MODELS FOR GROWTH OF HETEROGENEOUS SANDPILES VIA MOSCO CONVERGENCE

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ABSTRACT. In this paper we study the asymptotic behavior of several classes of power-law functionals involving variable exponents $p_n(\cdot) \to \infty$, via Mosco convergence. In the particular case $p_n(\cdot) = np(\cdot)$, we show that the sequence $\{H_n\}$ of functionals $H_n : L^2(\mathbb{R}^N) \to [0, +\infty]$ given by

$$H_n(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^N} \frac{\lambda(x)^n}{np(x)} |\nabla u(x)|^{np(x)}\, dx & \text{if } u \in L^2(\mathbb{R}^N) \cap W^{1,np(\cdot)}(\mathbb{R}^N) \\ +\infty & \text{otherwise,} \end{cases}$$

converges in the sense of Mosco to a functional which vanishes on the set

$$\left\{ u \in L^2(\mathbb{R}^N) \ : \ \lambda(x)|\nabla u|^{p(x)} \le 1 \text{ a.e. } x \in \mathbb{R}^N \right\}$$

and is infinite in its complement. We also provide an example of a sequence of functionals whose Mosco limit cannot be described in terms of the characteristic function of a subset of $L^2(\mathbb{R}^N)$.

As an application of our results we obtain a model for the growth of a sandpile in which the allowed slope of the sand depends explicitly on the position in the sample.

1. INTRODUCTION.

The main results of this paper are concerned with the asymptotic behavior of certain power-law functionals with variable exponents by means of Mosco convergence. This notion of variational convergence, introduced by Umberto Mosco in the 1960's [36], provides an appropriate framework for studying the asymptotic behavior of large classes of variational problems, and has been recognized as a powerful tool for the analysis of important problems in Calculus of Variations, Partial Differential Equations, and their applications [5], [17], [18], [34], [37], [38], [39], [45]. We refer to Section 2 for the definition of Mosco convergence, and to [6] for a detailed introduction to the theory.

The study of power-law functionals with variable exponents and the associated PDEs has received a great deal of attention in recent years. Partial differential equations involving variable exponents became popular during the last decade in relation to applications to elasticity and electrorheological fluids [43], [42], [44]. Meanwhile, the underlying functional analytical tools have been extensively developed and new applications, e.g. to image processing [19], have emerged. For general references on the $p(x)$-Laplacian we refer to [23], which includes a thorough bibliography, and to [33], a seminal paper where many of the basic properties of variable exponent spaces were established. The delicate regularity properties of $p(x)$-harmonic functions have been established in [1] and [2].

Recent results on the asymptotic behavior of power-law functionals are motivated by applications to the study of dielectric breakdown, electrical resistivity, and polycrystal plasticity (see, e.g., [31], [7], [8]).

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The asymptotics in the works just cited is undertaken in the framework of De Giorgi's Γ-convergence [21], [22].

In the present paper we are interested in applications to models of difussion in sandpiles (which are governed by parabolic problems), and we adopt instead the Mosco convergence as the main tool to study the asymptotic behavior of the functionals involved.

To give an idea of the type of convergence results which we pursue in the sequel, let us consider, for each $n \in \mathbb{N}$, the functionals $H_n, H_\infty : L^2(\mathbb{R}^N) \to [0, +\infty]$ defined by

$$H_n(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^N} \frac{\lambda(x)^n}{np(x)} |\nabla u(x)|^{np(x)}\, dx & \text{if } u \in L^2(\mathbb{R}^N) \cap W^{1,np(\cdot)}(\mathbb{R}^N), \\ +\infty & \text{otherwise}, \end{cases} \tag{1.1}$$

and

$$H_\infty(u) = \begin{cases} 0 & \text{if } \lambda(x)|\nabla u(x)|^{p(x)} \le 1 \text{ a.e. } x \in \mathbb{R}^N, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2}$$

Here $p : \mathbb{R}^N \to (1, \infty)$ is a bounded function and $\lambda \in L^\infty(\mathbb{R}^N)$ is such that $0 < \alpha \le \lambda(x) \le \beta$ a.e. $x \in \mathbb{R}^N$. With this notation, one of our results (see Section 4 for details) can be stated as follows:

H_n converges in the sense of Mosco to H_∞.

Let us now recall some known results on evolution problems. The limiting behavior as $p \to \infty$ of solutions to the quasilinear parabolic problem

$$\begin{cases} v_{p,t} - \Delta_p v_p = f & \text{in } (0,T) \times \mathbb{R}^N, \\ v_p(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

was investigated in [26] (see also [4]). Here, f is nonnegative, and represents a given source term which is interpreted physically as adding material to an evolving system within which mass particles are continually rearranged by diffusion. Let us consider the functional

$$F_p(v) := \begin{cases} \displaystyle\frac{1}{p}\int_\Omega |\nabla v(y)|^p\, dy & \text{if } u \in L^2(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \\ +\infty & \text{if } u \in L^2(\mathbb{R}^N) \setminus W^{1,p}(\mathbb{R}^N). \end{cases}$$

The problem (1.3) has the standard reformulation

$$\begin{cases} f - v_{p,t} = \partial F_p(v_p) & \text{a.e. } t \in (0,T), \\ v(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

where ∂F_p denotes the subdifferential of F_p (see Section 2 for the precise definition).

In [26], assuming that u_0 is a Lipschitz function with compact support satisfying $\|\nabla u_0\|_{L^\infty(\mathbb{R}^N)} \le 1$, and for f a smooth nonnegative function with compact support in $[0,T] \times \mathbb{R}^N$, it is shown that there exists a sequence $p_i \to +\infty$ and a limit function v_∞ such that, for each $T > 0$,

$$\begin{cases} v_{p_i} \to v_\infty & \text{a.e. and in } L^2(\mathbb{R}^N \times (0,T)), \\ Dv_{p_i} \rightharpoonup Dv_\infty, \ v_{p_i,t} \rightharpoonup v_{\infty,t} & \text{weakly in } L^2(\mathbb{R}^N \times (0,T)). \end{cases}$$

Moreover, v_∞ is a solution to the problem

$$\begin{cases} f(t) - v_{\infty,t} \in \partial F_\infty(v_\infty(t)) & \text{a.e. } t \in (0,T), \\ v_\infty(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

where

$$F_\infty(v) = \begin{cases} 0 & \text{if } |\nabla v| \le 1, \\ +\infty & \text{otherwise.} \end{cases}$$

The limiting problem (1.4) governs the movement of a sandpile, with $v_\infty(t,x)$ describing the amount of sand at the point x at time t, under the main assumption that the sandpile is stable if the slope is less or equal than one and unstable if not.

The model described above has been extensively studied in [26], [27], [30], [40] and [41] (see also [3], [4], [10], [11], [24] and, for numerical approximations, [28] and [29]). As an application of our Mosco-convergence results we extend this to models which take into account the fact that the admissible slopes may depend explicitly on the spatial location due, e.g., to the presence of heterogeneities (different types of sand at different places in the sandpile). When one considers, for example, the functional H_n defined by (1.1) on the Hilbert space $L^2(\mathbb{R}^N)$, the associated PDE reads as

$$\begin{cases} (u_n)_t + \operatorname{div}\left(\lambda(x)^n |\nabla u_n|^{np(x)-2}\nabla u_n\right) = f & \text{a.e. } t \in (0,T), \\ u_n(0) = u_0. \end{cases} \tag{1.5}$$

In this case, general results from [5] and [16] give

$$u_n \to u \qquad \text{in } C([0,T] : L^2(\mathbb{R}^N)),$$

where u is the solution to the problem

$$\begin{cases} u_t + \partial H_\infty(u) \ni f & \text{a.e. } t \in (0,T), \\ u(0) = u_0, \end{cases}$$

and where $H_\infty : L^2(\mathbb{R}^N) \to [0,+\infty]$ is given by (1.2). The above limiting problem can be seen as a model for the growth of a sandpile where the critical slope of the sand depends on the spatial location. In fact, note that the pointwise restriction in the definition of H_∞ reads as

$$|\nabla u(x)| \le \left(\frac{1}{\lambda(x)}\right)^{1/p(x)} =: A(x) \text{ a.e. } x \in \mathbb{R}^N.$$

We refer to Section 6 for some explicit examples of solutions to this evolution problem.

The paper is organized as follows: in Section 2 we collect some preliminary results and introduce the notation which will be used in the sequel; in Section 3, for a general sequence of variable exponents, we prove our first Mosco-convergence result; Section 4 deals with the particular case of variable exponents that will be of interest when studying the heterogeneous sandpile model; Section 5 contains an example of a Mosco limit of power-law functionals which is not given by the characteristic function of a set; finally, in Section 6 we apply our result to analyze the new sandpile model.

2. Preliminaries.

In this section we set up the notation which will be used throughout, and we present some preliminary results. We refer to [23], [33] and the survey [32] for more details. Given $\Omega \subseteq \mathbb{R}^N$ and $p : \Omega \to (1, \infty)$ a function in $L^\infty(\Omega)$, the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as follows:

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \right\},$$

and it is endowed with the norm

$$|u|_{p(x)} := \inf \left\{ \tau > 0 \ : \ \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \le 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is given by

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) \ : \ |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

and

$$\|u\| := \inf \left\{ \tau > 0 \ : \ \int_\Omega \left(\left| \frac{\nabla u(x)}{\tau} \right|^{p(x)} + \left| \frac{u(x)}{\tau} \right|^{p(x)} \right) dx \le 1 \right\}.$$

is a norm on this space. We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

For any function p as above, we define

$$p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x), \qquad \text{and} \qquad p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x)\,.$$

In this paper we will only deal with functions p which satisfy $1 < p^- \le p^+ < \infty$. The following result is well-known (see, e.g., [33]).

