AN EXISTENCE RESULT FOR THE INFINITY LAPLACIAN WITH NON-HOMOGENEOUS NEUMANN BOUNDARY CONDITIONS USING TUG-OF-WAR GAMES

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ABSTRACT. In this paper we show how to use a Tug-of-War game to obtain existence of a viscosity solution to the infinity laplacian with non-homogeneous mixed boundary conditions. For a Lipschitz and positive function g there exists a viscosity solution of the mixed boundary value problem,

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = g(x) & \text{on } \Gamma_N, \\ u(x) = 0 & \text{on } \Gamma_D. \end{cases}$$

1. INTRODUCTION

Our main goal in this paper is to show that Tug-of-War games are a useful tool to obtain existence of solutions for nonlinear elliptic PDEs. More concretely, we will show existence of a viscosity solution to

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = g(x) & \text{on } \Gamma_N, \\ u(x) = 0 & \text{on } \Gamma_D, \end{cases} \tag{1.1}$$

with $g > 0$ and Lipschitz. Here Ω is a bounded, convex and smooth domain, Γ_N a subdomain of $\partial\Omega$, and $\Gamma_D = \partial\Omega \setminus \Gamma_N$.

We have to emphasize that the proof of the existence of a solution of (1.1) is not trivial. Indeed, if we want to use PDE methods we realize that the problem is not variational and moreover, it is not clear if there is a maximum (or comparison) principle for viscosity solutions to this problem, therefore Perron's method is not applicable. Furthermore, this problem is not the limit as $p \to \infty$ of the natural mixed boundary value problem for the $p-$Laplacian, $-\Delta_p u_p = 0$ in Ω, $u_p = 0$ on Γ_D and $|\nabla u_p|^{p-2} \frac{\partial u_p}{\partial n} = g$ on Γ_N (note that this is the natural Neumann boundary condition associated to $-\Delta_p u_p$). It will be interesting to get a proof of existence for (1.1) using only PDE tools.

Through this paper we will use the $1-$homogeneous version of the infinity Laplacian, which is the operator that appears naturally in connection with

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game theory. Let us recall that it is given by

$$\Delta_\infty u(x) = \Big\langle D^2 u(x) \frac{\nabla u(x)}{|\nabla u(x)|}, \frac{\nabla u(x)}{|\nabla u(x)|} \Big\rangle. \tag{1.2}$$

This definition is valid when $\nabla u(x) \neq 0$ but it needs some technical extension when $\nabla u(x) = 0$, see Section 4.

Notice that, formally, $\Delta_\infty u$ is the second derivative of u in the direction of the gradient. In fact, if u is a C^2 function and we take a direction v, then the second derivative of u in the direction of v is

$$D_v^2 u(x) = \frac{d^2}{dt^2}\Big|_{t=0} u(x+tv) = \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(x) v_i v_j.$$

If $\nabla u(x) \neq 0$, we can take $v = \dfrac{\nabla u(x)}{|\nabla u(x)|}$, and get $\Delta_\infty u(x) = D_v^2 u(x)$.

Infinity harmonic functions (solutions to $-\Delta_\infty u = 0$) appear naturally as limits of $p-$harmonic functions (solutions to $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$) and have applications to optimal transport problems, image processing, etc. See [4], [7], [10] and references therein. In this case it is equivalent to take the $3-$homogeneous version of the infinity Laplacian, namely, $\Delta_\infty u(x) = \langle D^2 u(x) \nabla u(x), \nabla u(x)\rangle$. It is known that the problem $-\Delta_\infty u = 0$ with a Dirichlet datum, $u = F$ on $\partial\Omega$ has a unique viscosity solution, (as proved in [12], and in a more general framework, in [18]). Moreover, it is the unique AMLE (absolutely minimal Lipschitz extension) of $F : \Gamma_D \to \mathbb{R}$ in the sense that $Lip_U(u) = Lip_{\partial U \cap \Omega}(u)$ for every open set $U \subset \overline{\Omega} \setminus \Gamma_D$. AMLE extensions were introduced by Aronsson in [3], see the survey [4] for more references and applications of this subject.

