Sharp estimates for the arithmetic Nullstellensatz

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Abstract. We present sharp estimates for the degree and the height of the polynomials in the Nullstellensatz over the integer ring $\mathbb{Z}$. The result improves previous work of Philippon, Berenstein-Yger and Krick-Pardo.

We also present degree and height estimates of intrinsic type, which depend mainly on the degree and the height of the input polynomial system. As an application, we derive an effective arithmetic Nullstellensatz for sparse polynomial systems.

The proof of these results relies heavily on the notion of local height of an affine variety defined over a number field. We introduce this notion and study its basic properties.

Keywords. Height of varieties, Chow forms, arithmetic Nullstellensatz, intrinsic parameters, sparse elimination theory.

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Contents

Introduction

Hilbert Nullstellensatz is a cornerstone of algebraic geometry. Under a simplified form, its statement is the following:

Let $f_1, \ldots, f_s \in \mathbb{Z}[x_1, \ldots, x_n]$ be polynomials such that the equation system

$$f_1(x) = 0, \ldots, f_s(x) = 0 \tag{1}$$

has no solution in $\mathbb{C}^n$. Then there exist $a \in \mathbb{Z} \setminus \{0\}$ and $g_1, \ldots, g_s \in \mathbb{Z}[x_1, \ldots, x_n]$ satisfying the Bézout identity

$$a = g_1 f_1 + \cdots + g_s f_s. \tag{2}$$

As for many central results in commutative algebra and algebraic geometry, it is an existential non-effective statement. The estimation of both the degree and the height of polynomials satisfying identity (2) became an important and widely considered question. Effective versions of Hilbert Nullstellensatz apply to a wide range of situations in number theory and theoretical computer science. In particular, they decide the consistency of a given polynomial system. In their arithmetic presentation, they apply to Lojasiewicz inequalities [51], [26] and to the consistency problem over finite fields [28], [22].

Let $h(f)$ denote the *height* of an arbitrary polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$, defined as the logarithm of the maximum modulus of its coefficients. The main result of this paper is the following effective arithmetic Nullstellensatz:

Theorem 1 Let $f_1, \ldots, f_s \in \mathbb{Z}[x_1, \ldots, x_n]$ be polynomials without common zeros in $\mathbb{C}^n$. Set $d := \max_i \deg f_i$ and $h := \max_i h(f_i)$.
Then there exist $a \in \mathbb{Z} \setminus \{0\}$ and $g_1, \ldots, g_s \in \mathbb{Z}[x_1, \ldots, x_n]$ such that

- $a = g_1 f_1 + \cdots + g_s f_s$,

- $\deg g_i \le 4\, n\, d^n$,

- $h(a), h(g_i) \le 4\, n\, (n+1)\, d^n\, (h + \log s + (n+7) \log(n+1)\, d)$.

As we will see below, this result substantially improves all previously known estimates for the arithmetic Nullstellensatz.

The following variant of a well-known example due to Masser and Philippon [6] yields a lower bound for any general degree and height estimate. Set

$$f_1 := x_1^d, \ f_2 := x_1 \, x_n^{d-1} - x_2^d, \ \ldots, \ f_{n-1} := x_{n-2} \, x_n^{d-1} - x_{n-1}^d, \ f_n := x_{n-1} \, x_n^{d-1} - H$$

for any positive integers n, d and H. These are polynomials of degree d and height bounded by $h := \log H$ without common zeros in $\mathbb{C}^n$. Let $a \in \mathbb{Z} \setminus \{0\}$ and $g_1, \ldots, g_n \in \mathbb{Z}[x_1, \ldots, x_n]$ such that

$$a = g_1 f_1 + \cdots + g_n f_n.$$

Specializing this identity at $x_1 := H^{d^{n-2}} t^{d^{n-1}-1}, \ldots, x_{n-1} := H \, t^{d-1}, x_n := 1/t$ we obtain

$$a = g_1(H^{d^{n-2}} t^{d^{n-1}-1}, \ldots, H \, t^{d-1}, 1/t) \, H^{d^{n-1}} t^{d^n - d}.$$

We conclude that $\deg g_1 \geq d^n - d$ and $h(a) \geq d^{n-1} h$. In fact, a modified version of this example gives the improved lower bound $h(a) \geq d^n h$ (Example 3.10). This shows that our estimate is essentially optimal.

The earlier work on the effective Nullstellensatz dealt with the degree bounds. Let k be a field and $\overline{k}$ its algebraic closure, and let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be polynomials of degree bounded by d without common zeros in $\overline{k}^n$.
In 1926, Hermann [25] (see also [23], [43]) proved that there exist $g_1, \ldots, g_s \in k[x_1, \ldots, x_n]$ such that

$$1 = g_1 f_1 + \cdots + g_s f_s$$

with $\deg g_i f_i \leq 2 \, (2d)^{2^{n-1}}$.
After a conjecture of Keller and Gröbner, this estimate was dramatically improved by Brownawell [6] to $\deg g_i f_i \leq n^2 d^n + n \, d$ in case $\mathrm{char}\,(k) = 0$, while Caniglia, Galligo and Heintz [7] showed that $\deg g_i f_i \leq d^{n^2}$ holds in the general case.
These results were then independently refined by Kollár [29] and by Fitchas and Galligo [12] to

$$\deg g_i f_i \leq \max\{3, d\}^n,$$

which is optimal in case $d \geq 3$. For $d = 2$, Sombra [53] recently showed that the bound $\deg g_i f_i \leq 2^{n+1}$ holds.

Now, let us consider the height aspect: assume that $f_1, \ldots, f_s \in \mathbb{Z}[x_1, \ldots, x_n]$ are polynomials of degree and height bounded by d and h, respectively. The previous degree bound reduces Bézout identity (2) to a system of $\mathbb{Q}$-linear equations. Applying Cramer rule to this linear system one obtains an estimate for the height of a and the polynomials g_i of type $s \, d^{n^2} (h + \log s + d)$.

However, it was soon conjectured that the true height bound should be much smaller.
Philippon [48] obtained the following sharper estimate for the denominator a in the Bézout equation:

$$\deg g_i \leq (n+2) \, d^n \qquad , \qquad h(a) \leq \kappa(n) \, d^n (h + d),$$

where $\kappa(n)$ depends exponentially on n.
The first essential progress on height estimates for all the polynomials g_i was achieved by Berenstein and Yger [2], who obtained

$$\deg g_i \leq n \, (2\,n + 1) \, d^n \quad , \quad h(a), h(g_i) \leq \lambda(n) \, d^{8n+3} (h + \log s + d \, \log d),$$

where $\lambda(n)$ is a (non-explicit) constant which depends exponentially on n. Their proof relies on the previous work of Philippon and on techniques from complex analysis.
Later on, Krick and Pardo [31], [32] obtained

$$\deg g_i \leq (n \, d)^{c \, n} \qquad , \qquad h(a), h(g_i) \leq (n \, d)^{c \, n} (h + \log s + d),$$

where c is a universal constant ($c \leq 35$). Their proof, based on duality theory for Gorenstein algebras, is completely algebraic.

Finally, Berenstein and Yger [3] improved their height bound to $\lambda(n) \, d^{4n+2} \, (h + \log s + d)$, and extended it to the case when $\mathbb{Z}$ is replaced by an arbitrary diophantine ring. It should be said, however, that the possibility of such an extension was already clear from the arguments of [32].

We refer the reader to the surveys [58], [1], [45] for a broad introduction to the history of the effective Nullstellensatz, main results and open questions. Aside from degree and height estimates, there is a strong current area of research on computational issues [19], [13], [32], [18], [16], [22]. There are other results in the recent research papers [50], [30], [10].

With respect to previous work, in this paper we improve in an almost optimal way the dependence of the height estimate on d^n and we eliminate the extraneous exponential constants depending on n. We remark that the polynomials arising in Theorem 1 are a slight variant of the polynomials which appear in [32] and can thus be effectively computed by their algorithm.

Although the exponential behavior of the degree and height estimates is — in the worst-case — unavoidable, it has been observed that there are many particular instances in which these estimates can be essentially improved. This has motivated the introduction of parameters associated to the input system which identify special families whose behavior with respect to our problem is polynomial instead of exponential.

In this spirit, Giusti et al. [18] introduced the notion of *degree of a polynomial system* $f_1, \ldots, f_s$. Roughly speaking, this parameter measures the degree of the varieties cut out by $f_1, \ldots, f_i$ for $i = 1, \ldots, s - 1$. It was soon realized that the degrees in the Nullstellensatz can be controlled in terms of this parameter, giving rise to the so-called "intrinsic Nullstellensätze" [18], [33], [16], [52]. Recently Hägele, Morais, Pardo and Sombra [22] (see also [21]) obtained an arithmetic analogue of these intrinsic Nullstellensätze. To this aim, they introduced the notion of *height of a polynomial system*, the arithmetic analogue of the degree of the system. They obtained degree and height estimates which depend *polynomially* on the number of variables and on the degree, height and complexity of the input system. This result followed from their study of the computational complexity of the Nullstellensatz.

In this paper we obtain a dramatical improvement over this result, bringing it to an (apparently) almost optimal form. In particular, we show that the dependence on the degree and the height of the system is *linear*, and we eliminate the influence of the complexity of the input.

Theorem 2 *Let* $f_1, \ldots, f_s \in \mathbb{Z}[x_1, \ldots, x_n]$ *be polynomials without common zeros in* $\mathbb{C}^n$. *Set* $d := \max_i \deg f_i$ *and* $h := \max_i h(f_i)$. *Let* δ *and* η *denote the degree and the height of the polynomial system* $f_1, \ldots, f_s$.
Then there exist $a \in \mathbb{Z} \setminus \{0\}$ *and* $g_1, \ldots, g_s \in \mathbb{Z}[x_1, \ldots, x_n]$ *such that*

- $a = g_1 \, f_1 + \cdots + g_s \, f_s$,

- $\deg g_i \leq 2 \, n^2 \, d \, \delta$,

- $h(a), h(g_i) \leq (n+1)^2 \, d \, (2 \, \eta + (h + \log s) \, \delta + 21 \, (n+1)^2 \, d \, \log(d+1) \, \delta)$.

Since $\delta \leq d^{n-1}$ and $\eta \leq n \, d^{n-1} \, (h + \log s + 3 \, n \, (n+1) \, d)$ (Lemma 4.8) one recovers from this statement essentially the same estimates of Theorem 1. However, we remark that Theorem 2 is a more flexible result, as there are many situations in which the degree and the height of the input system are smaller than the Bézout bounds. When this is the case, it yields a much more accurate estimate (Subsection 4.2.2).

A more general example of the situation when both the degree and the height of the system are smaller than the expected worst-case bounds is the sparse case. To state the result, we first need to introduce some standard notation.

4

The *support* $\mathrm{Supp}(f_1, \ldots, f_s)$ of a polynomial system $f_1, \ldots, f_s \subset \mathbb{C}[x_1, \ldots, x_n]$ is defined as the set of exponents of all the non-zero monomials of all f_i's, and the *Newton polytope* $\mathcal{N}(f_1, \ldots, f_s)$ is the convex hull of this support. The *(normalized) volume* of $f_1, \ldots, f_s$ equals $n!$ times the volume of the corresponding Newton polytope.

The notions of Newton polytope and volume of a polynomial system give a sharper characterization of its monomial structure than the degree alone. These concepts were introduced in the context of root counting by Bernstein [4] and Kushnirenko [35], and are now in the basis of sparse elimination theory (see e.g. [56]).

As an application of Theorem 2 we derive the following arithmetic effective Nullstellensatz for sparse polynomial systems:

Corollary 3 *Let* $f_1, \ldots, f_s \in \mathbb{Z}[x_1, \ldots, x_n]$ *be polynomials without common zeros in* $\mathbb{C}^n$. *Set* $d := \max_i \deg f_i$ *and* $h := \max_i h(f_i)$. *Let* $\mathcal{V}$ *denote the volume of the polynomial system* $1, x_1, \ldots, x_n, f_1, \ldots, f_s$.
Then there exist $a \in \mathbb{Z} \setminus \{0\}$ *and* $g_1, \ldots, g_s \in \mathbb{Z}[x_1, \ldots, x_n]$ *such that*

- $a = g_1 f_1 + \cdots + g_s f_s$,

- $\deg g_i \leq 2 \, n^2 \, d \, \mathcal{V}$,

- $h(a), h(g_i) \leq 2 \, (n+1)^3 \, d \, \mathcal{V} \, (h + \log s + 2^{2n+3} d \, \log(d+1))$.

The crucial observation here is that both the degree and the height of a polynomial system are essentially controlled by the normalized volume. This follows from an adequate arithmetic version of the Bernstein-Kushnirenko theorem (Proposition 2.12). Our result follows then from Theorem 2 in a straightforward way.

As before, we can apply the worst-case bound $\mathcal{V} \leq d^n$ to recover from this result an estimate similar to the one presented in Theorem 1. However, this result gives sharper estimates for both the degree and the height when the input system is sparse (Example 4.13).

The sparse aspect in the Nullstellensatz was previously considered by Canny and Emiris [8, Thm. 8.2] for the case of $n+1$ n-variate Laurent polynomials without common roots at toric infinity. Their result is the sparse analogue of Macaulay's effective Nullstellensatz [40]. The first general sparse Nullstellensatz was obtained by Sombra [53]. In both cases the authors give bounds for the Newton polytopes of the output polynomials in terms of the Newton polytopes of the input ones. We refer to the original papers for the exact statements.

It is quite difficult to make a definite comparison between these results and ours. The latter does not give sharp bounds for Newton polytopes. But on the other hand, our degree estimate for the general case is better, while the height estimate is completely new.

The key ingredient in our treatment of the arithmetic Nullstellensatz is the notion of local height of a variety defined over a number field K.

Let $V \subset \mathbb{A}^n(\overline{\mathbb{Q}})$ be an equidimensional affine variety defined over K. For each absolute value v over K, we introduce the *local height* $h_v(V)$ of V at v as a Mahler measure of a suitable normalized Chow form of V. This is consistent with the Falting's height $h(V)$ of V, namely:

$$h(V) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} N_v \, h_v(V),$$

where M_K denotes the set of canonical absolute values of K, and N_v the multiplicity of v.
We study the basic properties of this notion. In particular we are able to estimate the local height of the trace and the norm of a polynomial $f \in K[x_1, \ldots, x_n]$ with respect to an integral extension $K[\mathbb{A}^r] \hookrightarrow K[V]$. We also obtain local analogues of many of the global results of Bost, Gillet and Soulé [5] and Philippon [49].

Our proof of the arithmetic Nullstellensatz is based on duality theory for Gorenstein algebras (trace formula). This technique was introduced in the context of the effective Nullstellensatz in [19], [13]. Here, we follow mostly the lines of Sabia–Solernó [50] and Krick–Pardo [32].

The trace formula allows to perform division modulo complete intersection ideals, with good control of the degree and height of the involved polynomials. The local arithmetic intersection theory plays, with respect to the height estimates, the role of the classical intersection theory with respect to the degree bounds.

Finally, we remark that all of our results are valid not just for $\mathbb{Q}$ but for arbitrary number fields. In fact, the general analysis over number fields is necessary to obtain the sharpest estimates for the case $K := \mathbb{Q}$. We also remark that the estimates in the general version of Theorem 1 (Theorem 3.6) do not depend on the involved number field.

The outline of the paper is the following:

In Chapter 1, we recall the basic definitions and properties of the height of polynomials, and we introduce the notion of local height of a variety defined over a number field.

In Chapter 2, we derive useful estimates for the local heights of the trace and the norm of a polynomial in $K[V]$, and we study the behavior of the local heights of the intersection of a variety with a hypersurface.

In Chapter 3, we recall the basic facts of duality theory which will be useful in our context, and we prove Theorem 1.

In Chapter 4, we focus on the intrinsic and sparse versions of the arithmetic Nullstellensatz.

1 Height of polynomials and varieties

Throughout this paper $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Z}$ the ring of rational integers, K a number field, and $\mathcal{O}_K$ its ring of integers. We also denote by $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers, k an arbitrary field, and $\overline{k}$ an algebraic closure of k. As usual, $\mathbb{A}^n$ and $\mathbb{P}^n$ will denote the affine and the projective space of n dimensions over $\overline{k}$, respectively.

For every rational prime p we denote by $|\cdot|_p$ the p-adic absolute value over $\mathbb{Q}$ such that $|p|_p = p^{-1}$. We also denote the ordinary absolute value over $\mathbb{Q}$ by $|\cdot|_\infty$ or simply by $|\cdot|$. These form a complete set of independent absolute values over $\mathbb{Q}$: we identify the set M_Q of these absolute values with the set $\{\infty, p;\ p\ \text{prime}\}$.

For $v \in M_Q$ we denote by $\mathbb{Q}_v$ the completion of $\mathbb{Q}$ with respect to the absolute value v. In case $v = \infty$ we have $\mathbb{Q}_\infty = \mathbb{R}$, while in case p is prime, we have that $\mathbb{Q}_p$ is the p-adic field. There exists a unique extension of v to an absolute value over the algebraic closure $\overline{\mathbb{Q}}_v$. We denote by $\mathbb{C}_v$ the completion of $\overline{\mathbb{Q}}_v$ with respect to this absolute value. This field is algebraically closed and complete with respect to the induced absolute value, which we also denote by v. We have $\mathbb{C}_\infty = \mathbb{C}$.

1.1 Height of polynomials

In this section we introduce the different measures for the size of a multivariate polynomial, both over $\mathbb{C}_v$ and over a number field. We establish the link between the different notions and study their basic properties.

1.1.1 Height of polynomials over $\mathbb{C}_v$

We fix an absolute value $v \in \{\infty, p;\ p\ \text{prime}\}$ for the rest of this subsection. Let $\mathcal{A} \subset \mathbb{C}_v$ be a finite set. We denote by $|\mathcal{A}|_v := \max\{|a|_v, a \in \mathcal{A}\}$ its *absolute value*. Then we define the

(logarithmic) height of $\mathcal{A}$ as
$$h_v(\mathcal{A}) := \max\{\,0, \log|\mathcal{A}|_v\,\},$$

that is $h_v(\mathcal{A}) = \log|\{1\} \cup \mathcal{A}|_v$.

For a polynomial $f = \sum_\alpha a_\alpha\, x^\alpha \in \mathbb{C}_v[x_1, \ldots, x_n]$, we define its *absolute value* $|f|_v$ as the absolute value of its set of coefficients, that is $|f|_v := \max_\alpha\{\,|a_\alpha|_v\,\}$. In the same way we define the *height* $h_v(f)$ of f as the height of its set of coefficients:

$$h_v(f) := \max\{\,0, \log|f|_v\,\}.$$

When $v = \infty$, i.e. when f has complex coefficients, we will make use of the *(logarithmic) Mahler measure* of f defined as

$$m(f) := \int_0^1 \cdots \int_0^1 \log|f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})|\, dt_1 \ldots dt_n.$$

This integral is well-defined, as $\log|f|$ is a plurisubharmonic function on $\mathbb{C}^n$ [39, Appendix I]. The Mahler measure was introduced by Lehmer [37] for the case of a univariate polynomial $f := a_d \prod_{i=1}^d (x - \alpha_i) \in \mathbb{C}[x]$ as

$$m(f) = \log|a_d| + \sum_{i=1}^d \max\{0, \log|\alpha_i|\,\}.$$

The link between both expressions of $m(f)$ is given by Jensen's formula. The general case was introduced and studied by Mahler [41].

The key property of the Mahler measure is its additivity:

$$m(f\,g) = m(f) + m(g).$$

We have the following relation between $\log|f|$ and $m(f)$:

$$-\log(n+1)\,\deg f \le m(f) - \log|f| \le \log(n+1)\,\deg f. \tag{1.1}$$

The right inequality follows from the definition of m and the fact that the number of monomials of f is bounded by $\binom{n+\deg f}{n} \le (n+1)^{\deg f}$. For the left inequality, we refer to [47, Lem. 1.13] and its proof.

When f has total degree bounded by 1, the inequality is refined to $\log|f| \le m(f)$. Also, for any degree, $m(f(x_1, \ldots, x_{n-1}, 0)) \le m(f)$.

We will make frequent use of the following more precise relation:

Lemma 1.1 *Let* $f \in \mathbb{C}[X_1, \ldots, X_r]$ *be a polynomial in* r *groups of* n_i *variables each, for* $i = 1, \ldots, r$. *Let* d_i *denote the degree of* f *in the group of variables* X_i. *Then*

$$-\sum_{i=1}^r \log(n_i+1)\,d_i \le m(f) - \log|f| \le \sum_{i=1}^r \log(n_i+1)\,d_i.$$

Proof.– The right inequality follows directly from the definition of $m(f)$ and the fact that we can bound by $\prod_i (n_i+1)^{d_i}$ the number of monomials of f. Thus we only consider the left inequality.

Let $f_{\alpha_1 \cdots \alpha_i} \in \mathbb{C}[X_{i+1}, \ldots, X_r]$ denote the coefficient of f with respect to the monomial $X_1^{\alpha_1} \cdots X_i^{\alpha_i}$. Applying inequality (1.1) we obtain for all $(\xi_{i+1}, \ldots, \xi_r) \in \mathbb{C}^{n_{i+1} + \cdots + n_r}$:

$$\log|f_{\alpha_1 \cdots \alpha_{i-1}}(X_i, \xi_{i+1}, \ldots, \xi_r)| \le m(f_{\alpha_1 \cdots \alpha_{i-1}}(X_i, \xi_{i+1}, \ldots, \xi_r)) + \log(n_i+1)\,d_i.$$

We have $|f_{\alpha_1\cdots\alpha_{i-1}}(X_i,\xi_{i+1},\ldots,\xi_r)| = \max_{\alpha_i}|f_{\alpha_1\cdots\alpha_i}(\xi_{i+1},\ldots,\xi_r)|$. We integrate both sides of the last inequality on $S_1^{n_{i+1}+\cdots+n_r}$ and we deduce

$$\max\{m(f_{\alpha_1\cdots\alpha_i})\,;\,\alpha_i\in\mathbb{Z}^{n_i}\} \leq m(f_{\alpha_1\cdots\alpha_{i-1}}) + \log(n_i+1)\,d_i$$

We apply this relation recursively and we obtain

$$\log|f| = \max\{m(f_{\alpha_1\cdots\alpha_r})\,;\,\alpha_1\in\mathbb{Z}^{n_1},\ldots,\alpha_r\in\mathbb{Z}^{n_r}\} \leq m(f) + \sum_{i=1}^{r}\log(n_i+1)\,d_i.$$

$\square$

Let $f\in\mathbb{C}[X_1,\ldots,X_r]$ be a multihomogeneous polynomial in r groups of n_i+1 each, and set f^a for a deshomogenization of f with respect to these groups of variables. Then $m(f^a)=m(f)$, $\log|f^a| = \log|f|$. Thus the estimates of the preceding lemma also hold for f.

Next we introduce the *(logarithmic) S_n-Mahler measure* of a polynomial $f\in\mathbb{C}[x_1,\ldots,x_n]$ as

$$m(f;S_n) := \int_{S_n}\log|f(x)|\,\mu_n(x),$$

where $S_n := \{(z_1,\ldots,z_n)\in\mathbb{C}^n : |z_1|^2+\cdots+|z_n|^2=1\}$ is the unit sphere in $\mathbb{C}^n$, and μ_n is the measure of total mass 1, invariant with respect to the unitary group $U(n)$.

More generally, let $f\in\mathbb{C}[X_1,\ldots,X_r]$ be a polynomial in r groups of n variables each. Its S_n^r-*Mahler measure* is then defined as

$$m(f;S_n^r) := \int_{S_n^r}\log|f(X)|\,\mu_n^r(X),$$

with $S_n^r := S_n\times\cdots\times S_n$. This alternative Mahler measure was introduced by Philippon [49, I]. With this notation, the ordinary Mahler measure $m(f)$ of $f\in\mathbb{C}[x_1,\ldots,x_n]$ coincides with $m(f;S_1^n)$. When $f\in\mathbb{C}$ is a constant, we agree that $m(f;S_n^0)=\log|f|$.

The S_n^r-Mahler measure is related to the ordinary Mahler measure by the following inequalities [38, Thm. 4]:

$$0 \leq m(f) - m(f;S_n^r) \leq r\,d\sum_{i=1}^{n-1}\frac{1}{2\,i}, \tag{1.2}$$

where d is a bound for the degree of f in each group of variables.

Finally, we summarize in the following lemma the basic properties of the notion of height of polynomials in $\mathbb{C}_v[x_1,\ldots,x_n]$.

Lemma 1.2 *Let $v\in M_\mathbb{Q}$ and $f_1,\ldots,f_s\in\mathbb{C}_v[x_1,\ldots,x_n]$.*

1. *If $v=\infty$ then*

   (a) $h_\infty(\sum_i f_i) \leq \max_i\{h_\infty(f_i)\} + \log s$.

   (b) $h_\infty(\prod_{i=1}^s f_i) \leq \sum_{i=1}^s h_\infty(f_i) + \log(n+1)\sum_{i=1}^{s-1}\deg f_i$.
   $h_\infty(f_1\,f_2) \leq h_\infty(f_1) + h_\infty(f_2) + \log(n+1)\min\{\deg f_1,\deg f_2\}$

   (c) *Let $g\in\mathbb{C}[y_1,\ldots,y_s]$. Set $d:=\max_i\{\deg f_i\}$ and $h_\infty:=\max_i\{h_\infty(f_i)\}$. Then*

   $$h_\infty(g(f_1,\ldots,f_s)) \leq h_\infty(g) + \deg g\,(h_\infty+\log(s+1)+\log(n+1)\,d).$$

   (d) $\log|\prod_i f_i|_\infty \geq \sum_i\log|f_i|_\infty - 2\log(n+1)\sum_i\deg f_i$.

2. *If $v=p$ for some prime p then*

(a) $h_p(\sum_i f_i) \leq \max_i \{h_p(f_i)\}$.

(b) $h_p(\prod_i f_i) \leq \sum_i h_p(f_i)$.

(c) Let $g \in \mathbb{C}_p[y_1, \ldots, y_s]$. Set $d := \max_i \{\deg f_i\}$ and $h_p := \max_i \{h_p(f_i)\}$. Then
     $h_p(g(f_1, \ldots, f_s)) \leq h_p(g) + \deg g \, h_p$.

(d) $\log |\prod_i f_i|_p = \sum_i \log |f_i|_p$.

Proof of Lemma 1.2.– The different behavior for $v = \infty$ or $v = p$ is simply due to the fact that $|\cdot|_p$ is non-archimedean, that is verifies the stronger inequality $|a + b|_p \leq \max\{|a|_p, |b|_p\}$ for any $a, b \in \mathbb{C}_p$.

Inequalities (1.a), (1.b), (2.a) and (2.b) are now immediate from the definition of h_v .

(1.c) and (2.c):
Let us consider the case $v = \infty$. Set $c(n) := \log(n + 1)$.
First we compute $h_v(f_1^{\alpha_1} \cdots f_s^{\alpha_s})$ for the exponent $(\alpha_1, \ldots, \alpha_s)$ of a monomial of g . Applying (1.b) we obtain

$$h_\infty(f_1^{\alpha_1} \cdots f_s^{\alpha_s}) \leq (c(n) \, d + h_\infty) \sum_i \alpha_i \leq (c(n) \, d + h_\infty) \deg g.$$

The polynomial g has at most $(s + 1)^{\deg g}$ monomials and so

$$h_\infty(g(f_1, \ldots, f_s)) \leq h_\infty(g) + (c(n) \, d + h_\infty) \deg g + c(s) \deg g.$$

The case $v \neq \infty$ follows in a similar way.