Proposition 1. *(i) The spaces* $\left(L^{p(x)}(\Omega), |\cdot|_{p(x)}\right)$, $\left(W^{1,p(x)}(\Omega),\ \|\cdot\|\right)$ *and* $\left(W_0^{1,p(x)}(\Omega),\ \|\cdot\|\right)$ *are separable and reflexive Banach spaces.*
(ii) Hölder inequality holds:

$$\int_\Omega |uv|\, dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(x)} |v|_{p'(x)}, \quad \forall\ u \in L^{p(x)}(\Omega),\ \forall\ v \in L^{p'(x)}(\Omega),$$

where $p'(x) := \frac{p(x)}{p(x)-1}$.

Next, we recall the definition of Mosco-convergence. If X is a metric space, and $\{A_n\}$ is a sequence of subsets of X, we define

$$\liminf_{n\to\infty} A_n := \{x \in X \ : \ \exists x_n \in A_n,\ x_n \to x\}, \text{ and } \limsup_{n\to\infty} A_n := \{x \in X \ : \ \exists x_{n_k} \in A_{n_k},\ x_{n_k} \to x\}.$$

If X is a normed space, we denote by $s - \lim$ and $w - \lim$ the above limits associated, respectively, to the strong and to the weak topology of X.

Definition 1. *Let H be a Hilbert space. Given $\Psi_n, \Psi : H \to (-\infty, +\infty]$ convex, lower-semicontinuous functionals, we say that Ψ_n converges to Ψ in the sense of Mosco if*

$$w - \limsup_{n\to\infty} \operatorname{Epi}(\Psi_n) \subset \operatorname{Epi}(\Psi) \subset s - \liminf_{n\to\infty} \operatorname{Epi}(\Psi_n), \tag{2.1}$$

where $\mathrm{Epi}(\Psi_n)$ *and* $\mathrm{Epi}(\Psi)$ *denote the epigraphs of the functionals* Ψ_n *and* Ψ*, defined by*

$$\mathrm{Epi}(\Psi_n) := \{(u,\lambda) \in L^2(\mathbb{R}^N) \times \mathbb{R} \ : \ \lambda \geq \Psi_n(u)\}, \text{ and } \mathrm{Epi}(\Psi) := \{(u,\lambda) \in L^2(\mathbb{R}^N) \times \mathbb{R} \ : \ \lambda \geq \Psi(u)\}.$$

Remark 1. We note that (2.1) is equivalent to the requirement that the following two conditions are simultaneously satisfied:

$$\forall\, u \in D(\Psi) \ \ \exists u_n \in D(\Psi_n) \ : \ u_n \to u \ \text{ and } \ \Psi(u) \geq \limsup_{n\to\infty} \Psi_n(u_n); \tag{2.2}$$

$$\text{for every subsequence } \{n_k\}, \ \Psi(u) \leq \liminf_k \Psi_{n_k}(u_k) \text{ whenever } u_k \rightharpoonup u. \tag{2.3}$$

Here $D(\Psi) := \{u \in H \ : \ \Psi(u) < \infty\}$ and $D(\Psi_n) := \{u \in H \ : \ \Psi_n(u) < \infty\}$ denote the domain of Ψ and Ψ_n, respectively.

3. A result for general sequences of variable exponents $p_n(x)$.

Consider a sequence $\{p_n\} \subset L^\infty(\mathbb{R}^N)$ with $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} p_n(x) > 1$ for each $n \in \mathbb{N}$, satisfying the conditions

$$p_n^- := \operatorname{ess}\inf_{x\in\mathbb{R}^N} p_n(x) \to \infty \text{ as } n \to \infty, \tag{3.1}$$

and

$$\text{there exists a real constant } \gamma > 1 \text{ such that } p_n^+ := \operatorname{ess}\sup_{x\in\mathbb{R}^N} p_n(x) \leq \gamma p_n^- \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

Let

$$p_n'(x) := \frac{p_n(x)}{p_n(x)-1}.$$

In particular, (3.1) and (3.2) imply that we have

$$p_n'^{\,+} \to 1 \text{ and } \left(p_n^+\right)^{\frac{1}{p_n^-}} \to 1 \text{ as } n \to \infty. \tag{3.3}$$

Let $\mu : \mathbb{R}^N \to \left[-1/\sqrt{2}, 1/\sqrt{2}\right]$ be a function with the property that there exists $q \geq 1$ such that $\mu \in L^q(\mathbb{R}^N)$. We note that such functions exist: $\mu \equiv 0$ and $\mu(x) = \frac{\exp(-|x|)}{\sqrt{2}}$ are just two examples.

For $n \in \mathbb{N}$, consider the functionals $F_n : L^2(\mathbb{R}^N) \to [0, +\infty]$ defined by

$$F_n(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^N} \frac{1}{p_n(x)} \left(\mu(x)^2 + |\nabla u(x)|^2\right)^{p_n(x)/2} dx & \text{if } u \in L^2(\mathbb{R}^N) \cap W^{1,p_n(\cdot)}(\mathbb{R}^N), \\ +\infty & \text{otherwise,} \end{cases}$$

and $F_\infty : L^2(\mathbb{R}^N) \to [0, +\infty]$ defined by

$$F_\infty(u) = \begin{cases} 0 & \text{if } |\nabla u(x)| \leq \sqrt{1-\mu(x)^2} \text{ a.e. } x \in \mathbb{R}^N, \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 1. *The sequence* F_n *converges in the sense of Mosco to* F_∞*.*

Proof. We will first show that (2.2) holds, that is

$$\forall\, u \in D(F_\infty) \ \ \exists u_n \in D(F_n) \ : \ u_n \to u \ \text{ and } \ F_\infty(u) \geq \limsup_{n\to\infty} F_n(u_n).$$

Note that, since $u \in D(F_\infty)$, we have $u \in L^2(\mathbb{R}^N)$, with $|\nabla u(x)| \leq \sqrt{1-\mu(x)^2}$ a.e. $x \in \mathbb{R}^N$ and $F_\infty(u) = 0$.

We claim that $u \in L^\infty(\mathbb{R}^N)$. To see this, first note that by an approximation argument we may assume that u is smooth and that $\|\nabla u\|_{L^\infty(\mathbb{R}^N)} \neq 0$ (otherwise $u \equiv c$ and since $u \in L^2(\mathbb{R}^N)$, $u \equiv 0$). Fix $x_0 \in \mathbb{R}^N$ such that $|u(x_0)| > \|\nabla u\|_{L^\infty(\mathbb{R}^N)}$. If there is no such x_0, then $|u| \leq \|\nabla u\|_{L^\infty(\mathbb{R}^N)}$ in $\mathbb{R}^N$, and thus there is nothing to prove. For any $y \in B_1(x_0)$, we have

$$||u(y)| - |u(x_0)|| \leq |u(y) - u(x_0)| \leq \|\nabla u\|_{L^\infty(\mathbb{R}^N)}|y - x_0| \leq \|\nabla u\|_{L^\infty(\mathbb{R}^N)}.$$

This gives

$$|u(y)| \geq |u(x_0)| - \|\nabla u\|_{L^\infty(\mathbb{R}^N)} \qquad \forall y \in B_1(x_0).$$

Therefore,

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{R}^N)} &\geq \int_{B_1(x_0)} |u(y)|^2 dy \geq \int_{B_1(x_0)} \Big(|u(x_0)| - \|\nabla u\|_{L^\infty(\mathbb{R}^N)}\Big)^2 dy \\ &= \Big(|u(x_0)| - \|\nabla u\|_{L^\infty(\mathbb{R}^N)}\Big)^2 |B_1(0)|.\end{aligned}$$

Thus,

$$|u(x_0)| \leq \frac{\|u\|_{L^2(\mathbb{R}^N)}}{|B_1(0)|^{1/2}} + \|\nabla u\|_{L^\infty(\mathbb{R}^N)}.$$

We conclude that $u \in L^\infty(\mathbb{R}^N)$, as claimed.

