CLOSED FORMULA FOR UNIVARIATE SUBRESULTANTS IN MULTIPLE ROOTS

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Abstract. We generalize Sylvester single sums to multisets (sets with repeated elements), and show that these sums compute subresultants of two univariate polyomials as a function of their roots independently of their multiplicity structure. This is the first closed formula for subresultants in terms of roots that works for arbitrary polynomials, previous efforts only handled special cases. Our extension involves in some cases confluent Schur polynomials, and is obtained by using a multivariate symmetric interpolation via an Exchange Lemma.

1. Introduction

Let $\mathbb{K}$ be a field. Given two finite sets $A, B \subset \mathbb{K}$ of cardinalities m and n respectively, and $0 \le p \le m$ and $0 \le q \le n$, J.J. Sylvester introduced in [Syl1853] the following *double sum*:

$$\mathrm{Syl}_{p,q}(A,B)(x) := \sum_{\substack{A'\subset A,\, B'\subset B\\ |A'|=p,\, |B'|=q}} \mathcal{R}(A',B')\,\mathcal{R}(A\backslash A', B\backslash B')\,\frac{\mathcal{R}(x,A')\,\mathcal{R}(x,B')}{\mathcal{R}(A',A\backslash A')\,\mathcal{R}(B',B\backslash B')},$$

where $\mathcal{R}(X,Y) := \prod_{\substack{x\in X\\ y\in Y}}(x-y)$, with the convention that $\mathcal{R}(X,Y)=1$ if $X=\emptyset$ or $Y=\emptyset$.

For $f := f_m x^m + \cdots + f_0$, $g := g_n x^n + \cdots + g_0 \in \mathbb{K}[x]$, and $0 \le d \le \min\{m,n\}$ when $m \ne n$ or $0 \le d < m = n$, Sylvester also introduced the *order d subresultant $Sres_d(f,g)(x) \in \mathbb{K}[x]$* :

$$\mathrm{Sres}_d(f,g)(x) := \det \begin{array}{c} \overbrace{\hphantom{f_m \cdots \cdots f_{d+1-(n-d-1)} \quad x^{n-d-1}f}}^{m+n-2d} \\ \left|\begin{array}{cccccc} f_m & \cdots & & \cdots & f_{d+1-(n-d-1)} & x^{n-d-1}f \\ & \ddots & & & \vdots & \vdots \\ & & f_m & \cdots & f_{d+1} & f \\ \hline g_n & \cdots & & \cdots & g_{d+1-(m-d-1)} & x^{m-d-1}g \\ & \ddots & & & \vdots & \vdots \\ & & g_n & \cdots & g_{d+1} & g \end{array}\right| \end{array} \begin{array}{l} \\ n-d \\ \\ m-d \end{array}.$$

When

$$f = \prod_{a\in A}(x-a) \quad \text{and} \quad g = \prod_{b\in B}(x-b),$$

2010 *Mathematics Subject Classification.* 13P15; 05E05.
Key words and phrases. Subresultants, Exchange Lemma, formulas in roots, Schur functions.

that is, all roots of f and g are simple roots, the following quite mysterious relation between double sums and subresultants was established by Sylvester [Syl1853]: set $d := p + q$. If $d \leq \min\{m, n\}$ when $m \neq n$, or $d < m = n$,

$$\mathrm{Syl}_{p,q}(A, B)(x) = (-1)^{p(m-d)} \binom{d}{p} \mathrm{Sres}_d(f, g)(x). \tag{1}$$

There are now different proofs of this fact in a more modern language, in the literature, though they are usually quite intricate. A. Lascoux and P. Pragacsz proved it in [LaPr2001] by using the theory of multi-Schur functions and divided differences. In [DHKS2007] the proof makes use of a slick manipulation of matrix multiplication and Vandermonde determinants, while M.-F. Roy and A. Szpirglas show in [RoSz2011] that the two objects satisfy the same recursion. The recent new proof in [KSV2016] is based on symmetric interpolation, which yields what is called an *Exchange Lemma* (see Lemma 2.2 below) that shows that in fact Sylvester double sum is just a rewriting of Sylvester *single sum* $\mathrm{Syl}_{d,0}(A, B)(x)$:

$$\mathrm{Syl}_{p,q}(A, B)(x) = (-1)^{q(m-d)} \binom{d}{p} \mathrm{Syl}_{d,0}(A, B)(x), \tag{2}$$

where we note that

$$\mathrm{Syl}_{d,0}(A, B)(x) = \sum_{\substack{A_1 \cup A_2 = A \\ |A_1| = d, |A_2| = m-d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(x, A_1)}{\mathcal{R}(A_1, A_2)}. \tag{3}$$

Relation (2) immediately implies Sylvester's statement (1) between double sums and subresultants, once it can be established that

$$\mathrm{Sres}_d(f, g)(x) = (-1)^{d(m-d)} \mathrm{Syl}_{d,0}(A, B)(x). \tag{4}$$

The latter is in fact easier to show than (1), and one can find some more different proofs of it in the literature, see for instance [Bor1860, ApJo2006, Cha1990, Hon1999, DTGV2004].

Note that (4) can be considered as a "Poisson formula" for the subresultant, as it describes it in terms of the values of g in the roots of f:

$$\mathrm{Syl}_{d,0}(A, B)(x) = (-1)^{m-d} \sum_{\substack{A_1 \cup A_2 = A \\ |A_1| = d, |A_2| = m-d}} \frac{\prod_{a \in A_2} g(a) \mathcal{R}(x, A_1)}{\mathcal{R}(A_1, A_2)},$$

but this equality only holds in the case where the roots of f are all simple.

Even though there is a long history in the study of the connection between subresultants and Sylvester sums in the simple root case, little is known about extensions when the roots of f and g have multiplicities. As noted above, the generalization is not straightforward, since some denominators in the Sylvester sums turn zero in case of root multiplicities. In [DKS2013], it is shown that the generalization of determinantal expressions for subresultants extend straightforwardly to the multiple root case, but only the special cases of $d = 1$ and $d = \min\{m, n\} - 1$ were treated to obtain a generalized Sylvester sum formula for multiple roots. In the recent [DKSV2016], we considered the extremal case when $f = (x - a)^m$ and $g = (x - b)^n$, and gave a generalization of Sylvester double sum formulae for the subresultants. The present paper is the first one to give expressions for subresultants of arbitrary polynomials

f and g and arbitrary values of $0 \le d \le \min\{m, n\}$ that are generalization of the classical Sylvester sums.

To do so, we first define a generalization $\mathrm{SylM}_{d,0}(A, B)(x)$ of the notion of Sylvester single sum $\mathrm{Syl}_{d,0}(A, B)(x)$, for multisets A and B and d big enough, which coincides with the usual notion of single sum when A and B are sets, and we show that our definition also satisfies -as desired- (4).

Given a multiset $X \subset \mathbb{K}$ (a set where repeating elements is allowed), we denote with $|X|$ its length (the number of elements counted with multiplicities).

Definition 1.1. *Let A, $B \subset \mathbb{K}$ be multisets with $|A| = m$, $|B| = n$ and let $\overline{A} \subset A$ and $\overline{B} \subset B$ be the sets of distinct elements in A and B respectively, with $|\overline{A}| = \overline{m}$, $|\overline{B}| = \overline{n}$. Set $m' := m - \overline{m}$ and $n' := n - \overline{n}$. For any d such that $m' + n' \le d \le \min\{m, n\}$ if $m \ne n$ or $m' + n' \le d < m = n$, we define*

$$\mathrm{SylM}_{d,0}(A, B)(x) :=$$
$$(-1)^{m'(m-d)} \sum_{\substack{A' \subset \overline{A} \\ |A'|=d-m'}} \sum_{\substack{B' \subset \overline{B} \\ |B'|=m'}} \frac{\mathcal{R}(A\backslash\overline{A}, \overline{B}\backslash B')\mathcal{R}(\overline{A}\backslash A', B\backslash B')\mathcal{R}(x, A')\mathcal{R}(x, B')}{\mathcal{R}(A', \overline{A}\backslash A')\mathcal{R}(B', \overline{B}\backslash B')}.$$

The previous definition only makes sense when A and B have few repeated elements and d is in the aforementioned range. For those values of d we have:

Theorem 1.2. *Let $f, g \in \mathbb{K}[x]$ be monic polynomials of degrees m and n, with multisets of roots A and B and sets of distinct roots $\overline{A}$ and $\overline{B}$ respectively, and set m', n' as in Definition 1.1. For any d such that $m' + n' \le d \le \min\{m, n\}$ if $m \ne n$, or $m' + n' \le d < m = n$, we have*

$$\mathrm{Sres}_d(f, g)(x) = (-1)^{d(m-d)}\mathrm{SylM}_{d,0}(A, B)(x).$$

One can wonder whether the lower bound stated for d in Theorem 1.2 is sharp since the definition of $\mathrm{SylM}_{d,0}(A, B)$ makes sense for $m' \le \min\{d, \overline{n}\}$. The next example illustrates that the result holds for d in the right range and shows that the constraint on d is necessary.

