees with the above. ■

Theorem 6.9 *Let w be a k-braid word with crossings ρ_ℓ, $\ell = 1, \ldots, h$, and $\vec{x}$ a bottom color vector by a quandle X. Let $\alpha_{x,y} = \eta_{x,y} + \tau_{x,y} + \kappa_{x,y}$ be a quandle 2-cocycle with coefficient group G, a G_X-module, and $\vec{b} = M(w, \vec{x}) \cdot \vec{a}$. Here, the map M corresponds to α with possibly non-zero κ. Then we have*

$$WS_{\vec{y}}(\vec{b}) - WS_{\vec{x}}(\vec{a}) = \sum_{\ell=1}^{h} B(\mathcal{C}, \rho_\ell).$$

Proof. Let $M^0(w, \vec{x})$ denote the map corresponding to the 2-cocycle $\alpha^0_{x,y} = \eta_{x,y} + \tau_{x,y}$, which is obtained from α by setting $\kappa = 0$. The theorem follows from induction once we prove it for a braid generator and its inverse, and we show the case $w = \sigma_i$, as the inverse case is similar. Then we compute

$$\begin{aligned} \mathrm{WS}_{\vec{y}}(\vec{b}) = \mathrm{WS}_{\vec{y}}(M(\sigma_i, \vec{x}) \cdot \vec{a}) &= \mathrm{WS}_{\vec{y}}(M^0(\sigma_i, \vec{x}) \cdot \vec{a}) + B(\mathcal{C}, \sigma_i) \\ &= \mathrm{WS}_{\vec{x}}(\vec{a}) - B(\mathcal{C}, \sigma_i), \end{aligned}$$

where the first equality follows from the definitions, and the second equality follows from Lemma 6.8. Note that the braid generator represents a negative crossing in the definition of the quandle module invariant, and a positive crossing in the cocycle invariant, so that there is a negative sign for the weight $B(\mathcal{C}, \sigma_i)$. ■

Theorem 6.10 *If a coloring $\mathcal{C}$ of L by X extends to a coloring of L by the extension $E = G \times_\alpha X$, then the coloring contributes a trivial term (i.e., an integer in $\mathbb{Z}G$) to the generalized cocycle invariant $\Phi_\kappa(L)$.*

Proof. A given coloring agrees on the bottom and top strings, so that $\vec{x} = \vec{y}$. If the given coloring extends to E, we have $\vec{b} = M(\sigma_i, \vec{x}) \cdot \vec{a} = \vec{a}$, and in particular, $\mathrm{WS}_{\vec{y}}(\vec{b}) = \mathrm{WS}_{\vec{x}}(\vec{a})$. Then this theorem follows from Theorem 6.9. ■

5Thus the invariant $\tilde{\Phi}(X, \alpha\ ; L)$ measures exactly

Remark 6.11 In Livingston [32], the following situation is examined. Suppose that E is a split extension of a group G by an abelian group A, so that $0 \to A \to E \to G \to 1$ is a split exact sequence. Let $\rho : \pi = \pi_1(S^3 \setminus K) \to G$ denote a homomorphism. Then since there is an action of G on A via conjugation in E, there is a corresponding action of π on A given via ρ. Livingston examines when there is a lift of ρ to $\tilde{\rho} : \pi \to E$ thereby generalizing Perko's theorem that any homomorphism $\rho : \pi \to \Sigma_3$ lifts to a homomorphism $\tilde{\rho} : \pi \to \Sigma_4$. In this situation, the permutation group Σ_3 acts on the Klein 4 group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(1), (12)(34), (13)(24), (14)(23)\}$ via conjugation and the obvious section $s : \Sigma_3 \to \Sigma_4$. Livingston shows that there is a one-to-one correspondence between A-conjugacy classes of lifts of ρ and elements in $H^1(\pi, \{A\})$ where the coefficients $\{A\}$ are twisted by the action of π on A.

Consider a quandle coloring of a knot K by a quandle $\mathrm{Conj}(G)$. Such a quandle coloring is a homomorphism from the fundamental quandle, $\pi_Q(K)$, of K, to $\mathrm{Conj}(G)$. The fundamental group π can be thought of as $\pi = G_{\pi_Q(K)}$. Thus a quandle coloring is a homomorphism ρ as above.

Thus, we have the following cohomological characterization of lifting colorings. If a coloring lifts, then the coloring contributes a constant term to the cocycle invariant by Theorem 6.10 above, and it corresponds to a 1-dimensional cohomology class of $H^1(\pi, \{A\})$.

Computations

In this section we present a computational method using closed braid form. *We take a positive crossing as a positive generator, in this section.* Let $w = w_1 \cdots w_h$ be a braid word, where $w_s = \sigma_{j(s)}^{\epsilon(s)}$ is a standard generator or its inverse for each $s = 1, \cdots, h$. The braids are oriented downward and the normal points to the right. Then the target region is to the right of each crossing.

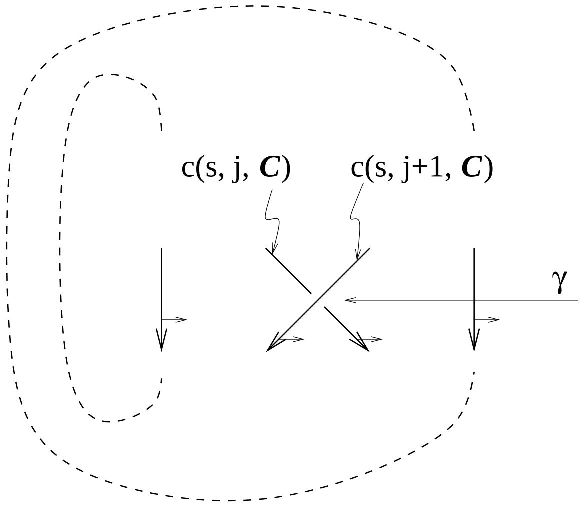

Figure 14: Colors and arcs for braids

Let X be a quandle and $\mathcal{C}$ be a coloring of $\hat{w}$ by X. Let $c(s,i,\mathcal{C})$ be the color of the ith string from the left, immediately above the sth crossing (w_s) for $\mathcal{C}$. Take the right-most region as the region at infinity, and take the closure of a given braid to the left, as depicted in Fig. 14. As an arc γ to the target region, take an arc that goes horizontally from the right to left to the crossing, see Fig. 14. Then the Boltzmann weight at the s-th crossing is given by

$$B(\mathcal{C}, w_s) = (c(s,n,\mathcal{C}) \cdots c(s,j+2,\mathcal{C}))\kappa_{c(s,j,\mathcal{C}),c(s,j+1,\mathcal{C})}$$

for a positive crossing (when $\epsilon(s) = 1$), and the 2-cocycle evaluation is replaced by $\kappa_{c(s+1,j,\mathcal{C}),c(s,j,\mathcal{C})}$ for a negative crossing, and with a negative sign in front. Note that the expression in front of κ represents the action of this group element on the coefficient group, so that in the case of wreath product, for example, this can be written in terms of matrices. Also, if $j = n-1$, then the expression in front is understood to be empty. This formula can be used to evaluate the invariant for knots in closed braid form, and can be implemented in a computer.

Example 6.12 We implemented the above formula for R_3 in *Maple* and obtained the following results, confirmed also by *Mathematica*. The action is in the wreath product; thus R_3 acts on $(\mathbb{Z}_q)^3$ by transpositions of factors. We also represent elements of R_3 by $\{1, 2, 3\}$ with the following correspondence in this example: $1 = (2\ 3)$, $2 = (1\ 3)$ and $3 = (1\ 2)$.

For $q = 0$, any 2-cocycle gave trivial invariant (each coloring contributes the zero vector, and the family of vectors is a family of zero vectors), for all 3-colorable knots in the knot table up to 9 crossings. Thus we conjecture that $q = 0$ gives rise to the trivial invariant.