(1.d) and (2.d):
In case $v = \infty$, we apply directly Inequality (1.1):

$$
\begin{aligned}
\sum_i \log |f_i|_\infty \quad &\leq \quad \sum_i (m(f_i) + c(n) \deg f_i) \\
&= \quad m(\prod_i f_i) + c(n) \sum_i \deg f_i \\
&\leq \quad \log |\prod_i f_i|_\infty + 2 \, c(n) \sum_i \deg f_i.
\end{aligned}
$$

In case $v = p$, Gauss Lemma implies that $\sum_i \log |f_i|_p = \log |\prod_i f_i|_p$.   □

We will make frequent use of the following particular case of the previous lemma:
Let $(f_{ij})_{ij}$ be a $s \times s$ -matrix of polynomials in $\mathbb{C}_v[x_1, \ldots, x_n]$ of degrees and heights bounded by d and h_v respectively. From Lemma 1.2(a,b) we obtain:

- $h_\infty(\det(f_{ij})_{ij}) \leq s \, (h_\infty + \log s + d \log(n + 1))$,

- $h_p(\det(f_{ij})_{ij}) \leq s \, h_p$.

1.1.2 Height of polynomials over a number field

The set M_K of absolute values over K which extend the absolute values in M_Q is called the *canonical set*. We denote by M_K^∞ the set of archimedean absolute values in M_K , i.e. the absolute values extending ∞ .
If $v \in M_K$ extends an absolute value $v_0 \in M_Q$ (which is denoted by $v \,|\, v_0$) there exists a (non necessarily unique) embedding $\sigma_v : K \hookrightarrow \mathbb{C}_{v_0}$ corresponding to v , i.e. such that $|a|_v = |\sigma_v(a)|_{v_0}$ for every $a \in K$.

In the p-adic case, there is a one-to-one correspondence $\mathcal{P} \mapsto v(\mathcal{P})$ between prime ideals of $\mathcal{O}_K$ which divide p, and absolute values extending p, defined by

$$|a|_{v(\mathcal{P})} := p^{-\mathrm{ord}_\mathcal{P}(a)/e_\mathcal{P}} = \mathrm{N}(\mathcal{P})^{-\mathrm{ord}_\mathcal{P}(a)/e_\mathcal{P} \, f_\mathcal{P}}$$

for $a \in K^*$. Here $\mathrm{ord}_\mathcal{P}(a)$ denotes the order of $\mathcal{P}$ in the factorization of a, and $\mathrm{N}(\mathcal{P})$ the norm of the ideal $\mathcal{P}$. Also $e_\mathcal{P} := \mathrm{ord}_\mathcal{P}(p)$ denotes the ramification index, and $f_\mathcal{P} := [\mathcal{O}_K/\mathcal{P} : \mathbb{Z}/(p)]$ the residual degree of the prime ideal $\mathcal{P}$.

Note that $a \in \mathcal{O}_K$ if and only if $\log |a|_v \leq 0$ for every $v \in M_K \setminus M_K^\infty$.

We denote by K_v the completion of K in $\mathbb{C}_{v_0}$. The *local degree* of K at v is defined as:

$$N_v := [K_v : \mathbb{Q}_{v_0}],$$

and it coincides with the number of different embeddings $\sigma : K \hookrightarrow \mathbb{C}_{v_0}$ which correspond to v. When v is archimedean, K_v is either $\mathbb{R}$ or $\mathbb{C}$, and N_v equals 1 or 2 accordingly. When v is non-archimedean, $N_v = e_\mathcal{P} f_\mathcal{P}$, where $\mathcal{P}$ is the prime ideal which corresponds to v.

In any case

$$[K : \mathbb{Q}] = \sum_{v \mid v_0} N_v$$

for $v_0 \in M_Q$. The canonical set M_K satisfies the *product formula* with multiplicities N_v:

$$\prod_{v \in M_K} |a|_v^{N_v} = 1, \qquad \forall \, a \in K^*. \tag{1.3}$$

Let $\mathcal{A} \subset K$ be a finite set. Let $v \in M_K$ be an absolute value which extends $v_0 \in M_Q$, and let σ_v be an embedding corresponding to v. The *local absolute value* of $\mathcal{A}$ at v is defined as

$$|\mathcal{A}|_v := |\sigma_v(\mathcal{A})|_{v_0} = \max\{|\sigma_v(a)|_{v_0}, a \in \mathcal{A}\}.$$

Then we define the *local height* of $\mathcal{A}$ as

$$h_v(\mathcal{A}) := \max\{0, \log |\mathcal{A}|_v\} = h_{v_0}(\sigma_v(\mathcal{A})).$$

We note that this notion behaves well with respect to extensions: let $K \hookrightarrow L$ be a finite extension, and let $w \in M_L$ be an absolute value extending v. Then $h_v(\mathcal{A}) = h_w(\mathcal{A})$.

For a polynomial $f = \sum_\alpha a_\alpha x^\alpha \in K[x_1, \ldots, x_n]$, we define the *local absolute value* of f at v (denoted by $|f|_v$) as the absolute value at v of its set of coefficients, and the *local height* of f at v (denoted by $h_v(f)$) as the local height at v of its set of coefficients.

Finally we define the *(global) height* of a finite set $\mathcal{A} \subset K$ as

$$h(\mathcal{A}) := \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} N_v \, h_v(\mathcal{A}).$$

In classical terms, this is the *affine* height of $\mathcal{A}$: if we set $\mathcal{A} := \{a_1, \ldots, a_N\}$ then $h(\mathcal{A})$ equals the Weil absolute height of the point $(1 : a_1 : \cdots : a_N) \in \mathbb{P}^N$.

Because of the imposed normalization, this quantity does not depend on the field K in which we consider the set $\mathcal{A}$. This allows us to extend the definition of h to subsets of $\overline{\mathbb{Q}}$.

We also define the *(global) height* of $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$ as the global height of its set of coefficients, that is

$$h(f_1, \ldots, f_s) := \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} N_v \, \max_i h_v(f_i). \tag{1.4}$$

We have $h_v(a) \leq h_v(\mathcal{A})$ for every $a \in \mathcal{A}$ and every $v \in M_K$, and so

$$\max_{a \in \mathcal{A}} h(a) \leq h(\mathcal{A}).$$

In case $\mathcal{A} \subset \mathcal{O}_K$, we have that $h_v(\mathcal{A}) = 0$ for every $v \in M_K \setminus M_K^\infty$ and so $h(\mathcal{A}) = (1/[K : \mathbb{Q}]) \sum_{v \in M_K^\infty} N_v h_v(\mathcal{A})$. We also have $h_v(\mathcal{A}) \leq [K : \mathbb{Q}] \max_{a \in \mathcal{A}} h(a)$ for all $v \in M_K$ and hence

$$h(\mathcal{A}) \leq [K : \mathbb{Q}] \max_{a \in \mathcal{A}} h(a).$$

Both inequalities are sharp. Equality is attained in the first one when, for instance, $\mathcal{A}$ has only one element.

For the second one, set $\mathcal{A} = \{1 + \sqrt{2}, 1 - \sqrt{2}\} \subset \mathbb{Q}[\sqrt{2}]$. Then $h(\mathcal{A}) = \log(1 + \sqrt{2})$ while $h(1 + \sqrt{2}) = h(1 - \sqrt{2}) = (1/2) \log(1 + \sqrt{2})$. Hence $h(\mathcal{A}) = 2 \max_{a \in \mathcal{A}} h(a)$.

More generally, if $a \in \mathbb{C}$ is a Pisot number, namely an algebraic integer such that $|a| > 1$ and all its conjugates lie inside the unit disk, and $K := \mathbb{Q}[a]$ is Galois, then, for $\mathcal{A} := \{\sigma(a) : \sigma \in \mathrm{Gal}(K/\mathbb{Q})\} \subset K$, we have $h(\mathcal{A}) = [K : \mathbb{Q}] h(a)$.

Let $a = m/n \in \mathbb{Q}^*$ be a rational number, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ are coprime. Then $h(a) = \max\{|m|, n\}$, that is, the height of a controls both the size of the minimal numerator and denominator of a. More generally, let $\mathcal{A} \subset \mathbb{Q}$ be a finite set, and let $b \in \mathbb{N}$ be a minimal common denominator for all the elements of $\mathcal{A}$. Then $h(\mathcal{A}) = \log \max\{|b\,\mathcal{A}|, b\}$. The following is the analogous statement for the general case:

Lemma 1.3 *Let $\mathcal{A} \subset K$ be a finite set. Then there exist $b \in \mathbb{Z} \setminus \{0\}$ and $\mathcal{B} \subset \mathcal{O}_K$ such that*

$$b\,\mathcal{A} = \mathcal{B} \qquad , \qquad h(\mathcal{A}) \leq h(\{b\} \cup \mathcal{B}) \leq [K : \mathbb{Q}] h(\mathcal{A}).$$

Proof.– Let $v \in M_K \setminus M_K^\infty$, and set $\mathcal{P}$ for the corresponding prime ideal of $\mathcal{O}_K$. Let $a_v \in \mathcal{A}$ such that $h_v(\mathcal{A}) = h_v(a_v)$ and set

$$c(\mathcal{P}) = \max\{0, -\mathrm{ord}_\mathcal{P}(a_v)\}.$$

Then $\mathrm{ord}_\mathcal{P}(a) \geq -c(\mathcal{P})$ for every $a \in \mathcal{A}$, and $h_v(\mathcal{A}) = c(\mathcal{P}) \log \mathrm{N}(\mathcal{P})/e_\mathcal{P} f_\mathcal{P}$. Set

$$b := \prod_\mathcal{P} \mathrm{N}(\mathcal{P})^{c(\mathcal{P})} \qquad , \qquad \mathcal{B} := \{b\,a \, ; \, a \in \mathcal{A}\},$$

where $\mathcal{P}$ runs over all prime ideals of $\mathcal{O}_K$. Clearly $b \in \mathbb{N} \setminus \{0\}$. We have $\mathrm{ord}_\mathcal{P}(b) = e_\mathcal{P} f_\mathcal{P} c(\mathcal{P})$ and so $\mathrm{ord}_\mathcal{P}(b\,a) \geq e_\mathcal{P} f_\mathcal{P} c(\mathcal{P}) - c(\mathcal{P}) \geq 0$ for every $a \in \mathcal{A}$. Hence $\mathcal{B} \subset \mathcal{O}_K$.

For $v \in M_K^\infty$ we have $h_v(\{b\} \cup \mathcal{B}) = h_v(\mathcal{A}) + \log b$ and so

$$h(\{b\} \cup \mathcal{B}) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K^\infty} N_v \, h_v(\{b\} \cup \mathcal{B}) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K^\infty} N_v \, h_v(\mathcal{A}) \; + \; \log b.$$

We have

$$\log b = \sum_\mathcal{P} c(\mathcal{P}) \log \mathrm{N}(\mathcal{P}) = \sum_{v \notin M_K^\infty} N_v h_v(\mathcal{A}).$$

and therefore:

$$h(\{b\} \cup \mathcal{B}) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K^\infty} N_v \, h_v(\mathcal{A}) \; + \sum_{v \notin M_K^\infty} N_v h_v(\mathcal{A}) \leq [K : \mathbb{Q}] h(\mathcal{A}).$$

On the other hand we have $h_v(\mathcal{A}) + \log |b|_v \leq h_v(\{b\} \cup B)$ for all $v \in M_K$. Applying the product formula (1.3) to b we obtain:

$$h(\mathcal{A}) = \frac{1}{[K : \mathbb{Q}]} \sum_v N_v \, (h_v(\mathcal{A}) + \log |b|_v) \leq \frac{1}{[K : \mathbb{Q}]} \sum_v N_v \, h_v(\{b\} \cup \mathcal{B}) = h(\{b\} \cup \mathcal{B}).$$

$\square$

Finally, let $a \in \overline{\mathbb{Q}}^*$ be a non-zero algebraic number, and set $p_a \in \mathbb{Z}[t]$ for its primitive minimal polynomial. We have $h(a) = m(p_a)/\deg a$. More generally, the height of a finite set can be seen as the height of the minimal polynomial of a generic linear combination of its elements. This gives a partial motivation for the notion of global height of a finite set.

Lemma 1.4 *Let* $\mathcal{A} := \{a_1, \ldots, a_N\} \subset K$ *be a finite set and set*

$$p_{\mathcal{A}} := \prod_{\sigma} (u_0 + \sigma(a_1)\, u_1 + \cdots + \sigma(a_N)\, u_N) \in \mathbb{Q}[u_0, \ldots, u_N],$$

where the product is taken over all $\mathbb{Q}$ *-embeddings* $\sigma : K \hookrightarrow \overline{\mathbb{Q}}$. *Then*

$$-\log(N+1) \leq h(\mathcal{A}) - h(p_{\mathcal{A}})/[K:\mathbb{Q}] \leq \log(N+1).$$

Proof.– Set $L(u) := u_0 + a_1 u_1 + \cdots + a_N u_N \in K[u]$ so that

$$p_{\mathcal{A}} = \prod_{\sigma} \sigma(L).$$

For $v_0 \in M_Q$ we choose an inclusion $\mathbb{Q} \hookrightarrow \mathbb{C}_{v_0}$. Then for each $v \in M_K$ such that $v|v_0$ there are N_v embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}}$ which correspond to it. We note that for each such σ, $\log|\sigma(L)| = h_v(\mathcal{A})$ holds. Applying Lemma 1.2(b) we obtain

$$h_\infty(p_{\mathcal{A}}) \leq \sum_{\sigma} \log|\sigma(L)| + [K:\mathbb{Q}]\log(N+1) = \sum_{v \in M_K^\infty} N_v\, h_v(\mathcal{A}) + [K:\mathbb{Q}]\log(N+1).$$

In the same way we obtain $h_p(p_{\mathcal{A}}) \leq \sum_{v|p} N_v\, h_v(\mathcal{A})$ for p prime and hence

$$h(p_{\mathcal{A}}) \leq \sum_{v \in M_K} N_v\, h_v(\mathcal{A}) + [K:\mathbb{Q}]\log(N+1) = [K:\mathbb{Q}]\,(h(\mathcal{A}) + \log(N+1)).$$

On the other hand $\log|\sigma(L)| \leq m(\sigma(L))$ for every σ, as L has total degree 1. Thus

$$
\begin{aligned}
[K:\mathbb{Q}]\, h(\mathcal{A}) &= \sum_{v \in M_K} N_v\, h_v(\mathcal{A}) \\
&= \sum_{\sigma} \log|\sigma(L)|_\infty + \sum_p \sum_\sigma \log|\sigma(L)|_p \\
&\leq m(p_{\mathcal{A}}) + \sum_p h_p(p_{\mathcal{A}}) \\
&\leq h(p_{\mathcal{A}}) + [K:\mathbb{Q}]\log(N+1)
\end{aligned}
$$

by application of Lemma 1.2(d) and inequality (1.1), and the definition of the height. □

1.2 Height of varieties

In this section we introduce the notions of local and global height of an affine variety defined over a number field. For this aim, we recall the basic facts of the degree and Chow form of varieties. As an important particular case, we study the height of an affine toric variety.

1.2.1 Degree of varieties

Let k be an arbitrary field and $V \subset \mathbb{A}^n$ be an affine equidimensional variety of dimension r. We recall that the degree of V is defined as the number of points in the intersection of V with a generic linear space of dimension $n-r$. This coincides with the sum of the degrees of its irreducible components.

For an arbitrary variety $V \subset \mathbb{A}^n$ we set $V = \cup_i V_i$ for its decomposition into equidimensional varieties. Following Heintz [23], we define the *degree* of V as

$$\deg V := \sum_i \deg V_i.$$

For $V = \emptyset$ we agree $\deg V := 1$.

This is a positive integer, and we have $\deg V = 1$ if and only V is a linear variety.

The degree of a hypersurface equals the degree of any generator of its defining ideal. The degree of a finite variety equals its cardinal.

For a linear morphism $\varphi : \mathbb{A}^n \to \mathbb{A}^m$ and a variety $V \subset \mathbb{A}^n$ we have $\deg \overline{\varphi(V)} \le \deg V$, where $\overline{\varphi(V)}$ denotes the Zariski closure of $\varphi(V)$ in $\mathbb{A}^m$.

The basic aspect of this notion of degree is its behavior with respect to intersections. It verifies the *Bézout inequality*:

$$\deg(V \cap W) \le \deg V \deg W$$

for $V, W \subset \mathbb{A}^n$, without any restriction on the intersection type of V and W [23, Thm. 1], [14, Example 8.4.6].

1.2.2 Normalization of Chow forms

Let $V \subset \mathbb{A}^n$ be an affine equidimensional variety of dimension r defined over a field k. Let $\mathcal{F}_V$ be a Chow form of V, that is a Chow form of its projective closure $\overline{V} \subset \mathbb{P}^n$. This is a squarefree polynomial over k in $r + 1$ groups $U_0, \ldots, U_r$ of $n + 1$ variables each. It is multihomogeneous of degree $D := \deg V$ in each group of variables, and is uniquely determined up to a scalar factor. In case V is irreducible, $\mathcal{F}_V$ is an irreducible polynomial, and in the general case of an equidimensional variety, the product of Chow forms of its irreducible components is a Chow form of V.

In order to avoid this indeterminacy of $\mathcal{F}_V$, we are going to fix one of its coefficients under a technical assumption on the variety V. For purpose of reference, we resume it in the following:

Assumption 1.5 *We assume that the projection* $\pi_V : V \to \mathbb{A}^r$ *defined by* $x \mapsto (x_1, \ldots, x_r)$ *verifies* $\#\pi_V^{-1}(0) = \deg V$.

This assumption implies that $\pi_V : V \to \mathbb{A}^r$ is a dominant map of degree $\deg V$, by the theorem of dimension of fibers. Later on, we will prove that in fact this assumption implies that the projection π_V is finite, that is, the variables $x_1, \ldots, x_r$ are in Noether normal position with respect to V (Lemma 2.14). We remark that the previous condition is satisfied by any variety under a generic linear change of variables.

Each group of variables U_i is associated to the coefficients of a generic linear form $L_i(U_i) := U_{i\,0} + U_{i\,1} x_1 + \cdots + U_{i\,n} x_n$. The main feature of a Chow form is that

$$\mathcal{F}_V(\nu_0, \ldots, \nu_r) = 0 \Leftrightarrow \overline{V} \cap \{L^h(\nu_0) = 0\} \cap \ldots \cap \{L^h(\nu_r) = 0\} \ne \emptyset$$

holds for $\nu_i \in \overline{k}^{n+1}$. Here $L_i^h := U_{i\,0} x_0 + \cdots + U_{i\,n} x_n$ stands for the homogenization of L_i.

Assumption 1.5 implies that $\overline{V} \cap \{x_1 = 0\} \cap \ldots \cap \{x_r = 0\}$ is a zero-dimensional variety of $\mathbb{P}^n$ lying in the affine space $\{x_0 \ne 0\}$. Set e_i for the $(i + 1)$-vector of the canonical basis of k^{n+1}. Then $\mathcal{F}_V(e_0, \ldots, e_r)$ — that is, the coefficient of the monomial $U_{0\,0}^D \cdots U_{r\,r}^D$ — is non-zero.

We then define *the (normalized) Chow form* Ch_V of V by fixing the election of $\mathcal{F}_V$ through the condition

$$Ch_V(e_0, \ldots, e_r) = 1.$$

Under this normalization, Ch_V equals the product of the normalized Chow forms of the irreducible components of V.

1.2.3 Height of varieties over $\mathbb{C}_v$

Let $v \in \{\infty, p; \ p \text{ prime}\}$ be an absolute value over $\mathbb{Q}$, and $V \subset \mathbb{A}^n(\mathbb{C}_v)$ an equidimensional variety of dimension r which satisfies Assumption 1.5. We introduce the height of V as a Mahler measure of its normalized Chow form:

Definition 1.6 *The* height *of the affine variety* $V \subset \mathbb{A}^n(\mathbb{C}_v)$ *is defined as*

$$h_v(V) := m(\mathcal{C}h_V; S_{n+1}^{r+1}) + (r+1)\left(\sum_{i=1}^n 1/2\, i\right) \deg V$$

in case $v = \infty$*, and as*

$$h_v(V) := h_v(\mathcal{C}h_V)$$

in case $v = p$ *for some prime* p*.*

This definition coincides in the non-archimedean cases with the local height of $\overline{V} \subset \mathbb{P}^n$ with respect to the divisors $\operatorname{div}(x_0), \ldots, \operatorname{div}(x_r) \in \operatorname{Div}(\mathbb{P}^n)$ as it is introduced in [20, Section 9]. In general, it is also closely related to Philippon's local height of a projective variety [49, II].

Let us consider some examples:

- We have that $h_\infty(\mathbb{A}^n(\mathbb{C}))$ equals the Stoll number $\sum_{i=1}^n \sum_{j=1}^i 1/2\, j$, while $h_p(\mathbb{A}^n(\mathbb{C}_p)) = 0$. This follows from [5, Lem. 3.3.1] and [55, Thm. 3], [49, I, Thm. 2], and the fact that $\mathcal{C}h_{\mathbb{A}^n} = \det(U_0, \ldots, U_n)$.

- Let $V \subset \mathbb{A}^n(\mathbb{C}_v)$ be a hypersurface verifying Assumption 1.5, defined by a squarefree polynomial $f \in \mathbb{C}_v[x_1, \ldots, x_n]$. Then the coefficient of the monomial $x_n^{\deg V}$ is non-zero, and we can suppose without loss of generality that it equals 1. Let f^h denote the homogenization of f. Then
$$h_v(V) = m(f^h; S_{n+1}) + \left(\sum_{i=1}^{n-1} \sum_{j=1}^i 1/2\, j\right) \deg V,$$
in case $v = \infty$, while in case $v = p$ for some prime p, $h_v(V) = h_v(f)$. [49, I, Cor. 4].

- In case $V = \{\xi\}$ for some $\xi \in \mathbb{A}^n$, we have (see e.g. [49, I, Prop. 4])
$$
\begin{aligned}
h_\infty(V) &= \frac{1}{2}\log(1 + |\xi_1|^2 + \cdots + |\xi_n|^2), \\
h_p(V) &= h_p(\xi).
\end{aligned}
$$

1.2.4 Height of varieties over a number field

Let $V \subset \mathbb{A}^n(\overline{\mathbb{Q}})$ be an equidimensional variety of dimension r defined over a number field K. We define the *(global) height* $h(V)$ of V as the Faltings' height [11] of its projective closure $\overline{V} \subset \mathbb{P}^n$. It verifies the identity

$$h(V) = \frac{1}{[K:\mathbb{Q}]}\left(\sum_{v \in M_K^\infty} N_v\, m(\sigma_v(\mathcal{F}_V); S_{n+1}^{r+1}) + \sum_{v \notin M_K^\infty} N_v \log|\mathcal{F}_V|_v \right) + (r+1)\left(\sum_{i=1}^n 1/2\, i\right) \deg V,$$

where $\mathcal{F}_V$ denotes any Chow form of V [55, Thm. 3], [49, I, Thm. 2]. Following Philippon [49, III], we introduce h through this identity, without appealing to Arakelov theory.

For an arbitrary affine variety, we define its *(global) height* as the sum of the heights of its equidimensional components. It coincides with the sum of the heights of its equidimensional components. We agree that $h(\emptyset) := 0$.

We also introduce the local counterpart of this notion. Let $v \in M_K$ be an absolute value over K, and suppose that V satisfies Assumption 1.5. Let $v_0 \in M_Q$ such that $v|v_0$, and let $\sigma_v : K_v \to \mathbb{C}_{v_0}$ be an embedding corresponding to v. We define the *local height* of V at v as

$$h_v(V) := h_{v_0}(\sigma_v(V)).$$

This is consistent with the global height:

$$h(V) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} N_v \, h_v(V).$$

The global height h is related to the height h_{BGS} of Bost, Gillet and Soulé by the formula [5, Prop. 4.1.2 (i)]:

$$h(V) = h_{\mathrm{BGS}}(V) + \left(\sum_{i=1}^{r} \sum_{j=1}^{i} 1/2 \, j \right) \deg V.$$

It is also related to the height $\overline{h}$ introduced by Giusti et al. [16] in terms of the so-called geometric solution of a variety. They are *polynomially* equivalent [54, Thm. 1.3.26], namely

$$h(V) \le (n \, \deg V \overline{h}(V))^c \qquad , \qquad \overline{h}(V) \le (n \, \deg V h(V))^c,$$

for some constant $c > 0$.

We have $h(V) \ge \left(\sum_{i=1}^{r} \sum_{j=1}^{i} 1/2 \, j \right) \deg V$, with equality only in case V is defined by the vanishing of $n - r$ standard coordinates. [5, Thm. 5.2.3]. For instance $h(\mathbb{A}^n) = \sum_{i=1}^{n} \sum_{j=1}^{i} 1/2 \, j$. In particular $h(V) \ge 0$.

This notion of height satisfies the *arithmetic Bézout inequality* [5, Thm. 5.5.1 (iii)], [49, III, Thm. 3]:

$$h(V \cap W) \ \le \ h(V) \deg W + \deg V \, h(W) + c \, \deg V \, \deg W,$$

for $V, W \subset \mathbb{A}^n(\overline{\mathbb{Q}})$, with $c := \left(\sum_{i=0}^{\dim V} \sum_{j=0}^{\dim W} 1/2(i + j + 1) \right) + \left(n - (\dim V + \dim W)/2 \right) \log 2$.

1.2.5 Height of affine toric varieties

Now we consider the case of affine toric varieties. The obtained height estimate is crucial in our treatment of the sparse arithmetic Nullstellensatz (Corollary 4.12).

In what follows we recall some basic notation and results of affine toric varieties and sparse resultants. References are [15], [57].

Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_N\} \subset \mathbb{Z}^n$ be a finite set of integer vectors. Let $r := \dim \mathcal{A}$ denote the *dimension* of $\mathcal{A}$, that is, the dimension of the free $\mathbb{Z}$-module $\mathbb{Z}\mathcal{A}$. We normalize the volume form of $\mathbb{R}\mathcal{A}$ in order that any elementary simplex of the lattice $\mathbb{Z}\mathcal{A}$ has volume 1. The *(normalized) volume* $\mathrm{Vol}(\mathcal{A})$ of $\mathcal{A}$ is defined as the volume of the convex hull $\mathrm{Conv}(\{0\} \cup \mathcal{A})$ with respect to this volume form. In case $\mathbb{Z}\mathcal{A} = \mathbb{Z}^n$, then $\mathrm{Vol}(\mathcal{A})$ equals $n!$ times the volume of $\mathrm{Conv}(\{0\} \cup \mathcal{A})$ with respect to the Euclidean volume form of $\mathbb{R}^n$.

We associate to the set $\mathcal{A}$ a map $(\overline{\mathbb{Q}}^*)^n \to \overline{\mathbb{Q}}^N$ defined by $\xi \mapsto (\xi^{\alpha_1}, \ldots, \xi^{\alpha_N})$. The Zariski closure of the image of this map is the *affine toric variety* $X_\mathcal{A} \subset \mathbb{A}^N$. This is an irreducible variety of dimension r and degree $\mathrm{Vol}(\mathcal{A})$.

For $i = 0, \ldots, r$, we denote by U_i a group of variables indexed by the elements of $\mathcal{A}$ and we set

$$F_i := \sum_{\alpha \in \mathcal{A}} U_{i\alpha} \, x^\alpha$$

for the generic Laurent polynomial with support contained in $\mathcal{A}$. Let $W \subset (\mathbb{P}^{N-1})^{r+1} \times (\overline{\mathbb{Q}}^*)^n$ be the incidence variety of $F_0, \ldots, F_r$ in $(\overline{\mathbb{Q}}^*)^n$, that is

$$W = \{(\nu_0, \ldots, \nu_r; \xi); \quad F_i(\nu_i)(\xi) = 0 \ \forall i\},$$

and let $\pi : (\mathbb{P}^{N-1})^{r+1} \times (\overline{\mathbb{Q}}^*)^n \to (\mathbb{P}^{N-1})^{r+1}$ be the canonical projection. Then $\overline{\pi(W)}$ is an irreducible variety of codimension 1. Any of its defining polynomial $R_{\mathcal{A}} \subset \mathbb{Q}[U_0, \ldots, U_r]$ is called the $\mathcal{A}$-*resultant* or *sparse resultant*, and it coincides with a Chow form of the affine toric variety $X_{\mathcal{A}}$ [27]. It is a multihomogeneous polynomial of degree $\mathrm{Vol}(\mathcal{A})$ in each group of variables, and it is uniquely defined up to its sign, if we assume it to be a primitive polynomial with integer coefficients.