Example 1.3. *Take $f = (x - \alpha_1)(x - \alpha_2)^2$ and $g = (x - \beta_1)^2$, so $A = (\alpha_1, \alpha_2, \alpha_2)$ with $\overline{A} = \{\alpha_1, \alpha_2\}$ and $B = (\beta_1, \beta_1)$ with $\overline{B} = \{\beta_1\}$. For $d = 2$, since $(3 - 2) + (2 - 1) \le 2 \le \min\{3, 2\}$, one has $\mathrm{Sres}_2(f, g)(x) = g(x)$ while $\mathrm{SylM}_{2,0}(A, B)(x)$ equals*

$$\begin{aligned} -\Big(&\frac{(\alpha_2 - \beta_1)(x - \alpha_1)(x - \beta_1)}{\alpha_1 - \alpha_2} + \frac{(\alpha_1 - \beta_1)(x - \alpha_2)(x - \beta_1)}{\alpha_2 - \alpha_1}\Big) \\ &= \frac{\big((\alpha_2 - \beta_1)(x - \alpha_1) - (\alpha_1 - \beta_1)(x - \alpha_2)\big)(x - \beta_1)}{\alpha_2 - \alpha_1} \\ &= (x - \beta_1)(x - \beta_1) = g(x), \end{aligned}$$

so Theorem 1.2 holds in this case.

Now take $f = (x - \alpha_1)(x - \alpha_2)^2$ and $g = (x - \beta_1)^3$. For this case, $A = (\alpha_1, \alpha_2, \alpha_2)$ with $\overline{A} = \{\alpha_1, \alpha_2\}$ and $B = (\beta_1, \beta_1, \beta_1)$ with $\overline{B} = \{\beta_1\}$. For $d = 2 < (3-2)+(3-1)$ we have $\mathrm{Sres}_2(f, g)(x) = g(x)-f(x)$ and $\mathrm{SylM}_{2,0}(A, B)(x)$ can still be defined according to Definition 1.1 since $m' = 1 \le \min\{d, \overline{n}\}$, but it is a multiple of $x - \beta_1$, so the two expressions obviously do not coincide.

We need to extend the definition of $\mathrm{SylM}_{d,0}(A,B)(x)$ for any d. We succeed doing it at the cost of introducing confluent Schur polynomials $S_k^{(R)}(X)$, which are defined in (9) for a multiset X of length r, by removing a subset R of $r-k$ rows in the confluent Vandermonde matrix of X of size $k \times r$.

Definition 1.4. *Let A, $B \subset \mathbb{K}$ be multisets with $|A| = m$, $|B| = n$ and let $\overline{A} \subset A$ and $\overline{B} \subset B$ be the sets of distinct elements in A and B respectively, with $|\overline{A}| = \overline{m}$, $|\overline{B}| = \overline{n}$. Set $m' := m - \overline{m}$ and $n' := n - \overline{n}$. and m', n' as in Definition 1.1. For $0 \le d \le \min\{m,n\}$ if $m \neq n$ or $0 \le d < m = n$, we define*

$$\mathrm{SylM}_{d,0}(A,B)(x) :=$$
$$\sum (-1)^{\sigma_R} \frac{\mathcal{R}(A\backslash\overline{A}, \overline{B}\backslash B')\mathcal{R}(\overline{A}\backslash A', B\backslash B')\mathcal{R}(x,A')\mathcal{R}(x,B')}{\mathcal{R}(A', \overline{A}\backslash A')\mathcal{R}(B', \overline{B}\backslash B')} \cdot$$
$$\cdot\ S_{d+1}^{(\widetilde{R}_1)}(A' \cup B' \cup x) S_{m+n-d}^{(R_2)}((\overline{A}\backslash A') \cup B) S_{m+n-d}^{(R_3)}(A \cup (\overline{B}\backslash B')),$$

where the sum is indexed by

- *all partitions $R := R_1 \sqcup R_2 \sqcup R_3$ of the set $\{1, \dots, m'+n'-d\}$ with $R_1 \subset \{m+n-2d, \dots, m'+n'-d\}$, $|R_1| \le d - (\overline{m}+\overline{n}) + 1$, $m'-d \le |R_2| \le m-d$ and $n'-d \le |R_3| \le n-d$,*
- *all subsets $A' \subset \overline{A}$, $|A'| = |R_2| + d - m'$,*
- *all subsets $B' \subset \overline{B}$, $|B'| = |R_3| + \min\{m', d-n'\}$,*

σ_R is specified in (10), *and $\widetilde{R}_1 := \{i - (m+n-2d-1) : i \in R_1\}$.*

It is easy to verify that this notion generalizes Definition 1.1, as when $m'+n' \le d$, $m'+n'-d \le 0$ so the the sets R_1, R_2 and R_3 in the sum above are empty, and $|B'| = m'$. In this way, one recovers the previous multiple sum straightforwardly. The main result of our paper is the following generalization of Theorem 1.2, which shows that $\mathrm{SylM}_{d,0}(A,B)(x)$ computes the subresultant in all the cases.

Theorem 1.5. *Let $f, g \in \mathbb{K}[x]$ be monic polynomials of degrees m and n, with multisets of roots A and B, respectively, and $0 \le d \le \min\{m,n\}$ if $m \neq n$ or $0 \le d < m = n$. Then,*

$$\mathrm{Sres}_d(f,g)(x) = (-1)^{d(m-d)}\mathrm{SylM}_{d,0}(A,B)(x).$$

We consider again Example 1.3 to illustrate how now, under Definition 1.4, Theorem 1.5 indeed holds.

Example 1.6. *Take $f = (x-\alpha_1)(x-\alpha_2)^2$ and $g = (x-\beta_1)^3$ associated to the multisets $A = (\alpha_1, \alpha_2, \alpha_2)$ with $\overline{A} = \{\alpha_1, \alpha_2\}$, $B = (\beta_1, \beta_1, \beta_1)$ with $\overline{B} = \{\beta_1\}$ and $d = 2$. We have $\mathrm{Sres}_2(f,g)(x) = g(x) - f(x)$ while in this case $\mathrm{SylM}_{2,0}(A,B)(x)$ equals*

$$(\alpha_2-\beta_1)(x-\alpha_1)(x-\alpha_2) - \frac{(\alpha_1-\beta_1)(\alpha_1-\beta_1)(x-\alpha_2)(x-\beta_1)}{\alpha_2-\alpha_1} -$$
$$- \frac{(\alpha_2-\beta_1)(\alpha_2-\beta_1)(x-\alpha_1)(x-\beta_1)}{\alpha_1-\alpha_2}.$$

It is easy to check that the two expressions coincide.

The paper is organized as follows: in Section 2 we describe the main ingredient in our proofs, Proposition 2.1, which is a generalization of a result by F. Apéry and J.-P. Jouanolou. In Section 3, we apply this tool to justify the definition of $\mathrm{SylM}_{d,0}(A,B)(x)$ and show its connection with the subresultant. For the sake of clarity, we first present our results for the case d big enough and then in the following subsection, we recall the definition of confluent Schur polynomial and extend our definition and result to arbitrary values of d.

2. A generalization of a result by Apéry & Jouanolou

The main result of this section is the following generalization of a result by Apéry and Jouanolou that appears in [ApJo2006, Prop.91]. No multisets are involved in this paragraph.

Proposition 2.1. *Let $A, B \subset \mathbb{K}$ be finite sets with $|A| = m$, $|B| = n$. Set $0 \le d \le m$. Let X be a set of variables and $E \subset \mathbb{K}$ be any finite set satisfying*

$$|E| \ge \max\{|X| + d, m + n - d, m\}.$$

Then

$$\sum_{\substack{A_1 \sqcup A_2 = A \\ |A_1|=d,\, |A_2|=m-d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(X, A_1)}{\mathcal{R}(A_1, A_2)} =$$

$$= \sum_{\substack{E_1 \sqcup E_2 \sqcup E_3 = E \\ |E_1|=d, |E_2|=m-d, |E_3|=|E|-m}} \frac{\mathcal{R}(A, E_3)\mathcal{R}(E_2, B)\mathcal{R}(X, E_1)}{\mathcal{R}(E_1, E_2)\mathcal{R}(E_1, E_3)\mathcal{R}(E_2, E_3)}.$$

The original result in [ApJo2006, Prop.91] states that for $|E| = m + n - d$ and $d \le \min\{m, n\}$ for $m \ne n$ or $d < m = n$, one has

$$\mathrm{Sres}_d(f, g)(x) = \sum_{\substack{E_1 \sqcup E_2 \sqcup E_3 = E \\ |E_1|=d, |E_2|=m-d, |E_3|=n-d}} \frac{\mathcal{R}(E_3, A)\mathcal{R}(E_2, B)\mathcal{R}(x, E_1)}{\mathcal{R}(E_2, E_1)\mathcal{R}(E_3, E_1)\mathcal{R}(E_3, E_2)}.$$

This is a particular case of our result by (4) and the definition of the single sum (3).

The rest of this section is devoted to the proof of Proposition 2.1, which will follow from a suitable extension of the next Exchange Lemma that appears in [KSV2016, Lem.3.1 & Cor.3.2].