Let $h(i, j) = {}^{T}(f_1(i, j), f_2(i, j), f_3(i, j))$ denote a vector valued 2-cochain. Then for $q = 3$, the following defines a 2-cocycle:

$$\begin{array}{ccccccccccc} f_1(3,1) & = & f_3(3,1) & = & f_1(2,3) & = & f_1(2,1) & = & f_3(2,1) & = & 1, \\ f_2(3,1) & = & f_2(3,2) & = & f_2(1,3) & = & f_1(1,2) & = & f_2(1,2) & = & 2 \end{array}$$

and all the other values are zeros. Then we obtain the following results.

- $\Phi_\kappa(K) = \{\sqcup_9(0, 0, 0)\}$ for $K = 6_1, 8_{10}, 8_{11}, 8_{20}, 9_1, 9_6, 9_{23}, 9_{24}$, where $\{\sqcup_9(0, 0, 0)\}$ represents the family consisting of 9 copies of $(0, 0, 0)$ (similar notations are used below).
- $\Phi_\kappa(K) = \{\sqcup_3(0, 0, 0), \sqcup_6(1, 1, 1)\}$ for $K = 3_1, 7_4, 7_7, 9_{10}, 9_{38}$.
- $\Phi_\kappa(K) = \{\sqcup_3(0, 0, 0), \sqcup_6(2, 2, 2)\}$ for $K = 8_5, 8_{15}, 8_{19}, 8_{21}, 9_2, 9_4, 9_{11}, 9_{15}, 9_{16}, 9_{17}, 9_{28} 9_{29}, 9_{34}, 9_{40}$.
- $\Phi_\kappa(K) = \{\sqcup_9(0, 0, 0), \sqcup_{18}(1, 1, 1)\}$ for $K = 9_{35}, 9_{47}, 9_{48}$.
- $\Phi_\kappa(K) = \{\sqcup_3(0, 0, 0), \sqcup_{12}(1, 1, 1), \sqcup_{12}(2, 2, 2)\}$ for $K = 8_{18}$.
- $\Phi_\kappa(K) = \{\sqcup_{15}(0, 0, 0), \sqcup_6(1, 1, 1), \sqcup_6(2, 2, 2)\}$ for $K = 9_{37}, 9_{46}$.

In the original cohomology theory with trivial action on the coefficient, the second cohomology group $H^2_{\mathrm{Q}}(R_3; G)$ was trivial for any coefficient group G [10], so that R_3 gave rise to a trivial cocycle invariant. This example shows that, with the wreath product action, R_3 has non-trivial second cohomology group, and gives rise to a non-trivial cocycle invariant, showing that the theories with actions on coefficients are strictly more general and stronger than the original case.

Remark 6.13 With the same cocycle as the preceding example, we computed the invariants for the mirror images with the same orientations. The results are such that the values 1 and 2 are exchanged in all values (thus we conjecture that this is the case in general, at least for R_3). For example, the mirror image of the trefoil has $\Phi_\kappa(K) = \{\sqcup_3(0, 0, 0), \sqcup_6(2, 2, 2)\}$ as its invariant. Hence those with asymmetric values of the invariant are proven to be non-amphicheiral by this invariant. Specifically, 22 knots among 33 are detected to be non-amphicheiral.

7 Invariants for knotted surfaces

Definitions

The cocycle invariants are defined for knotted surfaces in 4-space in exactly the same manner as in Section 6 as follows. Let X be a finite quandle and $\mathbb{Z}(X)$ be its quandle algebra with generators $\{\eta^{\pm 1}_{x,y}\}_{x,y \in X}$ and $\{\tau_{x,y}\}_{x,y \in X}$. Let G be an abelian group that is a G_X-module. Recall that this induces a $\mathbb{Z}(X)$-module structure given by $\eta_{x,y} g = yg$ and $\tau_{x,y}(g) = (1 - x * y)g$ for $g \in G$ and

$x, y \in X$. Let $\kappa_{x,y,z}$ be a generalized quandle 3-cocycle of X with the coefficient group G. Thus the generalized 3-cocycle condition, in this setting, is written as

$$\begin{aligned} & w\kappa_{x,y,z} + \kappa_{x*z,y*z,w} + ((y*z)*w)\kappa_{x,z,w} + \kappa_{y,z,w} \\ = \quad & ((x*(y*z))*w)\kappa_{y,z,w} + \kappa_{x*y,z,w} + (z*w)\kappa_{x,y,w} + \kappa_{x*w,y*w,z*w}. \end{aligned}$$

We require further that $\kappa_{x,x,y} = \kappa_{x,y,y} = 0$. A cocycle invariant of knotted surfaces will be defined using such a 3-cocycle.

A knotted surface diagram K is given in 3-space. We assume the surface is oriented and use orientation normals to indicate the orientation. In a neighborhood of each triple point, there are eight regions that are separated by the sheets of the surface since the triple point looks like the intersection of the 3-coordinate planes in some parametrization. The region into which all normals point is called the *target* region. Let γ be an arc from the region at infinity of the 3-space to the target region of a given triple point r. Assume that γ intersects K transversely in a finitely many points thereby missing double point curves, branch points, and triple points. Let a_i, $i = 1, \ldots, k$, in this order, be the sheets of K that intersect γ from the region at infinity to the triple point r. Let $\mathcal{C}$ be a coloring of K by a fixed finite quandle X.

Definition 7.1 The *Boltzmann weight* $B(\mathcal{C}, r, \gamma)$ for the triple point r, for a coloring $\mathcal{C}$, with respect to γ, is defined by

$$B(\mathcal{C}, r, \gamma) = \pm(\mathcal{C}(a_1)^{\epsilon(a_1)}\mathcal{C}(a_2)^{\epsilon(a_2)} \cdots \mathcal{C}(a_k)^{\epsilon(a_k)})\kappa_{x,y,z} \in \mathbb{Z}G,$$

where x, y, z are the colors at the given triple point r (x is assigned on the bottom sheet from which the normals of the middle and top sheets point, and y is assigned to the middle sheet from which the normal of the top sheet points, and z is assigned to the top sheet). The sign $\pm$ in front is determined by whether r is positive $(+)$ or negative $(-)$. The exponent $\epsilon(a_j)$ is 1 is the arc γ crosses the arc a_j against its normal, and is -1 otherwise, for $j = 1, \ldots, k$.

Lemma 7.2 *The Boltzmann weight does not depend on the choice of the arc γ, so that it will be denoted by $B(\mathcal{C}, r)$.*

Proof. It is sufficient to check what happens when an arc is homotoped through double point curves, and Fig. 11 can be regarded as a cross-sectional view of such a homotopy, and the same computation as in the classical case holds. ■

Definition 7.3 The family $\Phi_\kappa(K) = \{\sum_r B(\mathcal{C}, r)\}_{\mathcal{C} \in \mathrm{Col}_{\mathrm{X}}(K)}$ is called the quandle cocycle invariant with respect to the (generalized) 3-cocycle κ.