We obtain the following bound for the height of $X_{\mathcal{A}}$. Our argument relies on the Canny-Emiris determinantal formula for the sparse resultant [8].

Proposition 1.7 *Let* $\mathcal{A} \subset \mathbb{Z}^n$ *be a finite set of dimension* r *and cardinality* $\#\mathcal{A} \geq 2$. *Then* $h(X_{\mathcal{A}}) \leq 2^{2r+2} \log(\#\mathcal{A}) \mathrm{Vol}(\mathcal{A})$.

Proof.– Let $R_{\mathcal{A}}$ denote the $\mathcal{A}-$ resultant, which we assume to be primitive with integer coefficients. Thus

$$h(X_{\mathcal{A}}) = m(R_{\mathcal{A}}; S_{N+1}^{r+1}) + (r+1)\left(\sum_{i=1}^{N} 1/2\, i\right) \mathrm{Vol}(\mathcal{A})$$

and so it suffices to estimate the S_{N+1}^{r+1}-Mahler measure of $R_{\mathcal{A}}$.

Let $\mathcal{M}$ be the Canny-Emiris matrix associated to the generic polynomial system $F_0, \ldots, F_r$. This is a non-singular square matrix of order M, where M denotes the cardinality of the set

$$\mathcal{E} := ((r+1)Q + \varepsilon) \cap \mathbb{Z}^n.$$

Here $Q := \mathrm{Conv}(\{0\} \cup \mathcal{A})$, and $\varepsilon \in \mathbb{R}^n$ is any vector such that each point in $\mathcal{E}$ is contained in the interior of a cell in a given triangulation of the polytope $(r+1)Q$. In particular ε can be arbitrarily chosen in a non empty open set of $\mathbb{R}^n$.

Every non-zero entry of $\mathcal{M}$ is a variable $U_{i\alpha}$. In fact, each row has exactly N non-zero entries, which consist of the variables in some group U_i. We refer to [8, Section 4] for the precise construction.

Thus $\det \mathcal{M} \in \mathbb{Z}[U_0, \ldots, U_r]$ is a multihomogeneous polynomial of total degree M and height bounded by $M \log N$. This polynomial is a non-zero multiple of the sparse resultant $R_{\mathcal{A}}$ [8, Thm. 5.2]. The assumption that $R_{\mathcal{A}}$ is primitive implies that $\det \mathcal{M}/R_{\mathcal{A}}$ lies in $\mathbb{Z}[U_0, \ldots, U_r]$, and so $m(R_{\mathcal{A}}) \leq m(\det \mathcal{M})$.

Let $\{T_j\}_{j \in J}$ be a unimodular triangulation of Q, so that $\{(r+1)T_j\}_{j \in I}$ is a triangulation of $(r+1)Q$. For every $\varepsilon \in \mathbb{R}^n$, the set of integer points contained in $(r+1)T_j + \varepsilon$ is in correspondence with a subset of those of $(r+1)T_j$. Moreover, for a generic choice of ε we loose — at least — the set of integer points in a facet of codimension 1. Thus

$$\#((r+1)T_j + \varepsilon) \cap \mathbb{Z}^n \leq \#\, r\, T_j \cap \mathbb{Z}^n = \binom{2r}{r} \leq 2^{2r-1}$$

and so

$$M \leq \sum_{j \in J} \#((r+1)T_j + \varepsilon) \cap \mathbb{Z}^n \leq 2^{2r-1}\, \#J = 2^{2r-1} \mathrm{Vol}(\mathcal{A}).$$

Applying Lemma 1.1 we obtain

$$\begin{aligned} m(R_{\mathcal{A}}) &\leq& \log|\det \mathcal{M}| + \deg(\det \mathcal{M}) \log N \\ &\leq& 2M \log N \\ &\leq& 2^{2r} \log N \, \mathrm{Vol}(\mathcal{A}). \end{aligned}$$

We conclude

$$
\begin{aligned}
h(X_{\mathcal{A}}) &= m(R_{\mathcal{A}}; S_{N+1}^{r+1}) + (r+1)\,(\sum_{i=1}^{N} 1/2\,i)\,\mathrm{Vol}(\mathcal{A}) \\
&\leq m(R_{\mathcal{A}}) + 2\,(r+1)\,\log N\,\mathrm{Vol}(\mathcal{A}) \\
&\leq 2^{2\,r+1}\,\log N\,\mathrm{Vol}(\mathcal{A}),
\end{aligned}
$$

as $N = \#\mathcal{A} \geq 2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

In case $\mathcal{A} \subset (\mathbb{Z}_{\geq 0})^n$ — that is, when $F_0, \ldots, F_r$ are polynomials — we set $d := \max\{|\alpha| : \alpha \in \mathcal{A}\} = \deg F_0$. We have then $N = \binom{d+n}{n} \leq (n+1)^d$ and so

$$
h(X_{\mathcal{A}}) \leq 2^{2\,r+1}\,\log(n+1)\,d\,\mathrm{Vol}(\mathcal{A}).
$$

2 Estimates for local and global heights

In this chapter we study the basic properties of local and global heights that we will need for our purposes. The key result is a precise estimate for the local height of the trace and the norm of a polynomial $f \in K[x_1, \ldots, x_n]$ with respect to an integral extension $K[\mathbb{A}^r] \hookrightarrow K[V]$.

We also study some of the basic properties of the height of a variety, in particular its behavior under intersection with hypersurfaces and under affine maps.

2.1 Estimates for Chow forms

In this section we recall the notion of generalized Chow form of a variety in the sense of Philippon [47], and we prove a technical estimate for its local height.

2.1.1 Generalized Chow forms

Let $V \subset \mathbb{A}^n$ be an affine equidimensional variety of dimension r and degree D defined over a field k.

For $d \in \mathbb{N}$ we denote by $U(d)_0$ a group of $\binom{d+n}{n}$ variables. Also, for $1 \leq i \leq r$ we denote by U_i a group of $n+1$ variables, and we set $U(d) := \{U(d)_0, U_1, \ldots, U_r\}$. Set

$$
F := \sum_{|\alpha| \leq d} U(d)_{0\alpha}\,x^\alpha \qquad , \qquad L_i := U_{i\,0} + U_{i\,1}\,x_1 + \cdots + U_{i\,n}\,x_n
$$

for the generic polynomial in n variables of degree d and 1 associated to $U(d)_0$ and U_i respectively.

Set $N := \binom{d+n}{n} + r\,(n+1)$ and let $W \subset \mathbb{A}^N \times V$ be the incidence variety of $F, L_1, \ldots, L_r$ with respect to V, that is

$$
W := \{(\nu(d)_0, \nu_1, \ldots, \nu_r; \xi)\,;\ \ \xi \in V,\ \ F(\nu(d)_0)(\xi) = 0,\ \ L_i(\nu_i)(\xi) = 0,\ 1 \leq i \leq r\}.
$$

Let $\pi : \mathbb{A}^N \times \mathbb{A}^n \to \mathbb{A}^N$ denote the canonical projection. Then $\overline{\pi(W)} \subset \mathbb{A}^N$ is a hypersurface [47, Prop. 1.5] and any of its defining equations $\mathcal{F}_{d,V} \in k[U(d)]$ is called a *generalized Chow form* or a *d-Chow form* of V.

A d-Chow form is uniquely defined up to a scalar factor. It shares many properties with the usual Chow form, which corresponds to the case $d = 1$. We have

$$
\mathcal{F}_{d,V}(\nu(d)_0, \nu_1, \ldots, \nu_r) = 0 \Leftrightarrow \overline{V} \cap \{F^h(\nu(d)_0) = 0\} \cap \{L_1^h(\nu_1) = 0\} \cap \cdots \cap \{L_r^h(\nu_r) = 0\} \neq \emptyset
$$

for $\nu(d)_0 \in \overline{k}^{\binom{d+n}{n}}$ and $\nu_i \in \overline{k}^{n+1}$. Here $\overline{V} \subset \mathbb{P}^n$ denotes the projective closure of V, while F^h and L_i^h stand for the homogenization of F and L_i respectively.

A d-Chow form $\mathcal{F}_{d,V} \in k[U(d)]$ is a multihomogeneous polynomial of degree D in the group of variables $U(d)_0$ and of degree dD in each group U_i [47, Lem. 1.8]. When V is an irreducible variety, $\mathcal{F}_{d,V}$ is an irreducible polynomial of $k[U(d)]$. When V is equidimensional, it coincides with the product of d-Chow forms of its irreducible components.

Now, let U_0 be another group of $n+1$ variables, and consider the morphism

$$\varrho_d : k[U(d)] \to k[U_0, U_1, \ldots, U_r]$$

defined by $\varrho_d(F) = L_0^d$ and $\varrho_d(L_i) = L_i$ for $i = 1, \ldots, r$, where L_0 stands for the generic linear form associated to U_0.

In other terms

$$\varrho_d(U(d)_{0\alpha}) = \binom{d}{\alpha} U_{00}^{d-|\alpha|} U_{01}^{\alpha_1} \cdots U_{0n}^{\alpha_n} \qquad \text{where} \quad \binom{d}{\alpha} := \frac{d!}{(d-|\alpha|)! \, \alpha_1! \, \cdots \, \alpha_n!}$$

for $|\alpha| \le d$, and $\varrho_d(U_{ij}) = U_{ij}$ for $i = 1, \ldots, r$ and $j = 0, \ldots, n$.

This morphism gives the following relation between a d-Chow form $\mathcal{F}_{d,V}$ and the usual one [47, Proof of Prop. 2.8]:

Lemma 2.1 *Let* $V \subset \mathbb{A}^n$ *be an equidimensional variety. Then* $\varrho_d(\mathcal{F}_{d,V}) = \lambda \, \mathcal{F}_V^d$ *for some* $\lambda \in k^*$.

Proof. It is enough to consider the case when V is irreducible. Set $r := \dim V$.
The polynomials $\varrho_d(\mathcal{F}_{d,V})$ and $\mathcal{F}_V$ have both the same zero locus: let $\nu_i \in \mathbb{A}^{n+1}$ for $i = 0, \ldots, r$. As $\varrho_d(\mathcal{F}_{d,V}) = \mathcal{F}_{d,V}((\varrho_d(U(d)_0))_\alpha, U_1, \ldots, U_r)$, then $\varrho_d(\mathcal{F}_{d,V})(\nu_0, \ldots, \nu_r) = 0$ if and only if $\overline{V} \cap \{\varrho_d(F^h)(\nu_0) = 0\} \cap \{L_1^h(\nu_1) = 0\} \cap \cdots \cap \{L_r^h(\nu_r) = 0\} \ne \emptyset$, that is if and only if

$$\overline{V} \cap \{L_0^h(\nu_0)^d = 0\} \cap \{L_1^h(\nu_1) = 0\} \cap \cdots \cap \{L_r^h(\nu_r) = 0\} \ne \emptyset,$$

which is clearly equivalent to $\mathcal{F}_V(\nu_0, \ldots, \nu_r) = 0$.

On the other hand, as V is irreducible, $\mathcal{F}_V$ is an irreducible polynomial, and thus $\varrho_d(\mathcal{F}_{d,V})$ is a power of $\mathcal{F}_V$ (modulo a constant λ). Since $\deg \mathcal{F}_V = (r+1) \deg V$ and $\deg \varrho_d(\mathcal{F}_{d,V}) = (r+1) d \deg V$, we derive that $\varrho_d(\mathcal{F}_{d,V}) = \lambda \, \mathcal{F}_V^d$ for some $\lambda \in k^*$. $\qquad\square$

Now assume that V satisfies Assumption 1.5. Then $\overline{V} \cap \{x_0^d = 0\} \cap \{x_1 = 0\} \cap \ldots \cap \{x_r = 0\} = \emptyset$.
Setting $e(d)_\alpha$ and e_i for the α-vector and the $(i+1)$-vector of the canonical bases of $k^{\binom{d+n}{n}}$ and k^{n+1} respectively, we infer that $\mathcal{F}_{d,V}(e(d)_0, e_1, \ldots, e_r)$ — that is, the coefficient of the monomial $U(d)_{00}^D U_{11}^{dD} \cdots U_{rr}^{dD}$ — is non-zero.
We define *the (normalized) d-Chow form* $\mathcal{Ch}_{d,V}$ of V by fixing the election of $\mathcal{F}_{d,V}$ with the condition $\mathcal{Ch}_V(e(d)_0, e_1, \ldots, e_r) = 1$.
In the previous construction, $U(d)_{00}^D U_{11}^{dD} \ldots U_{rr}^{dD}$ is the only monomial of $k[U(d)]$ which maps through ϱ_d to $U_{00}^{dD} \cdots U_{rr}^{dD}$. The imposed normalizations imply then

$$\varrho_d(\mathcal{Ch}_{d,V}) = \mathcal{Ch}_V^d.$$

2.1.2 An estimate for generalized Chow forms

The following technical result is crucial to our local height estimates for the trace and the norm of a polynomial (Subsection 2.3.2), as well as for the intersection of a variety with an hypersurface (Subsection 2.2.2). The proof follows the lines of [47, Prop. 2.8].

We adopt the following convention:
Let $f \in k[x_1, \ldots, x_n]$ be a polynomial of degree d. We denote by $\mathcal{F}_{d,V}(f)$ and $\mathcal{Ch}_{d,V}(f)$ the specialization of $U(d)_0$ into the coefficients of f in $\mathcal{F}_{d,V}$ and $\mathcal{Ch}_{d,V}$ respectively.

18

Lemma 2.2 *Let* $V \subset \mathbb{A}^n(\mathbb{C}_v)$ *be an equidimensional variety of dimension* r *which satisfies Assumption 1.5. Let* $f \in \mathbb{C}_v[x_1, \ldots, x_n]$. *Then*

- $m(\mathcal{C}h_{\deg f, V}(f); S_{n+1}^r) + r\left(\sum_{i=1}^n 1/2\, i\right) \deg f \deg V$

$$\leq \deg f\, h_v(V) + h_v(f) \deg V + \log(n+1) \deg f \deg V$$

  for $v = \infty$,

- $h_v(\mathcal{C}h_{\deg f, V}(f)) \leq \deg f\, h_v(V) + h_v(f) \deg V$ *for* $v = p$ *for some prime* p.

We will need the following lemma in order to treat the non-archimedean case:

Lemma 2.3 *Let* $g \in \mathbb{C}_p[y_1, \ldots, y_m]$, *and let* $\Omega \subset \mathbb{A}^m(\mathbb{C}_p)$ *be a Zariski open set. Then*

$$|g|_p = \max\{|g(\nu)|_p; \ \nu \in \Omega, \ |\nu|_p = 1\}.$$

Proof.– For $q \in \mathbb{N}$ we denote by G_q the set of q-roots of 1 in $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$ such that $|\alpha_i| < q$. Then

$$\sum_{\xi \in G_q^m} \xi^\alpha = \begin{cases} 0 & \text{if} \quad \alpha \neq 0, \\ q^m & \text{if} \quad \alpha = 0. \end{cases}$$

Set $g = \sum_\alpha a_\alpha\, x^\alpha$. Let $q > \deg g$ such that $|q|_p = 1$, that is $p \nmid q$. Then for any $\omega = (\omega_1, \ldots, \omega_m) \in (\mathbb{C}_p^*)^m$ we have

$$a_\alpha = \frac{1}{\omega^\alpha\, q^m} \sum_{\xi \in G_q^m} g(\omega\,\xi)\,\xi^{-\alpha}.$$

From the previous expression we derive that for each $\omega \in S := \{\omega\, ; \, |\omega_i|_p = 1\}$ there exists $\xi_\omega \in G_q^m$ such that $|g|_p \leq \max_\xi |g(\omega\,\xi)|_p = |g(\omega\,\xi_\omega)|_p$. But on the other hand $|g(\omega\,\xi_\omega)|_p \leq \max_\alpha |a_\alpha|_p = |g|_p$. Thus $|g|_p = |g(\omega\,\xi_\omega)|_p$.

The set S is Zariski dense in $\mathbb{A}^m(\mathbb{C}_p)$, and as G_q^m is finite, the set $\{\omega\,\xi_\omega; \ \omega \in S\} \cap \Omega$ is also dense, and in particular it is non empty. For any ν_0 in this set we have $|g|_p = |g(\nu_0)|_p$ and therefore:

$$|g|_p \leq \max\{|g(\nu)|_p; \ \nu \in \Omega, \ |\nu|_p = 1\}.$$

The other inequality is straightforward. □

Proof of Lemma 2.2.– First we consider the case when V is a 0–dimensional variety. We may assume without loss of generality that V is irreducible, that is $V = \{\xi\}$ for some $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}_v^n$.
Set $d := \deg f$. Then

$$\mathcal{C}h_V = L(\xi) := U_0 + U_1\xi_1 + \cdots + U_n\xi_n \quad , \quad \mathcal{C}h_{d,V} = F(\xi) := \sum_\alpha U_\alpha\, \xi^\alpha,$$

where L and F denote generic polynomials in n variables of degree 1 and d respectively. Then

$$\begin{aligned} h_\infty(F(\xi)) &= \log \max_{|\alpha| \leq d}\{|\xi^\alpha|\} \\ &= \log \max_i\{1, |\xi_i|^d\} \\ &= d\, h_\infty(L(\xi)) \\ &\leq d\, m(L(\xi); S_{n+1}) + d\left(\sum_{i=1}^n 1/2\, i\right). \end{aligned}$$

This last line follows from inequality (1.2). Now, a direct computation shows that

$$h_\infty(\mathcal{C}h_{d,V}(f)) \le h_\infty(F(\xi)) + h_\infty(f) + \log(n+1)\, d.$$

In this case $\mathcal{C}h_{d,V}(f) \in \mathbb{C}$ and so

$$
\begin{aligned}
m(\mathcal{C}h_{d,V}(f)) &= h_\infty(\mathcal{C}h_{d,V}(f)) \\
&\le d\, m(L(\xi); S_{n+1}) + d\,\Big(\sum_{i=1}^{n} 1/2\, i\Big) + h_\infty(f) + \log(n+1)\, d \\
&\le d\, h_\infty(V) + h_\infty(f) + \log(n+1)\, d.
\end{aligned}
$$

Analogously, $h_p(F(\xi)) \le d\, h_p(L(\xi))$ and so $h_p(\mathcal{C}h_{d,V}(f)) \le d\, h_p(V) + h_p(f)$.

Now we consider the general case. Set $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{C}_v^{r\,(n+1)}$, $L(\nu_i) := \nu_{i\,0} + \nu_{i\,1}\, x_1 + \cdots + \nu_{i\,n}\, x_n$ and

$$V(\nu) := V \cap V(L(\nu_1), \ldots, L(\nu_r)) \subset \mathbb{A}^n(\mathbb{C}_v).$$

Then $V(\nu)$ is a 0-dimensional variety of degree $\deg V$ for ν in a Zariski open set Ω_v of $\mathbb{A}^{r\,(n+1)}(\mathbb{C}_v)$. Let $\nu \in \Omega$. By [47, Prop. 2.4] there exist $\lambda_\nu, \theta_\nu \in \mathbb{C}_v^*$ such that

$$\mathcal{C}h_{V(\nu)}(U_0) = \lambda_\nu\, \mathcal{C}h_V(U_0, \nu) \qquad , \qquad \mathcal{C}h_{d,V(\nu)}(U(d)_0) = \theta_\nu\, \mathcal{C}h_{d,V}(U(d)_0, \nu), \tag{2.1}$$

where $\mathcal{C}h_V(U_0, \nu), \mathcal{C}h_{d,V}(U(d)_0, \nu)$ stand for the specialization of $U_1, \ldots, U_r$ into $\nu_1, \ldots, \nu_r$. Applying the morphism ϱ_d linking the d-Chow form with the usual one we obtain

$$\mathcal{C}h^d_{V(\nu)} = \varrho_d(\mathcal{C}h_{d,V(\nu)}) = \theta_\nu\, \varrho_d(\mathcal{C}h_{d,V}(\nu)) = \theta_\nu\, \mathcal{C}h^d_V(\nu)$$

and so, $\theta_\nu = \lambda_\nu^d$ in identities (2.1).

We consider the case $v = \infty$. Any Zariski closed set of $\mathbb{A}^{r\,(n+1)}(\mathbb{C})$ intersects S_{n+1}^r in a set of μ_{n+1}^r-measure 0, and so the previous relation holds for almost every $\nu \in S_{n+1}^r$, which means that for those ν, $\mathcal{C}h_{d,V}(f, \nu) = \mathcal{C}h_{d,V(\nu)}(f)/\lambda_\nu^d$. Therefore

$$
\begin{aligned}
m(\mathcal{C}h_{d,V}(f); S_{n+1}^r) &= \int_{S_{n+1}^r} \log|\mathcal{C}h_{d,V}(f, \nu)|\, \mu_{n+1}^r \\
&= \int_{S_{n+1}^r} (\log|\mathcal{C}h_{d,V(\nu)}(f)| - d \log|\lambda_\nu|)\, \mu_{n+1}^r \\
&\le \int_{S_{n+1}^r} (d\, h_\infty(V(\nu)) + h_\infty(f) \deg V(\nu) + \log(n+1)\, d \deg V(\nu) - d \log|\lambda_\nu|)\, \mu_{n+1}^r \\
&= d \int_{S_{n+1}^r} (m(\mathcal{C}h_{V(\nu)}(U_0); S_{n+1}) - \log|\lambda_\nu|)\, \mu_{n+1}^r + \Big(\sum_{i=1}^{n} 1/2\, i\Big) d \deg V + h_\infty(f) \deg V \\
&\quad + \log(n+1)\, d \deg V \\
&= d \int_{S_{n+1}^r} m(\mathcal{C}h_V(U_0, \nu); S_{n+1})\, \mu_{n+1}^r + \Big(\sum_{i=1}^{n} 1/2\, i\Big) d \deg V + h_\infty(f) \deg V \\
&\quad + \log(n+1)\, d \deg V \\
&= d\, h_\infty(V) + h_\infty(f) \deg V + \log(n+1)\, d \deg V - r \Big(\sum_{i=1}^{n} 1/2\, i\Big) d \deg V.
\end{aligned}
$$

The case $v = p$ follows analogously from the previous lemma, identities (2.1) and the 0-dimensional case:

As before, let $\Omega_v \subset \mathbb{A}^{r\,(n+1)}(\mathbb{C}_v)$ be a Zariski open set such that $\nu \in \Omega_v$ implies that $V(\nu)$ is a 0-dimensional variety of degree $\deg V$. By Lemma 2.3 we can take $\nu \in \Omega_v$ such that $\log |\nu|_p = 1$ and

$$|\mathcal{C}h_{d,V}(f)|_p = |\mathcal{C}h_{d,V}(f,\nu)|_p.$$

Thus

$$
\begin{aligned}
\log |\mathcal{C}h_{d,V}(f)|_p &= \log |\mathcal{C}h_{d,V(\nu)}(f)|_p - d \log |\lambda_\nu|_p \\
&\leq d \log |\mathcal{C}h_{V(\nu)}|_p + h_p(f) \deg V - d \log |\lambda_\nu|_p \\
&= d \log |\mathcal{C}h_V(U_0,\nu)|_p + h_p(f) \deg V \\
&\leq d \log |\mathcal{C}h_V|_p + h_p(f) \deg V.
\end{aligned}
$$

$\square$

The hypothesis that V satisfies Assumption 1.5 is essential in order to properly normalize the involved Chow forms and to define the local height of V. If we disregard normalization, we obtain altogether the following global result:

Lemma 2.4 *Let $V \subset \mathbb{A}^n$ be an equidimensional variety of dimension r defined over a number field K, and let $\mathcal{F}_{d,V}$ be a d-Chow form of V. Let $f \in K[x_1,\ldots,x_n]$ be a polynomial of degree d. Then*

$$
\frac{1}{[K:\mathbb{Q}]}\Big(\sum_{v \in M_K^\infty} N_v\, m(\sigma_v(\mathcal{F}_{d,V}(f)); S_{n+1}^r) + \sum_{v \notin M_K^\infty} N_v \log |\mathcal{F}_{d,V}(f)|_v \Big) + r\left(\sum_{i=1}^n 1/2\,i\right) d \deg V
$$
$$
\leq d\,h(V) + h(f) \deg V + \log(n+1)\,d \deg V.
$$

Proof.– Note first that the product formula implies that the left hand side of the inequality does not depend on the choice of the d-Chow form $\mathcal{F}_{d,V}$.

In case V is 0-dimensional, it satisfies Assumption 1.5 trivially. Thus the result follows from direct application of the previous lemma.

For the general case, we let $\mathcal{F}_{d,V}$ be an arbitrary $d-$Chow form of V and we choose $\mathcal{F}_V$ so that $\varrho_d(\mathcal{F}_{d,V}) = \mathcal{F}_V^d$ holds.

Fix an absolute value $v \in M_K$. Following the notation in the proof of the previous lemma, for any $\nu \in \Omega_v$ there exists $\lambda_\nu \in \mathbb{C}_v^*$ such that

$$\mathcal{C}h_{V(\nu)}(U_0) = \lambda_\nu\, \mathcal{F}_V(U_0,\nu) \qquad , \qquad \mathcal{C}h_{d,V(\nu)}(U(d)_0) = \lambda_\nu^d\, \mathcal{F}_{d,V}(U(d)_0,\nu).$$

We then proceed as in the previous lemma, and we obtain the corresponding estimate for v. Adding up these estimates we derive the estimate in terms of the height of the variety. $\square$

2.2 Basic properties of the height

We derive some of the basic properties of the notion of height of a variety. In particular, we study the behavior of the height of a variety under intersection with a hypersurface and under an affine map.

We also obtain an arithmetic version of the Bernstein-Kushnirenko theorem.

2.2.1 Height of varieties under affine maps

Let $\varphi : \mathbb{A}^n \to \mathbb{A}^m$ be a regular map defined by polynomials $\varphi_1,\ldots,\varphi_m \in K[x_1,\ldots,x_n]$. We recall that the height of φ is defined as $h(\varphi) := h(\varphi_1,\ldots,\varphi_m)$.

We obtain the following estimate for the height of the image of a variety under an affine map:

Proposition 2.5 *Let $V \subset \mathbb{A}^n$ be a variety of dimension r, and let $\varphi : \mathbb{A}^n \to \mathbb{A}^N$ be an affine map. Then*

$$h(\varphi(V)) \leq h(V) + (r+1)\,(h(\varphi) + 8\,\log(n+N+1))\,\deg V.$$

The proof of this result follows from the study of the particular cases of a linear projection and an injective affine map.

The following estimate for the height of a linear projection of a variety generalizes [11, Prop. 2.10] and [5, 3.3.2]. Its proof is essentially based on the description of the Chow form of such projection variety, due to Pedersen and Sturmfels [46, Prop. 4.1].

Lemma 2.6 *Let $V \subset \mathbb{A}^n \times \mathbb{A}^m$ be a variety of dimension r, and let $\pi : \mathbb{A}^n \times \mathbb{A}^m \to \mathbb{A}^n$ denote the projection $(x,y) \mapsto x$. Then*

$$h(\overline{\pi(V)}) \leq h(V) + 3\,(r+1)\,\log(n+m+1)\,\deg V.$$

Proof. We assume without loss of generality that V is irreducible. Set $W := \overline{\pi(V)} \subset \mathbb{A}^n$ and $s := \dim W$.

The case $s = r$ follows directly from [46, Prop. 4.1]: in this case, there exists a partial monomial order $\prec$ such that

$$\mathcal{F}_W \,|\, \text{init}\mathcal{F}_V,$$

where $\text{init}\mathcal{F}_V$ denotes the initial polynomial of $\mathcal{F}_V$ with respect to $\prec$. In particular $\text{init}\mathcal{F}_V$ is the sum of some of the terms in the monomial expansion of $\mathcal{F}_V$.

The general case $s \leq r$ reduces to the previous one: we choose standard coordinates $z_{s+1}, \ldots, z_r$ of $\mathbb{A}^m$ such that the projection

$$\varpi : \mathbb{A}^n \times \mathbb{A}^m \to \mathbb{A}^n \times \mathbb{A}^{r-s} \qquad , \qquad (x,y) \mapsto (x,z)$$

verifies $\dim Z = r$ for $Z := \overline{\varpi(V)}$.