Lemma 2.2. *Set $d \ge 0$. Let A, $B \subset \mathbb{K}$ be finite sets with $|A|, |B| \ge d$, and X a set of variables with $|X| \le |A| - d$. Then*

$$\sum_{\substack{A' \subset A \\ |A'|=d}} \mathcal{R}(A \backslash A', B) \frac{\mathcal{R}(X, A')}{\mathcal{R}(A \backslash A', A')} = \sum_{\substack{B' \subseteq B \\ |B'|=d}} \mathcal{R}(A, B \backslash B') \frac{\mathcal{R}(X, B')}{\mathcal{R}(B', B \backslash B')}.$$

Lemma 2.2 turns out to be a consequence of the symmetric interpolation developed in [ChLo1996] (see also [KSV2016]) that we state here as we will need it for the proof of Lemma 2.4.

Proposition 2.3. *Let $E \subset \mathbb{K}$ be a finite set of size $|E| = e$. Set $0 \le d < e$, and let X be a set of variables with $|X| = e - d$. Then,*

$$\mathcal{B} := \left\{ \mathcal{R}(X, E') \,;\, E' \subseteq E, |E'| = d \right\}$$

is a basis of the $\mathbb{K}$-vector space $S_{(e-d,d)}$ of symmetric polynomials h in $X = \{x_1, \ldots, x_{e-d}\}$ over $\mathbb{K}$ such that $\deg_{x_i}(h) \le d$ for all $1 \le i \le e - d$. Moreover, any polynomial $h(X) \in S_{(e-d,d)}$ can be uniquely written as

$$h(X) = \sum_{E' \subseteq E, |E'| = d} h(E \backslash E') \frac{\mathcal{R}(X, E')}{\mathcal{R}(E \backslash E', E')}$$

where $h(E \backslash E') := h(e_1, \ldots, e_{e-d})$ for $E \backslash E' = \{e_1, \ldots, e_{e-d}\}$.

Our following extension of Lemma 2.2 relaxes slightly the condition on the size of X. Item (2) is presented for sake of completeness, we do not use it in the sequel.

Lemma 2.4. *Set $d \ge 0$. Let $A, B \subset \mathbb{K}$ be finite sets with $|A| \ge d$, and X be a set of variables with $|X| \le |A| + |B| - 2d$. Then*

(1) *If $|B| \ge d$, then*

$$\sum_{\substack{A_1 \sqcup A_2 = A \\ |A_1| = d, |A_2| = |A| - d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(X, A_1)}{\mathcal{R}(A_1, A_2)} =$$

$$= (-1)^{d(|A|-d)} \sum_{\substack{B_1 \sqcup B_2 = B \\ |B_1| = d, |B_2| = |B| - d}} \frac{\mathcal{R}(A, B_2)\mathcal{R}(X, B_1)}{\mathcal{R}(B_1, B_2)}.$$

(2) *If $|B| < d$, then*

$$\sum_{\substack{A_1 \sqcup A_2 = A \\ |A_1| = d, |A_2| = |A| - d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(X, A_1)}{\mathcal{R}(A_1, A_2)} = 0.$$

Proof. (1) When $|B| \ge d$, if $|X| \le |A| - d$ holds, we are in the conditions of Lemma 2.2 and the statement holds by simply correcting the sign.

Now assume $|B| \ge d$ and $r := |X| > |A| - d$. Write $X = Y \cup Z$, with $Y = \{x_1, \cdots, x_{|A|-d}\}$ and $Z = \{x_{|A|-d+1}, \cdots, x_r\}$. We define

$$h(Y, Z) = \sum_{\substack{A_1 \sqcup A_2 = A \\ |A_1| = d, |A_2| = |A| - d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(Y, A_1)\mathcal{R}(Z, A_1)}{\mathcal{R}(A_1, A_2)},$$

and

$$g(Y, Z) = \sum_{\substack{B_1 \sqcup B_2 = B \\ |B_1| = d, |B_2| = |B| - d}} \frac{\mathcal{R}(A, B_2)\mathcal{R}(Y, B_1)\mathcal{R}(Z, B_1)}{\mathcal{R}(B_1, B_2)},$$

and show that $h = (-1)^{d(|A|-d)} g$. For this purpose, we consider $g, h \in \mathbb{K}(Z)[Y]$, i.e. with coefficients in the field $\mathbb{K}(Z)$. Both polynomials are symmetric in Y and have multidegree in Y bounded by d. So $h, g \in S_{n-d,d}(\mathbb{K}(Z))$. Using Proposition 2.3, it is enough to verify that $h(A_2, Z) = (-1)^{d(|A|-d)} g(A_2, Z)$, for all $A_2 \subseteq A$ with $|A_2| = |A| - d$. Clearly, for given

A_2, $h(A_2,Z) = (-1)^{d(|A|-d)}\mathcal{R}(A_2,B)\mathcal{R}(Z,A_1)$ where $A_1 := A\backslash A_2$. Let us compute $g(A_2,Z)$:

$$\begin{aligned} g(A_2,Z) &= \sum_{\substack{B_1\sqcup B_2=B\\ |B_1|=d,\,|B_2|=|B|-d}} \frac{\mathcal{R}(A,B_2)\mathcal{R}(A_2,B_1)\mathcal{R}(Z,B_1)}{\mathcal{R}(B_1,B_2)} \\ &= \mathcal{R}(A_2,B) \sum_{\substack{B_1\sqcup B_2=B\\ |B_1|=d,\,|B_2|=|B|-d}} \frac{\mathcal{R}(A_1,B_2)\mathcal{R}(Z,B_1)}{\mathcal{R}(B_1,B_2)}. \end{aligned}$$

Thus it suffices to show that

$$\sum_{\substack{B_1\sqcup B_2=B\\ |B_1|=d,\,|B_2|=|B|-d}} \frac{\mathcal{R}(A_1,B_2)\mathcal{R}(Z,B_1)}{\mathcal{R}(B_1,B_2)} = \mathcal{R}(Z,A_1). \tag{5}$$

But this holds again by Lemma 2.2 for B instead of A, A_1 instead of B and Z instead of X, since $|Z| = |X| - (|A| - d) \le |B| - d$ by hypothesis (in this case the only subset of A_1 of size d is A_1 itself).

(2) When $|B| < d$, we enlarge B by adding variables Y so that $|B\cup Y| = d$, say $Y = \{y_1,\cdots,y_s\}$, with $s = d - |B|$. So we get, by applying the previous item, that

$$\begin{aligned} &\sum_{\substack{A_1\sqcup A_2=A\\ |A_1|=d,|A_2|=|A|-d}} \frac{\mathcal{R}(A_2,B)\mathcal{R}(X,A_1)}{\mathcal{R}(A_1,A_2)} = \\ &= (-1)^{(|A|-d)|Y|}\ \mathrm{coeff}_{y_1^{|A|-d}\cdots y_s^{|A|-d}}\Big(\sum_{\substack{A_1\sqcup A_2=A\\ |A_1|=d,|A_2|=|A|-d}} \frac{\mathcal{R}(A_2,B\cup Y)\mathcal{R}(X,A_1)}{\mathcal{R}(A_1,A_2)}\Big) \\ &= (-1)^{(|A|-d)|Y|}\ \mathrm{coeff}_{y_1^{|A|-d}\cdots y_s^{|A|-d}}\mathcal{R}(X,B\cup Y) \ = \ 0, \end{aligned}$$

since in this case the hypothesis $|X| \le |A| + |B| - 2d$ together with $|B| < d$ implies that $|X| < |A| - d$, and therefore there is no coefficient in y_i of degree $|A| - d$. □

Note that for $X = \{x\}$, setting $m = |A|$ and $n = |B|$, Lemma 2.4 boils down to

$$\mathrm{Syl}_{d,0}(A,B)(x) = (-1)^{d(|A|-d)}\mathrm{Syl}_{0,d}(A,B)(x),$$

for $d \le \min\{m,n\}$ when $m \ne n$ or $d < m = n$, which is (4) for the two single sums, and $\mathrm{Syl}_{d,0}(A,B)(x) = 0$ for $n < d \le m$.

Proof of Proposition 2.1. Set $e := |E|$, $m := |A|$ and $n := |B|$. The right-hand side of the equality we want to show can be rewritten as

$$\sum_{\substack{E_1\sqcup E'=E\\|E_1|=d,\,|E'|=e-d}}\ \sum_{\substack{E_2\sqcup E_3=E'\\|E_2|=m-d,\,|E_3|=e-m}} \frac{\mathcal{R}(A,E_3)\mathcal{R}(E_2,B)\mathcal{R}(X,E_1)}{\mathcal{R}(E_1,E')\mathcal{R}(E_2,E_3)}$$

$$= \sum_{\substack{E_1\sqcup E'=E\\|E_1|=d,\,|E'|=e-d}} \frac{\mathcal{R}(X,E_1)}{\mathcal{R}(E_1,E')} \sum_{\substack{E_2\sqcup E_3=E'\\|E_2|=m-d,\,|E_3|=e-m}} \frac{\mathcal{R}(A,E_3)\mathcal{R}(E_2,B)}{\mathcal{R}(E_2,E_3)}$$

$$= (-1)^{m(e-m)+n(m-d)} \sum_{\substack{E_1\sqcup E'=E\\|E_1|=d,\,|E'|=e-d}} \frac{\mathcal{R}(X,E_1)}{\mathcal{R}(E_1,E')} \sum_{\substack{E_2\sqcup E_3=E'\\|E_2|=m-d,\,|E_3|=e-m}} \frac{\mathcal{R}(E_3,A)\mathcal{R}(B,E_2)}{\mathcal{R}(E_2,E_3)}$$