Next, let Φ be a smooth function compactly supported in $B_2(0)$ such that $0 < \Phi < 1$ in $B_2(0)\setminus B_1(0)$, $\Phi \equiv 1$ in $B_1(0)$ and $\|\nabla\Phi\|_\infty \leq C$. For each $n \in \mathbb{N}$, define

$$\varphi_n(x) = (1-\varepsilon_n)\Phi\left(\frac{x}{R_n}\right),$$

with R_n and ε_n to be chosen later, satisfying

$$R_n \to \infty \text{ and } \varepsilon_n \to 0 \text{ as } n \to \infty. \tag{3.4}$$

Let us now define the sequence $v_n := u\varphi_n$. It is clear that $v_n \in D(F_n)$ and, in view of (3.4),

$$\begin{aligned}\int_{\mathbb{R}^N} |u - v_n|^2 dx &= \int_{\mathbb{R}^N\setminus B_{R_n}(0)} |u - v_n|^2 dx + \int_{B_{R_n}(0)} |u - v_n|^2 dx \\ &\leq 4\int_{\mathbb{R}^N\setminus B_{R_n}(0)} |u|^2 dx + \varepsilon_n^2 \int_{B_{R_n}(0)} |u|^2 dx \to 0 \text{ as } n \to \infty.\end{aligned}$$

On the other hand,

$$|\nabla v_n(x)| = |\nabla u(x)\varphi_n(x) + u(x)\nabla\varphi_n(x)| = \left|\nabla u(x)(1-\varepsilon_n)\Phi\left(\frac{x}{R_n}\right) + u(x)(1-\varepsilon_n)\nabla\Phi\left(\frac{x}{R_n}\right)\frac{1}{R_n}\right|.$$

Therefore, using the fact that $\nabla\Phi = 0$ in $B_{R_n}(0)$, and since $0 \le \Phi \le 1$, $|\nabla\Phi| \le C$ in the whole $\mathbb{R}^N$, $|\nabla u| \le \sqrt{1-\mu^2} \le 1$ a.e. in $\mathbb{R}^N$, we get

$$|\nabla v_n(x)| \le (1-\varepsilon_n)\chi_{B_{R_n}(0)}(x) + \left((1-\varepsilon_n) + C\|u\|_{L^\infty(\mathbb{R}^N)}\frac{(1-\varepsilon_n)}{R_n}\right)\chi_{B_{2R_n}(0)\setminus B_{R_n}(0)}(x).$$

Hence,

$$|\nabla v_n(x)|^{p_n(x)} \le 1\chi_{B_{R_n}(0)}(x) + 1\chi_{B_{2R_n}(0)\setminus B_{R_n}(0)}(x),$$

provided that we choose $R_n \to \infty$ and $\varepsilon_n \to 0$ such that

$$\left((1-\varepsilon_n) + C\|u\|_{L^\infty(\mathbb{R}^N)}\frac{(1-\varepsilon_n)}{R_n}\right) \le 1.$$

This can be achieved if

$$\varepsilon_n \ge C\|u\|_{L^\infty(\mathbb{R}^N)}\frac{(1-\varepsilon_n)}{R_n},$$

which clearly holds for $n \in \mathbb{N}$ large enough if we impose (in addition to (3.4)) that

$$\varepsilon_n R_n \to \infty \text{ as } n \to \infty. \tag{3.5}$$

Taking into account the previous estimates, and since by (3.1) we have $\max\{4,q\} < p_n^-$ for $n \in \mathbb{N}$ sufficiently large, we get

$$\begin{aligned}
F_n(v_n) &= \int_{\mathbb{R}^N} \frac{1}{p_n(x)}(\mu(x)^2 + |\nabla v_n(x)|^2)^{p_n(x)/2}\,dx \le \int_{\mathbb{R}^N}\frac{1}{p_n(x)}(\mu(x)^2 + \chi_{B_{2R_n}(0)}(x))^{p_n(x)/2}\,dx \\
&\le \int_{\mathbb{R}^N}\frac{1}{p_n(x)}2^{p_n(x)/2}(|\mu(x)|^{p_n(x)} + \chi_{B_{2R_n}(0)}(x))\,dx \\
&\le \frac{1}{p_n^-}\int_{\mathbb{R}^N}(\sqrt{2}|\mu(x)|)^{p_n(x)}\,dx + \int_{B_{2R_n}(0)}\frac{1}{p_n(x)}\,dx \\
&\le \frac{1}{p_n^-}\int_{\mathbb{R}^N}(\sqrt{2}|\mu(x)|)^{q}\,dx + C\frac{R_n^N}{p_n^-} \to 0 \text{ as } n \to \infty,
\end{aligned}$$

provided that we choose R_n such that

$$\frac{R_n^N}{p_n^-} \to 0 \text{ as } n \to \infty. \tag{3.6}$$

This shows that $\limsup_{n\to\infty} F_n(v_n) \le 0 = F_\infty(u)$, and thus (2.2) holds. Now, observe that the choices

$$R_n := (p_n^-)^{\frac{1}{2N}} \qquad \text{and} \qquad \varepsilon_n := (p_n^-)^{-\frac{1}{4N}},$$

ensure that (3.4), (3.5), and (3.6) are satisfied.

It remains to show that, whenever $u_{n_k} \in D(F_{n_k})$ is such that $u_{n_k} \rightharpoonup u$ weakly in $L^2(\mathbb{R}^N)$ as $k \to \infty$, then $\liminf_{k\to\infty} F_{n_k}(u_{n_k}) \ge F_\infty(u)$.

To proceed further, we cover $\mathbb{R}^N$ with a countable number of open and bounded domains Ω (not necessary disjoint) with $|\Omega| = 1$. Fix such an Ω. Let $x \in \Omega$ be a Lebesgue point for $\sqrt{\lambda^2 + |\nabla u|^2} \in L^1(\Omega)$. Then, for each $r > 0$ sufficiently small, we have $B_r(x) \subset \Omega$.

For each open set $\omega \subset \mathbb{R}^N$ and each $n_k \in \mathbb{N}$, define

$$\omega^+_{n_k,\mu} := \left\{x \in \omega;\ \sqrt{\mu(x)^2 + |\nabla u_{n_k}(x)|^2} > 1\right\} \quad \text{and} \quad \omega^-_{n_k,\mu} := \left\{x \in \omega;\ \sqrt{\mu(x)^2 + |\nabla u_{n_k}(x)|^2} \le 1\right\}.$$

We have

$$
\begin{aligned}
\int_{B_r(x)} \sqrt{|\mu(y)|^2 + |\nabla u_{n_k}(y)|^2}\ dy &= \int_{(B_r(x))^+_{n_k,\mu}} \sqrt{|\mu(y)|^2 + |\nabla u_{n_k}(y)|^2} dy \\
&\quad + \int_{(B_r(x)))^-_{n_k,\mu}} \sqrt{|\mu(y)|^2 + |\nabla u_{n_k}(y)|^2} dy \\
&\le \left(\int_{(B_r(x))^+_{n_k,\mu}} \left(|\mu(y)|^2 + |\nabla u_{n_k}(y)|^2\right)^{p^-_{n_k}/2} dy \right)^{\frac{1}{p^-_{n_k}}} |(B_r(x))^+_{n_k,\mu}|^{\frac{p^-_{n_k}-1}{p^-_{n_k}}} \\
&\quad + \left(\int_{(B_r(x))^-_{n_k,\mu}} \left(|\mu(y)|^2 + |\nabla u_{n_k}(y)|^2\right)^{p^+_{n_k}/2} dy \right)^{\frac{1}{p^+_{n_k}}} |(B_r(x))^-_{n_k,\mu}|^{\frac{p^+_{n_k}-1}{p^+_{n_k}}} \\
&\le \left(p^+_{n_k} \int_{(B_r(x))^+_{n_k,\mu}} \frac{1}{p_{n_k}(y)} \left(|\mu(y)|^2 + |\nabla u_{n_k}(y)|^2\right)^{p_{n_k}(y)/2} dy \right)^{\frac{1}{p^-_{n_k}}} |(B_r(x))^+_{n_k,\mu}|^{\frac{p^-_{n_k}-1}{p^-_{n_k}}} \\
&\quad + \left(p^+_{n_k} \int_{(B_r(x))^-_{n_k,\mu}} \frac{1}{p_{n_k}(y)} \left(|\mu(y)|^2 + |\nabla u_{n_k}(y)|^2\right)^{p_{n_k}(y)/2} dy \right)^{\frac{1}{p^+_{n_k}}} |(B_r(x))^-_{n_k,\mu}|^{\frac{p^+_{n_k}-1}{p^+_{n_k}}} \\
&\le \left(p^+_{n_k} F_{n_k}(u_{n_k})\right)^{\frac{1}{p^-_{n_k}}} |(B_r(x))^+_{n_k,\mu}|^{\frac{p^-_{n_k}-1}{p^-_{n_k}}} + \left(p^+_{n_k} F_{n_k}(u_{n_k})\right)^{\frac{1}{p^+_{n_k}}} |(B_r(x))^-_{n_k,\mu}|^{\frac{p^+_{n_k}-1}{p^+_{n_k}}} .
\end{aligned}
$$

Since

$$
\limsup_{k\to\infty} \left(p^+_{n_k} F_{n_k}(u_{n_k})\right)^{\frac{1}{p^-_{n_k}}} \le 1 \text{ and } \limsup_{k\to\infty} \left(p^+_{n_k} F_{n_k}(u_{n_k})\right)^{\frac{1}{p^+_{n_k}}} \le 1,
$$

we obtain that

$$
\limsup_{k\to\infty} \left(\left(p^+_{n_k} F_{n_k}(u_{n_k})\right)^{\frac{1}{p^-_{n_k}}} |(B_r(x))^+_{n_k,\mu}|^{\frac{p^-_{n_k}-1}{p^-_{n_k}}} + \left(p^+_{n_k} F_{n_k}(u_{n_k})\right)^{\frac{1}{p^+_{n_k}}} |(B_r(x))^-_{n_k,\mu}|^{\frac{p^+_{n_k}-1}{p^+_{n_k}}} \right) \le |B_r(x)|,
$$

and hence

$$
\limsup_{k\to\infty} \int_{B_r(x)} \sqrt{|\mu(y)|^2 + |\nabla u_{n_k}(y)|^2}\ dy \le |B_r(x)|. \tag{3.7}
$$