Theorem 7.4 *The family $\Phi_\kappa(K)$ does not depend on the choice of a diagram of a given knotted surface, so that it is a well-defined knot invariant.*

Proof. The proof is a routine check of Roseman moves, analogues of Reidemeister moves. In particular, the analogue of the type III Reidemeister move is called the tetrahedral move, and is a generic plane passing through the origin that is the triple point formed by coordinate planes. In

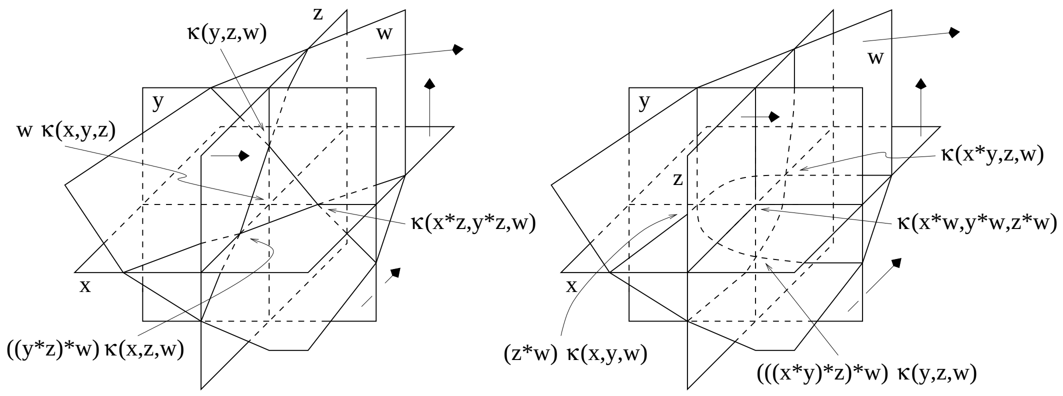

Figure 15: Contributions for the tetrahedral move

Fig. 15, such a move is depicted. A choice of normal vectors and quandle colorings are also depicted in the figure. The sheet labeled w is the top sheet, and the next highest sheet is labeled by z, the bottom is x. The region at infinity is chosen (for simplicity) to be the region at the top right, into which all normals point. The Boltzmann weight at each triple point is also indicated. The sum of the weights for the LHS and RHS are exactly those for the 3-cocycle condition. Other choices for orientations, and other moves, are checked similarly. In particular, by assuming that the cocycle κ satisfies the quandle condition $\kappa_{x,x,y} = \kappa_{x,y,y} = 0$, we ensure that the quantity is invariant under the Roseman move in which a branch point passes through another sheet. ■

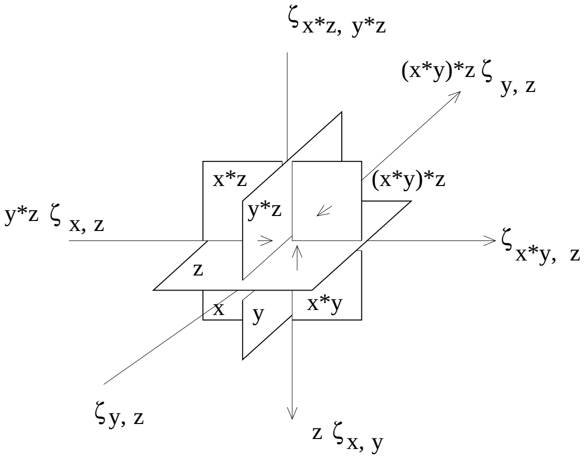

Figure 16: Coboundary terms distributed

A proof similar to that of Lemma 6.5 implies the following, where the distribution of coboundary terms is depicted in Fig. 16.

Lemma 7.5 *If $\kappa = \delta\zeta$ for some $\zeta \in C^2(X;A)$, then the cocycle invariant $\Phi_\kappa(F)$ is trivial for any knotted surface F. Moreover cohomologous cocycles yield the same invariants.*

Computations

We develop a computational method, based on Satoh's method [38], of computing this cocycle invariant for twist-spun knots using the closed braid form. First we review Satoh's method. A movie description of one full twist of a classical knot K is depicted in Fig. 17 from (1) through (5). Strictly speaking, K is a tangle with two end points, and the tangle is twisted about an axis containing these end points. For the twist-spun trefoil, a diagram of the trefoil (with two end points) goes in the place of K.

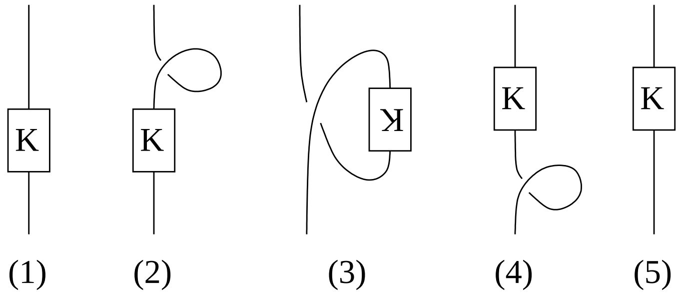

Figure 17: A movie of twist spinning

It is seen from this movie that branch points appear between (1) and (2), and (4) and (5), when type I Reidemeister moves occur in the movie. Triple points appear when K goes over an arc between (2) and (3), and goes under it between (3) and (4). Each triple point corresponds to a crossing of the diagram K. This movie will provide a broken surface diagram by taking the continuous traces of the movie, which is depicted in Fig. 18. Horizontal cross sections of Fig. 18 correspond to (1) through (5) as indicated at the top of the figure. Thus between (1) and (2) there is a branch point, for example. A choice of normal vectors is also depicted by short arrows.

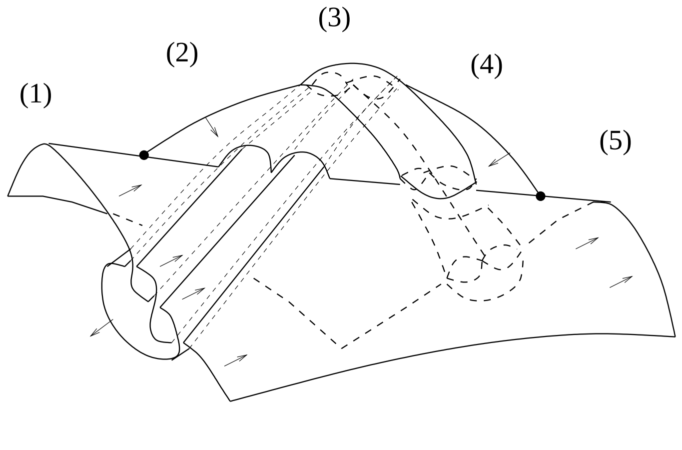

Figure 18: A part of a diagram of twist-spun trefoil

Now we apply this method to closed braid form. Let w be a diagram corresponding to a braid word of n strings. Then construct a tangle with two end points at top and bottom, by stretching the left-most end points and closing all the other end points of the braid, as depicted in Fig. 19.

Let K be this tangle, as well as the corresponding knot and knot diagram. A choice of normal vectors are also depicted. Let $\mathrm{Tw}^\ell(K)$ be the diagram of the ℓ-twist spun of K obtained by Satoh's method.

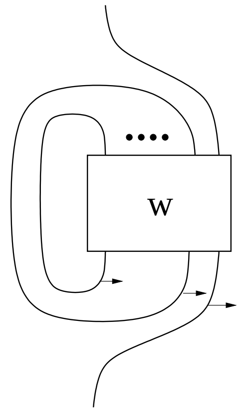

Figure 19: A tangle construction from a braid

Let w (as a braid word, written as the same letter as the diagram), be $w_1, \ldots, w_h$, where each w_s is a standard generator or its inverse, $\sigma_{j(s)}^{\epsilon(s)}$. In this section we use the positive crossing as a standard generator. Recall that each crossing gives rise to a triple point in the diagram of a twist-spun knot, when Satoh's method is applied. In Fig. 20, the triple point corresponding to $w_s = \sigma_j$ is depicted. This figure represents a triple point $T_1^l(s)$ that is formed when the crossing w_s goes through a sheet between steps (2) and (3) in Fig. 18 for $j = j(s)$ and $\epsilon(w_s) > 0$ (i.e., w_s is the j-th braid generator σ_j, and the crossing is positive). There is another triple point $T_1^r(s)$ formed by the same crossing between steps (3) and (4). The superscripts l and r represents that they appear in the left and right of Fig. 18, respectively. It is seen that $T_1^l(s)$ and $T_1^r(s)$ are negative and positive triple points with the right-hand rule, respectively, with respect to the normal vectors specified in Figs. 18 and 19. Then there are a pair of triple points $T_u^l(s)$ and $T_u^r(s)$ for the left and right, respectively, for the u-th twist.