(6)

$$= (-1)^{d(e-m)+n(m-d)} \sum_{\substack{E_1\sqcup E'=E\\|E_1|=d,\,|E'|=e-d}} \frac{\mathcal{R}(X,E_1)}{\mathcal{R}(E_1,E')} \sum_{\substack{A_2\sqcup A_1=A\\|A_2|=m-d,\,|A_1|=d}} \frac{\mathcal{R}(E',A')\mathcal{R}(B,A_2)}{\mathcal{R}(A_2,A_1)}$$

$$= (-1)^{d(e-m)} \sum_{\substack{A_2\sqcup A_1=A\\|A_2|=m-d,\,|A_1|=d}} \frac{\mathcal{R}(A_2,B)}{\mathcal{R}(A_2,A_1)} \sum_{\substack{E_1\sqcup E'=E\\|E_1|=d,\,|E'|=e-d}} \frac{\mathcal{R}(E',A_1)\mathcal{R}(X,E_1)}{\mathcal{R}(E_1,E')}$$

(7)

$$= (-1)^{d(m-d)} \sum_{\substack{A_2\sqcup A_1=A\\|A_2|=m-d,\,|A_1|=d}} \frac{\mathcal{R}(A_2,B)}{\mathcal{R}(A_2,A_1)}\mathcal{R}(X,A_1)$$

$$= \sum_{\substack{A_1\sqcup A_2=A\\|A_1|=d,\,|A_2|=m-d}} \frac{\mathcal{R}(A_2,B)\mathcal{R}(X,A_1)}{\mathcal{R}(A_1,A_2)},$$

where (6) is Lemma 2.4(1) applied to E' instead of A, A instead of B and B instead of X since $|B| \le |E'|+|A|-2(m-d)$, i.e. $n \le e-m+d$ by hypothesis, and (7) is the same corollary applied to E instead of A, A_1 instead of B and X since $|X| \le |E| + |A_1| - 2d$, i.e. $|X| \le e - d$ by hypothesis (note that in this case the only subset of A_1 of size d is A_1 itself and therefore the second sum in Lemma 2.4 simply equals $\mathcal{R}(X, A_1)$). $\square$

3. Application to subresultants

This section is devoted to motivate Definitions 1.1 and 1.4 and prove Theorems 1.2 and 1.5 of the introduction. This is done by proving Theorems 3.1 and 3.3 below, where A and B are assumed to be sets instead of multisets, and $\overline{A}$, $\overline{B}$ are arbitrary subsets of A, B respectively. Proposition 2.1, which can be interpreted as a multivariate version of $\mathrm{Syl}_{d,0}(A,B)(x)$ by means of an arbitrary auxiliary set E (where only the size of E matters), allows us to specialize E on sets in such a way that the denominators only depend on these $\overline{A}$ and $\overline{B}$. Then, in the proofs of Theorems 1.2 and 1.5, we let the elements of A or B collide, and our formulas remain well defined as long as we assume that the elements of $\overline{A}$ and $\overline{B}$ are all distinct.

We start with the easier case of multisets with few repeated elements and d big enough to be in the range of Definition 1.1.

3.1. **The case of d sufficiently large.**

Theorem 3.1. *Let $A, B \subset \mathbb{K}$ be sets with $|A| = m$ and $|B| = n$. Let $\overline{A} \subseteq A$ and $\overline{B} \subseteq B$ be any non-empty subsets of A and B respectively, with $|\overline{A}| = \overline{m}$ and $|\overline{B}| = \overline{n}$ and set $m' := m - \overline{m}$ and $n' := n - \overline{n}$. Assume d satisfies $m' + n' \leq d \leq \min\{m, n\}$ and let X be a set of variables with $|X| \leq m + n - 2d$. Then*

$$\sum_{\substack{A_1 \sqcup A_2 = A \\ |A_1|=d, |A_2|=m-d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(X, A_1)}{\mathcal{R}(A_1, A_2)} = (-1)^{m'(m-d)} \cdot$$

$$\cdot \sum_{\substack{A' \subset \overline{A} \\ |A'|=d-m'}} \sum_{\substack{B' \subset \overline{B} \\ |B'|=m'}} \frac{\mathcal{R}(A \backslash \overline{A}, \overline{B} \backslash B')\mathcal{R}(\overline{A} \backslash A', B \backslash B')\mathcal{R}(X, A')\mathcal{R}(X, B')}{\mathcal{R}(A', \overline{A} \backslash A')\mathcal{R}(B', \overline{B} \backslash B')}.$$

Proof. We first assume that $A \cap B = \emptyset$. By Corollary 2.1 applied to $E := \overline{A} \cup \overline{B}$, with $|E| = \overline{m} + \overline{n} \geq m + n - d$ by assumption, we have

$$\sum_{\substack{A_1 \sqcup A_2 = A \\ |A_1|=d, \\ |A_2|=m-d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(X, A_1)}{\mathcal{R}(A_1, A_2)} = \sum_{\substack{E_1 \sqcup E_2 \sqcup E_3 = \overline{A} \cup \overline{B} \\ |E_1|=d, |E_2|=m-d \\ |E_3|=\overline{m}+\overline{n}-m}} \frac{\mathcal{R}(A, E_3)\mathcal{R}(E_2, B)\mathcal{R}(X, E_1)}{\mathcal{R}(E_1, E_2)\mathcal{R}(E_1, E_3)\mathcal{R}(E_2, E_3)}.$$

Now, $\mathcal{R}(A, E_3) = \emptyset$ when $A \cap E_3 \neq \emptyset$ and $\mathcal{R}(E_2, B) = \emptyset$ when $E_2 \cap B \neq \emptyset$. Therefore $E_3 \subset \overline{B}$ and $E_2 \subset \overline{A}$. Setting $A' = \overline{A} \backslash E_2$ and $B' = \overline{B} \backslash E_3$, we get that $E_3 = \overline{B} \backslash B'$, $E_2 = \overline{A} \backslash A'$ and $E_1 = A' \cup B'$, and therefore we can rewrite the sum as

$$\sum_{\substack{A' \subset \overline{A} \\ |A'|=d-m'}} \sum_{\substack{B' \subset \overline{B} \\ |B'|=m'}} \frac{\mathcal{R}(A, \overline{B} \backslash B')\mathcal{R}(\overline{A} \backslash A', B)\mathcal{R}(X, A')\mathcal{R}(X, B')}{\mathcal{R}(A' \cup B', \overline{A} \backslash A')\mathcal{R}(A' \cup B', \overline{B} \backslash B')\mathcal{R}(\overline{A} \backslash A', \overline{B} \backslash B')}$$

$$= \sum_{\substack{A' \subset \overline{A} \\ |A'|=d-m'}} \sum_{\substack{B' \subset \overline{B} \\ |B'|=m'}} \frac{\mathcal{R}(A, \overline{B} \backslash B')\mathcal{R}(\overline{A} \backslash A', B)\mathcal{R}(X, A')\mathcal{R}(X, B')}{\mathcal{R}(A', \overline{A} \backslash A')\mathcal{R}(B', \overline{A} \backslash A')\mathcal{R}(\overline{A}, \overline{B} \backslash B')\mathcal{R}(B', \overline{B} \backslash B')}$$

$$= (-1)^{|B'| \cdot |\overline{A} \backslash A'|} \sum_{\substack{A' \subset \overline{A} \\ |A'|=d-m'}} \sum_{\substack{B' \subset \overline{B} \\ |B'|=m'}} \frac{\mathcal{R}(A \backslash \overline{A}, \overline{B} \backslash B')\mathcal{R}(\overline{A} \backslash A', B \backslash B')\mathcal{R}(X, A')\mathcal{R}(X, B')}{\mathcal{R}(A', \overline{A} \backslash A')\mathcal{R}(B', \overline{B} \backslash B')}$$

as desired, since $|B'| \cdot |\overline{A} \backslash A'| = m'(m - d)$.

The general statement follows from the fact that the two expressions generically coincide. □

We note that if in Theorem 3.1 we take $\overline{A} = A$, the right-hand side of the statement equals its left-hand side and we obtain nothing new. However the right-hand side of the statement makes sense even when A, B are multisets instead of sets, for one only needs $\overline{A}, \overline{B}$ to be sets. For $X = \{x\}$ we can then define the notion of single Sylvester sum for multisets A and B and d within the bounds of Theorem 3.1, which extends the usual notion of Sylvester single sums for sets, as stated in Definition 1.1.