Let $\varrho : \mathbb{A}^n \times \mathbb{A}^{r-s} \to \mathbb{A}^n$ denote the canonical projection. Then $\mathcal{F}_Z \,|\, \text{init}\mathcal{F}_V$, $\pi = \varrho \circ \varpi$ and $W = \overline{\varrho(Z)}$. We have that $\varrho^{-1}(\xi) = \{\xi\} \times \mathbb{A}^{r-s}$ for $\xi \in \varrho(Z)$ by the theorem of dimension of fibers. Thus $Z = W \times \mathbb{A}^{r-s}$, and in particular

$$i(W) = Z \cap V(z_{s+1}, \ldots, z_r) \subset \mathbb{A}^n \times \mathbb{A}^{r-s},$$

where i denotes the canonical inclusion $\mathbb{A}^n \hookrightarrow \mathbb{A}^n \times \mathbb{A}^{r-s}$. We have $\deg W = \deg Z$ and so $\mathcal{F}_W := \mathcal{F}_Z(z_{s+1}, \ldots, z_r)$ is a Chow form of W [47, Prop. 2.4].

Now we estimate the height of $\mathcal{F}_W$. Let K be a number field of definition of V, and set

$$\text{init}\mathcal{F}_V = Q\,\mathcal{F}_Z$$

for some polynomial Q. From the proof of [47, Lem. 1.12(v)], there is a non-zero coefficient λ of Q such that $\log|\lambda|_v \leq m(\sigma_v(Q))$ for all $v \in M_K^\infty$. Clearly $\log|\lambda|_v \leq \log|Q|_v$ also holds for all $v \notin M_K^\infty$. Thus

$$m(\sigma_v(\mathcal{F}_Z)) \leq m(\sigma_v(\text{init}\mathcal{F}_V)) - \log|\lambda|_v$$

for $v \in M_K^\infty$, while $\log|\mathcal{F}_Z|_v \leq \log|\text{init}\mathcal{F}_V|_v - \log|\lambda|_v$ for $v \notin M_K^\infty$.

Let $v \in M_K^\infty$. From [47, Lem. 1.13] we obtain $m(\sigma_v(\mathcal{F}_W)) \leq m(\sigma_v(\mathcal{F}_Z))$. Hence

$$
\begin{aligned}
m(\sigma_v(\mathcal{F}_W); S_{n+1}^{s+1}) &\leq m(\sigma_v(\mathcal{F}_W)) \\
&\leq m(\sigma_v(\text{init}\mathcal{F}_V)) - \log|\lambda|_v \\
&\leq \log|\text{init}\mathcal{F}_V|_v + (r+1)\log(n+m+1)\deg V - \log|\lambda|_v \\
&\leq \log|\mathcal{F}_V|_v + (r+1)\log(n+m+1)\deg V - \log|\lambda|_v \\
&\leq m(\sigma_v(\mathcal{F}_V); S_{n+m+1}^{r+1}) + (r+1)\,\Big(\sum_{i=1}^{n+m} 1/2\,i\Big)\deg V \\
&\qquad + 2\,(r+1)\log(n+m+1)\deg V - \log|\lambda|_v
\end{aligned}
$$

by application of Lemma 1.1 and inequality (1.2). In case $v \notin M_K^\infty$ we have analogously $\log |\mathcal{F}_W|_v \leq \log |\mathcal{F}_V|_v - \log |\lambda|_v$, and so

$$
\begin{aligned}
h(W) &\leq h(V) + (s+1)\,(\sum_{i=1}^{n} 1/2\,i)\,\deg V + 2\,(r+1)\,\log(n+m+1)\,\deg V \\
&\leq h(V) + 3\,(r+1)\,\log(n+m+1)\,\deg V.
\end{aligned}
$$

$\square$

The following is a variant of [49, I, Prop. 7]:

Lemma 2.7 *Let $V \subset \mathbb{A}^m$ be a variety of dimension r, and let $\psi : \mathbb{A}^m \to \mathbb{A}^n$ be an injective affine map. Then*

$$
h(\psi(V)) \leq h(V) + (r+1)\,(h(\psi) + 5\,\log(n+1))\,\deg V.
$$

Proof.– We assume again without loss of generality that V is irreducible. Let K be a number field of definition of both V and ψ, and set $\psi(x) = a + A\,x$ for some $m \times n-$ matrix A of maximal rank and $a \in K^n$. Then let $\psi^* : \mathbb{A}^{n+1} \to \mathbb{A}^{m+1}$ be the linear map $y \mapsto (a, A)^t\,y$ defined by the transpose of the matrix associated to ψ.

Set $W := \psi(V)$, and let $\overline{V} \subset \mathbb{P}^m$, $\overline{W} \subset \mathbb{P}^n$ denote the projective closures of V and W respectively.

For $i = 0, \ldots, r$ we let $\nu_i \in \overline{\mathbb{Q}}^{n+1}$, and we set $L^h(\nu_i) := \nu_{i\,0}\,x_0 + \ldots + \nu_{i\,n}\,x_n$ for the homogenization of the associated linear form. Then $\mathcal{F}_W(\nu_0, \ldots, \nu_r) = 0$ if and only if there exists $\xi \in \overline{V}$ such that $\psi(\xi)$ lies in the linear space determined by $\nu_0, \ldots, \nu_r$. Equivalently ξ lies in the linear space determined by $\psi^*(\nu_0), \ldots, \psi^*(\nu_r)$. We conclude that

$$
\mathcal{F}_W = \mathcal{F}_V \circ (\psi^*)^{r+1}.
$$

Let $v \in M_K^\infty$. Then

$$
\begin{aligned}
m(\sigma_v(\mathcal{F}_W), S_{n+1}^{r+1}) &\leq \log |\mathcal{F}_W|_v + (r+1)\,\log(n+1)\,\deg V \\
&\leq \log |\mathcal{F}_V|_v + (r+1)\,(h_v(\psi) + 2\,\log(n+1))\,\deg V + (r+1)\,\log(n+1)\,\deg V \\
&\leq m(\sigma_v(\mathcal{F}_V)) + (r+1)\,\log(m+1)\,\deg V + (r+1)\,(h_v(\psi) + 3\,\log(n+1))\,\deg V \\
&\leq m(\sigma(\mathcal{F}_V), S_{m+1}^{r+1}) + (\sum_{i=1}^{m} 1/2\,i)\,(r+1)\,\deg V \\
&\quad + (r+1)\,(h_v(\psi) + 4\,\log(n+1))\,\deg V.
\end{aligned}
$$

Here we have applied Lemma 1.1, inequality (1.2) and the proof of Lemma 1.2(c), using the fact that the number of monomials of $\mathcal{F}_V$ is bounded by $(n+1)^{(r+1)\,\deg V}$.

In case $v \notin M_K^\infty$ we obtain analogously $\log |\mathcal{F}_W|_v \leq \log |\mathcal{F}_V|_v + (r+1)\,h_v(\psi)\,\deg V$, and hence

$$
h(\psi(V)) \leq h(V) + (r+1)\,(h(\psi) + 5\,\log(n+1))\,\deg V.
$$

$\square$

Proof of Proposition 2.5.– Let $\psi : \mathbb{A}^n \to \mathbb{A}^N \times \mathbb{A}^n$ be the injective map $x \mapsto (\varphi(x), x)$. Then φ decomposes as

$$
\varphi = \pi \circ \psi,
$$

where $\pi : \mathbb{A}^N \times \mathbb{A}^n \to \mathbb{A}^N$ denotes the canonical projection. Thus

$$
\begin{aligned}
h(\varphi(V)) \;&\leq\; h(\psi(V)) + 3\,(r+1)\,\log(n+N+1)\,\deg\psi(V)\\[4pt]
&\leq\; h(V) + (r+1)\,(h(\psi) + 5\,\log(n+N+1))\,\deg V + 3\,(r+1)\,\log(n+N+1)\,\deg V\\[4pt]
&=\; h(V) + (r+1)\,(h(\varphi) + 8\,\log(n+N+1))\,\deg V.
\end{aligned}
$$

$\square$

2.2.2   Local height of the intersection of varieties

We obtain the following estimate for the local height of the intersection of a variety with an hypersurface. This is a consequence of our previous estimate for generalized Chow forms. This result can be seen as the local analogue of [47, Prop. 2.8], and its proof closely follows it.

Proposition 2.8 *Let $V \subset \mathbb{A}^n$ be an equidimensional variety of dimension r defined over a number field K. Let $f \in K[x_1, \ldots, x_n]$ be a polynomial which is not a zero-divisor in $K[V]$. We assume that both V and $V \cap V(f)$ satisfy Assumption 1.5.*
Then there exists $\lambda \in K^*$ such that

- $h_v(V \cap V(f)) \leq \deg f\, h_v(V) + h_v(f)\,\deg V + \log(n+1)\,\deg f\,\deg V - \log|\lambda|_v \quad$ *for $v \in M_K^\infty$,*

- $h_v(V \cap V(f)) \leq \deg f\, h_v(V) + h_v(f)\,\deg V - \log|\lambda|_v \quad$ *for $v \notin M_K^\infty$.*

Proof.– Set $d := \deg f$ and $W := V \cap V(f) \subset \mathbb{A}^n$. By [47, Prop. 2.4] there exists $Q \in K[U_1, \ldots, U_r] \setminus \{0\}$ such that $\mathcal{C}h_{d,V}(f) = Q\,\mathcal{C}h_W$. Then — as in the proof of Lemma 2.6 — there exists a non-zero coefficient λ of Q such that $\log|\lambda|_v \leq m(\sigma_v(Q))$ for all $v \in M_K^\infty$ and $\log|\lambda|_v \leq \log|Q|_v$ for all $v \notin M_K^\infty$.
Now let $v \in M_K^\infty$. From inequality (1.2) we obtain

$$
\log|\lambda|_v \leq m(\sigma_v(Q)) \leq m(\sigma_v(Q); S_{n+1}^r) + r\,(\sum_{i=1}^n 1/2\,i)\,(d\,\deg V - \deg W)
$$

since Q has degree $d\,\deg V - \deg W$ in each group of variables. Then

$$
\begin{aligned}
h_v(W) \;&=\; m(\sigma_v(\mathcal{C}h_W); S_{n+1}^r) + r\,(\sum_{i=1}^n 1/2\,i)\,\deg W\\[4pt]
&=\; m(\sigma_v(\mathcal{C}h_{d,V}(f)); S_{n+1}^r) + r\,(\sum_{i=1}^n 1/2\,i)\,d\,\deg V\\[4pt]
&\quad - m(\sigma_v(Q); S_{n+1}^r) - r\,(\sum_{i=1}^n 1/2\,i)\,(d\,\deg V - \deg W)\\[4pt]
&\leq\; d\,h_v(V) + h_v(f)\,\deg V + \log(n+1)\,d\,\deg V - \log|\lambda|_v.
\end{aligned}
$$

by straightforward application of Lemma 2.2. The case $v \notin M_K^\infty$ follows in an analogous way. $\square$

Proposition 2.8 can be immediately generalized to families of polynomials:

Corollary 2.9 *Let $V \subset \mathbb{A}^n$ be an equidimensional variety of dimension r defined over K. Let $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$ be polynomials which form a complete intersection in V. We assume that $V \cap V(f_1, \ldots, f_i)$ satisfies Assumption 1.5 for $i = 0, \ldots, s$. Set $d_i := \deg f_i$.*
Then there exists $\lambda \in K^*$ such that

- $h_v(V \cap V(f_1, \ldots, f_s)) \leq (h_v(V) + (\sum_i h_v(f_i)/d_i) \deg V + s \log(n+1) \deg V) \prod_i d_i - \log |\lambda|_v$

  for $v \in M_K^\infty$,

- $h_v(V \cap V(f_1, \ldots, f_s)) \leq (h_v(V) + (\sum_i h_v(f_i)/d_i) \deg V) \prod_i d_i - \log |\lambda|_v$ for $v \notin M_K^\infty$.

Proof.– We just consider the case when v is archimedean, as the other one follows similarly. From the preceding result we obtain

$$
\begin{aligned}
h_v(V \cap V(f_1, \ldots, f_i)) \quad \leq \quad & d_i\, h_v(V \cap V(f_1, \ldots, f_{i-1})) + h_v(f_i) \deg(V \cap V(f_1, \ldots, f_{i-1})) \\
& + \log(n+1)\, d_i\, \deg(V \cap V(f_1, \ldots, f_{i-1})) - \log |\lambda_i|_v
\end{aligned}
$$

for some $\lambda_i \in K^*$. For the final estimate we apply iteratively this inequality and we set $\lambda := \prod_{i=1}^s \lambda_i^{d_{i+1} \cdots d_s}$. $\qquad\square$

Corollary 2.10 *Let $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$ be polynomials which form a complete intersection in $\mathbb{A}^n$. We assume that $V(f_1, \ldots, f_i)$ satisfies Assumption 1.5 for $i = 1, \ldots, s$. Set $d_i := \deg f_i$. Then there exists $\lambda \in K^*$ such that:*

- $h_v(V(f_1, \ldots, f_s)) \leq (\sum_i h_v(f_i)/d_i + (n+s) \log(n+1)) \prod_i d_i - \log |\lambda|_v$ *for* $v \in M_K^\infty$,

- $h_v(V(f_1, \ldots, f_s)) \leq (\sum_i h_v(f_i)/d_i) \prod_i d_i - \log |\lambda|_v$ *for* $v \notin M_K^\infty$.

Proof.– We apply the previous result to $V := \mathbb{A}^n$, using the fact that

$$
h_\infty(\mathbb{A}^n) = \sum_{i=1}^n \sum_{j=1}^i 1/2\, j \leq n \log(n+1) \qquad , \qquad h_p(\mathbb{A}^n) = 0.
$$

$\qquad\square$

The following is the global counterpart of the previous results. It can be seen as an arithmetic analogue of [24, Prop. 2.3].

We remark that in the global situation, we do not need to assume Assumption 1.5 for the intermediate varieties. In particular $f_1, \ldots, f_s$ do not need to be a complete intersection in V.

Corollary 2.11 *Let $V \subset \mathbb{A}^n$ be a variety of dimension r, and let $f_1, \ldots, f_s \in \overline{\mathbb{Q}}[x_1, \ldots, x_n]$. Set $d_i := \deg f_i$, $h := h(f_1, \ldots, f_s)$ and $n_0 := \min\{r, s\}$. We assume that $d_1 \geq \ldots \geq d_s$ holds. Then:*

$$
h(V \cap V(f_1, \ldots, f_s)) \leq (h(V) + (\sum_{i=1}^{n_0} 1/d_i)\, h \deg V + n_0 \log(n+1) \deg V) \prod_{i=1}^{n_0} d_i.
$$

Proof.– We proceed by induction on (r, s) with respect to the product order on $\mathbb{N} \times \mathbb{N}$ defined by $(r, s) \succ (r', s') \Leftrightarrow r \geq r'$ and $s \geq s'$.

The case when $r = 0$ or $s = 0$ are both trivial. Now let $r, s \geq 1$, and we assume that the statement holds for all $(r', s') \prec (r, s)$ such that $(r', s') \neq (r, s)$.

Let $V = \cup_C C$ be the decomposition of V into irreducible components. In case $C \subset V(f_s)$ we have that $C \cap V(f_1, \ldots, f_s) = C \cap V(f_1, \ldots, f_{s-1})$ and by the inductive hypothesis:

$$
h(C \cap V(f_1, \ldots, f_s)) \leq (h(C) + (\sum_{i=1}^{m_0} 1/d_i)\, h \deg C + m_0 \log(n+1) \deg C) \prod_{i=1}^{m_0} d_i
$$

with $m_0 := \min\{r, s-1\}$.

In case $C \not\subset V(f_s)$ we have either $C \cap V(f_s) = \emptyset$ or $\dim(C \cap V(f_s)) \le r - 1$. The first case is trivial. For the second case, we proceed as in the proof of Lemma 2.8 applying Lemma 2.4 instead of Lemma 2.2, and we obtain

$$h(C \cap V(f_s)) \le d_s \, h(C) + h \deg C + \log(n+1) \, d \deg C.$$

Since then $\dim(C \cap V(f_s)) = r - 1$, we can apply the inductive hypothesis to the variety $C \cap V(f_s)$ and we obtain

$$
\begin{aligned}
h(C \cap V(f_1, \ldots, f_s)) &\le (h(C \cap V(f_s)) + (\sum_{i=1}^{n_0-1} 1/d_i) \, h \deg(C \cap V(f_s)) \prod_{i=1}^{n_0-1} d_i \\
&\quad + (n_0 - 1) \log(n+1) \deg(C \cap V(f_s))) \\
&\le (h(C) + (\sum_{i=1}^{n_0} 1/d_i) \, h \deg C + n_0 \log(n+1) \deg C) \prod_{i=1}^{n_0} d_i.
\end{aligned}
$$

Finally

$$
\begin{aligned}
h(V \cap V(f_1 \cap \cdots \cap f_s)) &\le \sum_C h(C \cap V(f_1 \cap \cdots \cap f_s)) \\
&\le \sum_C (h(C) + (\sum_{i=1}^{n_0} 1/d_i) \, h \deg C + n_0 \log(n+1) \deg C) \prod_{i=1}^{n_0} d_i \\
&= (h(V) + (\sum_{i=1}^{n_0} 1/d_i) \, h \deg V + n_0 \log(n+1) \deg V) \prod_{i=1}^{n_0} d_i.
\end{aligned}
$$

$\square$

With the same notations of Corollary 2.11, for $V := \mathbb{A}^n$ we obtain

$$h(V(f_1, \ldots, f_s)) \le ((\sum_{i=1}^{n_0} 1/d_i) \, h + (n + n_0) \log(n+1)) \prod_{i=1}^{n_0} d_i.$$

2.2.3 An arithmetic Bernstein-Kushnirenko theorem

From our estimate for the height of an affine toric variety (Proposition 1.7) and the previous results of this section we derive the following arithmetic version of the Bernstein-Kushnirenko theorem. We refer to Subsection 1.2.5 for the notation.

Proposition 2.12 *Let* $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$, *and set* $\mathcal{A} := \mathrm{Supp}(1, x_1, \ldots, x_n, f_1, \ldots, f_s) \subset (\mathbb{Z}_{\ge 0})^n$. *Also set* $d := \max_i \deg f_i$ *and* $h := h(f_1, \ldots, f_s)$. *Then*

- $\deg V(f_1, \ldots, f_s) \le \mathrm{Vol}(\mathcal{A})$,

- $h(V(f_1, \ldots, f_s)) \le (n \, h + 2^{2(n+1)} \log(n+1) \, d) \, \mathrm{Vol}(\mathcal{A})$.

Proof. – Set $\mathcal{A} := \{\alpha_1, \ldots, \alpha_N\}$. The case $N = 1$ is trivial, and so we assume $N \ge 2$. We also assume that $\alpha_1, \ldots, \alpha_n$ are the vectors of the canonical basis of $\mathbb{R}^n$.

The map $\varphi_{\mathcal{A}} : \mathbb{A}^n \to \mathbb{A}^N$ induces an isomorphism between $\mathbb{A}^n$ and the affine toric variety $X_{\mathcal{A}} \subset \mathbb{A}^N$. The projection map $\pi_{\mathcal{A}} : \mathbb{A}^N \to \mathbb{A}^n$ defined by $y \mapsto (y_1, \ldots, y_n)$ restricted to $X_{\mathcal{A}}$ is the inverse map of $\varphi_{\mathcal{A}}$.

For $i = 1, \ldots, s$ we set $f_i = \sum_{j=1}^N a_{ij} x^{\alpha_j}$ and we let

$$\ell_i := \sum_{j=1}^N a_{ij} y_j \in K[y_1, \ldots, y_N]$$

be the associated linear form. Set $V := V(f_1, \ldots, f_s) \subset \mathbb{A}^n$ and $W := X_{\mathcal{A}} \cap V(\ell_1, \ldots, \ell_s) \subset \mathbb{A}^N$. We have $\varphi_{\mathcal{A}}(V) = W$ and so $V = \pi_{\mathcal{A}}(W)$. Then

$$\deg V \le \deg W \le \deg X_{\mathcal{A}} = \mathrm{Vol}(\mathcal{A})$$

and

$$
\begin{aligned}
h(V) \quad &\le \quad h(W) + 3\,(n+1)\log(N+1)\deg W \\
&\le \quad h(X_{\mathcal{A}}) + n\,h\,\deg(X_{\mathcal{A}}) + 4\,(n+1)\log(N+1)\deg(X_{\mathcal{A}}) \\
&\le \quad (n\,h + (2^{2\,n+1}\log N + 4\,(n+1)\log(N+1)))\,\mathrm{Vol}(\mathcal{A}).
\end{aligned}
$$

by successive application of Lemma 2.6, Corollary 2.11 and Proposition 1.7. Finally $N \le \binom{d+n}{n}$ and so $h(V) \le (n\,h + 2^{2(n+1)}\log(n+1)\,d)\,\mathrm{Vol}(\mathcal{A})$. $\qquad\square$

It seems that the factor $2^{2\,n}$ in the estimate of $h(X_A)$ is superfluous. If this is the case, the above estimate can be considerably improved. Maillot has recently obtained another estimate for the height of the *isolated* points of $V(f_1, \ldots, f_s)$, which is more precise in some particular cases [42, Cor. 8.2.3].

2.3 Local height of norms and traces

Let $V \subset \mathbb{A}^n$ be an equidimensional variety of dimension r and degree D defined over a field k which satisfies Assumption 1.5. As we will see below, this implies that the projection $\pi_V : V \to \mathbb{A}^r$ defined by $x \mapsto (x_1, \ldots, x_r)$ is finite (Lemma 2.14). Set $L := k(\mathbb{A}^r)$ and $M := L \otimes_{k[\mathbb{A}^r]} k[V]$, so that M is a finite L-algebra of dimension D.

Let $f \in k[x_1, \ldots, x_n]$. We identify $\overline{f} \in k[V]$ with the multiplication map $M \to M$ defined by $q \mapsto f\,q$. ¿From Hamilton-Cayley theorem, we derive that the *characteristic polynomial* $\mathcal{X}_f \in L[t]$ of this map verifies $\mathcal{X}_f(\overline{f}) = 0$.

The fact that the inclusion $\pi_V^* : k[\mathbb{A}^r] \hookrightarrow k[V]$ is integral implies that the minimal polynomial m_f of this map lies in $k[\mathbb{A}^r][t]$. We have that $\mathcal{X}_f \mid m_f^D$ in $L[t]$, and so Gauss lemma implies that $\mathcal{X}_f$ lies in fact in $k[\mathbb{A}^r][t]$.

Moreover the natural map $k[V] \to M$ is an inclusion, as V is an equidimensional variety, and so $\mathcal{X}_f(\overline{f}) = 0$ in $k[V]$.

Set $\mathcal{X}_f = t^D + b_{D-1}\,t^{D-1} + \cdots + b_0 \in k[\mathbb{A}^r][t]$. Then the *norm* $\mathrm{N}_V(f)$ and the *trace* $\mathrm{Tr}_V(f)$ of f are defined as

$$\mathrm{N}_V(f) := (-1)^D\,b_0 \in k[\mathbb{A}^r] \qquad , \qquad \mathrm{Tr}_V(f) := -b_{D-1} \in k[\mathbb{A}^r].$$

They equal the determinant and the trace of the L-linear map $\overline{f} : M \to M$ respectively. We also define the *adjoint polynomial* f^* of f as

$$f^* := (-1)^{D-1}\,(f^{D-1} + b_{D-1}\,f^{D-2} + \cdots + b_1) \in k[x_1, \ldots, x_n].$$

From the identity $\mathcal{X}_f(\overline{f}) = 0$ we obtain that $\overline{f^* f} = \mathrm{N}_V(f)$ in $k[V]$.

The key result of this subsection is a precise bound for the height of the norm and the trace of a polynomial in case k is a number field.

2.3.1 Characteristic polynomials

Let $V \subset \mathbb{A}^n$ be an equidimensional variety of dimension r and degree D defined over k. We keep notations as in Subsection 2.1.1: for $d \in \mathbb{N}$ we denote by $F := \sum_{|\alpha| \le d} U(d)_{0\,\alpha}\, x^\alpha$ and $L_i := U_{i\,0} + U_{i\,1}\, x_1 + \cdots + U_{i\,n}\, x_n$ the generic polynomial of degree d and 1 associated to the group of variables $U(d)_0$ and U_i respectively.

As before, we set $U(d) := \{U(d)_0, U_1, \ldots, U_r\}$ and $N := \binom{d+n}{n} + r\,(n+1)$. Also we introduce an additional group $T := \{T_0, \ldots, T_r\}$ of $r+1$ variables which correspond to the coordinate functions of $\mathbb{A}^{r+1}$. We consider the map

$$\psi : \mathbb{A}^N \times \mathbb{A}^n \to \mathbb{A}^N \times \mathbb{A}^{r+1} \quad , \quad (\nu(d), \xi) \mapsto (\nu(d),\ F(\nu(d)_0)(\xi),\ L_1(\nu_1)(\xi), \ldots, L_r(\nu_r)(\xi)).$$

where $\nu(d) := (\nu(d)_0, \nu_1, \ldots, \nu_r) \in \mathbb{A}^N$ and $\xi \in \mathbb{A}^n$.

Then the Zariski closure $\overline{\psi(\mathbb{A}^N \times V)} \subset \mathbb{A}^N \times \mathbb{A}^{r+1}$ is a hypersurface, and any of its defining equations $\mathcal{P}_{d,V} \in k[U(d)][T]$ is called a d-*characteristic polynomial* of V. Also we define the *characteristic polynomial* of V by $\mathcal{P}_V := \mathcal{P}_{1,V}$.

A d-characteristic polynomial is uniquely defined up to a scalar factor. In case V is irreducible, $\overline{\psi(\mathbb{A}^N \times V)}$ is an irreducible hypersurface and thus $\mathcal{P}_{d,V}$ is an irreducible polynomial. When V is equidimensional, it coincides with the product of d-characteristic polynomials of its irreducible components.

The following construction links the characteristic polynomial of a variety with its generalized Chow form. Set

$$\zeta(d)_{0\alpha} := \begin{cases} U(d)_{00} - T_0 & \text{for } \alpha = 0 \\ U(d)_{0\alpha} & \text{for } \alpha \ne 0. \end{cases}$$

Analogously, for $i = 1, \ldots, r$ we set $\zeta_{i\,0} := U_{i\,0} - T_i$ and $\zeta_{i\,j} := U_{i\,j}$ for $j \ne 0$. Finally we set $\zeta(d) := (\zeta(d)_0, \zeta_1, \ldots, \zeta_r)$.

Lemma 2.13 *Let $V \subset \mathbb{A}^n$ be an equidimensional variety of dimension r and degree D. Let $\mathcal{F}_{d,V}$ be a d-Chow form of V. Then $\mathcal{F}_{d,V} \circ \zeta(d)$ is a d-characteristic polynomial of V.*

Proof.– It is enough to consider the case when V is irreducible.
Let $\mathcal{P}_{d,V}$ be a d-characteristic polynomial of V. For $(\nu(d), \xi) \in \mathbb{A}^N \times V$ we set

$$\vartheta := (F(\nu(d)_0)(\xi), L_1(\nu_1)(\xi), \ldots, L_r(\nu_r)(\xi)) \in \mathbb{A}^{r+1},$$

so that $\mathcal{P}_{d,V}(\nu(d))(\vartheta) = 0$. We observe that

$$\xi \in V \cap \{F(\nu(d)_0)(x) = \vartheta_0\} \cap \{L_1(\nu_1)(x) = \vartheta_1\} \cap \cdots \cap \{L_r(\nu_r)(x) = \vartheta_r\} \subset \mathbb{A}^n.$$

In particular, this variety is non-empty, and so we infer that $\mathcal{F}_{d,V} \circ \zeta(d)(\nu(d), \vartheta) = 0$. This implies that $\mathcal{P}_{d,V} | \mathcal{F}_{d,V} \circ \zeta(d)$ as $\mathcal{P}_{d,V}$ is an irreducible polynomial.