Tug-of-War games. A Tug-of-War is a two-person, zero-sum game, that is, two players are in contest and the total earnings of one are the losses of the other. Hence, one of them, say Player I, plays trying to maximize his expected outcome, while the other, say Player II is trying to minimize Player I's outcome (or, since the game is zero-sum, to maximize his own outcome). Recently, these type of games have been used in connection with some PDE problems, see [6], [15], [17], [18]. For the reader's convenience, let us first describe briefly the game introduced in [18] by Y. Peres, O. Schramm, S. Sheffield and D. Wilson. Consider a bounded domain $\Omega \subset \mathbb{R}^n$, and take $\Gamma_D \subset \partial\Omega$ and $\Gamma_N \equiv \partial\Omega \setminus \Gamma_D$. Let $F : \Gamma_D \to \mathbb{R}$ be a Lipschitz continuous function. At an initial time, a token is placed at a point $x_0 \in \overline{\Omega} \setminus \Gamma_D$. Then, a (fair) coin is tossed and the winner of the toss is allowed to move the game position to any $x_1 \in \overline{B_\epsilon(x_0)} \cap \overline{\Omega}$. At each turn, the coin is tossed again, and the winner chooses a new game state $x_k \in \overline{B_\epsilon(x_{k-1})} \cap \overline{\Omega}$. Once the token has reached some $x_\tau \in \Gamma_D$, the game ends and Player I earns $F(x_\tau)$ (while Player II earns $-F(x_\tau)$). This is the reason why we will refer to F as the *final payoff function.* It is considered also a *running payoff*, $f(x) > 0$, defined in Ω, which represents the reward (respectively, the cost) at each intermediate state x. This procedure gives a sequence of game states $x_0, x_1, x_2, \ldots, x_\tau$, where every x_k except x_0 are random variables, depending on the coin tosses and the strategies adopted by the players.

Now we want to give a definition of the *value of the game*. To this end we introduce some notation and the normal or strategic form of the game (see [17] and [16]). The initial state $x_0 \in \overline{\Omega} \setminus \Gamma_D$ is known to both players (public knowledge). Each player i chooses an *action* $a_0^i \in \overline{B_\epsilon(0)}$; this defines an action profile $a_0 = \{a_0^1, a_0^2\} \in \overline{B_\epsilon(0)} \times \overline{B_\epsilon(0)}$ which is announced to the other player. Then, the new state $x_1 \in \overline{B_\epsilon(x_0)}$ (namely, the current state plus the action) is selected according to the distribution $p(\cdot|x_0, a_0)$ in $\overline{\Omega}$. At stage k, knowing the history $h_k = (x_0, a_0, x_1, a_1, \dots, a_{k-1}, x_k)$, (the sequence of states and actions up to that stage), each player i chooses an action a_k^i. If the game ends at time $j < k$, we set $x_m = x_j$ and $a_m = 0$ for $j \le m \le k$. The current state x_k and the profile $a_k = \{a_k^1, a_k^2\}$ determine the distribution $p(\cdot|x_k, a_k)$ of the new state x_{k+1}. Denote $H_k = (\overline{\Omega} \setminus \Gamma_D) \times \big(\overline{B_\epsilon(0)} \times \overline{B_\epsilon(0)} \times \overline{\Omega}\big)^k$, the set of *histories up to stage* k, and by $H = \bigcup_{k \ge 1} H_k$ the set of all histories. Notice that H_k, as a product space, has a measurable structure. The *complete history space* H_∞ is the set of plays defined as infinite sequences $(x_0, a_0, \dots, a_{k-1}, x_k, \dots)$ endowed with the product topology. Then, the final payoff for Player I, i.e. F, induces a Borel-measurable function on H_∞. A *pure strategy* $S_i = \{S_i^k\}_k$ for Player i, is a sequence of mappings from histories to actions, namely, a mapping from H to $\overline{B_\epsilon(0)}$ such that S_i^k is a Borel-measurable mapping from H_k to $\overline{B_\epsilon(0)}$ that maps histories ending with x_k to elements of $\overline{B_\epsilon(0)}$ (roughly speaking, at every stage the strategy gives the next movement for the player, provided he win the coin toss, as a function of the current state and the past history). The initial state x_0 and a profile of strategies $\{S_I, S_{II}\}$ define (by Kolmogorov's extension theorem) a unique probability $\mathbb{P}^{x_0}_{S_I, S_{II}}$ on the space of plays H_∞. We denote by $\mathbb{E}^{x_0}_{S_I, S_{II}}$ the corresponding expectation.