Proof of Theorem 1.2. Assume $A = (\underbrace{a_1,\dots,a_1}_{j_1},\dots,\underbrace{a_{\overline{m}},\dots,a_{\overline{m}}}_{j_{\overline{m}}})$ with $m = j_1+\cdots+j_{\overline{m}}$ and $B = (\underbrace{b_1,\dots,b_1}_{\ell_1},\dots,\underbrace{b_{\overline{n}},\dots,b_{\overline{n}}}_{\ell_{\overline{n}}})$ with $n = \ell_1+\cdots+\ell_{\overline{n}}$, so that $f = \prod_{a\in A}(x-a)$ and $g = \prod_{b\in B}(x-b)$. Define sets of indeterminates $Y = \{y_{1,1},\dots,y_{1,j_1},\dots,y_{\overline{m},1},\dots,y_{\overline{m},j_{\overline{m}}}\}$ and $Z = \{z_{1,1},\dots,z_{1,\ell_1},\dots,z_{\overline{n},1},\dots,z_{\overline{n},\ell_{\overline{n}}}\}$, and set $f^y := (x-y_{1,1})\cdots(x-y_{\overline{m},j_{\overline{m}}})$ and $g^z := (x-z_{1,1})\cdots(x-z_{\overline{n},s\ell_{\overline{n}}})$. Then, if we set $\overline{Y} = \{y_{1,1},\dots,y_{\overline{m},1}\}$ and $\overline{Z} = \{z_{1,1},\dots,z_{\overline{n},1}\}$ and $m'+n' \le d \le \min\{m,n\}$ if $m \neq n$ or $m'+n' \le d < m = n$, according to Theorem 3.1 we have

$$\mathrm{Syl}_{d,0}(Y,Z)(x) = (-1)^{m'(m-d)} \sum_{\substack{Y'\subset\overline{Y}\\ |Y'|=d-m'}} \sum_{\substack{Z'\subset\overline{Z}\\ |Z'|=m'}} \frac{\mathcal{R}(Y\backslash\overline{Y},\overline{Z}\backslash Z')\mathcal{R}(\overline{Y}\backslash Y',Z\backslash Z')\mathcal{R}(x,Y')\mathcal{R}(x,Z')}{\mathcal{R}(Y',\overline{Y}\backslash Y')\mathcal{R}(Z',\overline{Z}\backslash Z')}.$$

On the other hand, by (4), $\mathrm{Sres}_d(f^y,g^z)(x) = (-1)^{d(m-d)}\mathrm{Syl}_{d,0}(Y,Z)(x)$. Therefore, for d within the stated bounds,

$$\mathrm{Sres}_d(f^y,g^z)(x) = (-1)^{(d-m')(m-d)} \sum_{\substack{Y'\subset\overline{Y}\\ |Y'|=d-m'}} \sum_{\substack{Z'\subset\overline{Z}\\ |Z'|=m'}} \frac{\mathcal{R}(Y\backslash\overline{Y},\overline{Z}\backslash Z')\mathcal{R}(\overline{Y}\backslash Y',Z\backslash Z')\mathcal{R}(x,Y')\mathcal{R}(x,Z')}{\mathcal{R}(Y',\overline{Y}\backslash Y')\mathcal{R}(Z',\overline{Z}\backslash Z')}.$$

We end the proof by making $y_{1,i}\to a_1,\dots,y_{\overline{m},i}\to a_{\overline{m}}$, $z_{1,i}\to b_1,\dots,z_{\overline{n},i}\to b_{\overline{n}}$ and note that both sides of the equality are well-defined. □

3.2. **The general case.**

In order to deal with the case when $0 \le d < m'+n'$ we need to recall the definition of Schur polynomials. Given a partition

$$\lambda = (\lambda_1,\lambda_2,\dots,\lambda_r),\ \lambda_i\in\mathbb{Z}_{\ge 0} \text{ for } 1\le i\le r, \text{ with } \lambda_1\ge\lambda_2\ge\cdots\ge\lambda_r,$$

the Schur polynomial $s_\lambda(X)$ for a set $X = \{x_1,\dots,x_r\}$ is defined as the ratio

$$s_\lambda(X) = \frac{\det\begin{pmatrix} x_1^{\lambda_1+r-1} & x_2^{\lambda_1+r-1} & \cdots & x_r^{\lambda_1+r-1}\\ x_1^{\lambda_2+r-2} & x_2^{\lambda_2+r-2} & \cdots & x_r^{\lambda_2+r-2}\\ \vdots & \vdots & \ddots & \vdots\\ x_1^{\lambda_r} & x_2^{\lambda_r} & \cdots & x_r^{\lambda_r}\end{pmatrix}}{\det\begin{pmatrix} x_1^{r-1} & \dots & x_r^{r-1}\\ \vdots & & \vdots\\ 1 & \dots & 1\end{pmatrix}}.$$

That is, Schur polynomials are ratios of determinants of Vandermonde matrices, where in the numerator some rows of a regular Vandermonde matrix are skipped, while in the denominator a regular Vandermonde matrix occurs. Note that Schur polynomials are symmetric in $x_1,\dots,x_r$, and thus it makes sense to write $s_\lambda(X)$ for a set X. For convenience here, we will not follow this usual notation for Schur polynomials given by partitions but introduce a notation with a set of exponents as follows: let

$$V_k(X) = \begin{pmatrix} x_1^{k-1} & \dots & x_r^{k-1} \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}$$

be a regular rectangular Vandermonde matrix of size $k \times r$ with $k \geq r$. When $k = r$ we write $V(X)$ for simplicity. Let $R = \{i_1, \dots, i_{k-r}\} \subset \{1, \dots, k\}$ be a subset of the row indexes, then we will denote by $V_k^{(R)}(X)$ the square submatrix of $V_k(X)$ obtained by removing from it the rows in R, that is, the rows corresponding to the indexes in R, and we define

$$S_k^{(R)}(X) := \frac{\det(V_k^{(R)}(X))}{\det(V(X))}, \tag{8}$$

that is $S_k^{(R)}(X)$ is the Schur polynomial associated to the set of indexes which are in $\{1, \dots, k\} \setminus R$.

In a more general setting, if $X = (\underbrace{x_1, \dots, x_1}_{j_1}, \dots, \underbrace{x_{\overline{r}}, \dots, x_{\overline{r}}}_{j_{\overline{r}}})$ is a multiset with $r = j_1 + \dots + j_{\overline{r}}$, one defines a generalized or confluent Vandermonde matrix instead of the regular Vandermonde matrix of size $k \times r$ as (c.f. [Kal1984])

$$V_k(X) = \begin{pmatrix} V_k(x_1, j_1) & \dots & V_k(x_{\overline{r}}, j_{\overline{r}}) \end{pmatrix}$$

where for any j, $V_k(x_i, j)$ of size $k \times j$ is defined by

$$V_k(x_i, j) := \begin{pmatrix} x_i^{k-1} & (k-1)x_i^{k-2} & (k-1)(k-2)x_i^{k-3} & \dots & \frac{(k-1)!}{(k-j)!}x_i^{k-j} \\ \vdots & \vdots & \vdots & & \vdots \\ x_i^2 & 2x_i & 2 & \dots & 0 \\ x_i & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where when $k = r$ one writes again $V(X)$ for simplicity. It is known that $V(X)$ is invertible when $x_i \neq x_j$ for $i \neq j$.

Then one can define confluent Schur polynomials in the same way as before: let $R = \{i_1, \dots, i_{k-r}\} \subset \{1, \dots, k\}$ be a subset of the row indexes, then we will denote by $V_k^{(R)}(X)$ the square submatrix of $V_k(X)$ obtained by removing from it the rows indexed by R, and define

$$S_k^{(R)}(X) := \frac{\det(V_k^{(R)}(X))}{\det(V(X))}. \tag{9}$$

Note that in principle $S_k^{(R)}(X)$ is a rational function. The next result shows that it is actually a polynomial.

Lemma 3.2. *$S_k^{(R)}(X)$ is a polynomial in the X-variables with coefficients in $\mathbb{K}$.*

Proof. When X is a set instead of being a multiset, the Schur function defined in (9) coincides with the Schur polynomial defined in (8), so the claim obviously holds in this situation. To prove the statement in the general case, consider a set $X = \{x_{1,1}, \dots, x_{1,j_1}, \dots, x_{\overline{r},1}, \dots, x_{\overline{r},i_{\overline{r}}}\}$ which will "converge" to a multiset Y by setting $x_{1,i} \to y_1$ for $1 \le i \le j_1$, $\dots$, $x_{\overline{r},i} \to y_{\overline{r}}$ for $1 \le i \le j_{\overline{r}}$. Then

$$S_k^{(R)}(X) \to S_k^{(R)}(Y)$$

as it can be seen for instance by computing the limits for $x_{1,2} \to x_{1,1}$ by L'Hopital rule, for $x_{i,3} \to x_{i,1}$ if necessary, and repeating the same for the other terms $x_{k,2} \to x_{k,1}$, etc. This shows that $S_k^{(R)}(Y)$ is actually a polynomial. $\square$

For a given (increasingly ordered) set $R \subset \{1, \dots, r\}$, we set $\mathrm{sg}_r(R) := (-1)^\sigma$, where σ is a number of transpositions needed to move this set to the first positions in $\{1, \dots, r\}$, i.e. if $R = \{i_1, \dots i_s\}$ with $1 \le i_1 < \cdots < i_s \le r$, then σ is the parity of the number of transpositions needed to bring $(1, \dots, r)$ to $(i_1, \dots i_s, \dots)$, without changing the relative order of the other elements.

Also, for a given partition $R := R_1 \sqcup R_2 \sqcup \dots \sqcup R_\ell$ of $\{1, \dots, r\}$, we denote $\mathrm{sg}(R) = \mathrm{sg}(R_1, \dots, R_\ell) := (-1)^\sigma$, where σ is the parity of the number of transpositions needed to bring the ordered set $(R_1, \dots, R_\ell)$ (we assume that each of them is also increasingly ordered) to $\{1, \dots, r\}$.