On the other hand $\mathcal{F}_{d,V} \circ \zeta(d)$ is also irreducible, as it is multihomogeneous and $\mathcal{F}_{d,V} \circ \zeta(d)(U(d), 0) = \mathcal{F}_{d,V}(U(d))$. We conclude that $\mathcal{P}_{d,V}$ and $\mathcal{F}_{d,V} \circ \zeta(d)$ coincide up to a factor in k^*. $\qquad \square$

The previous construction shows that a d-characteristic polynomial of V is multihomogeneous of degree D in the group of variables $U(d)_0 \cup \{T_0\}$ and of degree $d\,D$ in each group $U_i \cup \{T_i\}$.

Set $k_d := \overline{k(U(d))}$, and set

$$\phi : \mathbb{A}^n(k_d) \to \mathbb{A}^{r+1}(k_d) \quad , \quad x \mapsto (F(x), L_1(x), \ldots, L_r(x)).$$

Then $\mathcal{P}_{d,V} \in k_d[T]$ is also a minimal equation for the hypersurface $\overline{\phi(V)}$, and by Bézout inequality we have also $\deg_T \mathcal{P}_{d,V} \le d\,D$ (see e.g. [50, Prop.1]).

We assume from now on that V satisfies Assumption 1.5, that is $\#\pi_V^{-1}(0) = \deg V$. In order to avoid the indeterminacy of the d-characteristic polynomial, we fix it as

$$\mathcal{P}_{d,V} := (-1)^D \mathcal{C}h_{d,V} \circ \zeta(d).$$

In particular, we set $\mathcal{P}_V := (-1)^D \mathcal{C}h_V \circ \zeta(1)$ for *the* characteristic polynomial of V.

Set $\mathcal{P}_V := a_D T_0^D + \cdots + a_0$ for the expansion of $\mathcal{P}_V$ with respect to T_0. We have that $\mathcal{P}_V$ is multihomogeneous of degree D in each group $U_i \cup \{T_i\}$. This implies that a_D lies in fact in $k[U_1, \ldots, U_r]$ and is multihomogeneous of degree D in each U_i for $i = 1, \ldots, r$.

Moreover, a_D coincides with the coefficient of U_{00}^D in $\mathcal{C}h_V$, and the imposed normalization on $\mathcal{C}h_V$ implies that

$$a_D(e_1, \ldots, e_r) = \mathcal{C}h_V(e_0, e_1, \ldots, e_r) = 1.$$

We extend the morphism ϱ_d of Subsection 2.1.1 to a morphism $k[U(d)][T] \to k[U_0, \ldots, U_r][T]$ defining $\varrho_d(U(d)_{00} - T_0) := (U_{00} - T_0)^d$ and $\varrho_d(T_i) := T_i$ for $1 \le i \le r$. In other terms

$$\varrho_d(T_0) = \sum_{j=1}^{d} (-1)^{j-1} \binom{d}{j} U_{00}^{d-j} T_0^j.$$

From the previous lemma we obtain

$$\varrho_d(\mathcal{P}_{d,V}) = \varrho_d((-1)^D \mathcal{C}h_{d,V} \circ \zeta(d)) = (-1)^D (\mathcal{C}h_V \circ \zeta(1))^d = (-1)^{(d+1)D} \mathcal{P}_V^d.$$

Now set

$$\mathcal{P}_{d,V} = a_{d,D} T_0^D + \cdots + a_{d,0}$$

for the expansion of $\mathcal{P}_{d,V}$ with respect to T_0. The previous remark implies that $a_{d,D} = \varrho_d(a_{d,D}) = a_D^d$. In particular $a_{d,D} \in k[U_1, \ldots, U_r]$ and $a_{d,D}(e_1, \ldots, e_r) = 1$.

The following lemma allows us to obtain a characteristic polynomial of $f \in k[x_1, \ldots, x_n]$ from the d-characteristic polynomial of the variety V.

We introduce the following convention:
Given a polynomial $f \in k[x_1, \ldots, x_n]$ of degree d and linear forms $\ell_1, \ldots, \ell_r \in k[x_1, \ldots, x_n]$, we denote by $\mathcal{P}_{d,V}(f, \ell_1, \ldots, \ell_r)$ the specialization of the variables in $U(d)$ into the coefficients of $f, \ell_1, \ldots, \ell_r$.

Lemma 2.14 *Let $V \subset \mathbb{A}^n$ be an equidimensional variety of dimension r and degree D which satisfies Assumption 1.5. Then the projection $\pi_V : V \to \mathbb{A}^r$ is finite.*
Moreover, for a polynomial $f \in k[x_1, \ldots, x_n]$ of degree d, the characteristic polynomial of f is given by

$$\mathcal{X}_f = \mathcal{P}_{d,V}(f, e_1, \ldots, e_r)(t, x_1, \ldots, x_r) \in k[\mathbb{A}^r][t].$$

Proof.– We have that $\mathcal{P}_V(U_0, \ldots, U_r)(L_0, \ldots, L_r) = 0$ in $k[U] \otimes k[V]$ and so

$$\mathcal{P}_V(e_j, e_1, \ldots, e_r)(t, x_1, \ldots, x_r) \in k[\mathbb{A}^r][t]$$

is a monic equation for x_j in $k[V]$, for $j = r+1, \ldots, n$. Thus the projection π_V is finite.
For the second assertion, set

$$P_F(t) := \mathcal{P}_{d,V}(U(d)_0, e_1, \ldots, e_r)(t, x_1, \ldots, x_r) \in k[U(d)_0][\mathbb{A}^r][t].$$

This is a polynomial of degree D. It is monic with respect to t, as $a_{d,D} \in k[U_1, \ldots, U_r]$ and $a_{d,D}(e_1, \ldots, e_r) = 1$. We have $P_F(F) = 0$ in $k[U(d)_0] \otimes k[V]$.

Now let m_F be the monic minimal polynomial of F. Let $U'(d)_0$ be a group of $\binom{d+n-r}{n-r}$ variables and set F_0 for the generic polynomial of degree d in the variables $x_{r+1}, \ldots, x_n$.

29

Then
$$m_F(U'(d)_0, 0) \in k[U'(d)_0][t]$$

is an equation for F_0 over $\pi_V^{-1}(0)$. Since $\pi_V^{-1}(0)$ is a 0-dimensional variety of degree D and F_0 separates its points, we infer that $\deg_{T_0} m_F = D$, and so $P_F = m_F$.
Finally we obtain

$$\mathcal{X}_f = \mathcal{X}_F(f) = P_F(f) = \mathcal{P}_{d,V}(f, e_1, \ldots, e_r)(t, x_1, \ldots, x_r).$$

$\square$

2.3.2   Estimates for norms and traces

Finally we prove the announced estimates for the height of the norm and the trace of a polynomial.

Lemma 2.15 *Let $V \subset \mathbb{A}^n$ be an equidimensional variety of dimension r defined over K which satisfies Assumption 1.5. Let $f \in K[x_1, \ldots, x_n]$. Then*

- $\deg \mathrm{N}_V(f) \le \deg f \, \deg V$,

- $h_v(\mathrm{N}_V(f)) \le \deg f \, h_v(V) + h_v(f) \deg V + (r+1) \log(n+1) \deg f \deg V$ *for $v \in M_K^\infty$*,

- $h_v(\mathrm{N}_V(f)) \le \deg f \, h_v(V) + h_v(f) \deg V$ *for $v \notin M_K^\infty$*.

Proof.– We keep notations as in Subsection 2.3.1. Set $d := \deg f$ and $D := \deg V$. We have then

$$\mathrm{N}_V(f) = (-1)^D \, \mathcal{P}_{d,V}(f, e_1, \ldots, e_r)(0, x_1, \ldots, x_r) = \mathcal{C}h_{d,V}(f, e_1 - e_0 \, x_1, \ldots, e_r - e_0 \, x_r)$$

by Lemmas 2.14 and 2.13. Then

$$\deg \mathrm{N}_V(f) \le \deg_T \mathcal{P}_{d,V} \le d \, D.$$

From the previous expression we also obtain that the coefficients of $\mathrm{N}_V(f)$ are some of the coefficients of $\mathcal{C}h_{d,V}(f)$, and so $|\mathrm{N}_V(f)|_v \le |\mathcal{C}h_{d,V}(f)|_v$ for every absolute value v of K.
Let $v \in M_K^\infty$. Then

$$
\begin{aligned}
\log |\mathrm{N}_V(f)|_v &\le \log |\mathcal{C}h_{d,V}(f))|_v \\
&\le m(\sigma_v(\mathcal{C}h_{d,V}(f)); S_{n+1}^r) + r \, \Big(\sum_{i=1}^n 1/2 \, i\Big) d \, D + r \, \log(n+1) \, d \, D \\
&\le d \, h_v(V) + h_v(f) \, D + (r+1) \log(n+1) d \, D
\end{aligned}
$$

by inequalities (1.1) and (1.2), and Lemma 2.2. In a similar way we obtain $h_v(\mathrm{N}_V(f)) \le d \, h_v(V) + h_v(f) \, D$ for $v \notin M_K^\infty$. $\square$

The proof of the following lemma follows closely that of [50, Lem. 9]. We slightly improve the degree estimate obtained therein, and we get the corresponding height estimate.

Lemma 2.16 *Let $V \in \mathbb{A}^n$ be an equidimensional variety of dimension r defined over K which satisfies Assumption 1.5. Let $f, g \in K[x_1, \ldots, x_n]$ such that $\overline{f}$ is not a zero-divisor in $K[V]$. Set $d := \max\{\deg f, \deg g\}$ and $h_v := \max\{h_v(f), h_v(g)\}$ for $v \in M_K$. Then*

- $\deg \mathrm{Tr}_V(f^* g) \le d \, \deg V$,

- $h_v(\mathrm{Tr}_V(f^* g)) \le d \, h_v(V) + (h_v + \log 2) \deg V + (r+1) \log(n+1) d \, \deg V$ *for $v \in M_K^\infty$*,

- $h_v(\mathrm{Tr}_V(f^*g)) \leq d\, h_v(V) + h_v \deg V \quad$ *for* $v \notin M_K^\infty$.

Proof.– Let $D := \deg V$, and let t be a new variable. Then $K[x_1,\ldots,x_r,t] \hookrightarrow K[V \times \mathbb{A}^1]$ is again an integral inclusion and $\mathrm{N}_{V \times \mathbb{A}^1}(t - f^*g) = \mathcal{X}_{f^*g}(t)$. Set $Q(t) := \mathrm{N}_{V \times \mathbb{A}^1}(t\, f - g) \in K[x_1,\ldots,x_r,t]$. Since $f^* f = \mathrm{N}_V(f)$ we have that $\mathrm{N}_V(f^*) = \mathrm{N}_V(f)^{D-1}$, and so

$$\mathrm{N}_V(f)^{D-1}\, Q = \mathrm{N}_V(f^*)\, Q = \mathcal{X}_{f^*g}(\mathrm{N}_V(f)t).$$

Set $Q = c_D\, t^D + \cdots + c_0$ with $c_i \in K[\mathbb{A}^r]$. The last identity implies then $\mathrm{Tr}_V(f^*g) = -c_{D-1}$. Set $q > D$, and let G_q denote the group of q-roots of 1. Then $Q(\omega) = \mathrm{N}_V(\omega f - g)$ for $\omega \in G_q$, and so

$$\mathrm{Tr}_V(f^*g) = -\frac{1}{q} \sum_{\omega \in G_q} \mathrm{N}_V(\omega f - g)\, \omega^{1-D}.$$

From Lemma 2.15 we get $\deg \mathrm{Tr}_V(f^*g) \leq d\, D$.
For $v \in M_K^\infty$, we then obtain

$$h_v(\mathrm{Tr}_V(f^*g)) \leq \max_{\omega \in G_q} h_v(\mathrm{N}_V(\omega f - g)) \leq d\, h_v(V) + (h_v + \log 2)\, D + (r+1)\, \log(n+1)\, d\, D.$$

Analogously, for $v \notin M_K^\infty$ we take $q > D$ such that $|q|_v = 1$, and we obtain $h_v(\mathrm{Tr}_V(f^*g)) \leq d\, h_v(V) + h_v\, D$. $\qquad\square$

3   An effective arithmetic Nullstellensatz

In this chapter we obtain the announced estimates for the arithmetic Nullstellensatz over the ring of integers of a number field K . Theorem 1 in the Introduction corresponds to the case $K := \mathbb{Q}$. These estimates depend on the number of variables and on the degree and height of the input polynomials.

3.1   Division modulo complete intersection ideals

A crucial tool in our treatment of the arithmetic Nullstellensatz is the trace formula. One of its outstanding features is that it performs effective division modulo complete intersection ideals [19], [13], [32], [50], [16], [22]. In this section we apply the trace formula to obtain sharp height estimates in the division procedure.

3.1.1   Trace formula

We describe in what follows the basic aspects of duality theory for complete intersection algebras that we will need in the sequel. We refer to Kunz [34, Appendix F] for a more complete presentation of this theory.

Let k be a perfect field, and set $A := k[t_1,\ldots,t_r]$ and $A[x] := A[x_1,\ldots,x_n]$. Let $F := \{F_1,\ldots,F_n\} \subset A[x]$ be a reduced complete intersection which defines a radical ideal (F) of dimension r .
We consider the A-algebra

$$B := A[x]/(F) = A[x_1,\ldots,x_n]/(F_1,\ldots,F_n).$$

We assume that the inclusion $A \hookrightarrow B$ is finite, that is the variables $t_1,\ldots,t_r$ are in Noether normal position with respect to the variety $V := V(F) \subset \mathbb{A}^{r+n}$. This is the case, for instance, if V satisfies Assumption 1.5. Thus B is a projective A-module, which turns to be free of rank bounded by $\deg V$ by the Quillen-Suslin theorem.

The dual A-module $B^* := \operatorname{Hom}_A(B, A)$ can be seen as a B-module with scalar multiplication defined by $f \cdot \tau(g) := \tau(f g)$ for $f, g \in B$ and $\tau \in B^*$. It is a free B-module of rank 1 and any of its generators is called a *trace* of B.

The following construction yields a trace σ canonically associated to the complete intersection F. We take new variables $y := \{y_1, \ldots, y_n\}$, and we set $F_i^{(x)} := F_i(x) \in A[x]$ and $F_i^{(y)} := F_i(y) \in A[y]$. Then $F_i^{(y)} - F_i^{(x)}$ belongs to the ideal $(y_1 - x_1, \ldots, y_n - x_n)$ and so there exist (non unique) $P_{ij} \in A[x, y]$ such that

$$F_i^{(y)} - F_i^{(x)} = \sum_{j=1}^{n} P_{ij}(x, y) (y_j - x_j),$$

for $i = 1, \ldots, n$. We consider the determinant $\Delta \in A[x, y]$ of the square matrix $(P_{ij})_{ij}$, and we write it as

$$\Delta = \sum_m a_m b_m$$

with $a_m \in A[x]$ and $b_m \in A[y]$. Again, the polynomials a_m, b_m are not uniquely defined. The polynomial $\Delta \in A[x, y]$ is called a *pseudo-Jacobian determinant* of the complete intersection F. Set $c_m := b_m(x) \in A[x]$. Then there exists a unique trace $\sigma \in B^*$ such that for $g \in A[x]$

$$\overline{g} = \sum_m \sigma(\overline{g}\, \overline{a}_m)\, \overline{c}_m$$

where the bar denotes class modulo (F). This identity is known as the *trace formula*.

Let $J := \det(\partial F_i / \partial x_j)_{ij}$ be the Jacobian determinant of the complete intersection F with respect to the variables $x_1, \ldots, x_n$. Then the following identity —which justifies the name of pseudo-Jacobian for Δ— holds

$$\overline{J} = \sum_m \overline{a}_m\, \overline{c}_m.$$

The standard trace Tr_V is related to σ by the equality

$$\operatorname{Tr}_V(\overline{g}) = \sigma(\overline{J}\, \overline{g})$$

for all $g \in A[x]$.

3.1.2   A division lemma

Throughout this subsection we keep notations and assumptions as in the previous one, but we replace k by a number field K. Set $d := \max_i \deg F_i$ and $h_v := \max_i h_v(F_i)$ for $v \in M_K$. Here $\deg F_i$ denotes the *total* degree of F_i as an element of $K[t_1, \ldots, t_r][x_1, \ldots, x_n]$.

We will choose concrete polynomials a_m, c_m which satisfy the trace formula, and we will estimate their degree and local height.

First we choose the polynomials P_{ij}. Remarking that

$$F_i^{(y)} - F_i^{(x)} = \sum_{j=1}^{n} F_i(x_1, \ldots, x_{j-1}, y_j, \ldots, y_n) - F_i(x_1, \ldots, x_j, y_{j+1}, \ldots, y_n),$$

we set

$$P_{ij} := (F_i(x_1, \ldots, x_{j-1}, y_j, \ldots, y_n) - F_i(x_1, \ldots, x_j, y_{j+1}, \ldots, y_n))/(y_j - x_j)$$

Here we perform the division through the formula

$$(y_j^k - x_j^k)/(y_j - x_j) = y_j^{k-1} + y_j^{k-2} x_j + \cdots + y_j x_j^{k-2} + x_j^{k-1}.$$

We set $\Delta := \det(P_{ij})_{ij}$. Finally we choose $b_m \in A[y]$ as the monomials in the expansion of Δ with respect to y, $a_m \in A[x]$ as the corresponding coefficient, and we set $c_m := b_m(x)$.

Set $F_i = \sum_\alpha A_{i\,\alpha}\, x^\alpha$ with $A_{i\,\alpha} \in A$. Then

$$P_{ij} = \sum_\alpha A_{i\,\alpha}\, x_1^{\alpha_1} \cdots x_{j-1}^{\alpha_{j-1}} y_{j+1}^{\alpha_{j+1}} \cdots y_n^{\alpha_n} (y_j^{\alpha_j-1} + \cdots + x_j^{\alpha_j-1}) \in A[x,y].$$

We deduce that $\deg P_{ij} \le d-1$ and $h_v(P_{ij}) \le h_v$ for every $v \in M_K$. Then $\deg \Delta \le n\,(d-1)$ and so

$$\deg a_m + \deg c_m \le n\,(d-1).$$

We have also $h_v(c_m) = 0$ and $h_v(a_m) \le h_v(\Delta)$.

Finally we can write

$$P_{ij} = C_0 + \cdots + C_{d-1}\, y_j^{d-1},$$

where each $C_k \in A[x_1, \ldots, x_j, y_{j+1}, \ldots, y_n]$ is a polynomial in $n+r$ variables of total degree bounded by $\deg P_{ij} \le d-1$. This implies that the number of monomials of P_{ij} is bounded by $d\binom{n+r+d-1}{n+r} \le d\,(n+r+1)^{d-1}$.

Therefore, for $v \in M_K^\infty$ we have

$$
\begin{aligned}
h_v(a_m) &\le & h_v(\Delta) \\
&\le & n\, h_v + (n-1)\,(\log d + (d-1)\,\log(n+r+1)) + n \log n \\
&\le & n\,(h_v + d\,\log(n+r+1) + \log d).
\end{aligned}
\tag{3.1}
$$

Analogously we have $h_v(a_m) \le n\, h_v$ for $v \notin M_K^\infty$.

The following is a sharp estimate for the degree and local height of the polynomials in the division procedure. It is a substantial improvement over [32, Thm. 29].

We introduce the notation $\deg_t f$ and $\deg_x f$ for the degree of a polynomial $f \in A[x]$ with respect to the group of variables t and x, respectively.

Main Lemma 3.1 *(Division Lemma)*
Set $A := K[t_1, \ldots, t_r]$ and $A[x] := A[x_1, \ldots, x_n]$. Let $F := \{F_1, \ldots, F_n\} \subset A[x]$ be a reduced complete intersection defining a variety $V := V(F) \subset \mathbb{A}^{r+n}$ which satisfies Assumption 1.5. Set $B := K[V] = A[x]/(F)$.
Let $f, g \in A[x]$ be polynomials such that $\overline{f} \in B$ is a non-zero divisor and $\overline{f}\,|\,\overline{g}$ in B. Set $d := \max\{\deg f, \deg F_1, \ldots, \deg F_n\}$ and $h_v := \max\{h_v(f), h_v(F_1), \ldots, h_v(F_n)\}$ for $v \in M_K$.
Then there exist $q \in A[x]$ and $\xi \in K^*$ such that

- $\overline{q}\,\overline{f} = \overline{g}$,

- $\deg_x q \le n\,d$,

- $\deg q \le \deg_t g + (n\,d + \max\{(n+1)\,d, \deg_x g\})\,\deg V$,

- $h_v(q) \le h_v(g) + (n\,d + \max\{d, \deg_x g\})\,h_v(V)$
  $\qquad + ((n+1)\,h_v + (r+6)\,\log(n+r+1)\,(n\,d + \max\{(n+1)\,d, \deg_x g\}))\,\deg V$
  $\qquad + 2\,\log(r+1)\,\deg_t g - \log|\xi|_v$

  for $v \in M_K^\infty$,

- $h_v(q) \le h_v(g) + (n\,d + \max\{d, \deg_x g\})\,h_v(V) + (n+1)\,h_v\,\deg V - \log|\xi|_v$

  for $v \notin M_K^\infty$.

Proof.– Set $L := K(t_1, \ldots, t_r)$ for the quotient field of A and $M := L \otimes_A B$. Then M is a finite L-algebra of dimension $\deg V$ and σ can be uniquely extended to a L-linear map $\sigma : M \to M$. The fact that B is a torsion-free A-algebra implies that the canonical map $B \to M$ is an inclusion. We will only consider the case $n \geq 1$. For the case $n = 0$ we refer to Remark 3.2. Whenever it is clear from the context, we will avoid explicit reference to the ring in which we are considering a given element of $A[x]$.

Let $q_0 \in A[x]$ be any polynomial such that $q_0 \, f = g$ in B. We have that f is a non-zero divisor in B, and so it is invertible in M. Then $q_0 = f^{-1} g$ in M and therefore $\sigma(f^{-1} g \, p) = \sigma(q_0 \, p) \in A$ for all $p \in A[x]$. Then we set

$$q := \sum_m \sigma(f^{-1} g \, a_m) \, c_m \in A[x].$$

Trace formula implies that $q \equiv q_0 \pmod{(F)}$, and so $q \, f = g$ in B.

Let $J \in A[x]$ denote the Jacobian determinant of the complete intersection F with respect to the group of variables x. This is a non-zero divisor because of the Jacobian criterion (see for instance [9, Thm. 18.15]), and so it is also invertible in M.

Let $(J \, f)^*$ be the adjoint polynomial of $J \, f$ and set

$$\Lambda_m := \mathrm{Tr}_V((J \, f)^* g \, a_m) \in A.$$

We have $J \, f \, (J \, f)^* = \mathrm{N}(J \, f) \in A \setminus \{0\}$, and so

$$\Lambda_m / \mathrm{N}(J \, f) = \mathrm{Tr}((J \, f)^{-1} g \, a_m) = \sigma(f^{-1} g \, a_m) \in A.$$

In particular $\mathrm{N}(J \, f) \, | \, \Lambda_m$ in A, and we have the expression

$$q = \frac{1}{\mathrm{N}(J \, f)} \sum_m \Lambda_m \, c_m.$$

In the sequel, $\xi \in K^*$ will be any non-zero coefficient of $\mathrm{N}(J \, f)$.

Let us consider degrees. Clearly $\deg_x q \leq \max_m \deg c_m \leq n(d-1) \leq n \, d$.

Next we analyze the total degree of q. Let $g := \sum_\alpha p_\alpha \, x^\alpha$ be the monomial expansion of g with respect to x. Then

$$\Lambda_m = \sum_\alpha p_\alpha \, \mathrm{Tr}((Jf)^* x^\alpha \, a_m), \tag{3.2}$$

as Tr is a A-linear map. We have the estimates $\deg(J \, f) \leq n \, (d-1) + d \leq (n+1) \, d$ and $\deg(x^\alpha \, a_m) \leq \deg_x g + \deg a_m$, from where we get

$$\deg \mathrm{Tr}((Jf)^* x^\alpha \, a_m) \leq \max\{(n+1) \, d, \deg_x g + \deg a_m\} \deg V$$

by Lemma 2.16. Thus

$$
\begin{aligned}
\deg q \ &\leq \ \deg_t g + \max_m \{\max\{(n+1) \, d, \deg_x g + \deg a_m\} \deg V + \deg c_m\} \\
&\leq \ \deg_t g + \max_m \{\max\{(n+1) \, d + \deg c_m, \deg_x g + \deg a_m + \deg c_m\}\} \deg V \\
&\leq \ \deg_t g + \max\{(n+1) \, d + n \, d, \deg_x g + n \, d\} \deg V \\
&\leq \ \deg_t g + (n \, d + \max\{(n+1) \, d, \deg_x g\}) \deg V.
\end{aligned}
$$

For the rest of the proof, we will use several times the following basic estimates:

$$
\begin{aligned}
\max\{\deg(J \, f), \deg(x^\alpha \, a_m)\} \ &\leq \ n \, d + \max\{d, \deg_x g\}, \\
\deg \mathrm{Tr}((Jf)^* x^\alpha \, a_m) \ &\leq \ (n \, d + \max\{d, \deg_x g\}) \deg V.
\end{aligned}
$$

Finally we estimate the local height of q. Let $v \in M_K^\infty$. We have $h_v(\partial F_i/\partial x_j) \leq h_v + \log d$ and so

$$h_v(J) \leq n\,(h_v + \log d) + (n-1)\log(n+r+1)\,(d-1) + n\,\log n \leq n\,(h_v + \log(n+r+1)\,d + \log d).$$

Therefore

$$
\begin{aligned}
h_v(J\,f) &\leq\ n(h_v + \log(n+r+1)\,d + \log d) + h_v + \log(n+r+1)\,d \\
&\leq\ (n+1)\,h_v + (n+1)\,\log(n+r+1)\,d + n\,\log d \qquad\qquad (3.3)
\end{aligned}
$$

by Lemma 1.2(b). We recall that $\quad h_v(x^\alpha\,a_m) \leq n\,(h_v + \log(n+r+1)\,d + \log d)\quad$ by inequality (3.1) and so

$$\max\{h_v(J\,f), h_v(x^\alpha\,a_m)\} \leq (n+1)\,h_v + (n+1)\,\log(n+r+1)\,d + n\,\log d.$$

Then

$$
\begin{aligned}
h_v(\mathrm{Tr}((J\,f)^* x^\alpha a_m)) &\leq\ (n\,d + \max\{d, \deg_x g\})\,h_v(V) \\
&\quad + ((n+1)h_v + (n+1)\,\log(n+r+1)\,d + n\,\log d + \log 2)\,\deg V \\
&\quad + (r+1)\,\log(n+r+1)\,(n\,d + \max\{d, \deg_x g\})\,\deg V \\
&\leq\ (n\,d + \max\{d, \deg_x g\})\,h_v(V) + ((n+1)\,h_v + ((2\,n+1)\,\log(n+r+1)\,d)\,\deg V \\
&\quad + (r+1)\,\log(n+r+1)\,(n\,d + \max\{d, \deg_x g\})\,\deg V
\end{aligned}
$$

by Lemma 2.16.
By considering separately the cases $\deg_x g \leq (n+1)\,d$ and $\deg_x g > (n+1)\,d$ we obtain

$$(2\,n+1)\,d + (r+1)\,(n\,d + \deg_x g) \leq \deg_x g + n\,d + (r+1)\,(n\,d + \deg_x g) \leq (r+2)\,(n\,d + \deg_x g).$$

We conclude:

$$
\begin{aligned}
h_v(\mathrm{Tr}((J\,f)^* x^\alpha a_m)) &\leq\ (n\,d + \max\{d, \deg_x g\})\,h_v(V) \\
&\quad + ((n+1)\,h_v + (r+2)\,\log(n+r+1)\,(n\,d + \max\{(n+1)\,d, \deg_x g\}))\,\deg V.
\end{aligned}
$$

Hence

$$
\begin{aligned}
h_v(\Lambda_m) &\leq\ \max_\alpha\{h_v(p_\alpha\,\mathrm{Tr}((J\,f)^* x^\alpha a_m)\} + \log(n+1)\,\deg_x g \\
&\leq\ h_v(g) + \max_\alpha\{h_v(\mathrm{Tr}((J\,f)^* x^\alpha a_m))\} \\
&\quad + \log(r+1)\,(n\,d + \max\{d, \deg_x g\})\,\deg V + \log(n+1)\,\deg_x g \\
&\leq\ h_v(g) + (n\,d + \max\{d, \deg_x g\})h_v(V) \\
&\quad + ((n+1)\,h_v + (r+2)\,\log(n+r+1)\,(n\,d + \max\{(n+1)\,d, \deg_x g\}))\,\deg V \\
&\quad + \log(r+1)\,(n\,d + \max\{d, \deg_x g\})\,\deg V + \log(n+1)\,\deg_x g \\
&\leq\ h_v(g) + (n\,d + \max\{d, \deg_x g\})\,h_v(V) \\
&\quad + ((n+1)\,h_v + (r+4)\,\log(n+r+1)\,(n\,d + \max\{(n+1)d, \deg_x g\}))\,\deg V
\end{aligned}
$$

by application of identity (3.2) and Lemma 1.2(a,b). We have

$$h_v(q) \leq \max_m\{h_v(\Lambda_m/\mathrm{N}(J\,f))\}$$

as each c_m is a different monomial in x. Thus it only remains to estimate the local height of each $\Lambda_m/\mathrm{N}(Jf)$.