In view of (3.1), we have that $2 < p^-_{n_k}$ for sufficiently large $k \in \mathbb{N}$. Thus, using the classical Hölder's inequality we deduce that

$$
\begin{aligned}
\int_\Omega |\nabla u_{n_k}(x)|^2\ dx &\le \left(\int_\Omega |\nabla u_{n_k}(x)|^{p^-_{n_k}}\ dx \right)^{\frac{2}{p^-_{n_k}}} |\Omega|^{\frac{p^-_{n_k}-2}{p^-_{n_k}}} \\
&= \left(\int_{\Omega^-_{n_k,0}} |\nabla u_{n_k}(x)|^{p^-_{n_k}}\ dx + \int_{\Omega^+_{n_k,0}} |\nabla u_{n_k}(x)|^{p^-_{n_k}}\ dx \right)^{\frac{2}{p^-_{n_k}}} \\
&\le \left(1 + \int_\Omega |\nabla u_{n_k}(x)|^{p_{n_k}(x)}\ dx \right)^{\frac{2}{p^-_{n_k}}} \le \left(1 + p^+_{n_k} F_{n_k}(u_{n_k})\right)^{\frac{2}{p^-_{n_k}}} .
\end{aligned}
$$

It follows that the sequence $\{\nabla u_{n_k}\}$ is bounded in $L^2(\Omega;\mathbb{R}^N)$. Since $u_{n_k} \rightharpoonup u$ weakly in $L^2(\mathbb{R}^N)$, we deduce that $\{u_{n_k}\}$ is bounded in $L^2(\Omega)$. Overall, $\{u_{n_k}\}$ is bounded in $W^{1,2}(\Omega)$, and thus we may extract a subsequence (not relabelled) such that $u_{n_k} \rightharpoonup u$ weakly in $W^{1,2}(\Omega)$. Using a well-known lower semicontinuity result we find

$$\int_{B_r(x)} \sqrt{|\mu(y)|^2 + |\nabla u(y)|^2}\, dy \le \liminf_{k\to\infty} \int_{B_r(x)} \sqrt{|\mu(y)|^2 + |\nabla u_{n_k}(y)|^2}\, dy,$$

which implies, in view of (3.7), that

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} \sqrt{|\mu(y)|^2 + |\nabla u(y)|^2}\, dy \le 1.$$

Since almost every $x \in \Omega$ is a Lebesgue point for $\sqrt{\lambda^2 + |\nabla u|^2}$, passing to the limit $r \to 0^+$ in the above inequality yields $\sqrt{|\mu(x)|^2 + |\nabla u(x)|^2} \le 1$ for a.e. $x \in \Omega$, that is, $|\nabla u(x)| \le \sqrt{1-\mu(x)^2}$ for a.e. $x \in \Omega$.

Since $\mathbb{R}^N$ is covered by a countable number of sets of type Ω it follows that $|\nabla u(x)| \le \sqrt{1-\mu(x)^2}$ for a.e. $x \in \mathbb{R}^N$. Hence, $F_\infty(u) = 0$, and we deduce that $\liminf_{k\to\infty} F_{n_k}(u_{n_k}) \ge F_\infty(u)$. This concludes the proof of Theorem 1. □

Remark 2. The limiting functional in Theorem 1 coincides with the one obtained in the case of constant exponents, $p_n(x) = n$. Hence, in this case, the presence of the variable exponents does not induce any particular spatial dependence in the limit. This fact is even more transparent if we focus our attention on the particular case where $\mu \equiv 0$: the functionals F_n are then given by

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} \frac{1}{p_n(x)} |\nabla u(x)|^{p_n(x)}\, dx & \text{if } u \in L^2(\mathbb{R}^N) \cap W^{1,p_n(\cdot)}(\mathbb{R}^N), \\ +\infty & \text{otherwise,} \end{cases}$$

and the limiting functional F_∞ becomes

$$\begin{cases} 0 & \text{if } |\nabla u(x)| \le 1 \text{ a.e. } x \in \mathbb{R}^N, \\ +\infty & \text{otherwise.} \end{cases}$$

This observation is the starting point for the study undertaken in the next section.

4. The case $p_n(x) = np(x)$.

In this section we consider the particular case $p_n(x) = np(x)$, and we work with a sequence of functionals which will allow us to obtain a more involved dependence on x in the Mosco limit.

Theorem 2. *Let $p : \mathbb{R}^N \to (1, \infty)$ be a bounded function, and let $\lambda \in L^\infty(\mathbb{R}^N), 0 < \alpha \le \lambda(x) \le \beta$ for a.e. $x \in \mathbb{R}^N$. For each $n \in \mathbb{N}$, consider the functionals $H_n, H_\infty : L^2(\mathbb{R}^N) \to [0, +\infty]$ defined by*

$$H_n(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^N} \frac{\lambda(x)^n}{np(x)} |\nabla u(x)|^{np(x)}\, dx & \text{if } u \in L^2(\mathbb{R}^N) \cap W^{1,np(\cdot)}(\mathbb{R}^N), \\ +\infty & \text{otherwise}, \end{cases}$$

and

$$H_\infty(u) = \begin{cases} 0 & \text{if } \lambda(x)|\nabla u(x)|^{p(x)} \le 1 \text{ a.e. } x \in \mathbb{R}^N, \\ +\infty & \text{otherwise.} \end{cases}$$

Then H_n converges in the sense of Mosco to H_∞.

Proof. We divide the proof into two parts.

I. First, we show that

$$\forall\, u \in D(H_\infty) \;\; \exists u_n \in D(H_n) \;:\; u_n \to u \;\text{ and }\; H_\infty(u) \geq \limsup_{n\to\infty} H_n(u_n).$$

Let $u \in D(H_\infty)$, that is, $u \in L^2(\mathbb{R}^N)$ and $\lambda(x)|\nabla u(x)|^{p(x)} \leq 1$ a.e. $x \in \mathbb{R}^N$, in which case $H_\infty(u) = 0$. We proceed as in the proof of Theorem 1. Let $\{\varepsilon_n\}, \{R_n\}$ be the two sequences defined by

$$R_n := (np^-)^{\frac{1}{2N}} \qquad \text{and} \qquad \varepsilon_n := (np^-)^{-\frac{1}{4N}},$$

so that (3.4), (3.5), and (3.6) hold. Further, let Φ, φ_n be as in the proof of Theorem 1, and define $u_n \in L^2(\mathbb{R}^N) \cap W^{1,np(\cdot)}(\mathbb{R}^N)$ by $u_n := u\varphi_n$. Then, $u_n \to u$ in $L^2(\mathbb{R}^n)$, and we have

$$\begin{aligned} \lambda(x)^{\frac{1}{p(x)}}|\nabla u_n(x)| &= |\lambda(x)^{\frac{1}{p(x)}}\nabla u(x)\varphi_n(x) + \lambda(x)^{\frac{1}{p(x)}}u(x)\nabla\varphi_n(x)| \\ &= \left|\lambda(x)^{\frac{1}{p(x)}}\nabla u(x)(1-\varepsilon_n)\Phi\left(\frac{x}{R_n}\right) + \lambda(x)^{\frac{1}{p(x)}}u(x)(1-\varepsilon_n)\nabla\Phi\left(\frac{x}{R_n}\right)\frac{1}{R_n}\right|. \end{aligned} \tag{4.1}$$

Since $\nabla\Phi = 0$ in $B_{R_n}(0)$, $0 \leq \Phi \leq 1$, $|\nabla\Phi| \leq C$, and $\lambda(x)^{\frac{1}{p(x)}}|\nabla u(x)| \leq 1$ a.e. $x \in \mathbb{R}^N$, we deduce that there exists a constant $C_1 > 0$ such that

$$\lambda(x)^{\frac{1}{p(x)}}|\nabla u_n(x)| \leq (1-\varepsilon_n)\chi_{B_{R_n}(0)}(x) + \left((1-\varepsilon_n) + C_1|u|_\infty\frac{(1-\varepsilon_n)}{R_n}\right)\chi_{B_{2R_n}(0)\setminus B_{R_n}(0)}(x).$$

Since $\varepsilon_n \to 0$ and $\varepsilon_n R_n \to \infty$ as $n \to \infty$ we have, for $n \in \mathbb{N}$ sufficiently large,

$$(1-\varepsilon_n) + C_1|u|_\infty\frac{(1-\varepsilon_n)}{R_n} \leq 1.$$

Hence,

$$\lambda(x)|\nabla u_n(x)|^{p(x)} \leq \chi_{B_{2R_n}(0)}(x) \text{ a.e. } x \in \mathbb{R}^N.$$

Thus, there exists a constant $C_2 > 0$ such that, for $n \in \mathbb{N}$ sufficiently large,

$$H_n(u_n) = \int_{\mathbb{R}^N}\frac{\lambda(x)^n}{np(x)}|\nabla u_n(x)|^{np(x)}\,dx \leq \int_{B_{2R_n}(0)}\frac{1}{np(x)}\,dx \leq C_2\frac{R_n^N}{np^-} = C_2\frac{1}{\sqrt{np^-}}.$$

We deduce that $\limsup_{n\to\infty} H_n(u_n) = 0 = H_\infty(u)$.