Theorem 3.3. *Let $A, B \subset \mathbb{K}$ be sets with $|A| = m$ and $|B| = n$. Let $\overline{A} \subseteq A$ and $\overline{B} \subseteq B$ be any non-empty subsets of A and B respectively, with $|\overline{A}| = \overline{m}$ and $|\overline{B}| = \overline{n}$ and set $m' := m - \overline{m}$ and $n' := n - \overline{n}$. Assume that $0 \le d \le \min\{m, n\}$ if $m \ne n$ or $0 \le d < m = n$ satisfies in addition $d < m' + n'$. Then:*

(1) If $0 \le d < \overline{m} + \overline{n}$ then

$$\sum_{\substack{A_1 \sqcup A_2 = A \\ |A_1| = d, \\ |A_2| = m-d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(x, A_1)}{\mathcal{R}(A_1, A_2)} = \sum_{\substack{R_2 \sqcup R_3 = \{1,\dots,m'+n'-d\} \\ |R_2| = r_2,\, m'-d \le r_2 \le m-d \\ |R_3| = r_3,\, n'-d \le r_3 \le n-d}} (-1)^{\sigma_R} \cdot$$

$$\sum_{\substack{A' \subset \overline{A} \\ |A'| = r_2 - (m'-d)}} \sum_{\substack{B' \subset \overline{B} \\ |B'| = r_3 - (n'-d)}} \frac{\mathcal{R}(A \backslash \overline{A}, \overline{B} \backslash B')\mathcal{R}(\overline{A} \backslash A', B \backslash B')\mathcal{R}(x, A')\mathcal{R}(x, B')}{\mathcal{R}(A', \overline{A} \backslash A')\mathcal{R}(B', \overline{B} \backslash B')} \cdot$$

$$\cdot S_{m+n-d}^{(R_2)}((\overline{A} \backslash A') \cup B) S_{m+n-d}^{(R_3)}(A \cup (\overline{B} \backslash B')),$$

where for the partition $R := R_2 \sqcup R_3$ of $\{1, \dots, m' + n' - d\}$,

$$(-1)^{\sigma_R} = (-1)^{m'(m-d) + r_2(\overline{m}-1) + r_3(m'+n'-d-1) + r_2 r_3} \mathrm{sg}(R).$$

(2) If $\overline{m} + \overline{n} \le d < m' + n'$,

$$\sum_{\substack{A_1 \sqcup A_2 = A \\ |A_1| = d, \\ |A_2| = m-d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(x, A_1)}{\mathcal{R}(A_1, A_2)} = \sum_{\substack{R_1 \sqcup R_2 \sqcup R_3 = \{1,\dots,m'+n'-d\} \\ R_1 \subset \{m+n-2d, \dots, m'+n'-d\}, \\ |R_1| = r_1,\, r_1 \le d - (\overline{m}+\overline{n}) + 1 \\ |R_2| = r_2,\, m'-d \le r_2 \le m-d \\ |R_3| = r_3,\, n'-d \le r_3 \le n-d}} (-1)^{\sigma_R}$$

$$\sum_{\substack{A' \subset \overline{A} \\ |A'| = r_2 - (m'-d)}} \sum_{\substack{B' \subset \overline{B} \\ |B'| = r_3 - (n'-d)}} \frac{\mathcal{R}(A \backslash \overline{A}, \overline{B} \backslash B')\mathcal{R}(\overline{A} \backslash A', B \backslash B')\mathcal{R}(x, A')\mathcal{R}(x, B')}{\mathcal{R}(A', \overline{A} \backslash A')\mathcal{R}(B', \overline{B} \backslash B')} \cdot$$

$$\cdot S_{d+1}^{(\widetilde{R}_1)}(A' \cup B' \cup x) S_{m+n-d}^{(R_2)}((\overline{A} \backslash A') \cup B) S_{m+n-d}^{(R_3)}(A \cup (\overline{B} \backslash B')),$$

where for the partition $R = R_1 \sqcup R_2 \sqcup R_3$ *of* $\{1, \ldots, m' + n' - d\}$,

(10)
$$(-1)^{\sigma_R} = (-1)^{m'(m-d)+r_1(n-d+r_2+r_3)+r_2(\overline{m}-1)+r_3(m'+n'-d-1)+r_2r_3}\mathrm{sg}(R),$$

and $\widetilde{R}_1 := \{i - (m + n - 2d - 1) : i \in R_1\}$.

Proof. As in the proof of Theorem 3.1, we can assume that $A \cap B = \emptyset$. The idea of the proof is to add an auxiliary set of variables $T = \{t_1, \cdots, t_r\}$ with $r = m' + n' - d$ so that $E := \overline{A} \cup \overline{B} \cup T$ has size $|E| = m + n - d$, which allows us to apply Proposition 2.1 to E and $X = \{x\}$, and then to compare coefficients in the obtained expression.
Applying Proposition 2.1 we get

$$\sum_{\substack{A_1 \sqcup A_2 = A\\ |A_1|=d,\\ |A_2|=m-d}} \frac{\mathcal{R}(A_2, B)\mathcal{R}(x, A_1)}{\mathcal{R}(A_1, A_2)} = \sum_{\substack{E_1 \sqcup E_2 \sqcup E_3 = \overline{A} \cup \overline{B} \cup T\\ |E_1|=d, |E_2|=m-d\\ |E_3|=n-d}} \frac{\mathcal{R}(A, E_3)\mathcal{R}(E_2, B)\mathcal{R}(x, E_1)}{\mathcal{R}(E_1, E_2)\mathcal{R}(E_1, E_3)\mathcal{R}(E_2, E_3)}.$$

As in the proof of Theorem 3.1, $\mathcal{R}(A, E_3) = \emptyset$ when $E_3 \cap A \neq \emptyset$ and $\mathcal{R}(E_2, B) = \emptyset$ when $E_2 \cap B \neq \emptyset$. Therefore $E_3 \subset \overline{B} \cup T$ and $E_2 \subset \overline{A} \cup T$. Let us write $E_2 = (\overline{A} \backslash A') \cup T_2$ with $A' \subset \overline{A}$ and $T_2 \subset T$, $E_3 = (\overline{B} \backslash B') \cup T_3$ with $B' \subset \overline{B}$ and $T_3 \subset T$ with $T_2 \cap T_3 = \emptyset$. Then $E_1 = (A' \cup B') \cup T_1$ where $T_1 = T \backslash (T_2 \cup T_3)$, and we can rewrite the sum as we did in Theorem 3.1:

$$\sum_{\substack{T_1 \sqcup T_2 \sqcup T_3 = T\\ |T_1|=r_1, 0 \le r_1 \le d\\ |T_2|=r_2, 0 \le r_2 \le m-d\\ |T_3|=r_3, 0 \le r_3 \le n-d}} \sum_{\substack{A' \subset \overline{A}\\ |A'|=r_2+d-m'\\ 0 \le |A'| \le \overline{m}}} \sum_{\substack{B' \subset \overline{B}\\ |B'|=r_3+d-n'\\ 0 \le |B'| \le \overline{n}}}$$
$$\frac{\mathcal{R}(A, (\overline{B} \backslash B') \cup T_3)\mathcal{R}((\overline{A} \backslash A') \cup T_2, B)\mathcal{R}(x, (A' \cup B') \cup T_1)}{\mathcal{R}((A' \cup B') \cup T_1, (\overline{A} \backslash A') \cup T_2)\mathcal{R}((A' \cup B') \cup T_1, (\overline{B} \backslash B') \cup T_3)\mathcal{R}((\overline{A} \backslash A') \cup T_2, (\overline{B} \backslash B') \cup T_3)}$$
$$= \sum_{\substack{T_1 \sqcup T_2 \sqcup T_3 = T\\ |T_1|=r_1, 0 \le r_1 \le d\\ |T_2|=r_2, \max\{0, m'-d\} \le r_2 \le m-d\\ |T_3|=r_3, \max\{0, n'-d\} \le r_3 \le n-d}} \sum_{\substack{A' \subset \overline{A}\\ |A'|=r_2-(m'-d)}} \sum_{\substack{B' \subset \overline{B}\\ |B'|=r_3-(n'-d)}}$$
$$\frac{\mathcal{R}(A, (\overline{B} \backslash B') \cup T_3)\mathcal{R}((\overline{A} \backslash A') \cup T_2, B)\mathcal{R}(x, (A' \cup B') \cup T_1)}{\mathcal{R}((A' \cup B') \cup T_1, (\overline{A} \backslash A') \cup T_2)\mathcal{R}((A' \cup B') \cup T_1, (\overline{B} \backslash B') \cup T_3)\mathcal{R}((\overline{A} \backslash A') \cup T_2, (\overline{B} \backslash B') \cup T_3)}$$