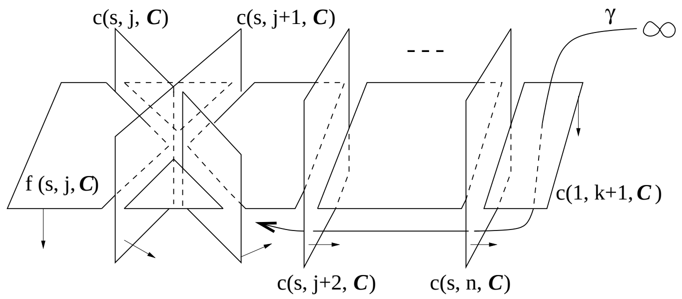

Figure 20: The weight at a triple point

Let a coloring $\mathcal{C}$ by a quandle X of the diagram $\mathrm{Tw}^\ell(K)$ be given. At the triple point $T_1^l(s)$, the triple of colors that contribute to the cocycle invariant are depicted in Fig. 20 as $f(s, j, \mathcal{C})$, $c(s, j, \mathcal{C})$ and $c(s, j+1, \mathcal{C})$ for the top, middle, and bottom sheet, respectively. Here f represents a color of the face left to the crossing w_s, and j in $c(s,j,\mathcal{C})$ represents the j-th string from the left in the braid w (as $w_s = \sigma_j$). The color of the horizontal sheet in Fig. 20 at the right-most region outside of the braid is the same as the color $c(1,k,\mathcal{C})$ of the k-th string of the braid w at the top and bottom, since the tangle goes through the k-th string, see Figs. 19 and 17. Thus we obtain

$$f(s,j,\mathcal{C}) = (\cdots(c(1,k,\mathcal{C})\bar{*}c(s,k,\mathcal{C}))\bar{*}c(s,k-1,\mathcal{C}))\bar{*}\cdots)\bar{*}c(s,j,\mathcal{C}))\cdots),$$

where for any $b,c \in X$ the unique element $a \in X$ with $a * b = c$ is denoted by $a = c\bar{*}b$. From Figs. 18 and 20, it is seen that the target region of $T_1^l(s)$ is to the bottom right. Thus from the region at infinity, we choose an arc γ as depicted in Fig. 20, that goes through the sheet with color $c(1,k,\mathcal{C})$, then hits all the sheets corresponding to the n-th through $(j+2)$-th braid strings, to the target region. Only the first sheet is oriented coherently with the direction of the arc we chose. Hence the sequence of quandle elements in the Boltzmann weight that corresponds to the arc γ is

$$c(1,k,\mathcal{C})^{-1}c(s,n,\mathcal{C})c(s,n-1,\mathcal{C})\cdots c(s,j+2,\mathcal{C}).$$

Similar arcs can be chosen for $T_1^r(s)$ as well, giving the same sequence. Thus we obtain the Boltzmann weights as follows.

$$B(\mathcal{C},T_1^l(s)) = \begin{cases} -(c(1,k,\mathcal{C})^{-1}c(s,n,\mathcal{C})\cdots c(s,j+2,\mathcal{C}))\kappa_{f(s,j,\mathcal{C}),\ c(s,j,\mathcal{C}),\ c(s,j+1,\mathcal{C})} & \text{if } \epsilon(T_1^l(s)) < 0 \\ c(1,k,\mathcal{C})^{-1}c(s,n,\mathcal{C})\cdots c(s,j+2,\mathcal{C})\kappa_{f(s,j,\mathcal{C}),\ c(s+1,j,\mathcal{C}),\ c(s,j,\mathcal{C})} & \text{if } \epsilon(T_1^l(s)) > 0, \end{cases}$$

$$B(\mathcal{C},T_1^r(s)) = \begin{cases} c(1,k,\mathcal{C})^{-1}c(s,n,\mathcal{C})\cdots c(s,j+2,\mathcal{C})\kappa_{c(s,j,\mathcal{C}),\ c(s,j+1,\mathcal{C}),\ c(1,k,\mathcal{C})} & \text{if } \epsilon(T_1^r(s)) > 0 \\ -(c(1,k,\mathcal{C})^{-1}c(s,n,\mathcal{C})\cdots c(s,j+2,\mathcal{C}))\kappa_{c(s+1,j,\mathcal{C}),\ c(s,j,\mathcal{C}),\ c(1,k,\mathcal{C})} & \text{if } \epsilon(T_1^r(s)) < 0. \end{cases}$$

For the u-th twist, the cocycle evaluation is replaced by

$$\kappa_{f(s,j,\mathcal{C})*c(1,k,\mathcal{C})^u,\ c(s,j,\mathcal{C})*c(1,k,\mathcal{C})^u,\ c(s,j+1,\mathcal{C})*c(1,k,\mathcal{C})^u}$$

for a positive triple point $T_u^l(s)$, and similar changes (multiplication by $*c(1,k,\mathcal{C})^u$ for every term) are made for the other triple points accordingly, where $x * y^\ell$ denotes the ℓ-fold product $(\cdots(x * y) * y)\cdots) * y$.

With these weights the cocycle invariant is computed by

$$\Phi_\kappa(\mathrm{Tw}^\ell(K)) = \left\{ \sum_{u=1}^{\ell}\sum_{s=1}^{h} \left[B(\mathcal{C},T_u^l(s)) + B(\mathcal{C},T_u^r(s)) \right] \right\}_{\mathcal{C}}.$$

This formula, again, can be implemented in computer calculations.

Example 7.6 We obtained the following calculations by *Maple* and *Mathematica*. Let $X = R_3$, the coefficient group $\mathbb{Z}^3$ with the wreath product action (elements of $R_3 = \{1,2,3\}$ acting on $\mathbb{Z}^3$ by transpositions of factors, $1 = (2\ 3)$, $2 = (1\ 3)$, and $3 = (1\ 2)$). Let $h(i,j,k) = {}^T(f_1(i,j,k), f_2(i,j,k), f_3(i,j,k))$ denote a vector valued 3-cochain, where $f_\ell(i,j,k) \in \mathbb{Z}$. Let $q_1, q_2 \in \mathbb{Z}$ be arbitrary elements. Then the following values, with all the other unspecified ones being zero, defined a 3-cocycle h:

$$\begin{aligned} f_1(1,2,1) = f_2(1,2,3) = f_3(3,1,2) = -f_1(1,3,1) = -f_1(1,3,2) &= q_1, \\ f_1(2,1,3) = f_1(3,1,2) = -f_1(2,3,2) &= q_2, \\ f_1(3,2,3) &= -q_1 - q_2 \end{aligned}$$

Knot K	$\Phi_\kappa(\mathrm{Tw}^2(K))$
3_1	$\sqcup_9(0,0,0)$.
6_1	$\sqcup_3(0,0,0),\ (-q_2,0,q_2),\ (q_2,q_1,-q_1-q_2),\ (q_1,0,-q_1),$
	$(0,-q_1,q_1),\ (-q_1+q_2,q_1-q_2,0),\ (-q_2,-q_1+q_2,q_1)$.
7_4	$\sqcup_3(0,0,0),\ (-q_2,-2q_1,2q_1+q_2),\ (-q_1+q_2,q_1,-q_2),$
	$(-q_1,2q_1,-q_1)\ ,(-q_1,-q_1,2q_1),\ (q_2,-q_2,0),\ (-q_2,q_2,0)$.
7_7	$\sqcup_3(0,0,0),\ \sqcup_2(q_1,0,-q_1),\ \sqcup_2(0,-q_1,q_1),\ \sqcup_2(-q_1,q_1,0)$.
8_5	$\sqcup_9(0,0,0)$.
8_{10}	$\sqcup_9(0,0,0)$.
8_{11}	$\sqcup_3(0,0,0),\ (q_2,0,-q_2),\ (-q_2,-q_1,q_1+q_2),\ (-q_1,0,q_1),$
	$(0,q_1,-q_1),\ (q_1-q_2,-q_1+q_2,0),\ (q_2,q_1-q_2,-q_1)$.
8_{15}	$\sqcup_3(0,0,0),\ \sqcup_3(0,-q_1,q_1),\ \sqcup_3(-q_1,q_1,0)$.
8_{18}	$\sqcup_9(0,0,0),\ \sqcup_6(q_1,0,-q_1),\ \sqcup_6(-q_1,q_1,0),\ \sqcup_6(0,-q_1,q_1)$.
8_{19}	$\sqcup_9(0,0,0)$.
8_{20}	$\sqcup_9(0,0,0)$.
8_{21}	$\sqcup_9(0,0,0)$.