Recall that $\xi \in K^*$ is any non-zero coefficient of $\mathrm{N}(Jf)$. Then

$$
\begin{aligned}
\log |\Lambda_m/\mathrm{N}(Jf)|_v \ \leq \ & h_v(\Lambda_m) + 2\log(r+1)\left(\deg_t g + (n\,d + \max\{d, \deg_x g\})\deg V\right) - \log |\mathrm{N}(Jf)|_v \\
\leq \ & h_v(g) + (n\,d + \max\{d, \deg_x g\})\, h_v(V) \\
& + ((n+1)\, h_v + (r+6)\log(n+r+1)\,(n\,d + \max\{(n+1)d, \deg_x g\}))\deg V \\
& + 2\log(r+1)\deg_t g - \log |\xi|_v
\end{aligned}
\tag{3.4}
$$

by Lemma 1.2(d) and the fact that $\log |\xi|_v \leq \log |\mathrm{N}(Jf)|_v$. From Lemma 2.15 and inequality (3.3) we obtain

$$
\begin{aligned}
\log |\xi|_v \ \leq \ & h_v(\mathrm{N}(Jf)) \\
\leq \ & (n+1)\,d\,h_v(V) + ((n+1)\,h_v + (n+1)\log(n+r+1)\,d + n\log d)\deg V \\
& + (r+1)(n+1)\log(n+r+1)\,d \deg V \\
\leq \ & (n+1)\,d\,h_v(V) + ((n+1)\,h_v + (r+3)(n+1)\log(n+r+1)\,d)\deg V
\end{aligned}
\tag{3.5}
$$

This implies that the right hand side of inequality (3.4) is non-negative. So the inequality also holds for $h_v(\Lambda_m/\mathrm{N}(Jf))$, and thus for $h_v(q)$.

The case $v \notin M_K^\infty$ is treated exactly in the same way. The obtained estimates do not involve any constant terms with respect to h_v, $h_v(g)$ and $h_v(V)$, in particular $\deg_t g$ does not appear in the estimate. This follows simply from 1.2.

In this case, inequality (3.5) reads as follows:

$$
\log |\xi|_v \leq (n+1)\,d\,h_v(V) + (n+1)\,h_v \deg V.
\tag{3.6}
$$

We remark that the election of ξ is independent of v, and so it can be done uniformly.

$\square$

Remark 3.2 *Let notations be as in the previous lemma. In case $n = 0$ we have the sharper estimates*

- *$\deg q \leq \deg g$,*

- *$h_v(q) \leq h_v(g) + h_v + 2\log(r+1)\deg g - \log |\xi|_v \quad$ for $v \in M_K^\infty$,*

- *$h_v(q) \leq h_v(g) + h_v - \log |\xi|_v \quad$ for $v \notin M_K^\infty$.*

Here $\xi \in K^*$ denotes any non-zero coefficient of f. The local height estimates follow from Lemma 1.2(d) and the fact that $h_v - \log |\xi|_v \geq 0$.

3.2 An effective arithmetic Nullstellensatz

3.2.1 Estimates for the complete intersection case

The following result gives estimates for the degree and local height of the polynomials arising in the Nullstellensatz over a number field K in case the input is a reduced weak regular sequence. It is a direct consequence of the division lemma above.

These estimates depend mainly on the degree and height of the varieties successively cut out by the input polynomials. They are quite flexible, and they apply to other situations as well, as we will see in Chapter 4.

We recall that $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$ is a *weak regular sequence* if f_{i+1} is not a zero-divisor modulo the ideal $(f_1, \ldots, f_{i-1})$ for $i = 0, \ldots, s-1$. Furthermore, it is called *reduced* when all these ideals are radical.

Lemma 3.3 Let $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$ be polynomials without common zeros in $\mathbb{A}^n$ which form a reduced weak regular sequence. Assume that $V_j := V(f_1, \ldots, f_j)$ satisfies Assumption 1.5 for $j = 1, \ldots, s-1$. Set $d := \max_i \deg f_i$ and $h_v := \max_i h_v(f_i)$ for $v \in M_K$. Assume also $n, d \geq 2$.

Then there exist $p_1, \ldots, p_s \in K[x_1, \ldots, x_n]$ and $\xi \in K^*$ such that

- $1 = p_1 f_1 + \cdots + p_s f_s$,

- $\deg p_i \leq 2\, n\, d\, (1 + \sum_{j=1}^{\min\{n,s\}-1} \deg V_j)$,

- $h_v(p_i) \leq 2\, n\, d\, \sum_{j=1}^{s-1} h_v(V_j) + ((n+1)\, h_v + 2\, n\, (2\, n+5)\, \log(n+1)\, d)\, (1 + \sum_{j=1}^{s-1} \deg V_j) - \log|\xi|_v$

  for $v \in M_K^\infty$,

- $h_v(p_i) \leq 2\, n\, d\, \sum_{j=1}^{s-1} h_v(V_j) + (n+1)\, h_v\, (1 + \sum_{j=1}^{s-1} \deg V_j) - \log|\xi|_v \quad$ for $v \notin M_K^\infty$.

Proof.– Set $I_i := I(V_i) = (f_1, \ldots, f_i)$ for $i = 1, \ldots, s-1$. Also set $f_0 := 0$, $V_0 := V(f_0) = \mathbb{A}^n$ and $I_0 := I(V_0) = (0)$. Finally set $A_i := K[x_1, \ldots, x_{n-i}]$ and $B_i := K[V_i] = K[x_1, \ldots, x_n]/I_i$ for $0 \leq i \leq s-1$. The fact that V_i satisfies Assumption 1.5 implies that the inclusion $A_i \hookrightarrow B_i$ is integral.

We note that the sets of free and dependent variables of B_i have cardinality $n-i$ and i respectively. Also the set of dependent variables of B_j is contained in that of B_i for $i \leq j$.

For $f \in K[x_1, \ldots, x_n]$ we denote by $\deg_{x(i)} f$ the degree of f in the dependent variables $x_{n-i+1}, \ldots, x_n$ of B_i with respect to the integral inclusion $A_i \hookrightarrow B_i$. For $i \leq j$, the previous observation implies that $\deg_{x(j)} f \leq \deg_{x(i)} f$.

Applying the Division Lemma 3.1, we will construct inductively polynomials $p_1, \ldots, p_s$: first we take p_s such that
$$p_s\, f_s \equiv 1 \pmod{I_{s-1}}.$$
For $0 \leq i \leq s-2$ we assume that $p_{i+2}, \ldots, p_s$ are already constructed and we set
$$b_{i+1} := 1 - (p_{i+2} f_{i+2} + \cdots + p_s f_s).$$

Then f_{i+1} is a non-zero divisor and $f_{i+1} \mid b_{i+1}$ in B_i. We apply again Division Lemma to obtain p_{i+1} such that
$$p_{i+1}\, f_{i+1} \equiv b_{i+1} \pmod{I_i},$$
Continuing this procedure until $i = 0$, we get $1 = p_1 f_1 + \cdots + p_s f_s$ in $K[x_1, \ldots, x_n]$.

Let us analyze degrees.

First we consider the case $s \leq n$. Again we proceed by induction.

The estimates from the Division Lemma for $A_{s-1} := K[x_1, \ldots, x_{n-(s-1)}]$, $g := 1$ and $f := f_s$ give $\deg_{x(s-1)} p_s \leq (s-1)\, d \leq (n-1)\, d$ and $\deg p_s \leq (2\, s-1)\, d\, \deg V_{s-1}$.

Now let $1 \leq i \leq s-2$. Then $\deg_{x(i)} p_{i+1} \leq i\, d$ and
$$\deg p_{i+1} \leq \deg b_{i+1} + (i\, d + \max\{(i+1)\, d, \deg_{x(i)} b_{i+1}\})\, \deg V_i.$$
where
$$\deg_{x(i)} b_{i+1} \leq \max_{j \geq i+2} \{\deg_{x(i)} p_j + \deg f_j\} \leq \max_{j \geq i+2} \deg_{x(j-1)} p_j + d \leq s\, d.$$
Applying recursively the previous inequality we obtain
$$
\begin{aligned}
\deg p_{i+1} &\leq \max_{j \geq i+2} \deg p_j + d + (s+i)\, d\, \deg V_i \\
&\leq (2\, s-1)\, d\, \deg V_{s-1} + \sum_{j=i}^{s-2} (d + (s+j)\, d\, \deg V_j) \\
&= (s-i-1)\, d + \sum_{j=i}^{s-1} (s+j)\, d\, \deg V_j.
\end{aligned}
$$

For $i = 0$ we have $p_1 \mid b_1$ and therefore $\deg p_1 \le \deg b_1 \le \max_{j \ge 2} \deg p_j + d$. Then for all i:

$$\deg p_i \le (s-1)\,d + \sum_{j=1}^{s-1} (s+j)\,d \, \deg V_j \le 2\,n\,d\,(1 + \sum_{j=1}^{s-1} \deg V_j).$$

Next we consider the case $s = n+1$. In this case V_s is a 0-dimensional variety and so

$$\deg p_{n+1} = \deg_{x(n)} p_{n+1} \le n\,d.$$

Let $1 \le i \le n-1$. Then $\deg_{x(i)} p_{i+1} \le i\,d$ and, again we apply recursively the previous inequality, and we get

$$
\begin{aligned}
\deg p_{i+1} \quad \le \quad & \max_{j \ge i+2} \deg p_j + d + (n+1+i)\,d \, \deg V_i \\
\le \quad & n\,d + \sum_{j=i}^{n-1} (d + (n+1+j)\,d \, \deg V_j) \\
= \quad & (2\,n - i)\,d + \sum_{j=i}^{n-1} (n+1+j)\,d \, \deg V_j.
\end{aligned}
$$

We have also $\deg p_1 \le \deg b_1 \le \max \deg_{j \ge 2} \deg p_j + d$. We conclude for all i:

$$\deg p_i \le 2\,n\,d + \sum_{j=1}^{n-1} (n+1+j)\,d \, \deg V_j \le 2\,n\,d\,(1 + \sum_{j=1}^{n-1} \deg V_j).$$

Finally we estimate the local height of these polynomials. In the rest of the proof we will make repeated use of the following degree bounds:

$$
\begin{aligned}
\deg_{x(i-1)} p_i \quad \le \quad & n\,d, \\
\deg p_i \quad \le \quad & 2\,n\,d\,(1 + \sum_{j=i-1}^{\min\{n,s\}-1} \deg V_j).
\end{aligned}
$$

As usual, we concentrate in the case $v \in M_K^\infty$, the case $v \notin M_K^\infty$ can be treated analogously. We apply the Division Lemma to $A_{s-1} := K[x_1, \ldots, x_{n-(s-1)}]$, $g := 1$ and $f := f_s$ and we obtain

$$h_v(p_s) \quad \le \quad s\,d\,h_v(V_{s-1}) + (s\,h_v + (n-(s-1)+6)\,(s+(s-1))\,\log(n+1)\,d)\,\deg V_{s-1} - \log|\xi_{s-1}|_v$$

for some $\xi_{s-1} \in K^*$.

Let $1 \le i \le s-2$ and set $n_0 := \min\{n,s\}$. Then there exists $\xi_i \in K^*$ such that

$$
\begin{aligned}
h_v(p_{i+1}) \quad \le \quad & h_v(b_{i+1}) + (i\,d + \max\{d, \deg_{x(i)} b_{i+1}\})\,h_v(V_i) \\
& + ((i+1)\,h_v + (n-i+6)\,\log(n+1)\,(i\,d + \max\{(i+1)\,d, \deg_{x(i)} b_{i+1}\}))\,\deg V_i \\
& + 2\,\log(n-i+1)\,\deg b_{i+1} - \log|\xi_i|_v \\
\le \quad & \max_{j \ge i+2} h_v(p_j) + h_v + \log(n+1)\,d + \log(s-i) + (s+i)\,d\,h_v(V_i) + (i+1)\,h_v\,\deg V_i \\
& + (n-i+6)\,(s+i)\,\log(n+1)\,d\,\deg V_i + 2\,\log(n+1)\,(2\,n\,d\,(1 + \sum_{j=i+1}^{n_0-1} \deg V_j) + d) \\
& - \log|\xi_i|_v.
\end{aligned}
$$

Applying the inductive hypothesis we obtain

$$
\begin{aligned}
h_v(p_{i+1}) \;\leq\; & s\,d\,h_v(V_{s-1}) + d\sum_{j=i}^{s-2}(s+j)\,h_v(V_j) + (s-i-1)\,h_v + h_v\sum_{j=i}^{s-1}(j+1)\,\deg V_j \\
& + 4\,(s-i-1)\,(n+1)\,\log(n+1)\,d + \log(n+1)\,d\sum_{j=i}^{s-1}(n-j+6)\,(s+j)\,\deg V_j \\
& + 4\,n\,\log(n+1)\,d\sum_{j=i+1}^{n_0-1}(j-i)\,\deg V_j - \sum_{j=i}^{s-1}\log|\xi_j|_v.
\end{aligned}
$$

For $i=0$ we apply Remark 3.2: there exists $\xi_0\in K^*$ such that

$$
\begin{aligned}
h_v(p_1) \;\leq\; & h_v(b_1) + h_v + 2\,\log(n+1)\,\deg b_1 - \log|\xi_0|_v \\
\;\leq\; & \max_{j\geq 2} h_v(p_j) + 2\,h_v + \log(n+1)\,d + \log s + 2\,\log(n+1)\,\Big(2\,n\,d\,(1+\sum_{j=1}^{n_0-1}\deg V_j) + d\Big) \\
& - \log|\xi_0|_v.
\end{aligned}
$$

We set $\xi := \prod_{j=0}^{s-1}\xi_j$. As $2\leq s\leq n+1$ we have

$$
\begin{aligned}
h_v(p_1) \;\leq\; & 2\,n\,d\sum_{j=1}^{s-1}h_v(V_j) + (n+1)\,h_v\,\Big(1+\sum_{j=1}^{s-1}\deg V_j\Big) + 4\,n\,(n+1)\,\log(n+1)\,d \\
& + \log(n+1)\,d\sum_{j=1}^{s-1}(n-j+6)\,(n+1+j)\,\deg V_j + 4\,n\,\log(n+1)\,d\sum_{j=1}^{n_0-1}j\,\deg V_j - \log|\xi|_v \\
\;\leq\; & 2\,n\,d\sum_{j=1}^{s-1}h(V_j) + \big((n+1)\,h_v + 2\,n\,(2\,n+5)\,\log(n+1)\,d\big)\,\Big(1+\sum_{j=1}^{s-1}\deg V_j\Big) - \log|\xi|_v.
\end{aligned}
$$

This last inequality follows from the facts that $\;4\,n\,j + (n-j+6)\,(j+s) \leq 2\,n\,(2\,n+5)\;$ for $j\leq n-1$, and $6\,(2\,n+1)\leq 2\,n\,(2\,n+5)\;$ as $n\geq 2$.

To conclude the proof, observe that for $i=1,\dots,s-1$, inequality (3.5) guarantees that the obtained estimate for p_i differs from the one for p_{i+1} by a positive term. Thus, the same estimate holds for $h_v(p_i)$, $1\leq i\leq s$.

The non-archimedean case is treated in exactly the same way. The conclusion of the proof comes in this case from inequality (3.6). $\qquad\square$

By means of Bézout inequality, we can now estimate the degree and height of the varieties V_j. In this way we obtain an estimate which only depends on the degree and height of the input polynomials.

Corollary 3.4 *Let notations and assumptions be as in Lemma 3.3, and assume $n, d\geq 2$. Then there exist $p_1,\dots,p_s\in K[x_1,\dots,x_n]$ and $\gamma\in K^*$ such that*

- $1 = p_1\,f_1 + \cdots + p_s\,f_s$,

- $\deg p_i \leq 4\,n\,d^n$,

- $h_v(p_i) \leq 4\,n\,(n+1)\,d^n\,h_v + 4\,n\,(4\,n+5)\,\log(n+1)\,d^{n+1} - \log|\gamma|_v\quad$ *for $v\in M_K^\infty$*,

- $h_v(p_i) \leq 4\,n\,(n+1)\,d^n\,h_v - \log|\gamma|_v\quad$ *for $v\notin M_K^\infty$*.

Proof.– Let us first consider degrees. From the preceding result we obtain

$$\deg(p_i) \leq 2\, n\, d\, (1 + \sum_{j=1}^{\min\{n,s\}-1} \deg V_j) \leq 2\, n\, d\, (1 + \cdots + d^{n-1}) \leq 4\, n\, d^n$$

Here we applied the inequality $1 + \ldots + d^{n-1} \leq 2\, d^{n-1}$ to obtain the last estimate.
Next we consider the local height estimates. Let $v \in M_K^\infty$. We have

$$h_v(p_i) \leq 2\, n\, d \sum_{j=1}^{s-1} h_v(V_j) + ((n+1)\, h_v + 2\, n\, (2\, n + 5)\, \log(n+1)d)\, (1 + \sum_{j=1}^{s-1} \deg V_j) - \log |\xi|_v$$

for some $\xi \in K^*$. Applying Corollary 2.10, $h_v(V_j) \leq j\, d^{j-1}\, h_v + (n+j) \log(n+1)\, d^j - \log|\lambda_j|_v$ for some $\lambda_j \in K^*$. Therefore

$$
\begin{aligned}
h_v(p_i) \quad \leq \quad & 2\, n\, d \sum_{j=1}^{s-1} (j\, d^{j-1}\, h_v + (n+j)\log(n+1)\, d^j - \log|\lambda_j|_v) \\
& + ((n+1)\, h_v + 2\, n\, (2\, n + 5)\, \log(n+1)\, d)\, \sum_{j=0}^{n} d^j - \log|\xi|_v \\
\leq \quad & 4\, n^2\, d^n\, h_v + 8\, n^2\, \log(n+1)\, d^{n+1} \\
& + 2\, (n+1)\, d^n\, h_v + 4\, n\, (2\, n + 5)\, \log(n+1)\, d^{n+1} - 2\, n\, d \sum_{j=1}^{s-1} \log|\lambda_j|_v - \log|\xi|_v \\
\leq \quad & 4\, n\, (n+1)\, d^n\, h_v + 4\, n\, (4\, n + 5)\, \log(n+1)\, d^{n+1} - \log|\gamma|_v,
\end{aligned}
$$

where $\gamma \in K^*$ is defined as $\gamma := \xi \prod_{j=1}^{s-1} \lambda_j^{2\, n\, d}$.
The case $v \notin M_K^\infty$ follows analogously:

$$h_v(p_i) \leq 2\, n\, d \sum_{j=1}^{s-1} h_v(V_j) + (n+1)\, h_v\, (1 + \sum_{j=1}^{s-1} \deg V_j) - \log |\xi|_v.$$

We have $h_v(V_j) \leq j\, d^{j-1}\, h_v - \log|\lambda_j|_v$, and therefore

$$
\begin{aligned}
h_v(p_i) \quad \leq \quad & 2\, n\, d \sum_{j=1}^{s-1} (j\, d^{j-1}\, h_v - \log|\lambda_j|_v) + (n+1)\, h_v \sum_{j=0}^{n} d^j - \log|\xi|_v \\
\leq \quad & 4\, n\, (n+1)\, d^n\, h_v - \log|\gamma|_v,
\end{aligned}
$$

<div align="right">□</div>

3.2.2 Proof of Theorem 1

In order to prove Theorem 1, it only remains to put the general case into the hypothesis of Corollary 3.4. This is accomplished by replacing the input polynomials and variables by generic linear combinations. The coefficients of the linear combinations will be chosen to be roots of 1. Amazingly enough, we do not need any control on the degree of the involved finite extension.

Let L be a finite extension of K, and let $\mathcal{B} := \{e_1, \ldots, e_N\}$ be a basis of L as a K-linear space. We recall that $\mathcal{B}^* := \{e_1^*, \ldots, e_N^*\}$ is the *dual basis* of $\mathcal{B}$ if $\operatorname{Tr}_K^L(e_i\, e_j^*) = 1$ for $i = j$ and 0 otherwise.

Lemma 3.5 *Let* $\omega \in \overline{\mathbb{Q}}$ *be a primitive* p-*root of 1 for some prime* p. *Then the basis* $\mathcal{B}^* := \{(\omega^{-j} - \omega)/p : j = 0, \ldots, p-2\}$ *of* $\mathbb{Q}(\omega)$ *is dual to* $\mathcal{B} := \{\omega^i : i = 0, \ldots, p-2\}$.

Proof.– A direct computation shows that for $i, j = 0, \ldots, p - 2$

$$\operatorname{Tr}(\omega^i \, (\omega^{-j} - \omega)) = \sum_{l=1}^{p-1} \omega^{l\,i} \, (\omega^{-l\,j} - \omega^l) = \begin{cases} p, & \text{for} \quad i = j, \\ 0, & \text{for} \quad i \neq j. \end{cases}$$

$\square$

We will use this result in the following way: let ω be a primitive p-root of 1 and set $L := K(\omega)$. Let us assume that $\mathbb{Q}(\omega)$ and K are linearly independent and that p does not divide the discriminant of K. Both conditions are satisfied by all but a finite number of p. Then $[L : K] = p - 1$ and $\mathcal{O}_L = \mathcal{O}_K[\omega]$ [36, Ch. III, Prop. 17].
Now, let $\nu \in \mathcal{O}_L \setminus \{0\}$. By the preceding lemma

$$\nu = \frac{1}{p} \operatorname{Tr}(\nu \, (1 - \omega)) + \cdots + \frac{1}{p} \operatorname{Tr}(\nu \, (\omega^{2-p} - \omega)) \, \omega^{2-p} \ \in \mathcal{O}_K[\omega] \setminus \{0\}$$

and so there exists $0 \leq j \leq p - 2$ such that $\operatorname{Tr}(\nu \, (\omega^{-j} - \omega)) / p \ \in \mathcal{O}_K \setminus \{0\}$.

Theorem 3.6 *(Effective arithmetic Nullstellensatz)*
Let K be a number field and let $f_1, \ldots, f_s \in \mathcal{O}_K[x_1, \ldots, x_n]$ be polynomials without common zeros in $\mathbb{A}^n$. Set $d := \max_i \deg f_i$ and $h := h(f_1, \ldots, f_s)$.
Then there exist $a \in \mathcal{O}_K \setminus \{0\}$ and $g_1, \ldots, g_s \in \mathcal{O}_K[x_1, \ldots, x_n]$ such that

- $a = g_1 \, f_1 + \cdots + g_s \, f_s$,

- $\deg g_i \leq 4 \, n \, d^n$,

- $h(a, g_1, \ldots, g_s) \leq 4 \, n \, (n + 1) \, d^n \, (h + \log s + (n + 7) \log(n + 1) \, d)$.

Theorem 1 in the Introduction corresponds to the case $K := \mathbb{Q}$.

The extremal cases $n = 1$ and $d = 1$ are both simple. We treat them directly in the following lemmas:

Lemma 3.7 *Let $\ell_1, \ldots, \ell_s \in \mathcal{O}_K[x_1, \ldots, x_n]$ be polynomials of degree bounded by 1 without common zeros in $\mathbb{A}^n$. Set $h := h(\ell_1, \ldots, \ell_s)$.*
Then there exist $a \in \mathcal{O}_K \setminus \{0\}$ and $a_1, \ldots, a_s \in \mathcal{O}_K$ such that

- $a = a_1 \, \ell_1 + \cdots + a_s \, \ell_s$,

- $h(a, a_1, \ldots, a_s) \leq (n + 1) \, (h + \log(n + 1))$.

Proof.– Equation $a = a_1 \, \ell_1 + \cdots + a_s \, \ell_s$ is equivalent to a $\mathcal{O}_K$-linear system of $n + 1$ equations in s unknowns. The height estimate follows then from application of Cramer rule.

$\square$

Lemma 3.8 *Let $f_1, \ldots, f_s \in \mathcal{O}_K[x]$ be polynomials without common zeros in $\mathbb{A}^1$. Set $d := \max_i \deg f_i$ and $h := h(f_1, \ldots, f_s)$.*
Then there exist $a \in \mathcal{O}_K \setminus \{0\}$ and $g_1, \ldots, g_s \in \mathcal{O}_K[x]$ such that

- $a = g_1 \, f_1 + \cdots + g_s \, f_s$,

- $\deg g_i \leq d - 1$,

- $h(a, g_1, \ldots, g_s) \leq 2 \, d \, (h + d)$.

Proof.– Let $f := \sum_i a_i f_i$, $g := \sum_i b_i f_i \in K[x]$ be generic linear combinations of $f_1, \ldots, f_s$. Then f and g are coprime polynomials, and so there exist $p, q \in K[x]$ with $\deg p < \deg g$ and $\deg q < \deg f$ such that $1 = p\, f + q\, g$.

Expanding this identity we see that there exist $p_1, \ldots, p_s \in K[x]$ with $\deg p_i \le d - 1$ such that

$$1 = p_1\, f_1 + \cdots + p_s\, f_s.$$

Thus the above Bézout identity translates to a *consistent* system of K-linear equations. The number of equations and variables equal $2\,d$ and $s\,d$ respectively. This system can be solved by Cramer rule. The integer a is the determinant of a non-singular $2\,d \times 2\,d-$ submatrix of the matrix of the linear system. $\quad\square$

Proof of Theorem 3.6.– We assume $n, d \ge 2$.

Let $G_p \subset \overline{\mathbb{Q}}$ denote the group of p-roots of 1, for a prime p. For $a_{ij} \in G_p$ and $i = 1, \ldots, \min\{n + 1, s\}$ we set

$$q_i := a_{i\,1} f_1 + \ldots + a_{i\,s} f_s.$$

Also, for $b_{kl} \in G_p$ and $k = 1, \ldots, n$ we set

$$y_k := b_{k\,0} + b_{k\,1} x_1 + \cdots + b_{k\,n} x_n.$$

We will assume that for a specific choice of p, a_{ij} and b_{kl} there exists $t \le \min\{n + 1, s\}$ such that $(q_1, \ldots, q_i) \subset K[x_1, \ldots, x_n]$ is a radical ideal of dimension $n - i$ for $i = 1, \ldots, t - 1$ and $1 \in (q_1, \ldots, q_t)$. We also assume that $y_1, \ldots, y_n$ is a linear change of variables, and that $V_i := V(q_1, \ldots, q_i) \subset \mathbb{A}^n$ satisfies Assumption 1.5 for $i = 1, \ldots, t-1$ with respect to $y_1, \ldots, y_{n-i}$.