II. Let $u \in L^2(\mathbb{R}^N)$. We will show that $H_\infty(u) \leq \liminf_{n\to\infty} H_n(u_n)$ whenever $\{u_n\} \subset L^2(\mathbb{R}^N)$ is such that $u_n \rightharpoonup u$ weakly in $L^2(\mathbb{R}^N)$.

We may assume, without loss of generality, that $u_n \in L^2(\mathbb{R}^N) \cap W^{1,np(\cdot)}(\mathbb{R}^N)$, and that we have

$$\liminf_{n\to\infty} H_n(u_n) = \lim_{n\to\infty} H_n(u_n) < +\infty. \tag{4.2}$$

Let $\{\Omega_j \;:\; j = 1, 2, \cdots\}$ be a collection of open sets with sufficiently smooth boundaries such that $|\Omega_j| = 1$ $\forall\, j \in \mathbb{N}$, and $\bigcup_{j=1}^{n}\Omega_j = \mathbb{R}^N$. Fix $j \in \mathbb{N}$, and let $q \geq 2$ be arbitrary. For $n \in \mathbb{N}$ sufficiently large, we have

$$\int_{\Omega_j}|\nabla u_n(x)|^q dx \leq 2||\nabla u_n|^q|_{np(\cdot)/q}, \tag{4.3}$$

and note that if $||\nabla u_n|^q|_{np(\cdot)/q} > 1$, we obtain

$$||\nabla u_n|^q|_{np(\cdot)/q}^{np^-/q} \leq \int_{\Omega_j} |\nabla u_n(x)|^{np(x)} dx \leq \frac{np^+}{\alpha^n} \int_{\Omega_j} \frac{\lambda(x)^n}{np(x)} |\nabla u_n(x)|^{np(x)} dx \leq \frac{np^+}{\alpha^n} H_n(u_n).$$

Thus,

$$||\nabla u_n|^q|_{np(\cdot)/q} \leq \max\left\{1, \left(\frac{np^+}{\alpha^n}\right)^{q/np^-} H_n(u_n)^{q/np^-}\right\}.$$

In view of (4.2) and (4.3) we deduce that $\{\nabla u_n\}$ is bounded in $L^q(\Omega_j;\mathbb{R}^N)$. Since $u_n \rightharpoonup u$ weakly in $L^2(\mathbb{R}^N)$ and $q \geq 2$ we have that the sequence $\{u_n\}$ is bounded in $L^q(\Omega_j)$. Hence, $\{u_n\}$ is bounded in $W^{1,q}(\Omega_j)$, and we may extract a subsequence (not relabelled), such that $u_n \rightharpoonup u$ weakly in $W^{1,q}(\Omega_j)$. In particular, for each $j \in \mathbb{N}$ fixed, we have $\nabla u \in L^q(\Omega_j;\mathbb{R}^N)$ for all $q \geq 2$. Thus, since $\lambda \in L^\infty(\mathbb{R}^N)$, we obtain that $\lambda(\cdot)|\nabla u(\cdot)|^{p(\cdot)} \in L^1(\Omega_j)$. Let $x \in \Omega_j$ be a Lebesgue point for this map, and let $r > 0$ be small enough so that $B_r(x) \subset \Omega_j$. We have (by arguments similar to those following (5.7) in the proof of Theorem 3 in the next section)

$$\int_{B_r(x)} \lambda(y)|\nabla u(y)|^{p(y)} dy \leq \liminf_{n\to\infty} \int_{B_r(x)} \lambda(y)|\nabla u_n(y)|^{p(y)} dy. \tag{4.4}$$

On the other hand,using Hölder's inequality,

$$\begin{aligned}\int_{B_r(x)} \lambda(y)|\nabla u_n(y)|^{p(y)} dy &\leq \|\lambda(\cdot)|\nabla u_n(\cdot)|^{p(\cdot)}\|_{L^n(B_r(x))} |B_r(x)|^{(n-1)/n} \\ &\leq \left(np^+ \int_{B_r(x)} \frac{\lambda(y)^n}{np(y)} |\nabla u(y)|^{np(y)}\, dy\right)^{1/n} |B_r(x)|^{(n-1)/n} \\ &\leq \left(np^+ H_n(u_n)\right)^{1/n} |B_r(x)|^{(n-1)/n}.\end{aligned}$$

Passing to the limit as $n \to \infty$ we deduce that

$$\limsup_{n\to\infty} \int_{B_r(x)} \lambda(y)|\nabla u_n(y)|^{p(y)} dy \leq |B_r(x)|.$$

Taking into account (4.4), we obtain

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} \lambda(y)|\nabla u(y)|^{p(y)} dy \leq 1.$$

Since $x \in \Omega_j$ was a Lebesgue point for $\lambda(\cdot)|\nabla u(\cdot)|^{p(\cdot)}$, we have that $\lambda(x)|\nabla u(x)|^{p(x)} \leq 1$. Thus, this inequality holds for a.e. $x \in \Omega_j$ $(j = 1, 2, \cdots)$. Since $\{\Omega_j\}_{j\in\mathbb{N}}$ is a countable covering of $\mathbb{R}^N$, we deduce that $\lambda(x)|\nabla u(x)|^{p(x)} \leq 1$ for a.e. $x \in \mathbb{R}^N$. We conclude that $H_\infty(u) = 0$, and thus

$$H_\infty(u) \leq \liminf_{n\to\infty} H_n(u_n).$$

This concluses the proof. □

5. An example of a nondegenerate Mosco limit

The aim of our next result is to show that a suitable scaling of our previous energy (with $\lambda \equiv 1$) gives rise to a nondegenerate Mosco-limit.

Theorem 3. *Let $p : \mathbb{R}^N \to (1, \infty)$ be a bounded function. For $n \in \mathbb{N}$, consider the functionals $G_n, G_\infty : L^2(\mathbb{R}^N) \to [0, +\infty]$ defined by*

$$G_n(u) = \begin{cases} \left(\int_{\mathbb{R}^N} \frac{1}{np(x)} |\nabla u(x)|^{np(x)} \, dx \right)^{1/n} & \text{if } u \in L^2(\mathbb{R}^N) \cap W^{1,np(\cdot)}(\mathbb{R}^N), \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$G_\infty(u) = \begin{cases} \| \, |\nabla u|^{p(\cdot)} \, \|_{L^\infty(\mathbb{R}^N)} & \text{if } \nabla u \in L^\infty(\mathbb{R}^N; \mathbb{R}^N), \\ +\infty & \text{otherwise.} \end{cases}$$

Then G_n converges in the sense of Mosco to G_∞, i.e. the following hold:

$$\forall \, u \in D(G_\infty) \;\; \exists u_n \in D(G_n) \; : \; u_n \to u \;\; \text{and} \;\; G_\infty(u) \geq \limsup_{n\to\infty} G_n(u_n); \tag{5.1}$$

$$\text{for every subsequence } \{n_k\}, \; G_\infty(u) \leq \liminf_{k\to\infty} G_{n_k}(u_k) \; \text{whenever } u_k \rightharpoonup u \text{ weakly in } L^2(\mathbb{R}^N). \tag{5.2}$$

Proof. **I.** We show first that (5.1) holds. Let $u \in D(G_\infty)$. Then, $G_\infty(u) = \| \, |\nabla u|^{p(\cdot)} \, \|_{L^\infty(\mathbb{R}^N)}$ and $u \in L^2(\mathbb{R}^N)$, $|\nabla u| \in L^\infty(\mathbb{R}^N)$. A similar proof to the one given in Theorem 1 shows that we can obtain estimates for $|u(x)|$ in terms of $\|u\|_{L^2(\mathbb{R}^N)}$ and $\| \, |\nabla u| \, \|_{L^\infty(\mathbb{R}^N)}$. Next, let Φ, ε_n, R_n, φ_n, v_n be defined as in the proof of Theorem 1. We have $v_n \in L^2(\mathbb{R}^N) \cap W^{1,np(\cdot)}(\mathbb{R}^N) = D(G_n)$, and the same proof as in Theorem 1 shows that $v_n \to u$ in $L^2(\mathbb{R}^N)$.