Table 3: A table of cocycle invariants for twist spun knots, part I

With this 3-cocycle the cocycle invariant for the 2-twist spun knots for 3-colorable knots in the table are evaluated as listed in Table 3 (up to 8 crossings) and Table 4 (9 crossing knots). The notations used in the table for families of vectors are similar to those in Example 6.12.

Knot K	$\Phi_\kappa(\mathrm{Tw}^2(K))$
9_1	$\sqcup_9(0,0,0)$.
9_2	$\sqcup_3(0,0,0),\ \sqcup_2(0,-q_1,q_1),\ \sqcup_2(-q_1,q_1,0),\ \sqcup_2(q_1,0,-q_1)$.
9_4	$\sqcup_3(0,0,0),\ \sqcup_2(0,q_1,-q_1),\ \sqcup_2(q_1,-q_1,0),\ \sqcup_2(q_1,-q_1,0)$.
9_6	$\sqcup_3(0,0,0),\ \sqcup_2(2q_1,0,-2q_1),\ \sqcup_2(0,-2q_1,2q_1),\ \sqcup_2(-2q_1,2q_1,0)$.
9_{10}	$\sqcup_4(0,0,0),\ (-q_2-q_1,0,q_1+q_2),\ (q_2,-q_1,q_1-q_2),$
	$(-2q_1,q_1,q_1),\ (-q_1+q_2,2q_1-q_2,-q_1),\ (q_1-q_2,-2q_1+q_2,q_1)$.
9_{11}	$\sqcup_3(0,0,0),\ \sqcup_6(q_1,0,-q_1)$.
9_{15}	$\sqcup_3(0,0,0),\ \sqcup_2(0,-q_1,q_1),\ \sqcup_2(-q_1,q_1,0),\ \sqcup_2(q_1,0,-q_1)$.
9_{16}	$\sqcup_3(0,0,0),\ \sqcup_2(-q_1,0,q_1),\ \sqcup_2(q_1,-q_1,0),\ \sqcup_2(0,q_1,-q_1)$.
9_{17}	$\sqcup_3(0,0,0),\ \sqcup_6(-q_1,0,q_1)$.
9_{23}	$\sqcup_3(0,0,0),\ \sqcup_2(0,-q_1,q_1),\ \sqcup_2(-q_1,q_1,0),\ \sqcup_2(q_1,0,-q_1)$.
9_{24}	$\sqcup_9(0,0,0)$.
9_{28}	$\sqcup_3(0,0,0),\ \sqcup_3(-q_1,0,q_1),\ \sqcup_3(0,q_1,-q_1)$.
9_{29}	$\sqcup_3(0,0,0),\ (-q_2,-q_1,q_1+q_2),\ (q_2,0,-q_2),\ (0,q_1,-q_1),$
	$(-q_1,0,q_1),\ (q_2,q_1-q_2,-q_1),\ (q_1-q_2,-q_1+q_2,0)$.
9_{34}	$\sqcup_3(0,0,0),\ \sqcup_6(-q_1,0,q_1)$.
9_{35}	$\sqcup_3(0,0,0),\ \sqcup_8(-q_1,0,q_1),\ \sqcup_3(0,q_1,-q_1),\ (q_1,0,-q_1),\ (q_1,-q_1,0),\ (0,-q_2,q_2),$
	$(0,2q_2,-2q_2),\ (-2q_1,0,2q_1),\ (-q_2,-2q_1-q_2,2q_1+2q_2),\ (q_1,q_2,-q_1-q_2),$
	$(0,q_1+q_2,-q_1-q_2),\ (-q_1,-2q_2,q_1+2q_2),\ (-q_1,2q_1-q_2,-q_1+q_2),\ (q_1,-q_1+q_2,-q_2),$
	$(0,q_1-q_2,-q_1+q_2),\ (q_1-q_2,-q_1+2q_2,-q_2)$.
9_{37}	$\sqcup_9(0,0,0),\ \sqcup_5(0,-q_1,q_1),\ \sqcup_5(q_1,0,-q_1),\ \sqcup_2(-q_2,0,q_2),\ (-q_2,q_2,0),$
	$(-q_2,q_1,-q_1+q_2),\ (q_1+q_2,-q_1,-q_2),\ (q_2,q_1,-q_1-q_2),\ (q_2,q_1,-q_1-q_2),$
	$(q_1,-2q_1-2q_2,q_1+2q_2),\ (q_1,2q_2,-q_1-2q_2),\ (-q_1+q_2,q_1-q_2,0),$
	$(-q_2,-q_1+q_2,q_1),\ (-q_2,-q_1+q_2,q_1),\ (-q_1+q_2,q_1-q_2,0),\ (q_2,-q_1-q_2,q_1)$.
9_{38}	$\sqcup_3(0,0,0),\ (-q_2,-2q_1,2q_1+q_2),\ (-q_1+q_2,q_1,-q_2),$
	$(-q_1,2q_1,-q_1),\ (-q_1,-q_1,2q_1),\ (q_2,-q_2,0),\ (-q_2,q_2,0)$.
9_{40}	$\sqcup_3(0,0,0),\ \sqcup_3(-q_1,0,q_1),\ \sqcup_3(0,q_1,-q_1)$.
9_{46}	$\sqcup_{17}(0,0,0),\ \sqcup_3(q_1,-q_1,0),\ \sqcup_3(0,q_1,-q_1),\ (-q_2,q_2,0),\ (q_2,-q_2,0),$
	$(q_1+q_2,0,-q_1-q_2),\ (-q_1-q_2,0,q_1+q_2)$.
9_{47}	$\sqcup_4(0,0,0),\ \sqcup_{12}(q_1,0,-q_1),\ \sqcup_3(0,q_1,-q_1),\ \sqcup_3(q_1,-q_1,0),$
	$(q_2,-q_2,0),\ (-q_2,q_2,0),\ (-q_1-q_2,0,q_1+q_2),\ (q_1+q_2,0,-q_2-q_1)$.
9_{48}	$\sqcup_3(0,0,0),\ \sqcup_4(0,-q_1,q_1),\ \sqcup_3(q_1,-q_1,0),\ \sqcup_3(-q_1,q_1,0),\ \sqcup_2(0,q_1,-q_1),\ \sqcup_2(q_1,0,-q_1),$
	$(q_2,-q_2,0),\ (-q_2,q_2,0),(-q_1,2q_1,-q_1),\ (-q_1,-q_1,2q_1),\ (-q_2,-2q_1,2q_1+q_2),$
	$(q_1+q_2,-q_1,-q_2),\ (-2q_1-q_2,q_1,q_1+q_2),\ (-q_1+q_2,q_1,-q_2),$
	$(-q_2,-q_1+q_2,q_1),\ (-q_1+q_2,q_1-q_2,0)$.

Table 4: A table of cocycle invariants for twist spun knots, part II

Non-invertibility of knotted surfaces

The computations of the cocycle invariant imply non-invertibility of twist-spun knots. Here is a brief overview on non-invertibility of knotted surfaces:

- Fox [18] presented a non-invertible knotted sphere using knot modules as follows. The first homology $H_1(\tilde{X})$ of the infinite cyclic cover $\tilde{X}$ of the complement X of the sphere in S^4 of Fox's Example 10 is $\mathbb{Z}[t, t^{-1}]/(2-t)$ as a $\Lambda = \mathbb{Z}[t, t^{-1}]$-module. Fox's Example 11 can be recognized as the same sphere as Example 10 in [18] with its orientation reversed. Its Alexander polynomial is $(1-2t)$, and therefore, not equivalent to Example 10.