This is guaranteed by the fact that these conditions are generically satisfied: there exists a hypersurface H of the coefficient space such that $(a_{i\,j}, b_{k\,l}) \notin H$ implies that $q_1, \ldots, q_s$ satisfy the stated conditions with respect to the variables $y_1, \ldots, y_n$ (see for instance [17, Thm. 3.5 and Thm. 3.7.2] [19, Section 3.2], [50, Prop. 18 and Proof of Thm. 19]). As $\cup_p G_p$ is Zariski dense in $\mathbb{A}^1$, it follows that these coefficients can be chosen to lie in G_p for some p. Moreover, p can be chosen such that for ω a primitive p-root of 1 and $L := K(\omega)$, $\mathbb{Q}(\omega)$ and K are linearly independent and p does not divide the discriminant of K.

We also refer the reader to Section 4.1, where we give a self-contained treatment of this topic.

Set $b := (b_{k\,0})_k \in G_p^n$ and $B := (b_{k\,l})_{k,l \ge 1} \in \mathrm{GL}_n(\overline{\mathbb{Q}})$, so that $x = B^{-1}(y - b)$. For $j = 1, \ldots, t$ set

$$F_j(y) := q_j(B^{-1}(y - b)) \in L[y_1, \ldots, y_n].$$

Then $F_1, \ldots, F_t$ satisfy the hypothesis of Corollary 3.4. Let $\gamma \in L^*$ and $P_1, \ldots, P_t \in L[y_1, \ldots, y_n]$ be the non-zero element and the polynomials satisfying Bézout identity we obtain there.

Now, for $i = 1, \ldots, s$, set

$$p_i := \sum_{j=1}^t a_{i\,j}\, P_j(B\,x + b) \in L[x_1, \ldots, x_n]$$

so that $1 = p_1 f_1 + \cdots + p_s f_s$ holds.

Finally set $\mu := (\det B)^{4\,n\,(n+1)\,d^{n+1}} \gamma \in L^*$. By Lemma 3.5 there exists $0 \le \ell \le p - 2$ such that $\mathrm{Tr}(\mu\,(\omega^{-\ell} - \omega)) \ne 0$.

We define

$$a := \mathrm{Tr}(\mu\,(\omega^{-\ell} - \omega))/p \in K^* \qquad , \qquad g_i := \mathrm{Tr}(\mu\,p_i\,(\omega^{-\ell} - \omega))/p \in K[x_1, \ldots, x_n]$$

for $i = 1, \ldots, s$.

Then

$$a = g_1\, f_1 + \cdots + g_s\, f_s$$

as $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$ and Tr is a K-linear map.

Aside from the degree and height bounds, we will show that since $f_1, \ldots, f_s \in \mathcal{O}_K[x_1, \ldots, x_n]$, $a \in \mathcal{O}_K$ and $g_i \in \mathcal{O}_K[x_1, \ldots, x_n]$.

Let us first analyze degrees and local heights.

As $\deg F_j \leq d$, $\deg g_i \leq \deg p_i \leq \max_j \deg P_j \leq 4\,n\,d^n$.

Now let $v \in M_K^\infty$ and let $w \in M_L$ such that $w \,|\, v$. We have $h_w(B^{-1}(y - b)) \leq n \log n - \log|\det B|_w$ and so

$$
\begin{aligned}
h_w(F_j) &\leq h_w(q_j) + (n \log n - \log|\det B|_w + 2 \log(n+1))\,d \\
&\leq h_v + \log s + (n+2) \log(n+1)\,d - \log|\det B|_w\,d
\end{aligned}
$$

by Lemma 1.2(c). ¿From Corollary 3.4

$$
\begin{aligned}
h_w(P_j) &\leq 4\,n\,(n+1)\,d^n \max_k h_w(F_k) + 4\,n\,(4\,n+5) \log(n+1)\,d^{n+1} - \log|\gamma|_w \\
&\leq 4\,n\,(n+1)\,d^n (h_v + \log s + (n+2) \log(n+1)\,d - \log|\det B|_w\,d) \\
&\quad + 4\,n\,(4\,n+5) \log(n+1)\,d^{n+1} - \log|\gamma|_w \\
&= 4\,n\,(n+1)\,d^n\,(h_v + \log s) + 4\,n\,(n^2 + 7\,n + 7) \log(n+1)\,d^{n+1} - \log|\mu|_w.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
h_w(\mu\,p_i) &\leq \max_j h_w(P_j) + 2 \log(n+1) \max_j \deg P_j + \log t + \log|\mu|_w \\
&\leq 4\,n\,(n+1)\,d^n\,(h_v + \log s) + 4\,n\,(n^2 + 7\,n + 7) \log(n+1)\,d^{n+1} \\
&\quad + 8\,n \log(n+1)\,d^n + \log(n+1) \\
&\leq 4\,n\,(n+1)\,d^n\,(h_v + \log s + (n+7) \log(n+1)\,d) - \log 2 \qquad (3.7)
\end{aligned}
$$

again by Lemma 1.2(c) and the fact $d, n \geq 2$. We have

$$
g_i = \frac{1}{p} \operatorname{Tr}(\mu\,p_i\,(\omega^{-\ell} - \omega)) = \frac{1}{p} \sum_{\sigma \in \operatorname{Gal}_{L/K}} \sigma(\mu\,p_i\,(\omega^{-\ell} - \omega))
$$

and so

$$
\begin{aligned}
h_v(g_i) &\leq \max_{w \,|\, v} h_w(\mu\,p_i) + \log 2 \\
&\leq 4\,n\,(n+1)\,d^n\,(h_v + \log s + (n+7) \log(n+1)\,d).
\end{aligned}
$$

We have $h_w(\mu) \leq 4\,n\,(n+1)\,d^n\,(h_v + \log s) + 4\,n\,(n^2 + 7\,n + 7) \log(n+1)\,d^{n+1}$ and so the previous estimate also holds for $h_v(a)$.

Now let $v \notin M_K^\infty$ and $w \,|\, v$. Analogously we have

$$
h_w(\mu), \; h_w(\mu\,p_i) \leq 4\,n\,(n+1)\,d^n\,h_v = 0
$$

as $f_1, \ldots, f_s \in \mathcal{O}_K[x_1, \ldots, x_n]$. Then $\mu \in \mathcal{O}_L \setminus \{0\}$ and $\mu\,p_i \in \mathcal{O}_L[x_1, \ldots, x_n]$, which in term implies that $a \in \mathcal{O}_K \setminus \{0\}$ and $g_i \in \mathcal{O}_K[x_1, \ldots, x_n]$ as desired.

The global height estimate follows then from the expression

$$
h(a, g_1, \ldots, g_s) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K^\infty} N_v \max\{h_v(a), h_v(g_1), \ldots, h_v(g_s)\}.
$$

$\square$

Remark 3.9 *The fact that the bound (3.7) is uniform on w for $w \,|\, v$ is the key that allows us to get rid of the roots of 1. This will no longer be the case in our treatment of the more refined arithmetic Nullstellensätze in Chapter 4.*

The following example improves the lower bound for a general height estimate given in the introduction, and thus shows that the term $d^n h$ is unavoidable.

Example 3.10 *Set*

$$f_1 := x_1 - H, \quad f_2 := x_2 - x_1^d, \dots, \quad f_n := x_n - x_{n-1}^d, \quad f_{n+1} := x_n^d$$

for any positive integers d, H. These are polynomials without common zeros in $\mathbb{A}^n$ of degree and height bounded by d and $h := \log H$ respectively.

Let $a \in \mathbb{Z} \setminus \{0\}$ and $g_1, \dots, g_{n+1} \in \mathbb{Z}[x_1, \dots, x_n]$ such that $a = g_1 \, f_1 + \cdots + g_{n+1} \, f_{n+1}$. We evaluate this identity in $(H, H^d, \cdots, H^{d^{n-1}})$ and we obtain

$$a = g_{n+1}(H, H^d, \cdots, H^{d^{n-1}}) \, H^{d^n}$$

from where we deduce $h(a) \geq d^n \, h$.

4   Intrinsic type estimates

Theorem 1 is essentially optimal in the general case. There are however many particular instances in which these estimates can be improved. Consider the following example:

$$f_1 := x_1 - 1, \quad f_2 := x_2 - x_1^d, \dots, \quad f_n := x_n - x_{n-1}^d, \quad f_{n+1} := H - x_n^d$$

for any positive integers d and H. These are polynomials without common zeros in $\mathbb{A}^n$ of degree and height bounded by d and $h := \log H$ respectively. Theorem 1 says there exist $a \in \mathbb{Z} \setminus \{0\}$ and $g_1, \dots, g_{n+1} \in \mathbb{Z}[x_1, \dots, x_n]$ such that

$$a = g_1 \, f_1 + \cdots + g_{n+1} \, f_{n+1}$$

with $\deg g_i \leq 4 \, n \, d^n$ and $h(a), h(g_i) \leq 4 \, n \, (n+1) \, d^n \, (h + (n+7) \, \log(n+1) \, d)$. However the following Bézout identity holds:

$$H - 1 = \frac{x_1^d - 1}{x_1 - 1} \cdots \frac{x_n^d - 1}{x_n - 1} \, f_1 + \cdots + \frac{x_n^d - 1}{x_n - 1} \, f_n + f_{n+1}.$$

Note that the polynomials arising in this identity have degree and height bounded by $n \, (d - 1)$ and h respectively.

There is in this case an exponential gap between the a priori general estimates and the actual ones. The explanation is somewhat simple: for $i = 1, \dots, n$, the varieties

$$V_i := V(f_1, \dots, f_i) = V(x_1 - 1, x_2 - 1, \dots, x_i - 1) \ \subset \mathbb{A}^n,$$

verify $\deg(V_i) = 1$ and $h(V_i) \leq 2 \, n \, \log(n+1)$. Namely, both the degree and the height of the varieties successively cut out by the input polynomials are much smaller than the corresponding Bézout estimate.

As the varieties V_i verify the assumptions of Lemma 3.3, a direct application together with Lemma 1.3 produces the more realistic estimates:

$$\deg g_i \leq 2 \, n^2 \, d \quad , \quad h(a), h(g_i) \leq (n+1)^2 (h + 8 \, n \, \log(n+1) \, d).$$

Based on this idea, we devote this chapter to the study of more refined arithmetic Nullstellensätze which can deal with such situations.

4.1 Equations in general position

This section deals with the preparation of the input data. To apply Lemma 3.3, we need to prepare the polynomials and the variables of the ambient space.

Let $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$ be polynomials without common zeros in $\mathbb{A}^n$. For $i = 1, \ldots, s$ and $a_{ij} \in \mathbb{Z}$ we set

$$q_i := a_{i\,1}\, f_1 + \cdots + a_{i\,s}\, f_s.$$

We will estimate the height of rational integers a_{ij} in order that there exists $t \leq \min\{n+1, s\}$ such that $(q_1, \ldots, q_i) \subset K[x_1, \ldots, x_n]$ is a radical ideal of dimension $n - i$ for $i = 1, \ldots, t-1$ and $1 \in (q_1, \ldots, q_t)$.
Also we set

$$y_k := b_{k\,0} + b_{k\,1}\, x_1 + \cdots + b_{k\,n}\, x_n$$

for $k = 1, \ldots, n$ and $b_{k\,l} \in \mathbb{Z}$. Again we want to estimate the height of rational integers b_{kl} such that $V_i := V(q_1, \ldots, q_i) \subset \mathbb{A}^n$ satisfies Assumption 1.5 with respect to this set of variables for $i = 1, \ldots, t-1$. Namely, the projection

$$\pi_i : V_i \to \mathbb{A}^{n-i} \qquad, \qquad x \mapsto (y_1, \ldots, y_{n-i})$$

must verify $\#\pi_i^{-1}(0) = \deg V_i$, that is $\# V_i \cap V(y_1, \cdots, y_{n-i}) = \deg V_i$ for $i = 1, \ldots, t-1$. Lemma 2.14 implies that the variables $y_1, \ldots, y_{n-i}$ are in Noether normal position with respect to V_i.

It is well-known that these conditions are satisfied by a generic election of a_{ij} and b_{kl}, see for instance [19, Section 3.2] and [50, Prop. 18 and Proof of Thm. 19]).

We have already applied such a preparation to obtain the classic style version of the effective arithmetic Nullstellensatz presented in Theorem 3.6. There, we chose roots of 1 as coefficients of the linear combinations since their existence was sufficient in our proof. However, technical reasons (see Remark 3.9) prevent us to apply the same principle in this chapter, and we need to carry out a more careful analysis.

We note that all aspects of this preparation were previously covered in the research papers [2, Section 4], [19, Section 3.2], [32, Section 6], [22, Section 5.2]. However the bounds presented therein are either non-explicit or not precise enough for our purposes. Here we chose to give a self-contained presentation, which yields another proof of the existence of such linear combinations. The obtained estimates substantially improve the previously known ones.

4.1.1 An effective Bertini theorem

This subsection is devoted to the preparation of the polynomials. We will first establish some auxiliary results.

The following is a version of the so-called shape lemma representation of a 0-dimensional radical ideal. The main difference here is that we choose a generic linear form — instead of a particular one — as a primitive element.

For a polynomial $f = c_D\, t^D + \cdots + c_0 \in k[t]$ we denote its discriminant by $\mathrm{discr}(f) \in k$. We recall that $\mathrm{discr}(f) \neq 0$ if and only if $c_D \neq 0$ and f is squarefree, that is f has exactly D distinct roots.

Lemma 4.1 *(Shape Lemma)*
Let $V \subset \mathbb{A}^n$ be a 0-dimensional variety defined over k. Let $U := (U_0, \ldots, U_n)$ be a group of $n+1$ variables, and set $L := U_0 + U_1\, x_1 + \cdots + U_n\, x_n$ for the associated generic linear form. Let $P := P_V \in k[U][T]$ be a characteristic polynomial of V. Set $P' := \partial P / \partial T \in k[U][T]$ and $\rho := \mathrm{discr}_T P \in k[U] \setminus \{0\}$. Also set I for the extension of $I(V)$ to $k[U][x]$.

Then there exist $v_1, \ldots, v_n \in k[U][T]$ with $\deg v_i \leq \deg V - 1$ such that

$$I_\rho = (P(L), P'(L)\, x_1 - v_1(L), \ldots, P'(L)\, x_n - v_n(L))_\rho \ \subset k[U]_\rho[x].$$

Here I_ρ denotes the localization of I at ρ.

Proof.– We note first that $I(V)$ is a radical ideal, and so $I = k[U] \otimes_k I(V)$ is also radical. We readily obtain from the definition of $P := P_V$ that $I \cap k[U][L] = (P(L))$, and so $P(L) \in I$.

We can write $P(L) = \sum_\alpha a_\alpha(x)\, U^\alpha$ with $a_\alpha(x) \in I(V)$. Therefore $\partial P(L)/\partial U_i$ also lies in I for all i. A direct computation shows that for $i = 1, \ldots, n$

$$\partial P(L)/\partial U_i = P'(L)\, x_i - v_i(L)$$

for some $v_i \in k[U][T]$ with $\deg v_i \leq \deg P - 1 = \deg V - 1$.
Set

$$J := (P(L), P'(L)\, x_1 - v_1(L), \ldots, P'(L)\, x_n - v_n(L)) \subset k[U][x].$$

The previous argument shows the inclusion $I \supset J$.

On the other hand, $\rho = A\,P + B\,P'$ for some $A, B \in k[U][T]$. Set $w_i := B\,v_i$. Then $x_i \equiv w_i(L)/\rho$ (mod J_ρ) and so for every $f \in k[U][x]$ we have that $f \equiv f(U, w_1(L)/\rho, \ldots, w_n(L)/\rho)$ modulo J_ρ, and hence modulo I_ρ.
For $f \in I$,

$$\rho^{\deg f}\, f(U, w_1(L)/\rho, \ldots, w_n(L)/\rho) \in I \cap k[U][L] = (P(L))$$

which implies $I_\rho \subset J_\rho$ as desired. $\qquad\square$

Let $\nu \in k^{n+1}$ such that $\rho(\nu) \neq 0$. It follows that $I(V)$ can be represented as

$$I(V) = (P(L), P'(L)\, x_1 - v_1(L), \ldots, P'(L)\, x_n - v_n(L))(\nu) \ \subset k[x].$$

Now let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be polynomials without common zeros in $\mathbb{A}^n$. For $i = 1, \ldots, s$ we let $Z_i := (Z_{i1}, \ldots, Z_{is})$ denote a group of s variables, and we set

$$Q_i := Z_{i1}\, f_1 + \cdots + Z_{is}\, f_s \in k[Z][x]$$

for the associated generic linear combination of $f_1, \ldots, f_s$.

Lemma 4.2 *For $\ell = 1, \ldots, s$, the ideal $(Q_1, \ldots, Q_\ell)$ is a complete intersection prime ideal of $k[Z][x]$.*

Proof.– Set $I := (Q_1, \ldots, Q_\ell)$ and $V := V(I) \subset \mathbb{A}^{s\ell} \times \mathbb{A}^n$. First we observe that V is a linear bundle over $\mathbb{A}^n$: the projection

$$\pi : V \to \mathbb{A}^n \qquad , \qquad (z, x) \mapsto x$$

is surjective, and the fibers are affine spaces of dimension $(s-1)\,\ell$. This follows from the assumption that the f_j have no common zeros. This implies that

$$\dim V = (s - 1)\, \ell + n$$

because of the theorem of dimension of fibers. Namely $Q_1, \ldots, Q_\ell$ is a complete intersection, and in particular the ideal I is unmixed.

Set $I = I_1 \cap \ldots \cap I_m$ for the primary decomposition of this ideal. We will show that I_j is prime for all j, and then that $m = 1$.

First we have that $I_{f_j} = (Q_1/f_j, \ldots, Q_\ell/f_j) = (Z_{1j} + H_{1j}, \ldots, Z_{\ell j} + H_{\ell j})$ where $H_{ij} \in k[Z_i][x]_{f_j}$ does not depend on Z_{ij}. Therefore

$$(k[\mathbb{A}^{s\ell} \times \mathbb{A}^n]/I)_{f_j} \cong k[\mathbb{A}^{(s-1)\ell} \times \mathbb{A}^n]_{f_j}$$

is a domain, that is I_{f_j} is prime. We have $I_{f_j} = (I_1)_{f_j} \cap \ldots \cap (I_m)_{f_j}$, and so there exists $1 \le n(j) \le m$ such that

$$I_{f_j} = (I_{n(j)})_{f_j} \qquad , \qquad V(I_i) \subset \{f_j = 0\} \quad \text{for } i \ne n(j).$$

In particular $I_{n(j)} = I_{f_j} \cap k[\mathbb{A}^{s\ell} \times \mathbb{A}^n]$ is prime. The fact that $\cap_j \{f_j = 0\} = \emptyset$ ensures that $n(j)$ runs over all $1 \le i \le m$, and so I is radical.

The expression $I_{f_j} = (Z_{1j} + H_{1j}, \ldots, Z_{\ell j} + H_{\ell j})$ implies that $\pi(V(I_{f_j})) \subset \mathbb{A}^n$ contains the dense open set $\{f_j \ne 0\}$. In particular $V(I_{f_j})$ is not contained in any of the hypersurfaces $\{f_i = 0\}$ and so $n(j) = n(1)$ for all j. This implies that $m = 1$, and so $I = I_1$ is prime. $\qquad\square$

The following proposition shows that $(Q_1(a_1), \ldots, Q_\ell(a_\ell))$ is a radical ideal for a generic election of $a_i := (a_{i1}, \ldots, a_{is})$. Unlike Lemmas 4.1 and 4.2, this result does note hold for arbitrary characteristic. For instance let $x^p, 1 - x^p \in \mathbb{F}_p[x]$ for some prime p. Then $Q_1(a_1) = b + c\, x^p$ for some $b, c \in \overline{\mathbb{F}_p}$ and so

$$Q_1(a_1) = (b^{1/p} + c^{1/p}\, x)^p$$

is not squarefree.

Proposition 4.3 *Let* $\mathrm{char}\,(k) = 0$ *and set* $I := (Q_1, \ldots, Q_\ell) \subset k[Z][x]$.

- *In case* $I \cap k[Z] \ne \{0\}$ *there exists* $F \in k[Z] \setminus \{0\}$ *with* $\deg F \le (d+1)^\ell$ *such that* $F(a_1, \ldots, a_\ell) \ne 0$ *for* $a_1, \ldots, a_\ell \in k^s$ *implies that* $1 \in (Q_1(a_1), \ldots, Q_\ell(a_\ell))$.

- *In case* $I \cap k[Z] = \{0\}$ *there exists* $F \in k[Z] \setminus \{0\}$ *with* $\deg F \le 2\,(d+1)^{2\,\ell}$ *such that* $F(a_1, \ldots, a_\ell) \ne 0$ *for* $a_1, \ldots, a_\ell \in k^s$ *implies that* $(Q_1(a_1), \ldots, Q_\ell(a_\ell)) \subset k[x]$ *is a radical ideal of dimension* $n - \ell$.

Proof.– Set $V := V(I) \subset \mathbb{A}^{s\ell} \times \mathbb{A}^n$. We have $\dim V = (s-1)\,\ell + n$ and $\deg V \le (d+1)^\ell$.

First we consider the case $I \cap k[Z] \ne \{0\}$. This occurs, for instance, when $\ell \ge n+1$, since then $\dim I = s\,\ell + n - \ell < \dim k[Z] = s\,\ell$.
Let $\pi : \mathbb{A}^{s\ell} \times \mathbb{A}^n \to \mathbb{A}^{s\ell}$ be the canonical projection. Then $\overline{\pi(V)}$ is a proper subvariety of $\mathbb{A}^{s\ell}$, and thus it is contained in a hypersurface of degree bounded by $\deg V$. This can be seen by taking a generic projection of this variety into an affine space of dimension $s\,\ell + n - \ell + 1$ [23, Remark 4]. Let $F \in k[Z]$ be a defining equation of this hypersurface. Then $F \in I$ as I is prime, and we have $\deg F \le (d+1)^\ell$. Thus

$$1 \in I_F \subset k[Z]_F[x],$$

and therefore $1 \in I(a) := (Q_1(a_1), \ldots, Q_\ell(a_\ell))$ for $a \in k^{s\ell}$ such that $F(a) \ne 0$.

Next we consider the case $I \cap k[Z] = \{0\}$.
We adopt the following convention: for an ideal $J \subset k[x]$ and for ζ any new group of variables, we denote by $J^{[\zeta]}$ and $J^{(\zeta)}$ the extension of J to the polynomial rings $k[\zeta][x]$ and $k(\zeta)[x]$ respectively.

We assume for the moment $\ell = n$. Then $\dim I = s\,\ell$ and so the extended ideal $I^{(Z)} \subset k(Z)[x]$ is a 0-dimensional prime ideal. We have then that $\overline{k(Z)} \otimes_k I^{(Z)} \subset \overline{k(Z)}[x]$ is a radical ideal, as $\mathrm{char}\,(k) = 0$ [44, Thm. 26.3].
Our approach to this case is based on Shape Lemma 4.1. We will determine a polynomial $F \in k[Z]$ such that $F(a) \ne 0$ implies that the shape lemma representation of $I^{(Z)}$ can be transferred to a shape lemma representation of $I(a)$.

Let U be a group of $n + 1$ variables and set

$L := U_0 + U_1 x_1 + \cdots + U_n x_n$ for the associated generic linear form. Consider the morphism

$$\Psi : \mathbb{A}^{n+1} \times \mathbb{A}^{s\ell} \times \mathbb{A}^n \to \mathbb{A}^{n+1} \times \mathbb{A}^{s\ell} \times \mathbb{A}^1 \quad , \quad (u, z, x) \mapsto (u, z, L(x)).$$

and let W be the variety defined by I in $\mathbb{A}^{n+1} \times \mathbb{A}^{s\ell} \times \mathbb{A}^n$, that is $W = \mathbb{A}^{n+1} \times V$. The Zariski closure $\overline{\Psi(W)}$ is then an irreducible hypersurface. We set $P \in k[U, Z][T]$ for one of its defining equations.

If $I^{[U](Z)}$ is the extension of $I^{(Z)}$ to $k[U](Z)[x]$, the polynomial P can be equivalently defined through the condition that $P(L)$ is a generator of the principal ideal $I^{[U](Z)} \cap k[U, Z][L]$. Namely, P is a characteristic polynomial of the $0-$ dimensional variety W_0 defined by $I^{(Z)}$ in $\mathbb{A}^n(\overline{k(Z)})$.

Let $v_1, \ldots, v_n \in k[U](Z)[T]$ denote the polynomials arising in Shape Lemma applied to W_0. From the proof of this lemma we have that

$$\partial P(L)/\partial U_i = P'(L)\, x_i - v_i(L) \in k[U, Z][L]$$

and so $v_i \in k[U, Z][T]$. Set $J := (P(L), P'(L)\, x_1 - v_1(L), \ldots, P'(L)\, x_n - v_n(L)) \subset k[U, Z][x]$ and $\rho := \mathrm{discr}_T P \in k[U, Z] \setminus \{0\}$. Then

$$(I^{[U](Z)})_\rho = (J^{[U](Z)})_\rho \ \subset k[U](Z)_\rho[x].$$

We have that both $I_\rho^{[U,Z]}$ and $J_\rho^{[U,Z]}$ are prime ideals of $k[U, Z]_\rho[x]$ with trivial intersection with the ring $k[U, Z]$. Thus they coincide with the contraction of $I_\rho^{[U](Z)}$ and $J_\rho^{[U](Z)}$ to $k[U, Z]_\rho[x]$ respectively, and so

$$I_\rho^{[U,Z]} = J_\rho^{[U,Z]} \ \subset k[U, Z]_\rho[x].$$

Define $F \in k[Z] \setminus \{0\}$ as any of the non-zero coefficients of the monomial expansion of ρ with respect to U. Let $a \in k^{s\ell}$ such that $F(a) \neq 0$. Then $\rho(U, a) \neq 0$ and so $P(U, a)[T]$ is squarefree. Then

$$(I(a)^{[U]})_{\rho(U,a)} = (P(L), P'(L)\, x_1 - v_1(L), \ldots, P'(L)\, x_n - v_n(L))(a) \ \subset k[U][x]$$

is radical, which implies in turn that $I(a) = (I(a)^{[U]})_{\rho(U,a)} \cap k[x]$ is a radical ideal of $k[x]$ as desired.

It remains to estimate the degree of F. To this end, it suffices to bound the degree of ρ with respect to the group of variables Z. We recall that P was defined as a defining equation of the hypersurface $\overline{\Psi(W)}$. The map Ψ is linear in the variables Z and x, and so

$$\deg_Z P \leq \deg W = \deg V \leq (d+1)^n.$$

This implies that $\deg F \leq \deg_Z \rho \leq \deg_Z P\, (2\, \deg_Z P - 1) \leq 2\, (d+1)^{2n}$.

Finally we consider the case $\ell < n$ for $I \cap k[Z] = \{0\}$.

Let $U_1, \ldots, U_{n-\ell}$ be groups of $n + 1$ variables each, and set

$$L_i := U_{i\,0} + U_{i\,1} x_1 + \cdots + U_{i\,n} x_n.$$

for $i = 1, \ldots, n - \ell$. Set $U := (U_1, \ldots, U_{n-\ell})$, $L := (L_1, \ldots, L_{n-\ell})$ and $k_0 := k(U, L)$. The extended ideal $I_0 \subset k_0[Z][x_1, \ldots, x_\ell]$ verifies $I_0 \cap k_0[Z] = \{0\}$ and thus falls into the previously considered case.