It remains to show that $G_\infty(u) \geq \limsup_{n\to\infty} G_n(v_n)$. To this aim, we establish first the following: for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$(1+t)^{p(x)} - t^{p(x)} < \varepsilon t^{p(x)} + (1 + \delta(\varepsilon))^{p^+}, \quad \forall \, x \in \mathbb{R}^N, \; \forall \, t \geq 0. \tag{5.3}$$

Let $\varepsilon > 0$ be fixed. Since

$$\lim_{t\to\infty} \left[\left(1 + \frac{1}{t} \right)^{p^+} - 1 \right] = 0 \, ,$$

it follows that there exists $\delta(\varepsilon) > 0$ such that

$$\left(1 + \frac{1}{t} \right)^{p^+} - 1 < \varepsilon, \quad \forall \, t > \delta(\varepsilon).$$

Hence

$$\left(1 + \frac{1}{t} \right)^{p(x)} - 1 < \varepsilon, \quad \forall \, t > \delta(\varepsilon), \; \forall \, x \in \mathbb{R}^N,$$

or, equivalently,

$$(1+t)^{p(x)} - t^{p(x)} < \varepsilon t^{p(x)}, \quad \forall \, t > \delta(\varepsilon), \; \forall \, x \in \mathbb{R}^N.$$

On the other hand, for any $x \in \mathbb{R}^N$ and any $t \in [0, \delta(\varepsilon)]$ we have

$$(1+t)^{p(x)} - t^{p(x)} < (1+t)^{p^+} < (1 + \delta(\varepsilon))^{p^+}.$$

The last two inequalities show that (5.3) holds. Furthermore, (5.3) implies that for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$(a+b)^{p(x)} < (\varepsilon+1)a^{p(x)} + (1+\delta(\varepsilon))^{p^+} b^{p(x)}, \quad \forall\, x \in \mathbb{R}^N,\ \forall\, a,\ b \geq 0. \tag{5.4}$$

Now, the same computations as in the proof of Theorem 1 yield that for each $x \in \mathbb{R}^N$ we have

$$|\nabla v_n(x)| = |\nabla u(x)\varphi_n(x) + u(x)\nabla\varphi_n(x)| = \left|\nabla u(x)(1-\varepsilon_n)\Phi\left(\frac{x}{R_n}\right) + u(x)(1-\varepsilon_n)\nabla\Phi\left(\frac{x}{R_n}\right)\frac{1}{R_n}\right|,$$

and it follows that

$$|\nabla v_n(x)| \leq |\nabla u(x)|(1-\varepsilon_n)\chi_{B_{2R_n}(0)} + C\|u\|_{L^\infty(\mathbb{R}^N)}\frac{(1-\varepsilon_n)}{R_n}\chi_{B_{2R_n}(0)\setminus B_{R_n}(0)}.$$

Thus,

$$|\nabla v_n(x)|^{p(x)} \leq \left(|\nabla u(x)|(1-\varepsilon_n)\chi_{B_{2R_n}(0)} + C\|u\|_{L^\infty(\mathbb{R}^N)}\frac{1-\varepsilon_n}{R_n}\chi_{B_{2R_n}(0)\setminus B_{R_n}(0)}\right)^{p(x)}, \quad \forall\, x \in \mathbb{R}^N.$$

Let $\varepsilon > 0$ be arbitrary but fixed, and let $\delta(\varepsilon) > 0$ be such that (5.4) holds. Using the previous estimates and (5.4) we find that for each $x \in \mathbb{R}^N$ we have

$$\begin{aligned}
|\nabla v_n(x)|^{p(x)} &\leq (1+\varepsilon)|\nabla u(x)|^{p(x)}(1-\varepsilon_n)^{p(x)}\chi_{B_{2R_n}(0)} + \\
&\quad (1+\delta(\varepsilon))^{p^+}(C\|u\|_{L^\infty(\mathbb{R}^N)})^{p(x)}\left(\frac{1-\varepsilon_n}{R_n}\right)^{p(x)}\chi_{B_{2R_n}(0)\setminus B_{R_n}(0)} \\
&\leq (1+\varepsilon)\|\,|\nabla u|^{p(\cdot)}\,\|_{L^\infty(\mathbb{R}^N)}(1-\varepsilon_n)\chi_{B_{2R_n}(0)} + \\
&\quad C_\varepsilon^{p^+}\left(\frac{1-\varepsilon_n}{R_n}\right)^{p^-}\chi_{B_{2R_n}(0)\setminus B_{R_n}(0)} \\
&= (1+\varepsilon)\|\,|\nabla u|^{p(\cdot)}\,\|_{L^\infty(\mathbb{R}^N)}(1-\varepsilon_n)\chi_{B_{R_n}(0)} + \\
&\quad \left[(1+\varepsilon)\|\,|\nabla u|^{p(\cdot)}\,\|_{L^\infty(\mathbb{R}^N)}(1-\varepsilon_n) + C_\varepsilon^{p^+}\left(\frac{1-\varepsilon_n}{R_n}\right)^{p^-}\right]\chi_{B_{2R_n}(0)\setminus B_{R_n}(0)},
\end{aligned}$$

where $C_\varepsilon := (1+\delta(\varepsilon))\max\{1, C\|u\|_{L^\infty(\mathbb{R}^N)}\}$. Next, we show that for $n \in \mathbb{N}$ sufficiently large we have

$$(1+\varepsilon)\|\,|\nabla u|^{p(\cdot)}\,\|_{L^\infty(\mathbb{R}^N)}(1-\varepsilon_n) + C_\varepsilon^{p^+}\left(\frac{1-\varepsilon_n}{R_n}\right)^{p^-} \leq (1+\varepsilon)\|\,|\nabla u|^{p(\cdot)}\,\|_{L^\infty(\mathbb{R}^N)},$$

or, equivalently,

$$C_\varepsilon^{p^+}\left(\frac{1-\varepsilon_n}{R_n}\right)^{p^-} \leq (1+\varepsilon)\varepsilon_n\|\,|\nabla u|^{p(\cdot)}\,\|_{L^\infty(\mathbb{R}^N)}.$$

This inequality holds since

$$\lim_{n\to\infty}(1+\varepsilon)\varepsilon_n R_n^{p^-} = \lim_{n\to\infty}(1+\varepsilon)(np^-)^{-\frac{1}{4N}}(np^-)^{\frac{p^-}{2N}} = C\lim_{n\to\infty} n^{\frac{p^-}{2N}-\frac{1}{4N}} = \infty,$$

where here, and in what follows, $C > 0$ is a real constant which may vary from line to line and expression to expression. Hence, for $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$ sufficiently large, we have

$$|\nabla v_n(x)|^{p(x)} \leq (1+\varepsilon)\|\,|\nabla u|^{p(\cdot)}\,\|_{L^\infty(\mathbb{R}^N)}\chi_{B_{2R_n}(0)},$$

which gives

$$|\nabla v_n(x)|^{np(x)} \leq (1+\varepsilon)^n\|\,|\nabla u|^{p(\cdot)}\,\|^n_{L^\infty(\mathbb{R}^N)}\chi_{B_{2R_n}(0)}\,.$$

It follows that

$$\begin{aligned} G_n(v_n) &= \left(\int_{\mathbb{R}^N} \frac{1}{np(x)}|\nabla v_n(x)|^{np(x)}\,dx\right)^{1/n} \\ &\leq \left(\int_{\mathbb{R}^N} \frac{1}{np(x)}(1+\varepsilon)^n \|\,|\nabla u|^{p(\cdot)}\,\|^n_{L^\infty(\mathbb{R}^N)}\chi_{B_{2R_n}(0)}\,dx\right)^{1/n} \\ &\leq \left(\frac{1}{np^-}\right)^{1/n}(1+\varepsilon)\|\,|\nabla u|^{p(\cdot)}\,\|_{L^\infty(\mathbb{R}^N)}|B_{2R_n}(0)|^{1/n} \\ &= \left(\frac{1}{np^-}\right)^{1/n}(1+\varepsilon)\|\,|\nabla u|^{p(\cdot)}\,\|_{L^\infty(\mathbb{R}^N)}(CR_n^N)^{1/n} \\ &= \left(\frac{1}{np^-}\right)^{1/n}(1+\varepsilon)\|\,|\nabla u|^{p(\cdot)}\,\|_{L^\infty(\mathbb{R}^N)}C^{1/n}n^{1/(2n)}. \end{aligned}$$

Since the right hand side converges to $(1+\varepsilon)\|\,|\nabla u|^{p(\cdot)}\,\|_{L^\infty(\mathbb{R}^N)}$ as $n\to\infty$, we deduce that for each $\varepsilon>0$ we have

$$\limsup_{n\to\infty} G_n(v_n) \leq (1+\varepsilon)\|\,|\nabla u|^{p(\cdot)}\,\|_{L^\infty(\mathbb{R}^N)}.$$

Letting $\varepsilon\to 0$, we obtain

$$G_\infty(u) \geq \limsup_{n\to\infty} G_n(v_n).$$

II. We now show that (5.2) holds. Let $\{n_k\}$ be a given subsequence of $\{k\}$(clearly, $n_k\geq k$) and let $u_k \rightharpoonup u$ in $L^2(\mathbb{R}^N)$. Without loss of generality, we may assume that $u_k\in W^{1,n_kp(\cdot)}(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$ (which, in particular, implies that $|\nabla u_k|^{p(\cdot)}\in L^{n_k}(\mathbb{R}^N)$), and that

$$\liminf_{k\to\infty} G_{n_k}(u_k) = \lim_{k\to\infty} G_{n_k}(u_k) =: L < \infty. \tag{5.5}$$

We can cover the space $\mathbb{R}^N$ with a countable number of open sets Ω_j with $|\Omega_j|=1$ for each $j\in\mathbb{N}$, i.e.

$$\mathbb{R}^N = \bigcup_{j=1}^\infty \Omega_j, \quad |\Omega_j|=1,\ \forall\, j\in\mathbb{N}.$$