Then, if S_I and S_{II} denote the strategies adopted by Player I and II respectively, we define the expected payoff for Player I as

$$V_{x_0,I}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I, S_{II}}[F(x_\tau) + \sum\limits_i f(x_i)], & \text{if the game terminates a.s.} \\ +\infty, & \text{otherwise.} \end{cases}$$

Analogously, we define the expected payoff for Player II as

$$V_{x_0,II}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I, S_{II}}[F(x_\tau) + \sum\limits_i f(x_i)], & \text{if the game terminates a.s.} \\ -\infty, & \text{otherwise.} \end{cases}$$

The ϵ-value of the game for Player I is given by

$$u_I^\epsilon(x_0) = \sup_{S_I} \inf_{S_{II}} V_{x_0,I}(S_I, S_{II}),$$

while the ϵ-value of the game for Player II is defined as

$$u_{II}^\epsilon(x_0) = \inf_{S_{II}} \sup_{S_I} V_{x_0,II}(S_I, S_{II}).$$

In some sense, $u_I^\epsilon(x_0), u_{II}^\epsilon(x_0)$ are the least possible outcomes that each player expects to get when the ϵ-game starts at x_0. Notice that, as in [18], we penalize severely the games that never end.

If $u_I^\epsilon = u_{II}^\epsilon := u_\epsilon$, we say that *the game has a value.* In [18] it is shown that under very general hypotheses, that are fulfilled in the present setting, the ϵ-Tug-of-War game has a value.

Note that it is essential that the running payoff is strictly positive or identically zero in Ω in order to have a value of the game, see [18] for a counterexample. This fact imposes the restriction that g must be positive on Γ_N. We have to mention that in [2] the authors propose a suitable modification of the game that gives existence of a continuous solution assuming only that the payoff is nonnegative. Uniqueness is only known for strictly positive (or zero) running payoff (from the results in [18]).

In [18], [17] and [8], the limit as $\epsilon \to 0$ is studied and it is proved there that when the running payoff is of order ϵ^2, they consider $f(x) = \epsilon^2 a(x)$, then $u(x) = \lim_{\epsilon\to 0} u_\epsilon(x)$ exists and is a solution to

$$\begin{cases} -\Delta_\infty u(x) = a(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{on } \Gamma_N, \\ u(x) = F(x) & \text{on } \Gamma_D. \end{cases}$$

Let us to point out that the arguments that we develop here allow to fix an easily solvable mistake in our previous work [8], see Remark 11.

As we have mentioned our main task here is to obtain a non-homogeneous Neumann boundary condition, hence we consider a running payoff of the form

$$f(x) = \begin{cases} \epsilon \dfrac{g(x)}{2} & \text{for } d(x,\Gamma_N) \le \epsilon, \\ \epsilon^3 & \text{for } d(x,\Gamma_N) > \epsilon. \end{cases} \tag{1.3}$$

Note that $f(x)$ is strictly positive in Ω and concentrates as $\epsilon \to 0$ in a small strip of width ϵ near Γ_N. Also note that our running cost $f(x)$ has size $\epsilon \times \epsilon^2$ for $d(x,\Gamma_N) > \epsilon$ (this will give a solution to $\Delta_\infty u(x) = 0$ in Ω) and $\epsilon^{-1}\frac{g(x)}{2}\epsilon^2$ for $d(x,\Gamma_N) \le \epsilon$ (this part will become singular giving $\frac{\partial u}{\partial n}(x) = g(x)$ on Γ_N when passing to the limit). Finally let us observe that ϵ^2 is the homogeneity associated to a change of scale in the 1-homogeneous infinity Laplacian, while ϵ is the homogeneity associated with $\frac{\partial u}{\partial n}$.

For technical reasons we also have to assume that the final payoff F is zero, $F = 0$. This hypothesis is used to find the estimates for u_ϵ collected in Section 3. We also use there the convexity of Ω.

Note that, since the running payoff is strictly positive, $f > 0$, and the final payoff is zero, $F = 0$, we have that the value of the game is nonnegative, $u_\epsilon \ge 0$.

One of the main properties of the ϵ−values of the game is the so called *Dynamic Programming Principle*, that in this case reads as follows:

$$u_\epsilon(x) = \frac{1}{2} \sup_{y\in \overline{B_\epsilon(x)}\cap\bar\Omega} u_\epsilon(y) + \frac{1}{2} \inf_{y\in \overline{B_\epsilon(x)}\cap\bar\Omega} u_\epsilon(y) + f(x)$$

for every $x \in \bar{\Omega} \setminus \Gamma_D$, where $B_\epsilon(x)$ denotes the open ball of radius ϵ centered at x and f is given by (1.3).