Here for each choice of T_1, T_2, T_3 and A', B', the numerator equals

$$\mathcal{R}(A, \overline{B} \backslash B')\mathcal{R}(A, T_3)\mathcal{R}(\overline{A} \backslash A', B)\mathcal{R}(T_2, B)\mathcal{R}(x, A')\mathcal{R}(x, B')\mathcal{R}(x, T_1)$$

while the denominator can be rewritten as

$$\begin{aligned}&\mathcal{R}(A' \cup B', \overline{A} \backslash A')\mathcal{R}(A' \cup B', T_2)\mathcal{R}(T_1, \overline{A} \backslash A')\mathcal{R}(T_1, T_2)\cdot\\ &\quad\cdot \mathcal{R}(A' \cup B', \overline{B} \backslash B')\mathcal{R}(A' \cup B', T_3)\mathcal{R}(T_1, \overline{B} \backslash B')\mathcal{R}(T_1, T_3)\cdot\\ &\quad\cdot \mathcal{R}(\overline{A} \backslash A', \overline{B} \backslash B')\mathcal{R}(\overline{A} \backslash A', T_3)\mathcal{R}(T_2, \overline{B} \backslash B')\mathcal{R}(T_2, T_3)\end{aligned}$$

Therefore, the part of the quotient which is free of T_ℓ's equals, as in Theorem 3.1,

$$= (-1)^{\sigma_1} \sum_{\substack{A'\subset\overline{A}\\ |A'|=r_2-(m'-d)}} \sum_{\substack{B'\subset\overline{B}\\ |B'|=r_3-(n'-d)}} \frac{\mathcal{R}(A\backslash\overline{A},\overline{B}\backslash B')\mathcal{R}(\overline{A}\backslash A',B\backslash B')\mathcal{R}(x,A')\mathcal{R}(x,B')}{\mathcal{R}(A',\overline{A}\backslash A')\mathcal{R}(B',\overline{B}\backslash B')}$$

where $\sigma_1 := |B'|\,|\overline{A}\backslash A'|$.
We deal now with the part of the quotient that involves some T_ℓ. Multiplying and dividing by $\mathcal{R}(T_1,A'\cup B')\mathcal{R}(T_2,\overline{A}\backslash A')\mathcal{R}(T_3,\overline{B}\backslash B')$, we get that this quotient equals

$$(-1)^{\sigma_2}\frac{\mathcal{R}(T_3,A\cup(\overline{B}\backslash B'))\mathcal{R}(T_2,(\overline{A}\backslash A')\cup B)\mathcal{R}(T_1,A'\cup B'\cup x)}{\mathcal{R}(T,\overline{A}\cup\overline{B})\mathcal{R}(T_1,T_2)\mathcal{R}(T_1,T_3)\mathcal{R}(T_2,T_3)},$$

where $\sigma_2 := |T_3|\,|\overline{A}\backslash A'| + (|T_2|+|T_3|)|A'\cup B'| + |T_3|\,|A| + |T_1|$.
Next we multiply and divide by the product of Vandermonde determinants $\det(V(T_1))\det(V(T_2))\det(V(T_3))$, where we consider in each T_ℓ the elements t_i with the indices i in increasing order, and get

$$\begin{aligned}
&\frac{\mathcal{R}(T_3,A\cup(\overline{B}\backslash B'))\mathcal{R}(T_2,(\overline{A}\backslash A')\cup B)\mathcal{R}(T_1,A'\cup B'\cup x)\det(V(T_1))\det(V(T_2))\det(V(T_3))}{\mathcal{R}(T,\overline{A}\cup\overline{B})\mathcal{R}(T_1,T_2)\mathcal{R}(T_1,T_3)\mathcal{R}(T_2,T_3)\det(V(T_1))\det(V(T_2))\det(V(T_3))} =\\
&\quad = \mathrm{sg}(T_1,T_2,T_3)\cdot\\
&\quad\cdot\frac{\mathcal{R}(T_3,A\cup(\overline{B}\backslash B'))\mathcal{R}(T_2,(\overline{A}\backslash A')\cup B)\mathcal{R}(T_1,A'\cup B'\cup x)\det(V(T_1))\det(V(T_2))\det(V(T_3))}{\mathcal{R}(T,\overline{A}\cup\overline{B})\det(V(T))},
\end{aligned}$$

where $\mathrm{sg}(T_1,T_2,T_3) := (-1)^\sigma$ where σ is the parity of the number of transpositions needed to bring the ordered set $T_1\sqcup T_2\sqcup T_3$ to $\{t_1,\dots,t_r\}$.

Since the denominator is independent of the choices of T_ℓ, going back to the first expression, we have

$$\begin{aligned}
&\mathcal{R}(T,\overline{A}\cup\overline{B})\det(V(T)) \sum_{\substack{A_1\sqcup A_2=A\\ |A_1|=d,\,|A_2|=m-d}} \frac{\mathcal{R}(A_2,B)\mathcal{R}(x,A_1)}{\mathcal{R}(A_1,A_2)} =\\
&\qquad = \sum_{\substack{T_1\sqcup T_2\sqcup T_3=T\\ |T_1|=r_1,0\le r_1\le d\\ |T_2|=r_2,\max\{0,m'-d\}\le r_2\le m-d\\ |T_3|=r_3,\max\{0,n'-d\}\le r_3\le n-d}} (-1)^{\sigma'}\mathrm{sg}(T_1,T_2,T_3)\cdot\\
&\qquad\cdot \sum_{\substack{A'\subset\overline{A}\\ |A'|=r_2-(m'-d))}} \sum_{\substack{B'\subset\overline{B}\\ |B'|=r_3-(n'-d)}} \frac{\mathcal{R}(A\backslash\overline{A},\overline{B}\backslash B')\mathcal{R}(\overline{A}\backslash A',B\backslash B')\mathcal{R}(x,A')\mathcal{R}(x,B')}{\mathcal{R}(A',\overline{A}\backslash A')\mathcal{R}(B',\overline{B}\backslash B')}\cdot\\
&\qquad\cdot \mathcal{R}(T_3,A\cup(\overline{B}\backslash B'))\mathcal{R}(T_2,(\overline{A}\backslash A')\cup B)\mathcal{R}(T_1,A'\cup B'\cup x)\cdot\\
&\qquad\cdot \det(V(T_1))\det(V(T_2))\det(V(T_3)),
\end{aligned}$$

where

$$\begin{aligned}
\sigma' :&= \sigma_1+\sigma_2\\
&= (|B'|+|T_3|)|\overline{A}\backslash A'| + (|T_2|+|T_3|)|A'\cup B'| + |T_3|\,|A| + |T_1|\\
&\equiv (n'-d)(m-d) + r_1 + r_2(m'-d+1) + r_3(n'-\overline{m}+1) \pmod 2\\
&\equiv m'(m-d) + r_1(m-d-1) + r_2(\overline{m}-1) + r_3(m'+n'-d-1) \pmod 2.
\end{aligned}$$

(The last row is written in a way that it coincides with the exponent in Theorem 3.1, when $r<0$ is interpreted as $r_1=r_2=r_3=0$.)

To recover the sum we are looking for, we take a specific coefficient in $(t_1,\dots,t_r)$ in both sides. Note that the leading coefficient of $\mathcal{R}(T,\overline{A}\cup\overline{B})\det(V(T))$ w.r.t. the lexicographic term order $t_1>\cdots>t_r$ equals

$$\mathrm{coeff}_{t_1^{m+n-d-1}t_2^{m+n-d-2}\cdots t_r^{m+n-d-r}}\left(\mathcal{R}(T,\overline{A}\cup\overline{B})\det(V(T))\right)=1.$$

We look now at this coefficient on the right hand side of the whole expression: we do it considering the variables t_i that belong to each T_ℓ. We look first at variables in T_2, and then in T_3, since they behave similarly. Observe that

$$\mathcal{R}(T_2,(\overline{A}\backslash A')\cup B)\det(V(T_2))=\frac{\det(V(T_2\cup(\overline{A}\backslash A')\cup B))}{\det(V((\overline{A}\backslash A')\cup B))},$$

and

$$\mathcal{R}(T_3,A\cup(\overline{B}\backslash B'))\det(V(T_3))=\frac{\det(V(T_3\cup A\cup(\overline{B}\backslash B')))}{\det(V(A\cup(\overline{B}\backslash B')))},$$

where the matrices in the numerator of the right-hand sides are both of size $(m+n-d)\times(m+n-d)$. The coefficient of the monomial $\prod_{t_i\in T_2}t_i^{m+n-d-i}$ corresponds in the numerator to the submatrix of $V_{m+n-d}((\overline{A}\backslash A')\cup B)$ where the rows indexed by $R_2:=\{i:\,t_i\in T_2\}$ have been erased. Then

$$\begin{aligned}\mathrm{coeff}_{\prod_{t_i\in T_2}t_i^{m+n-d-i}}\left(\frac{\det(V(T_2\cup(\overline{A}\backslash A')\cup B))}{\det(V((\overline{A}\backslash A')\cup B))}\right)&=\mathrm{sg}_{m+n-d}(R_2)S^{(R_2)}_{m+n-d}((\overline{A}\backslash A')\cup B)\\&=\mathrm{sg}_{m'+n'-d}(R_2)S^{(R_2)}_{m+n-d}((\overline{A}\backslash A')\cup B)\end{aligned}$$

since $R_2\subset\{1,\dots,m'+n'-d\}$. Analogously

$$\mathrm{coeff}_{\prod_{t_i\in T_3}t_i^{m+n-d-i}}\left(\frac{\det(V(T_3\cup A\cup(\overline{B}\backslash B')))}{\det(V(A\cup(\overline{B}\backslash B')))}\right)=\mathrm{sg}_{m'+n'-d}(R_3)S^{(R_3)}_{m+n-d}(A\cup(\overline{B}\backslash B')),$$

where $R_3:=\{i:\,t_i\in T_3\}$.