Fix $j\in\mathbb{N}$, and let $q\geq 1$ be arbitrary. For $k\in\mathbb{N}$ sufficiently large, we have

$$\begin{aligned} \int_{\Omega_j}|\nabla u_k(x)|^{qp^+}dx &\leq |\Omega_j| + \int_{\Omega_j}|\nabla u_k(x)|^{\frac{p(x)qp^+}{p^-}}dx \leq |\Omega_j| + \left(\int_{\Omega_j}|\nabla u_k(x)|^{n_kp(x)}dx\right)^{\frac{qp^+}{n_kp^-}}|\Omega_j|^{1-\frac{qp^+}{n_kp^-}} \\ &\leq 1 + \big(n_kp^+\big)^{\frac{qp^+}{n_kp^-}}\,G_{n_k}(u_k)^{\frac{qp^+}{p^-}}, \end{aligned} \tag{5.6}$$

where we have used Hölder's inequality. Thus, $\{\nabla u_k\}$ is bounded in $L^{qp^+}(\Omega_j;\mathbb{R}^N)$. Since $u_k\rightharpoonup u$ weakly in $L^2(\mathbb{R}^N)$ and $u_k\to u$ in $L^1(\Omega_j)$, we deduce by Poincaré-Wirtinger's inequality that $\{u_k\}$ is bounded in $L^{qp^+}(\Omega_j)$. Thus, $\{u_k\}$ is bounded in $W^{1,qp^+}(\Omega_j)$. It follows that we can extract a subsequence (not relabelled) such that $u_k\rightharpoonup u$ weakly in $W^{1,qp^+}(\Omega_j)$. Since $p(x)\leq p^+$ for any $x\in\Omega_j$, $W^{1,qp^+}(\Omega_j)$ is continuously embedded in $W^{1,qp(\cdot)}(\Omega_j)$, and we deduce that $u_k\rightharpoonup u$ weakly in $W^{1,qp(\cdot)}(\Omega_j)$. Then [35, Lemma 3.4] yields

$$\int_{\Omega_j}|\nabla u(x)|^{qp(x)}dx \leq \liminf_{k\to\infty}\int_{\Omega_j}|\nabla u_k(x)|^{qp(x)}dx. \tag{5.7}$$

An alternative argument for (5.7) is as follows: Let $f : \Omega_j \times \mathbb{R}^N \to [0, +\infty)$ be defined by $f(x, v) := |v|^{qp(x)}$. Note that f is continuous, and that it satisfies the growth condition $0 \le f(x, v) \le C(1 + |v|^{qp^+})$ for all $(x, v) \in \Omega_j \times \mathbb{R}^N$. Also, $f(x, \cdot)$ is convex for all $x \in \Omega_j$. Since $u_k \rightharpoonup u$ weakly in $W^{1,qp^+}(\Omega_j)$, (5.7) now follows from well-known weak lower semicontinuity results for functionals of the form

$$u \mapsto \int_{\Omega_j} f(x, \nabla u(x))\, dx$$

(see, e.g., [20]). Applying again Hölder's inequality, we find

$$\int_{\Omega_j} |\nabla u_k|^{qp(x)} dx \le \left(\int_{\Omega_j} |\nabla u_k|^{n_k p(x)} dx \right)^{\frac{q}{n_k}} |\Omega_j|^{1 - \frac{q}{n_k}} \le (n_k p^+)^{\frac{q}{n_k}} G_{n_k}(u_k)^q |\Omega_j|^{1 - \frac{q}{n_k}}.$$

Thus, taking into account (5.5),

$$\limsup_{k\to\infty} \left(\int_{\Omega_j} |\nabla u_k|^{qp(x)} dx \right)^{\frac{1}{q}} \le |\Omega_j|^{\frac{1}{q}} \liminf_{k\to\infty} G_{n_k}(u_k). \tag{5.8}$$

Finally, using the fact that

$$\left(\liminf_{k\to\infty} \int_{\Omega_j} |\nabla u_k|^{qp(x)} dx \right)^{\frac{1}{q}} \le \limsup_{k\to\infty} \left(\int_{\Omega_j} |\nabla u_k|^{qp(x)} dx \right)^{\frac{1}{q}},$$

and in view of (5.7) and (5.8), we obtain that

$$\| |\nabla u|^{p(\cdot)} \|_{L^q(\Omega_j)} \le |\Omega_j|^{\frac{1}{q}} \liminf_{k\to\infty} G_{n_k}(u_k).$$

Letting $q \to \infty$, we obtain

$$\| |\nabla u|^{p(\cdot)} \|_{L^\infty(\Omega_j)} \le L\,.$$

The above inequality holds for each $j \in \mathbb{N}$, and since $\mathbb{R}^N$ is a countable union of sets Ω_j we deduce that

$$|\nabla u(x)|^{p(x)} \le L \text{ a.e. } x \in \mathbb{R}^N.$$

Thus,

$$\| |\nabla u|^{p(\cdot)} \|_{L^\infty(\mathbb{R}^N)} \le L = \liminf_{k\to\infty} G_{n_k}(u_k)\,.$$

This concludes the proof of Theorem 3. ☐

6. A model for sandpiles

To identify the limit of the solutions u_n of problem (1.5) (see the Introduction), we will use the methods of Convex Analysis, and so we must first recall some terminology (see [25], [15] and [5]).

If H is a real Hilbert space with inner product $(\cdot, \cdot)$ and $\Psi : H \to (-\infty, +\infty]$ is convex, then the subdifferential of Ψ is defined as the multivalued operator $\partial\Psi$ given by

$$v \in \partial\Psi(u) \iff \Psi(w) - \Psi(u) \ge (v, w - u) \quad \forall\, w \in H.$$

Recall that the epigraph of Ψ is defined by

$$\mathrm{Epi}(\Psi) = \{(u, \lambda) \in H \times \mathbb{R} \;:\; \lambda \ge \Psi(u)\}.$$

Given K a closed convex subset of H, we define the indicator function of K by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

Then the subdifferential is characterized by

$$v \in \partial I_K(u) \iff u \in K \text{ and } (v, w-u) \le 0 \quad \forall\, w \in K.$$

When the convex functional $\Psi : H \to (-\infty, +\infty]$ is proper, lower-semicontinuous, and such that $\min \Psi = 0$, it is well known (see [15]) that the abstract Cauchy problem

$$\begin{cases} u_t + \partial\Psi(u) \ni f, & \text{a.e } t \in (0,T), \\ u(0) = u_0, \end{cases}$$

has a unique solution for any $f \in L^2(0,T;H)$ and $u_0 \in \overline{D(\partial\Psi)}$.

The Mosco convergence is a very useful tool to study convergence of solutions of parabolic problems. The following theorem is a consequence of results in [16] and [5].

Theorem 4. *Let $\Psi_n, \Psi : H \to (-\infty, +\infty]$ be convex and lower semicontinuous functionals. Then the following statements are equivalent:*

(i) *Ψ_n converges to Ψ in the sense of Mosco.*

(ii) $(I + \lambda\partial\Psi_n)^{-1}u \to (I + \lambda\partial\Psi)^{-1}u \quad \forall\, \lambda > 0,\ u \in H.$

Moreover, either one of the above conditions, (i) or (ii), imply that

(iii) *for every $u_0 \in \overline{D(\partial\Psi)}$ and $u_{0,n} \in \overline{D(\partial\Psi_n)}$ such that $u_{0,n} \to u_0$, and for every $f_n, f \in L^1(0,T;H)$ with $f_n \to f$, if $u_n(t)$, $u(t)$ are solutions of the abstract Cauchy problems*

$$\begin{cases} (u_n)_t + \partial\Psi_n(u_n) \ni f_n & \text{a.e. } t \in (0,T) \\ u_n(0) = u_{0,n}, \end{cases}$$

and

$$\begin{cases} u_t + \partial\Psi(u) \ni f & \text{a.e. } t \in (0,T) \\ u(0) = u_0, \end{cases}$$

respectively, then

$$u_n \to u \qquad \text{in } C([0,T] : H).$$

Now we observe that if we take $H = L^2(\mathbb{R}^N)$ and

$$\Psi_n(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} \frac{\lambda(x)^n}{np(x)} |\nabla u(x)|^{np(x)}\, dx & \text{if } u \in L^2(\mathbb{R}^N) \cap W^{1,np(\cdot)}(\mathbb{R}^N) \\ +\infty & \text{otherwise,} \end{cases}$$

then the associated PDE reads:

$$\begin{cases} (u_n)_t + \operatorname{div}\left(\lambda(x)^n |\nabla u_n|^{np(x)-2} \nabla u_n\right) = f & \text{a.e. } t \in (0,T) \\ u_n(0) = u_0. \end{cases}$$

In this case, in view of our Theorem 2, together with Theorem 4, we deduce that

$$u_n \to u \qquad \text{in } C([0,T] : L^2(\mathbb{R}^N)),$$

where u is the solution to

$$\begin{cases} u_t + \partial\Psi_\infty(u) \ni f & \text{a.e. } t \in (0,T) \\ u(0) = u_0, \end{cases}$$

and $\Psi_\infty : L^2(\mathbb{R}^N) \to [0,+\infty]$ is given by

$$\Psi_\infty(u) = \begin{cases} 0 & \text{if } \lambda(x)|\nabla u(x)|^{p(x)} \le 1 \text{ a.e. } x \in \mathbb{R}^N \\ +\infty & \text{otherwise.} \end{cases}$$

As already mentioned in the Introduction, this limit can be seen as a model for the growth of a sandpile in which the critical slope of the sand depends explicitly on the spatial location. This dependence can be explained by differences in the sand composition or humidity.

We present below, in the one-dimensional case $N = 1$, some explicit examples of solutions to the limiting evolution problem in the particular case where $f = \delta_0$ and $u_0 = 0$, subject to pointwise constraints on the derivatives.