  Knot modules, however, fail to detect non-invertibility of the 2-twist spun trefoil (whose knot module is $\Lambda/(2-t, 1-2t)$).
- Farber [15] showed that the 2-twist spun trefoil was non-invertible using the Farber-Levine pairing (see also Hillman [22]).
- Ruberman [37] used Casson-Gordon invariants to prove the same result, with other new examples of non-invertible knotted spheres.
- Neither technique applies directly to the same knot with trivial 1-handles attached (in this case the knot is a surface with a higher genus). Kawauchi [27, 28] has generalized the Farber-Levine pairing to higher genus surfaces, showing that such a surface is also non-invertible.
- Gordon [19] showed that a large family of knotted spheres are indeed non-invertible. His extensive lists are: (1) the 2-twist spin of a rational knot K is invertible if and only if K is amphicheiral; (2) if m, p, q are > 1, then the m-twist spin of the (p, q) torus knot is non-invertible, (3) if $m \geq 3$ then the m-twist spin of a hyperbolic knot K is invertible if and only if K is $(+)$-amphicheiral. His topological argument uses the fact that the twist spun knots are fibered 2-knots. In particular, the corresponding results are unknown for surfaces of higher genuses.

Then the cocycle invariant provided a diagrammatic method of detecting non-invertibility of knotted surfaces. In [10], it was shown using the cocycle invariant that the 2-twist spun trefoil is non-invertible. Furthermore, as was mentioned in Section 1, all the surfaces that are obtained from the 2-twist spun trefoil by attaching trivial 1-handles, called its *stabilized* surfaces, are also non-invertible, since the stabilized surfaces have the same cocycle invariant as the original knotted sphere. The result was generalized by Satoh [2] to an infinite family of twist spins of torus knots.

The computations given in Example 7.6 can be carried out for the 2-twist spun knots with orientations reversed. Specifically, if the orientation of the surface is reversed, then the computations change in the following manner. First, the face colors are determined by

$$f(s, j, \mathcal{C}) = (\cdots (c(1, k, \mathcal{C}) * c(s, k, \mathcal{C})) * c(s, k-1, \mathcal{C})) * \cdots) * c(s, j, \mathcal{C})) \cdots).$$

The target region of $T_1^l(s)$ depicted in Fig. 20 is the top left region in the figure, so that the sequence of colors that an arc γ meets is

$$c(s, n, \mathcal{C})^{-1} c(s, n-1, \mathcal{C})^{-1} \cdots c(s, j, \mathcal{C})^{-1},$$

where γ does not cross the horizontal sheet in the figure but goes through k-th through j-th vertical sheets from left. Thus we obtain

$$B(\mathcal{C}, T_1^l(s)) = \begin{cases} c(s,n,\mathcal{C})^{-1} \cdots c(s,j,\mathcal{C})^{-1} \kappa_{f(s,j+2,\mathcal{C}),\ c(s+1,j+1,\mathcal{C}),\ c(s,j+1,\mathcal{C})} & \text{if} \quad \epsilon(T_1^l(s)) > 0 \\ -(c(s,n,\mathcal{C})^{-1} \cdots c(s,j,\mathcal{C})^{-1}) \kappa_{f(s,j+2,\mathcal{C}),\ c(s,j+1,\mathcal{C}),\ c(s,j,\mathcal{C})} & \text{if} \quad \epsilon(T_1^l(s)) < 0, \end{cases}$$

$$B(\mathcal{C}, T_1^r(s)) = \begin{cases} -(c(s,n,\mathcal{C})^{-1} \cdots c(s,j,\mathcal{C})^{-1}) \kappa_{c(s+1,j+1,\mathcal{C})\ c(s,j+1,\mathcal{C})\ c(1,k,\mathcal{C})} & \text{if} \quad \epsilon(T_1^r(s)) < 0 \\ c(s,n,\mathcal{C})^{-1} \cdots c(s,j,\mathcal{C})^{-1} \kappa_{c(s,j+1,\mathcal{C})\ c(s,j,\mathcal{C})\ c(1,k,\mathcal{C})} & \text{if} \quad \epsilon(T_1^r(s)) > 0. \end{cases}$$

The computational results are presented in Tables 5 and 6. By comparing the computational results, we conclude that such 2-twist spun knots $\tau^2(K)$ are non-invertible, for those knots that give rise to distinct values for the cocycle invariants, as well as all of their stabilized surfaces. This list (with stabilized surfaces) of non-invertible surfaces has not been obtained by any other method.

Furthermore, it is easily seen that if the contribution to the invariant for a coloring $\mathcal{C}$ is a vector $(a,b,c) \in \mathbb{Z}^3$ for the 2-twist spun $\tau^2(K)$ of a classical knot K, then the contribution for the corresponding coloring $\mathcal{C}$ is the vector $k(a,b,c)$ for the $2k$-twist spun $\tau^{2k}(K)$ of K. Hence non-invertibility determined by this invariant for 2-twist spuns can be applied to $2k$-twist spuns for all positive integer k as well.

We summarize the result in the following theorem.

Theorem 7.7 *For any positive integer k, the $2k$-twist spun of all the 3-colorable knots in the table up to 9 crossings excluding 8_{20}, as well as their stabilized surfaces of any genus, are non-invertible.*

The cocycle invariant in Example 7.6 fails to detect non-invertibility of 8_{20} and we obtain no conclusion.

The 2-twist spins of 24 of these 3-colorable knots can be distinguished from their inverses by the invariant of [10]. Their $6k$-twist spins, however, as well as the $2k$-twist spins of the following list and their stabilizations, can be distinguished from their inverses only by the current invariant: 6_1, 8_{10}, 8_{11}, 8_{18}, 9_1, 9_6, 9_{23}, 9_{24}, 9_{37}, 9_{46}.

Furthermore, the original invariant for R_3 with trivial action has non-trivial cohomology only with $\mathbb{Z}_3$, so that the invariant is trivial for $6k$-twist spuns, and in particular, unable to detect non-invertibility in this case. The current invariant with non-trivial action, having a free abelian coefficient group $\mathbb{Z}^3$, is non-trivial for any $2k$-twist spuns, if the invariant is non-trivial for the 2-twist spun.

Acknowledgement

We thank Pat Gilmer, Seiichi Kamada, Dan Silver, and Susan Williams for valuable comments and discussions.