Thus there exists $F_0 \in k_0[Z] \setminus \{0\}$ with $\deg F_0 \leq 2\, (d+1)^{2\ell}$ such that $F_0(a) \neq 0$ for $a \in k^{s\ell}$ implies that $I_0(a)$ is a radical ideal of $k_0[Z][x_1, \ldots, x_\ell]$. This implies in turn that $I(a)$ is a radical ideal of $k[x]$, as

$$I(a) = I_0(a) \cap k[x].$$

We can assume without loss of generality that F_0 lies in $k[U, L][Z]$. We conclude by taking F as any non-zero coefficient of the monomial expansion of F_0 with respect to the variables U and L.
$\square$

Corollary 4.4 *Let* $\text{char}(k) = 0$ *, and let* $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ *be polynomials without common zeros in* $\mathbb{A}^n$ *. Set* $d := \max_i \deg f_i$ *.*
Then there exist $t \leq \min\{n + 1, s\}$ *and* $a_1, \ldots, a_t \in \mathbb{Z}^s$ *such that*

- $(Q_1(a_1), \ldots, Q_i(a_i))$ *is a radical ideal of dimension* $n - i$ *for* $1 \leq i \leq t - 1$ *,*

- $1 \in (Q_1(a_1), \ldots, Q_t(a_t))$ *,*

- $h(a_i) \leq 2(n + 1) \log(d + 1)$ *.*

Proof.– Set t for the minimal i such that $I_i := (Q_1, \ldots, Q_i) \cap k[Z] \neq \{0\}$. Then $t \leq n + 1$, and by the previous result there exists $F_t \in k[Z]$ with $\deg F_t \leq (d + 1)^t$ such that $F_t(a) \neq 0$ implies that $1 \in (Q_1(a_1), \ldots, Q_t(a_t))$.
On the other hand, for $i < t$ we take a polynomial $F_i \in k[Z]$ of degree bounded by $2(d + 1)^{2i}$ such that $F_i(a) \neq 0$ implies that $(Q_1(a_1), \ldots, Q_i(a_i))$ is a radical ideal of dimension $n - i$. Then we take $F := F_1 \cdots F_t$ and so

$$
\begin{aligned}
\deg F &\leq 2(d + 1)^2 + \cdots + 2(d + 1)^{2(t-1)} + (d + 1)^t \\
&\leq (d + 1)^{2n} + 2(d + 1)^{2n} + (d + 1)^{n+1} \\
&\leq 4(d + 1)^{2n}.
\end{aligned}
$$

Finally, $F \neq 0$ implies there exist $a_1, \ldots, a_t \in \mathbb{Z}^s$ such that $h(a_i) \leq \log(\deg F)$ and $F(a) \neq 0$. $\square$

4.1.2 Effective Noether normal position

Now we devote to the preparation of the variables. For $k = 0, \ldots, n$ we let $U_k := (U_{k0}, \ldots, U_{kn})$ be a group of $n + 1$ variables and we set

$$
Y_k := U_{k\,0} + U_{k\,1}\, x_1 + \cdots + U_{k\,n}\, x_n.
$$

Proposition 4.5 *Let* $V \subset \mathbb{A}^n$ *be an equidimensional variety of dimension* r *defined over* k *.*
Then there exists $G \in k[U_1, \ldots, U_r] \setminus \{0\}$ *with* $\deg_{U_k} G \leq 2(\deg V)^2$ *such that* $G(b_1, \ldots, b_r) \neq 0$ *for* $b_1, \ldots, b_r \in k^{n+1}$ *implies that*

$$
\# \, V \cap V(Y_1(b_1), \ldots, Y_r(b_r)) = \deg V.
$$

Proof.– Let f_V be a Chow form of V and $P_V \in k[U, T]$ be the characteristic polynomial of V associated to f_V given by Lemma 2.13.
Set $D := \deg V$ and let $P_V = c_D T_0^D + \cdots + c_0$ be its expansion with respect to T_0 . Also set

$$
\rho := \text{discr}_{T_0} P_V \in k[U_0, \ldots, U_r][T_1, \ldots, T_r] \setminus \{0\}
$$

for the discriminant of P_V with respect to T_0 .
Observe that as P_V is multihomogeneous of degree D in each group of variables $U_i \cup \{T_i\}$, the degree of ρ in each of these group of variables is bounded by $D(2D - 1)$.
Now let $\nu_1, \ldots, \nu_r \in \overline{k}^{n+1}$ such that $V(\nu) := V \cap V(Y_1(\nu_1), \ldots, Y_r(\nu_r))$ is a 0-dimensional variety of cardinality D , and $f_{V(\nu)}$ be a Chow forms of $V(\nu)$.
Set $\zeta_0 := (T_0 - U_{00}, U_{01}, \ldots, U_{0n})$. Then applying [47, Prop. 2.4], there exists $\lambda \in k^*$ such that:

$$
P_V(U_0, \nu_1, \ldots, \nu_r)(T_0, 0, \ldots, 0) = f_V(\zeta_0(U_0, T_0), \nu_1, \ldots, \nu_r) = \lambda\, f_{V(\nu)}(\zeta_0(U_0, T_0)) = \lambda\, P_{V(\nu)}(U_0)(T_0)
$$

where $P_{V(\nu)}$ is a characteristic polynomial of $V(\nu)$.
This implies $P_V(U)(T_0, 0, \ldots, 0) \in k[U][T_0]$ is a squarefree polynomial and so $\rho(U)(0) \neq 0$.

We take $G \in k[U_1, \ldots, U_r]$ as any non-zero coefficient of the expansion of $\rho(U)(0)$ with respect to U_0. Therefore

$$\deg G \le \deg_{U_i} \rho(U)(0) \le D\,(2\,D - 1).$$

The condition $G(b) \ne 0$ implies that $\rho(U_0, b_1, \ldots, b_r)(0) \ne 0$, and so $\#V(b) = D$. $\qquad\square$

As we noted before, this implies that the variables $Y_1(b_1), \ldots, Y_r(b_r)$ are in Noether normal position with respect to the variety V.

Corollary 4.6 *Let* $\mathrm{char}\,(k) = 0$ *and let* $q_1, \ldots, q_t \in k[x_1, \ldots, x_n]$ *be polynomials without common zeros in* $\mathbb{A}^n$ *which form a reduced weak regular sequence. Set* $d := \max_i \deg f_i$.
Then there exist $b_1, \ldots, b_n \in \mathbb{Z}^{n+1}$ *such that* $V(q_1, \ldots, q_i)$ *satisfies Assumption 1.5 with respect to the variables* $Y_1(b_1), \ldots, Y_{n-i}(b_{n-i})$ *for* $i = 1, \ldots, t$, *and*

$$h(b_k) \le 2\,(n+1)\,\log(d+1).$$

Proof.– This follows readily from the previous result. We take G_i as the polynomial corresponding to the variety $V(q_1, \ldots, q_i)$ and we set $G := G_1 \cdots G_{t-1} \in k[U_1, \ldots, U_n]$. We have $\deg_{U_j} G_i \le 2\,d^{2\,i}$ and so

$$\deg_{U_j} G \le 2\,d^2 + \cdots + 2\,d^{2\,(t-1)} \le 4\,d^{2\,(t-1)} \le 4\,d^{2\,n}.$$

We conclude by taking $b_1, \ldots, b_n \in \mathbb{Z}^{n+1}$ such that $h(b_i) \le \log(\deg G)$ and $G(b) \ne 0$. $\qquad\square$

4.2   An intrinsic arithmetic Nullstellensatz

In this section we introduce the notions of degree and height of a polynomial system defined over a number field K. Modulo setting the input equations in general position, these parameters measure the degree and height of the varieties successively cut out.
The resulting estimates for the arithmetic Nullstellensatz are *linear* in these parameters. As an important particular case, we derive a sparse arithmetic Nullstellensatz.

4.2.1   Intrinsic parameters

Let $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$ be polynomials of degree bounded by d without common zeros in $\mathbb{A}^n$. For $i = 1, \ldots, s$ we let Z_i denote a group of s variables and we set

$$Q_i(Z) := Z_{i1}\,f_1 + \cdots + Z_{is}\,f_s \in K[Z][x]$$

for the associated generic linear combination of $f_1, \ldots, f_s$.
Let Γ be the set of integer $s \times s-$matrices $a = (a_{ij})_{ij} \in \mathbb{Z}^{s \times s}$ of height bounded by $2\,(n+1)\,\log(d+1)$ such that

$$I_i(a) := (Q_1(a_1), \ldots, Q_i(a_i)) \subset K[x_1, \ldots, x_n]$$

is a radical ideal of dimension $n - i$ for $i = 1, \ldots, t - 1$ and $1 \in I_t(a)$ for some $t \le \min\{n+1, s\}$. Corollary 4.4 implies that $\Gamma \ne \emptyset$.
For $a \in \Gamma$ we set

$$\delta(a) \quad := \quad \max\,\{\,\deg V(I_i(a))\,;\, 1 \le i \le \min\{t, n\} - 1\,\},$$
$$\eta(a) \quad := \quad \max\,\{\,h(V(I_i(a)))\,;\, 1 \le i \le t - 1\,\}.$$

We set $\Gamma_{\min} \subset \mathbb{Z}^{s \times s}$ for the subset of matrices $a \in \Gamma$ such that $\eta(a) + d\,\delta(a)$ is minimum. Finally let $a_{\min} \in \Gamma_{\min}$ be a matrix which attains the minimum of $\delta(a)$ for $a \in \Gamma_{\min}$.

Definition 4.7 *Let notations be as in the previous paragraph. Then we define the* degree *and the* height *of the polynomial system* $f_1, \ldots, f_s$ *respectively as*

$$\delta(f_1, \ldots, f_s) := \delta(a_{\min}) \qquad , \qquad \eta(f_1, \ldots, f_s) := \eta(a_{\min}).$$

We restrict ourselves to integer matrices of bounded height in order to keep control of the height of $Q_1(a_1), \ldots, Q_t(a_t)$. The election of $\eta(a) + d\,\delta(a)$ as the defining invariant comes from the need of estimating the degree and height simultaneously.

We note that in case $f_1, \ldots, f_s$ is already a reduced weak regular sequence we have

$$\eta(f_1, \ldots, f_s) + d\,\delta(f_1, \ldots, f_s) \leq \eta(\mathrm{Id}) + d\,\delta(\mathrm{Id}).$$

We can estimate this parameters through the arithmetic Bézout inequality:

Lemma 4.8 *Let* $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$ *be polynomials without common zeros in* $\mathbb{A}^n$. *Set* $d_i := \deg f_i$ *and assume that* $d_1 \geq \cdots \geq d_s$ *holds. Set* $d := d_1 = \max_i \deg f_i$ *and* $h := h(f_1, \ldots, f_s)$. *Also set* $n_0 := \min\{n, s\}$ *and* $n_1 := \min\{n+1, s\}$. *Then*

- $\delta(f_1, \ldots, f_s) \leq \prod_{j=1}^{n_0-1} d_j$,

- $\eta(f_1, \ldots, f_s) \leq n\,(h + \log s + 3\,n\,(n+1)\,d)\,\prod_{j=1}^{n_1-2} d_j$.

Proof.– Let $a := a_{\min} = (a_{ij})_{ij} \in \mathbb{Z}^{s \times s}$ be a coefficient matrix such that $\delta(f_1, \ldots, f_s) = \delta(a)$ and $\eta(f_1, \ldots, f_s) = \eta(a)$, and set

$$q_i := a_{i1}\,f_1 + \cdots + a_{is}\,f_s\,, \qquad 1 \leq i \leq s.$$

Let $t \leq n_1 = \min\{n+1, s\}$ be minimum such that $1 \in (q_1, \ldots, q_t)$. Let $\widetilde{a} \in \mathbb{Z}^{(t-1) \times s}$ be the matrix formed by the first $t-1$ rows of a and let $c \in \mathbb{Z}^{(t-1) \times s}$ be a staircase matrix equivalent to $\widetilde{a}$.

The polynomial system

$$\widetilde{q}_i := c_{i1}\,f_1 + \cdots + c_{is}\,f_s$$

is then equivalent to $q_1, \ldots, q_{t-1}$, that is $(\widetilde{q}_1, \ldots, \widetilde{q}_i) = (q_1, \ldots, q_i)$ for $i = 1, \ldots, t-1$. Also we have $\deg \widetilde{q}_i \leq d_i$, and so

$$\delta := \max\{\deg V_i;\ 1 \leq i \leq \min\{n, t\} - 1\} \leq \prod_{j=1}^{n_0-1} d_j.$$

We have also that each coefficient of c is a subdeterminant of $\widetilde{a}$. Thus

$$
\begin{aligned}
\widetilde{h} := h(\widetilde{q}_1, \ldots, \widetilde{q}_{t-1}) \quad &\leq \quad h + \log s + h(c) \\
&\leq \quad h + \log s + (t-1)\,(2\,(n+1)\,\log(d+1) + \log(t-1)) \\
&\leq \quad h + \log s + n\,(3\,n+1)\,d
\end{aligned}
$$

and so, applying Corollary 2.11,

$$
\begin{aligned}
\eta \quad &\leq \quad \max\{h(V_i) : 1 \leq i \leq \min\{n+1, t\} - 1\} \\
&\leq \quad \Big(\sum_{l=1}^{n_1-1} \widetilde{h}/d_l + (n + n_1 - 1)\,\log(n+1)\Big) \prod_{j=1}^{n_1-1} d_j \\
&\leq \quad (n\,(h + \log s + n\,(3\,n+1)\,d) + 2\,n\,\log(n+1)) \prod_{j=1}^{n_1-2} d_j \\
&\leq \quad n\,(h + \log s + 3\,n\,(n+1)\,d) \prod_{j=1}^{n_1-2} d_j.
\end{aligned}
$$

$\square$

We can also estimate these parameters through the arithmetic Bernstein-Kushnirenko inequality:

Lemma 4.9 Let $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$ be polynomials without common zeros in $\mathbb{A}^n$. Set $d := \max_i \deg f_i$ and $h := h(f_1, \ldots, f_s)$. Also let $\mathcal{V}$ denote the volume of $1, x_1, \ldots, x_n, f_1, \ldots, f_s$. Then

- $\delta(f_1, \ldots, f_s) \leq \mathcal{V}$,

- $\eta(f_1, \ldots, f_s) \leq n \mathcal{V} (h + \log s + 2^{2n+3} d)$.

Proof.– Let $a := a_{\min} = (a_{ij})_{ij} \in \mathbb{Z}^{s \times s}$ and set $q_i := a_{i1} f_1 + \cdots + a_{is} f_s$ for $i = 1, \ldots, s$. Then $\operatorname{Supp}(q_i) \subset \operatorname{Supp}(f_1, \ldots, f_s)$ and so $\mathcal{V}(1, x_1, \ldots x_n, q_1, \ldots, q_s) \leq \mathcal{V}$.
Applying Proposition 2.12 we obtain $\delta \leq \mathcal{V}$ and

$$
\begin{aligned}
\eta &\leq (n \max_i h(q_i) + 2^{2n+2} \log(n+1) d) \mathcal{V} \\
&\leq (n (h + \log s + 2(n+1) \log(d+1)) + 2^{2n+2} \log(n+1) d) \mathcal{V} \\
&\leq n \mathcal{V} (h + \log s + 2^{2n+3} d).
\end{aligned}
$$

$\square$

4.2.2 Proof of Theorem 2

Modulo the preparation of the input data, the proof of Theorem 2 follows the lines of the example introduced at the beginning of Chapter 4.
The following is the general version of Theorem 2 over number fields:

Theorem 4.10 *(Intrinsic arithmetic Nullstellensatz)*
Let K be a number field and let $f_1, \ldots, f_s \in \mathcal{O}_K[x_1, \ldots, x_n]$ be polynomials without common zeros in $\mathbb{A}^n$. Set $d := \max_i \deg f_i$ and $h := h(f_1, \ldots, f_s)$. Also let δ and η denote the degree and the height of the polynomial system $f_1, \ldots, f_s$.
Then there exist $a \in \mathbb{Z} \setminus \{0\}$ and $g_1, \ldots, g_s \in \mathcal{O}_K[x_1, \ldots, x_n]$ such that

- $a = g_1 f_1 + \cdots + g_s f_s$,

- $\deg g_i \leq 2 n^2 d \delta$,

- $h(a, g_1, \ldots, g_s) \leq (n+1)^2 [K : \mathbb{Q}] d (2\eta + (h + \log s) \delta + 21 (n+1)^2 d \log(d+1) \delta)$.

Proof.– Let $a_{\min} = (a_{ij})_{ij} \in \mathbb{Z}^{s \times s}$ be a coefficient matrix such that $\delta = \delta(a_{\min})$, $\eta = \eta(a_{\min})$ and $h(a_{\min}) \leq 2(n+1) \log(d+1)$. We set

$$
q_i := a_{i1} f_1 + \cdots + a_{is} f_s
$$

for $i = 1, \ldots, s$. Then $(q_1, \ldots, q_i)$ is a radical ideal of dimension $n - i$ for $i = 1, \ldots, t-1$ and $1 \in (q_1, \ldots, q_t)$ for some $t \leq \min\{n+1, s\}$.
For $1 \leq k \leq n$, $0 \leq l \leq n$, we also let $b_{kl} \in \mathbb{Z}$ be integers with $h(b_{kl}) \leq 2(n+1) \log(d+1)$ such that $V_i := V(q_1, \ldots, q_i)$ satisfies Assumption 1.5 with respect to the variables

$$
y_k := b_{k0} + b_{k1} x_1 + \cdots + b_{kn} x_n
$$

for $i = 1, \ldots, t-1$. Set $b := (b_{k0})_k \in \mathbb{Z}^n$ and $B := (b_{kl})_{k,l \geq 1} \in \operatorname{GL}_n(\mathbb{Q})$, and set $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ for the affine map $\varphi(x) := B x + b$. For $j = 1, \ldots, t$ we then set

$$
F_j(y) := q_j(x) = q_j(\varphi^{-1}(y)) \in K[y_1, \ldots, y_n]
$$

Thus $F_1, \ldots, F_t$ are in the hypothesis of Lemma 3.3 with respect to $y_1, \ldots, y_n$ and we let $P_1, \ldots, P_t \in K[x_1, \ldots, x_n]$ be the polynomials satisfying Bézout identity we obtain there. Finally, for $i = 1, \ldots, s$, we set

$$p_i := \sum_{j=1}^{t} a_{ij} P_j(\varphi(x)) \ \in K[x_1, \ldots, x_n].$$

We have $\ 1 = p_1 f_1 + \cdots + p_s f_s$.

Now we analyze the degree and the height of these polynomials. We will assume $n, d \geq 2$ as the remaining cases have already been considered in Lemmas 3.7 and 3.8.

Set $W_l := V(F_1, \ldots, F_l) \subset \mathbb{A}^n$ for $l = 1, \ldots, t-1$. We have $W_l = \varphi(V_l)$ and so $\deg W_l = \deg V_l$. We have also $\deg F_j = \deg q_j \leq d$ and so

$$\deg p_i \leq \max_j \deg P_j \leq 2\, n\, d\, (1 + \sum_{l=1}^{\min\{n,s\}-1} \deg W_l) \leq 2\, n^2\, d\, \delta$$

as $\deg W_l \leq \delta$ for $l \leq n-1$.
Now let $v \in M_K^\infty$. We have $h_\infty(\varphi) \leq 2\,(n+1)\,\log(d+1)$ and so

$$
\begin{aligned}
h_\infty(\varphi^{-1}) \ &\leq \ n\,(h_\infty(\varphi) + \log n) - \log|\det B|_\infty \\
&\leq \ n\,(2\,(n+1)\,\log(d+1) + \log n) - \log|\det B|_\infty \\
&\leq \ 3\,n\,(n+1)\,\log(d+1) - \log|\det B|_\infty
\end{aligned}
$$

Set $h_v := \max_i h_v(f_i)$. Then

$$
\begin{aligned}
h_v(F_i) \ &\leq \ h_v(q_i) + (h_\infty(\varphi^{-1}) + 2\,\log(n+1))\,\deg q_i \\
&\leq \ h_v + 2\,(n+1)\,\log(d+1) + \log s + (3\,n\,(n+1)\,\log(d+1) - \log|\det B|_\infty + 2\,\log(n+1))\,d \\
&\leq \ h_v + \log s + (n+1+3\,n\,(n+1)+2\,n)\,d\,\log(d+1) - \log|\det B|_\infty\, d \\
&\leq \ h_v + \log s + 3\,(n+1)^2\,d\,\log(d+1) - \log|\det B|_\infty\, d.
\end{aligned}
$$

by Lemma 1.2(c) and the facts that $\log(n+1) \leq n$ and $\log(d+1) \geq 1$ for $d \geq 2$. Next, applying Lemma 2.7, we obtain

$$
\begin{aligned}
h(W_l) \ &\leq \ h(V_l) + (n-l+1)(h(\varphi) + 5\,\log(n+1))\,\deg V_l \\
&\leq \ h(V_l) + n\,(2\,(n+1)\,\log(d+1) + 5\,\log(n+1))\,\deg V_l \\
&\leq \ \eta + n\,(7\,n+2)\,d\,\log(d+1)\,\delta
\end{aligned}
$$

as $\deg W_l = \deg V_l \leq d\,\delta$ and $h(V_l) \leq \eta$ for $l = 1, \ldots, t-1$. By Lemma 3.3 there exists $\xi \in K^*$ such that

$$
\begin{aligned}
h_v(P_j) \ &\leq \ 2\,n\,d \sum_{l=1}^{t-1} h_v(W_l) + ((n+1)\max_l h_v(F_l) + 2\,n\,(2\,n+5)\,\log(n+1)\,d)\,(1 + \sum_{l=1}^{t-1} \deg W_l) \\
&\quad - \log|\xi|_v \\
&\leq \ 2\,n\,d \sum_{l=1}^{t-1} h_v(W_l) + (n+1)^2\,(h_v + \log s)\,d\,\delta \\
&\quad + (3\,(n+1)^4 + 2\,n^2\,(2\,n+5)\,(n+1))\,d^2\,\log(d+1)\,\delta - \log|\mu|_v
\end{aligned}
$$

with $\mu := (\det B)^{(n+1)^2 d^2 \delta} \xi \in K^*$. From the previous estimates we deduce

$$
\begin{aligned}
h_v(p_i) \;\leq\; & \max_j h_v(P_j) + (h_\infty(\varphi) + 2\log(n+1)) \max_j \deg P_j + 2(n+1)\log(d+1) + \log t \\[2mm]
\leq\; & 2\,n\,d \sum_l h_v(W_l) + (n+1)^2\,(h_v + \log s)\,d\,\delta \\
& + (3\,(n+1)^4 + 2\,n^2\,(2\,n+5)\,(n+1))\,d^2\,\log(d+1)\,\delta - \log|\mu|_v \\
& + (2\,(n+1)\log(d+1) + 2\log(n+1))\,2\,n^2\,d\,\delta + 2\,(n+1)\log(d+1) + \log(n+1) \\[2mm]
\leq\; & 2\,n\,d \sum_l h_v(W_l) + (n+1)^2\,(h_v + \log s)\,d\,\delta + 7\,(n+1)^3\,(n+2)\,d^2\,\log(d+1)\,\delta - \log|\mu|_v .
\end{aligned}
$$

Analogously $\;h_v(p_i) \leq 2\,n\,d \sum_l h_v(W_l) + (n+1)^2\,h_v\,d\,\delta - \log|\mu|_v\;$ for $v \notin M_K^\infty$.
Hence

$$
\begin{aligned}
h(p_1,\ldots,p_s) \;\leq\; & 2\,n\,d \sum_l h(W_l) + (n+1)^2\,(h + \log s)\,d\,\delta + 7\,(n+1)^3\,(n+2)\,d^2\,\log(d+1)\,\delta \\[2mm]
\leq\; & 2\,n^2\,d\,\eta + 2\,n^3\,(7\,n+2)\,d^2\,\log(d+1)\,\delta + (n+1)^2\,(h_v + \log s)\,d\,\delta \\
& + 7\,(n+1)^3\,(n+2)\,d^2\,\log(d+1)\,\delta \\[2mm]
\leq\; & 2\,n^2\,d\,\eta + (n+1)^2\,(h + \log s)\,d\,\delta + 21\,(n+1)^4\,d^2\,\log(d+1)\,\delta .
\end{aligned}
$$

Finally we apply Lemma 1.3 to obtain $a \in \mathbb{Z} \setminus \{0\}$ such that $g_i := a\,p_i \in \mathcal{O}_K[x_1,\ldots,x_n]$. Thus

$$
a = g_1\,f_1 + \cdots + g_s\,f_s
$$

and the corresponding height estimates are multiplied by $[K:\mathbb{Q}]$. $\qquad\square$

We derive from this result and Lemma 4.8 the following estimate in terms of the degree and the height of the input polynomials:

Corollary 4.11 Let $f_1,\ldots,f_s \in \mathcal{O}_K[x_1,\ldots,x_n]$ be polynomials without common zeros in $\mathbb{A}^n$. Set $d_i := \deg f_i$ and assume that $d_1 \geq \cdots \geq d_s$ holds. Also set $d := d_1 = \max_i \deg f_i$, $h := h(f_1,\ldots,f_s)$, and $n_0 := \min\{n,s\}$.
Then there exist $a \in \mathbb{Z} \setminus \{0\}$ and $g_1,\ldots,g_s \in \mathcal{O}_K[x_1,\ldots,x_n]$ such that

- $a = g_1\,f_1 + \cdots + g_s\,f_s$,

- $\deg g_i \leq 2\,n^2\,d \prod_{j=1}^{n_0-1} d_j$,

- $h(a,g_1,\ldots,g_s) \leq 2\,(n+1)^3\,[K:\mathbb{Q}]\,(h + \log s + 3\,n(n+7)\,d\,\log(d+1))\,d \prod_{j=1}^{n_0-1} d_j$.

4.2.3 Estimates for the sparse case

Our arithmetic Bernstein-Kushnirenko inequality (Proposition 2.12 and Lemma 4.9) shows that both the degree and the height of a system are controlled by its volume. We then derive from Theorem 4.10 the following arithmetic Nullstellensatz for sparse polynomial systems. Corollary 3 in the Introduction corresponds to the case $K := \mathbb{Q}$.

Corollary 4.12 *(Sparse arithmetic Nullstellensatz)*
Let $f_1,\ldots,f_s \in \mathcal{O}_K[x_1,\ldots,x_n]$ be polynomials without common zeros in $\mathbb{A}^n$. Set $d := \max_i \deg f_i$ and $h := h(f_1,\ldots,f_s)$. Also let $\mathcal{V}$ denote the volume of the polynomial system $1, x_1,\ldots,x_n, f_1,\ldots,f_s$.
Then there exist $a \in \mathbb{Z} \setminus \{0\}$ and $g_1,\ldots,g_s \in \mathcal{O}_K[x_1,\ldots,x_n]$ such that

- $a = g_1 f_1 + \cdots + g_s f_s$

- $\deg g_i \leq 2 \, n^2 \, d \, \mathcal{V}$,

- $h(a, g_1, \ldots, g_s) \leq 2 \, (n+1)^3 \, [K : \mathbb{Q}] \, d \, \mathcal{V} \, (\, h + \log s + 2^{2n+3} d \log(d+1))$.

Example 4.13 *For* $1 \leq i \leq s$ *we let*

$$f_i := a_{i\,0} + a_{i\,1} x_1 + \cdots + a_{i\,n} x_n + b_{i\,1} x_1 \cdots x_n + \cdots + b_{i\,d}(x_1 \cdots x_n)^d \in \mathbb{Z}[x_1, \ldots, x_n]$$

be polynomials of degree bounded by $n\,d$ *without common zeros in* $\mathbb{A}^n$. *Set* $h := \max_i h(f_i)$. *Also set* $\mathcal{P}_d := \mathrm{Conv}(0, e_1, \ldots, e_n, d\,(e_1 + \cdots + e_n)) \subset \mathbb{R}^n$, *so that* $\mathcal{P}_d$ *contains the Newton polytope of the polynomials* $1, x_1, \ldots, x_n, f_1, \ldots, f_s$. *Then*

$$\mathcal{V} \leq \mathrm{Vol}(\mathcal{P}_d) = n!\,d/(n-1)! \,=\, n\,d.$$

We conclude that there exist $a \in \mathbb{Z} \setminus \{0\}$ *and* $g_1, \ldots, g_s \in \mathbb{Z}[x_1, \ldots, x_n]$ *such that*

- $a = g_1 f_1 + \cdots + g_s f_s$,

- $\deg g_i \leq 2 \, n^4 \, d^2$,

- $h(a), h(g_i) \leq 2 \, n^2 \, (n+1)^3 \, d^2 \, (\, h + \log s + n \, 2^{2\,n+3} \, d \, \log(n\,d+1))$.

This estimate is sharper than the one given by Theorem 1.

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