First, let us consider the case in which the restriction on the derivative reads as

$$|u_x|(x) \le 1 \quad \text{for } x \le 0, \qquad \text{and} \qquad |u_x|(x) \le 1/2 \quad \text{for } x > 0,$$

that is,

$$|u_x(x)| \le A(x) := \begin{cases} 1 & \text{if } x \le 0 \\ 1/2 & \text{if } x > 0. \end{cases}$$

Now, let

$$u(x,t) = \begin{cases} x + z(t) & \text{if} \quad 0 > x > -z(t) \\ z(t) - \frac{1}{2}x & \text{if} \quad 2z(t) > x \ge 0 \\ 0 & \text{otherwise,} \end{cases} \tag{6.1}$$

with $z(t) = \sqrt{\frac{2t}{3}}$ the solution to

$$z'(t) = \frac{1}{3z(t)}, \qquad z(0) = 0.$$

The function $u(x,t)$ defined by (6.1) is the unique solution to the problem

$$\begin{cases} u_t + \partial\Psi_\infty(u) \ni \delta_0 & \text{a.e. } t \in (0,T) \\ u(0) = 0, \end{cases} \tag{6.2}$$

where $\Psi_\infty : L^2(\mathbb{R}) \to [0,+\infty]$ is given by

$$\Psi_\infty(u) = \begin{cases} 0 & \text{if } |u'(x)| \le A(x) \text{ a.e. } x \in \mathbb{R} \\ +\infty & \text{otherwise.} \end{cases}$$

Let us prove this fact. We need to show that for every $v \in L^2(\mathbb{R})$ we have

$$\Psi_\infty(v) \ge \Psi_\infty(u(\cdot,t)) + \int_{\mathbb{R}} (\delta_0 - u_t(\cdot,t))(v - u(\cdot,t))dx.$$

As $|u_x(x,t)| \leq A(x),\ x \in \mathbb{R}$, we have $\Psi_\infty(u(\cdot,t)) = 0$, and thus we can restrict our attention to functions $v \in L^2(\mathbb{R})$ such that $\Psi_\infty(v) = 0$, that is, $|v'(x)| \leq A(x)$ (otherwise $\Psi_\infty(v) = +\infty$, and there is nothing to prove). Hence, we are left with

$$\int_{\mathbb{R}} u_t(x,t)(v(x) - u(x,t))\, dx \geq v(0) - u(0,t).$$

Now, since

$$u_t(x,t) = \begin{cases} z'(t) & \text{if } -z(t) \leq x \leq 2z(t) \\ 0 & \text{otherwise,} \end{cases}$$

we need to verify:

$$\int_0^{2z(t)} (v(x) - u(x,t))\, dx + \int_{-z(t)}^0 (v(x) - u(x,t))\, dx \geq 3z(t)(v(0) - u(0,t)).$$

We will show that

$$\int_0^{2z(t)} (v(x) - u(x,t))\, dx \geq 2z(t)(v(0) - u(0,t)), \tag{6.3}$$

and

$$\int_{-z(t)}^0 (v(x) - u(x,t))\, dx \geq z(t)(v(0) - u(0,t)) \tag{6.4}$$

hold. The fact that (6.3) or, equivalently,

$$\int_0^{2z(t)} (v(0) - v(x))\, dx \leq \int_0^{2z(t)} (u(0,t) - u(x,t))\, dx,$$

holds follows by taking into account the fact that v satisfies $|v'(x)| \leq A(x) = \frac{1}{2}$ for $x \in (0, 2z(t))$, and observing that this gives

$$u(0,t) - u(x,t) = \frac{1}{2}x \geq v(0) - v(x) \qquad \text{for } x \in (0, 2z(t)).$$

Similarly, one can also show (6.4). We conclude that $u(x,t)$ given by (6.1) solves (6.2).

The above discussion can be adapted to treat the general case where A is only required to satisfy

$$0 < c_1 \leq A(x) \leq c_2 < +\infty. \tag{6.5}$$

In this case, for every $z > 0$ there exists $s_-(z) < 0$ and $s_+(z) > 0$ such that

$$\int_{s_-(z)}^0 A(s)\, ds + z = 0, \text{ and } -\int_0^{s_+(z)} A(s)\, ds + z = 0.$$

The solution $u = u(x,t)$ to (6.2) (we keep the data $f = \delta_0$ and $u_0 = 0$) is now given by

$$u(x,t) = \begin{cases} \displaystyle\int_0^x A(s)\, ds + z(t) & \text{if } s_-(z(t)) \leq x \leq 0 \\ \displaystyle -\int_0^x A(s)\, ds + z(t) & \text{if } 0 < x \leq s_+(z(t)) \\ 0 & \text{otherwise,} \end{cases} \tag{6.6}$$

with $z = z(t)$ being the solution to the problem

$$z'(t) = \frac{1}{s_+(z(t)) - s_-(z(t))}, \qquad z(0) = 0.$$

To prove this, we need to show, as before, that

$$\Psi_\infty(v) \geq \Psi_\infty(u(\cdot,t)) + \int_{\mathbb{R}} (\delta_0 - u_t(\cdot,t))(v - u(\cdot,t))dx \ \forall \ v \in L^2(\mathbb{R}).$$

Since $|u_x(x,t)| \leq A(x)$, we have $\Psi_\infty(u(\cdot,t)) = 0$, and thus, without loss of generality, we only consider functions $v \in L^2(\mathbb{R})$ such that $\Psi_\infty(v) = 0$, that is, $|v'(x)| \leq A(x)$ (otherwise $\Psi_\infty(v) = +\infty$, and there is nothing to prove). Hence, we need to show that

$$\int_{\mathbb{R}} u_t(x,t)(v(x) - u(x,t))\, dx \geq v(0) - u(0,t).$$

Since

$$u_t(x,t) = \begin{cases} z'(t) & s_-(z(t)) \leq x \leq s_+(z(t)), \\ 0 & \text{otherwise,} \end{cases}$$

this reduces to showing that

$$\int_0^{s_+(z(t))} (v(x) - u(x,t))\, dx + \int_{s_-(z(t))}^0 (v(x) - u(x,t))\, dx \geq (s_+(z(t)) - s_-(z(t)))(v(0) - u(0,t)).$$

To this aim, it is enough to prove that

$$\int_0^{s_+(z(t))} (v(x) - u(x,t))\, dx \geq s_+(z(t))(v(0) - u(0,t)), \tag{6.7}$$

and

$$\int_{s_-(z(t))}^0 (v(x) - u(x,t))\, dx \geq -s_-(z(t))(v(0) - u(0,t)) \tag{6.8}$$

hold. Using the fact that v satisfies $|v'(x)| \leq A(x)$ for $x \in (0, s_+(z(t))$, we have

$$u(0,t) - u(x,t) = \int_0^x A(s)\, ds \geq v(0) - v(x) \qquad \text{for all } x \in (0, s_+(z(t)),$$

which implies that

$$\int_0^{s_+(z(t))} (v(0) - v(x))\, dx \leq \int_0^{s_+(z(t))} (u(0,t) - u(x,t))dx.$$

This shows that (6.7) holds. The remaining inequality, (6.8), follows similarly. We conclude that $u = u(x,t)$ given by (6.6) is the solution to (6.2) for functions A satisfying (6.5).

Finally, let us consider a nontrivial initial condition $u_0 = u_0(x)$ satisfying the restriction $|(u_0)'(x)| \leq A(x)$. We assume again that (6.5) holds, and we keep the data $f = \delta_0$.

In this case, for every $z > u_0(0)$ there exists $s_-(z) < 0$ and $s_+(z) > 0$ such that

$$\int_{s_-(z)}^0 A(s)\, ds + z = u_0(s_-(z)), \quad \text{and} \quad -\int_0^{s_+(z)} A(s)\, ds + z = u_0(s_+(z)).$$

The solution $u = u(x,t)$ to (6.2) (with $u(0) = u_0$) is given by (6.6), that is,

$$u(x,t) = \begin{cases} \displaystyle\int_0^x A(s)\, ds + z(t) & \text{if } s_-(z(t)) \leq x \leq 0, \\ \displaystyle -\int_0^x A(s)\, ds + z(t) & \text{if } 0 < x \leq s_+(z(t)), \\ 0 & \text{otherwise,} \end{cases} \tag{6.9}$$

but now $z = z(t)$ is the solution to the ODE

$$z'(t) = \frac{1}{s_+(z(t)) - s_-(z(t))}, \qquad z(0) = u_0(0).$$

To prove this, we need to show, arguing as before, that

$$\int_{\mathbb{R}} u_t(x,t)(v(x) - u(x,t))\,dx \geq v(0) - u(0,t).$$

Since

$$u_t(x,t) = \begin{cases} z'(t) & s_-(z(t)) \leq x \leq s_+(z(t)) \\ 0 & \text{otherwise,} \end{cases}$$

this reduces to showing that

$$\int_0^{s_+(z(t))} (v(x) - u(x,t))\,dx + \int_{s_-(z(t))}^0 (v(x) - u(x,t))\,dx \geq (s_+(z(t)) - s_-(z(t)))(v(0) - u(0,t)).$$

The proof of this fact runs exactly as before, we show that (6.7) and (6.8) hold, following the same steps performed in the case $u_0 = 0$. We conclude that $u = u(x,t)$ given by (6.9) is the solution to (6.2).

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