Knot K	$\Phi_\kappa(\mathrm{Tw}^2(K))$
3_1	$\sqcup_3(0,0,0),\ \sqcup_2(q_1,0,-q_1),\ \sqcup_2(0,-q_1,q_1),\ \sqcup_2(-q_1,q_1,0).$
6_1	$\sqcup_3(0,0,0),\ (0,q_1,-q_1),\ (q_1,-q_1,0),\ (0,-q_2,q_2),$
	$(q_1,q_2,-q_1-q_2),\ (q_1,-q_1+q_2,-q_2),\ (0,q_1-q_2,-q_1+q_2).$
7_4	$\sqcup_3(0,0,0),\ (0,q_1+q_2,-q_1-q_2),\ (q_1,-q_2,-q_1+q_2),$
	$(2q_1,-q_1,-q_1),\ (q_1,-2q_1+q_2,q_1-q_2),\ (-q_1,2q_1-q_2,-q_1+q_2).$
7_7	$\sqcup_3(0,0,0),\ (q_1,q_1+q_2,-2q_1-q_2),\ (0,-2q_1-q_2,2q_1+q_2),\ (0,-q_1,q_1),$
	$(q_1,0,-q_1),\ (q_1,q_2,-q_1-q_2),\ (0,-q_1-q_2,q_1+q_2).$
8_5	$\sqcup_3(0,0,0),\ \sqcup_3(2q_1,-2q_1,0),\ \sqcup_3(0,2q_1,-2q_1).$
8_{10}	$\sqcup_3(0,0,0),\ \sqcup_2(-q_1,0,q_1),\ (0,q_1,-q_1),\ (q_1,-q_1,0),$
	$(-2q_1,2q_1,0),\ (0,-2q_1,2q_1).$
8_{11}	$\sqcup_3(0,0,0),\ \sqcup_2(0,-q_2,q_2),\ (q_1,-2q_1,q_1),\ (q_1,q_1,-2q_1),$
	$(-q_1,q_1-q_2,q_2),\ (2q_1,q_2,-2q_1-q_2).$
8_{15}	$\sqcup_3(0,0,0),\ \sqcup_2(q_1,0,-q_1),\ \sqcup_2(-q_1,q_1,0),\ \sqcup_2(0,-q_1,q_1).$
8_{18}	$\sqcup_9(0,0,0),\ \sqcup_3(0,-q_1-q_2,q_1+q_2),\ \sqcup_2(q_1,0,-q_1),\ \sqcup_2(0,-q_1,q_1),\ (-q_1,2q_1,-q_1),$
	$\sqcup_2(q_1,q_1+q_2,-2q_1-q_2),\ \sqcup_2(0,-2q_1-q_2,2q_1+q_2),\ (q_1,-q_1+q_2,-q_2),$
	$(q_1,q_2,-q_2-q_1),\ (2q_1,-q_1+q_2,-q_1-q_2),\ (q_1,q_1-q_2,-2q_1+q_2),\ (q_1,q_2,-q_1-q_2).$
8_{19}	$\sqcup_5(0,0,0),\ (-q_1,2q_1,-q_1),\ (2q_1,-q_1,-q_1),\ (q_1,-2q_1,q_1),\ (q_1,q_1,-2q_1).$
8_{20}	$\sqcup_9(0,0,0).$
8_{21}	$\sqcup_5(0,0,0)\ (q_1,q_1,-2q_1),\ (2q_1,-q_1,-q_1),\ (-q_1,2q_1,-q_1),\ (q_1,-2q_1,q_1).$

Table 5: A table of cocycle invariants with orientations reversed, part I

Knot K	$\Phi_\kappa(\mathrm{Tw}^2(K))$
9_1	$\sqcup_3(0,0,0),\ \sqcup_2(3q_1,0,-3q_1),\ \sqcup_2(-3q_1,3q_1,0),\ \sqcup_2(0,-3q_1,3q_1).$
9_2	$\sqcup_3(0,0,0),\ (-q_2,0,q_2),\ (q_1,0,-q_1),\ (2q_1,-2q_1,0),\ (q_2,-q_1-q_2,q_1),$
	$(-q_1-q_2,q_1+q_2,0),\ (q_1+q_2,2q_1,-3q_1-q_2).$
9_4	$\sqcup_3(0,0,0),\ (0,-2q_1,2q_1),\ (0,q_1,-q_1),$
	$(-2q_1,q_1-q_2,q_1+q_2),\ (q_1,q_2,-q_1-q_2),\ (-q_1,-q_2,q_1+q_2),\ (-q_1,q_2,q_1-q_2).$
9_6	$\sqcup_3(0,0,0),\ (0,-3q_1,3q_1),\ (3q_1,0,-3q_1),\ (q_1,2q_1+q_2,-3q_1-q_2),$
	$(q_1,-3q_1-q_2,2q_1+q_2),\ (2q_1,-q_1+q_2,-q_1-q_2),\ (-q_1,-q_1-q_2,2q_1+q_2).$
9_{10}	$\sqcup_3(0,0,0),\ (2q_1,q_2,-q_2-2q_1),\ (-q_1,-q_2+q_1,q_2),$
	$(q_1,q_1,-2q_1),(q_1,-2q_1,q_1),(0,q_2,-q_2),(0,-q_2,q_2).$
9_{11}	$\sqcup_3(0,0,0),\ \sqcup_3(q_1,-q_1,0),\ \sqcup_3(0,q_1,-q_1).$
9_{15}	$\sqcup_3(0,0,0),\ \sqcup_2(0,q_1,-q_1),\ \sqcup_2(2q_1,0,-2q_1),\ \sqcup_2(q_1,-q_1,0).$
9_{16}	$\sqcup_3(0,0,0),\ \sqcup_2(q_1,0,-q_1),\ \sqcup_2(-q_1,q_1,0),\ \sqcup_2(0,-q_1,q_1).$
9_{17}	$\sqcup_9(0,0,0).$
9_{23}	$\sqcup_5(0,0,0),\ (q_1,q_1,-2q_1),\ (2q_1,-q_1,-q_1),\ (q_1,-2q_1,q_1),\ (-q_1,2q_1,-q_1).$
9_{24}	$\sqcup_3(0,0,0),\ \sqcup_2(-q_1,0,q_1),\ (q_1,-q_1,0),\ (0,q_1,-q_1),\ (0,-2q_1,2q_1),\ (-2q_1,2q_1,0).$
9_{28}	$\sqcup_6(0,0,0),\ (-q_1,2q_1,-q_1),\ (2q_1,-q_1,-q_1),\ (-q_1,-q_1,2q_1).$
9_{29}	$\sqcup_3(0,0,0),\ (0,q_1,-q_1),\ (0,-q_2,q_2),\ (q_1,-q_1,0),$
	$(q_1,q_2,-q_2-q_1),\ (q_1,q_2-q_1,-q_2),\ (0,-q_2+q_1,q_2-q_1).$
9_{34}	$\sqcup_9(0,0,0).$
9_{35}	$\sqcup_4(0,0,0),\ \sqcup_3(0,-q_1,q_1),\ \sqcup_3(-q_1,q_1,0),\ \sqcup_3(0,2q_1,-2q_1),\ \sqcup_3(2q_1,-2q_1,0),$
	$(q_1,q_1,-2q_1),\ (q_1,-2q_1,q_1),\ (2q_1,-q_1,-q_1),\ (-q_2,-q_1,q_1+q_2),$
	$(q_1,2q_2,-q_1-2q_2),\ (q_1-q_2,-q_1+q_2,0),\ (q_1+q_2,-q_1-q_2,0),\ (q_1+q_2,q_1+q_2,-2q_1-2q_2),$
	$(q_1,-2q_1-2q_2,q_1+2q_2),\ (2q_1+q_2,-q_1,-q_1-q_2),\ (-q_1+q_2,2q_1-2q_2,-q_1+q_2).$
9_{37}	$\sqcup_3(0,0,0),\ \sqcup_2(0,q_1,-q_1),\ \sqcup_2(q_1,-q_1,0),\ \sqcup_2(0,-q_2,q_2),\ \sqcup_3(0,-q_2+q_1,-q_1+q_2),$
	$(0,-q_1,q_1),\ (-q_1,q_1,0),\ (0,q_2,-q_2),\ (0,2q_2,-2q_2),\ (0,-q_1+q_2,-q_2+q_1),$
	$(q_1,-q_2,-q_1+q_2),\ (q_1,q_2,-q_1-q_2),\ (2q_1+q_2,0,-2q_1-q_2),\ (q_1,q_2,-q_1-q_2),$
	$(-q_1-q_2,-q_1,2q_1+q_2),\ (-q_1,-2q_2,q_1+2q_2),\ (q_1,-q_1+q_2,-q_2),\ (-q_1-q_2,q_1+q_2,0),$
	$(q_1+q_2,-q_2,-q_1),\ (q_1,-q_1+q_2,-q_2).$
9_{38}	$\sqcup_4(0,0,0),\ (0,q_1+q_2,-q_1-q_2),\ (q_1,-q_2,-q_1+q_2),\ (2q_1,-q_1,-q_1),$
	$(q_1,-2q_1+q_2,q_1-q_2),\ (-q_1,2q_1-q_2,-q_1+q_2).$
9_{40}	$\sqcup_6(0,0,0),\ (-q_1,2q_1,-q_1),\ (2q_1,-q_1,-q_1),\ (-q_1,-q_1,2q_1).$
9_{46}	$\sqcup_{15}(0,0,0),\ \sqcup_2(q_1,-q_1,0),\ \sqcup_2(0,-q_1,q_1),\ \sqcup_2(-q_1,0,q_1),\ (-q_1,q_1,0),\ (0,q_1,-q_1),\ (0,q_2,-q_2),$
	$(-q_1,-q_2,q_1+q_2),\ (-q_1,q_1-q_2,q_2),\ (0,-q_1+q_2,q_1-q_2).$
9_{47}	$\sqcup_{11}(0,0,0),\ \sqcup_3(0,-q_2,q_2),\ \sqcup_2(0,q_1,-q_1),\ \sqcup_2(q_1,-q_1,0),\ \sqcup_2(0,q_1-q_2,-q_1+q_2),\ (0,q_2,-q_2),$
	$\sqcup_2(q_1,q_2,-q_1-q_2),\ \sqcup_2(q_1,-q_1+q_2,-q_2),\ (0,q_1+q_2,-q_1-q_2),\ (0,-q_1-q_2,q_1+q_2).$
9_{48}	$\sqcup_3(0,0,0),\ \sqcup_6(0,-q_1,q_1),\ \sqcup_4(q_1,0,-q_1),\ \sqcup_4(-q_1,q_1,0),$
	$(0,q_1,-q_1),\ (q_1,-q_1,0),\ (0,q_2,-q_2),\ (0,-q_2,q_2),$
	$(q_1,q_2,-q_1-q_2),\ (q_1,-q_1-q_2,q_2),\ (q_1,-q_1+q_2,-q_2),$
	$(0,2q_1-q_2,-2q_1+q_2),\ (2q_1,-2q_1+q_2,-q_2),\ (0,q_1-q_2,-q_1+q_2).$