Now we deal with variables in T_1. Note that

$$\mathcal{R}(T_1,A'\cup B'\cup x)\det(V(T_1))=\frac{\det(V(T_1\cup A'\cup B'\cup x))}{\det(V(A'\cup B'\cup x))}.$$

Here the matrix in the numerator is a Vandermonde matrix of size $(d+1)\times(d+1)$ and the maximal exponent of t_i for $t_i\in T_1$ that can appear equals t_i^d. Set $R_1:=\{i:\,t_i\in T_1\}$. Therefore, for all $i\in R_1$ one needs $m+n-d-i\subset\{0,1,\dots,d\}$, i.e. $m+n-2d\le i\le m+n-d$. Since i satisfies $i\le r=m'+n'-d$ one needs $m+n-2d\le m'+n'-d$ and $R_1\subset\{m+n-2d,\dots,m'+n'-d\}$.
In particular, when $m+n-2d>m'+n'-d$, i.e. when $d<\overline{m}+\overline{n}$ there is no choice of R_1. In that case, we conclude

$$\sum_{\substack{A_1\sqcup A_2=A\\|A_1|=d,\\|A_2|=m-d}}\frac{\mathcal{R}(A_2,B)\mathcal{R}(x,A_1)}{\mathcal{R}(A_1,A_2)} = \sum_{\substack{R_2\sqcup R_3=\{1,\dots,m'+n'-d\}\\|R_2|=r_2,\max\{0,m'-d\}\le r_2\le m-d\\|R_3|=r_3,\max\{0,n'-d\}\le r_3\le n-d}}(-1)^{\sigma}\,\mathrm{sg}(R_2,R_3)$$

$$\sum_{\substack{A'\subset\overline{A}\\|A'|=r_2-(m'-d)}}\ \sum_{\substack{B'\subset\overline{B}\\|B'|=r_3-(n'-d)}}\frac{\mathcal{R}(A\backslash\overline{A},\overline{B}\backslash B')\mathcal{R}(\overline{A}\backslash A',B\backslash B')\mathcal{R}(x,A')\mathcal{R}(x,B')}{\mathcal{R}(A',\overline{A}\backslash A')\mathcal{R}(B',\overline{B}\backslash B')}\cdot$$

$$\cdot\, S^{(R_2)}_{m+n-d}((\overline{A}\backslash A')\cup B)S^{(R_3)}_{m+n-d}(A\cup(\overline{B}\backslash B')),$$

where

$$\sigma = m'(m-d)+r_2(\overline{m}-1)+r_3(m'+n'-d-1)+r_2r_3,$$

since it is easy to check that $\mathrm{sg}_{m'+n'-d}(R_2)\mathrm{sg}_{m'+n'-d}(R_3)=(-1)^{r_2r_3}$ as R_2 and R_3 are complementary sets in $\{1,\dots,m'+n'-d\}$ (or see Lemma 3.4 below).

Now, when $d\ge\overline{m}+\overline{n}$ and $R_1=\{i:\ t_i\in T_1\}\subset\{m+n-2d,\dots,m'+n'-d\}$ we have

$$\mathrm{coeff}_{\prod_{t_i\in T_1}t_i^{m+n-d-i}}\left(\frac{\det(V(T_1\cup A'\cup B'\cup x))}{\det(V(A'\cup B'\cup x))}\right)=\mathrm{sg}_{d+1}(\widetilde{R}_1)S^{(\widetilde{R}_1)}_{d+1}(A'\cup B'\cup x),$$

where $\widetilde{R}_1:=\{i-(m+n-2d-1):\ i\in R_1\}\subset\{1,\dots,d+1-(\overline{m}+\overline{n})\}$. We prove in Lemma 3.4 below that

$$\mathrm{sg}_{d+1}(\widetilde{R}_1)\mathrm{sg}_{m'+n'-d}(R_2)\mathrm{sg}_{m'+n'-d}(R_3)=(-1)^{r_1(r_2+r_3+m+n-1)+r_2r_3}.$$

Therefore we get

$$\sum_{\substack{A_1\sqcup A_2=A\\|A_1|=d,\\|A_2|=m-d}}\frac{\mathcal{R}(A_2,B)\mathcal{R}(x,A_1)}{\mathcal{R}(A_1,A_2)} = \sum_{\substack{R_1\sqcup R_2\sqcup R_3=\{1,\dots,m'+n'-d\}\\R_1\subset\{m+n-2d,\dots,m'+n'-d\},\\|R_1|=r_1,0\le r_1\le d-(\overline{m}+\overline{n})+1\\|R_2|=r_2,\max\{0,m'-d\}\le r_2\le m-d\\|R_3|=r_3,\max\{0,n'-d\}\le r_3\le n-d}}(-1)^{\sigma}\,\mathrm{sg}(R_1,R_2,R_3)$$

$$\sum_{\substack{A'\subset\overline{A}\\|A'|=r_2-(m'-d)}}\ \sum_{\substack{B'\subset\overline{B}\\|B'|=r_3-(n'-d)}}\frac{\mathcal{R}(A\backslash\overline{A},\overline{B}\backslash B')\mathcal{R}(\overline{A}\backslash A',B\backslash B')\mathcal{R}(x,A')\mathcal{R}(x,B')}{\mathcal{R}(A',\overline{A}\backslash A')\mathcal{R}(B',\overline{B}\backslash B')}\cdot$$

$$\cdot\ S^{(\widetilde{R}_1)}_{d+1}(A'\cup B'\cup x)\,S^{(R_2)}_{m+n-d}((\overline{A}\backslash A')\cup B)\,S^{(R_3)}_{m+n-d}(A\cup(\overline{B}\backslash B')),$$

where

$$\sigma=m'(m-d)+r_1(n-d+r_2+r_3)+r_2(\overline{m}-1)+r_3(m'+n'-d-1)+r_2r_3.$$

□

Lemma 3.4. *Let $R_1\sqcup R_2\sqcup R_3$ be a partition of $\{1,\dots,r\}$ with $|R_i|=r_i$ for $1\le i\le 3$, and $0\le s\le r$ be such that $\widetilde{R}_1=\{i-s:\ i\in R_1\}\subset\{1,\dots,r-s\}$. Then*

$$\mathrm{sg}_{r-s}(\widetilde{R}_1)\,\mathrm{sg}_r(R_2)\,\mathrm{sg}_r(R_3)=(-1)^{r_1(r_2+r_3+s)+r_2r_3}.$$

Proof. We set $R_1 = \{i_1, \ldots, i_{r_1}\}$, $R_2 = \{j_1, \ldots, j_{r_2}\}$ and $R_3 = \{k_1, \ldots, k_{r_3}\}$. Then

$$\begin{aligned}
\mathrm{sg}_{r-s}(\widetilde{R}_1)\,\mathrm{sg}_r(R_2)\,\mathrm{sg}_r(R_3) &= \sum_{1\le \ell\le r_1}(i_\ell - s - \ell) + \sum_{1\le \ell\le r_2}(j_\ell - \ell) + \sum_{1\le \ell\le r_3}(k_\ell - \ell)\\
&= \frac{r(r+1)}{2} - r_1 s - \frac{r_1(r_1+1)}{2} - \frac{r_2(r_2+1)}{2} - \frac{r_3(r_3+1)}{2}\\
&= \frac{r^2 - r_1^2 - r_2^2 - r_3^2}{2} - r_1 s\\
&= \frac{r^2 - (r_1+r_2+r_3)^2 + 2r_1r_2 + 2r_1r_3 + 2r_2r_3}{2} - r_1 s\\
&\equiv r_1r_2 + r_1r_3 + r_2r_3 + r_1 s \pmod 2.
\end{aligned}$$

□

We are ready now to conclude the proof of Theorem 1.5.

Proof of Theorem 1.5. First we note that the definition of $\mathrm{SylM}_{d,0}(A,B)(x)$ in Definition 1.4 is not only a generalization of Definition 1.1 as mentioned in the introduction but also generalizes the term in the right-hand side of Theorem 3.3(1) for sets, since when $d < \overline{m} + \overline{n}$, $R_1 \subset \{m+n-2d, \ldots, m' + n' - d\} = \emptyset$. Therefore, thanks to Theorems 3.1 and 3.3, one has that the following equality holds for sets A and B, any subsets $\overline{A} \subset A$ and $\overline{B} \subset B$ and any $0 \le d \le \min\{m,n\}$ if $m \ne n$ or $0 \le d < m = n$:

$$\mathrm{Sres}_d(f,g)(x) = (-1)^{d(m-d)}\mathrm{SylM}_{d,0}(A,B)(x).$$

The transition from sets to multisets is then straightforward taking limits of sets to multisets, as in the proof of Theorem 1.2, thanks to Lemma 3.2 and its proof, since both quantities are well defined for multisets. □

Acknowledgements: The authors would like to thank the Department of Mathematics at the Universidad de Buenos Aires for their hospitality, where the work on this project started during a visit of Carlos D'Andrea and Agnes Szanto in 2015. We would also like to thank the Department de Matemàtiques i Informàtica at the Universitat de Barcelona for hosting Marcelo Valdettaro to work on this project in July 2016.

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