Table 6: A table of cocycle invariants with orientation reversed, part II

References

[1] N. Andruskiewitsch and M. Graña, *From racks to pointed Hopf algebras*, Adv. in Math., to appear, available at: http://xxx.lanl.gov/math/abs/QA0202084.

[2] S. Asami and S. Satoh, *An infinite family of non-invertible surfaces in* 4*-space*, Preprint.

[3] R. Bowen and J. Franks, *Homology for zero-dimensional nonwandering sets*, Ann. Math. (2) **106** (1977) 73–92.

[4] E. Brieskorn, *Automorphic sets and singularities*, in "Contemporary math." **78** (1988) 45–115.

[5] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics **87**, Springer-Verlag, New York-Berlin (1982).

[6] J. S. Carter; M. Elhamdadi; M. A. Nikiforou; M. Saito, *Extensions of quandles and cocycle knot invariants*, J. of Knot Theory and Ramifications, to appear, available at: http://xxx.lanl.gov/math/abs/GT0107021.

[7] J. S. Carter; M. Elhamdadi; M. Saito, *Twisted Quandle homology theory and cocycle knot invariants*, Algebraic and Geometric Topology (2002) 95–135.

[8] J. S. Carter; A. Harris; M. A. Nikiforou; M. Saito, *Cocycle knot invariants, quandle extensions, and Alexander matrices*, Suurikaisekikenkyusho Koukyuroku, (Seminar note at RIMS, Kyoto) **1272** (2002) 12–35, available at http://xxx.lanl.gov/math/abs/GT0204113.

[9] J. S. Carter; D. Jelsovsky; S. Kamada; L. Langford; M. Saito, *State-sum invariants of knotted curves and surfaces from quandle cohomology*, Electron. Res. Announc. Amer. Math. Soc. **5** (1999) 146–156.

[10] J. S. Carter; D. Jelsovsky; S. Kamada; L. Langford; M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc., to appear, available at http://xxx.lanl.gov/abs/math.GT/9903135.

[11] J. S. Carter; D. Jelsovsky; S. Kamada; M. Saito, *Computations of quandle cocycle invariants of knotted curves and surfaces*, Advances in math **157** (2001) 36–94.

[12] J. S. Carter; M. Saito, *Knotted surfaces and their diagrams*, Surveys and monographs, Amer. Math. Soc. **55** (1998).

[13] R. Dijkgraaf; E. Witten, *Topological gauge theories and group cohomology*, Comm. Math. Phys. **129** (1990) 393–429.

[14] P. Etingof; M. Graña, *On rack cohomology*, J. of Pure and Applied Algebra **177** (2003) 49–59.

[15] M. S. Farber, *Linking coefficients and two-dimensional knots*, Soviet. Math. Doklady **16** (1975) 647–650.

[16] R. Fenn; C. Rourke, *Racks and links in codimension two*, J. Knot Theory Ramifications **1** (1992) 343-406.

[17] R. Fenn; C. Rourke; B. Sanderson, *James bundles and applications*, Preprint, available at: http://www.maths.warwick.ac.uk/~bjs/.

[18] R. H. Fox, *A quick trip through knot theory*, in "Topology of 3-manifolds and related topics (Georgia, 1961)," Prentice-Hall (1962) 120–167.

[19] C. McA. Gordon, *On the reversibility of twist-spun knots*, Preprint (2002).

[20] M. Graña, *Quandle knot invariants are quantum knot invariants*, J. Knot Theory Ramifications **11** (2002) 673–681.

[21] M. Graña, *Indecomposable racks of order p^2*, Preprint, available at: http://xxx.lanl.gov/abs/math.QA/0203157.

[22] J.A. Hillman, *Finite knot modules and the factorization of certain simple knots*, Math. Ann. **257** (1981) 261–274.

[23] V.F.R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. **126** (1989) 335-388.

[24] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Alg. **23** (1982) 37–65.

[25] L. H. Kauffman, *Knots and Physics*, Series on knots and everything **1**, World Scientific (1991).

[26] Kauffman, L. H. and Radford, D. E., *Oriented quantum algebras, categories and invariants of knots and links*, J. Knot Theory Ramifications **10** (2001) 1047–1084.

[27] A. Kawauchi, *Three dualities on the integral homology of infinite cyclic coverings of manifolds*, Osaka J. Math. **23** (1986) 633–651.

[28] A. Kawauchi, *The first Alexander modules of surfaces in 4-sphere*, in "Algebra and Topology" (Taejon, 1990), Proc. KAIST Math. Workshop **5**, KAIST, Taejon, Korea (1990) 81–89.

[29] J. Levine, *Polynomial invariants of knots of codimension two*, Ann. of Math. **84** (1966) 537–554.

[30] X.-S. Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Mathematica Sinica, English series **17** (2001) 361–380.

[31] R. A. Litherland; S. Nelson, *The Betti numbers of some finite racks*, J. Pure Appl. Algebra **178** (2003) 187–202.

[32] C. Livingston, *Lifting representations of knot groups*, J. Knot Theory Ramifications **4** (1995) 225–234.

[33] S. Matveev, *Distributive groupoids in knot theory (Russian)*, Math. USSR-Sbornik **47** (1982) 73–83.

[34] T. Mochizuki, *Some calculations of cohomology groups of finite Alexander quandles*, J. Pure Appl. Algebra **179** (2003) 287–330.

[35] T. Ohtsuki (ed), *Problems on invariants of knots and 3-manifolds*, Preprint.

[36] C. Rourke; B. Sanderson, *A new classification of links and some calculations using it*, Preprint, available at: http://xxx.lanl.gov/abs/math.GT/0006062.

[37] D. Ruberman, *Doubly slice knots and the Casson-Gordon invariants*, Trans. Amer. Math. Soc. **279** (1983) 569–588.

[38] S. Satoh, *Surface diagrams of twist-spun 2-knots*, in "Knots 2000 Korea, Vol. 1 (Yongpyong)" J. Knot Theory Ramifications **11** (2002) 413–430.

[39] S. Satoh; A. Shima, *The 2-twist-spun trefoil has the triple point number four*, Trans. Amer. Math. Soc., to appear.

[40] M. Wada, *Twisted Alexander polynomials for finitely presented groups*, Topology **33** (1994) 241–256.