This property follows from the probabilistic interpretation of the u_ϵ, since Player I and Player II have the same probability of winning the coin toss at the point x, and the running payoff gives the amount received at the point x. In particular, when the running payoff is zero, the Dynamic Programming Principle is some kind of mean value property for the $\epsilon-$ values of the game.

As we will prove, these $\epsilon-$values converge uniformly (along subsequences) when $\epsilon \to 0$. The uniform limit as $\epsilon \to 0$ of the game values u_ϵ is called *the continuous value* of the game, that we will denote by u.

One of the most delicate points in this paper is to show that this continuos value of the game exists. To this end, since the ϵ-values of the game have an *inf-sup* or *sup-inf* expression, we can get the required estimates considering prescribed strategies for Player II (or Player I) and using a comparison argument from [18].

In the analysis performed in [18] this convergence result is proved considering modified dyadic games (on scales ϵ and 2ϵ) that are favorable to one of the two players and then proving that monotonicity holds for those modified games. In our case it is difficult to adapt this argument because the size of the strip where the running payoff is of size ϵ, $d(x, \Gamma_N) \le \epsilon$, changes with ϵ. This fact implies that the dyadic modifications of the game as in [18] no longer yield a monotone sequence. Hence we have to introduce a different argument that is based on a uniform bound for the u_ϵ (note that this is not immediate since we have a running payoff with $2\epsilon^{-2} \max_x f(x) = \epsilon^{-1} \max_x g$, that is unbounded as $\epsilon \to 0$) together with a modification of the classical Ascoli-Arzela's Lemma (taking into account that in general the involved functions u_ϵ are not continuous).

From [18], it turns out that u is a viscosity solution to the problem $-\Delta_\infty u = 0$ in Ω and clearly, $u = 0$ on Γ_D. The second main point of this paper is to show that in the limit we get the nonhomogeneous Neumann boundary condition $\frac{\partial u}{\partial n} = g$ on Γ_N.

We have the following result

Theorem 1. *Let Ω be a bounded, convex and smooth domain, Γ_N a subdomain of $\partial\Omega$, and $\Gamma_D = \partial\Omega \setminus \Gamma_N$. Let $g(x)$ a positive Lipschitz function defined on Γ_N. Consider u_ϵ the value of the Tug-of-War game described above. Then, there exists a subsequence, u_{ϵ_j}, that converges uniformly to a Lipschitz continuous limit u that is a viscosity solution to the mixed boundary value problem*

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = g(x) & \text{on } \Gamma_N, \\ u(x) = 0 & \text{on } \Gamma_D. \end{cases} \tag{1.4}$$

The rest of the paper is organized as follows: in Section 2 we analyze this Tug-of-War games in an interval (this analysis will provide some insight on why the size of the running payoff is the right one), in Section 3 we prove that the sequence u_ϵ has a compactness property that allows us to pass to the limit, and finally in Section 4 we prove that this limit is a viscosity solution of (1.1). We leave to the Appendix the proof of a compactness result (a variant of Ascoli-Arzela's Lemma).

2. The $1-d$ game

Let us analyze in detail the one-dimensional game and its limit as $\epsilon \to 0$.

We set $\Omega = (0,1)$ and play the $\epsilon-$game. To simplify we assume that the running payoff is concentrated at one end. That is, we end the game at $x = 0$ (with zero final payoff) and we impose a running payoff at the interval $(1-\epsilon, 1]$ of amount $\epsilon g(1)/2 > 0$ and, to begin with, zero in the rest of the interval. Note that, since the running payoff is not strictly positive, the general result from [18] does not apply and hence we cannot assert the existence of a value for this game. Nevertheless, in this simple $1-d$ case we can obtain the existence of such value by direct computations. For the moment, let us assume that there exists a value that we call u_ϵ and proceed, in several steps, with the analysis of this sequence of functions u_ϵ for ϵ small. All the calculations below hold both for u_I^ϵ and for u_{II}^ϵ.

Step 1. $u_\epsilon(0) = 0$.

Step 2. u_ϵ is increasing in x and strictly positive in $(0, 1]$.

Indeed, if $x < y$ then for every pair of strategies S_I, S_{II} for Player I and II beginning at x we can construct strategies beginning at y in such a way that

$$x_{i,x} \le x_{i,y}$$

(here $x_{i,x}$ and $x_{i,y}$ are the positions of the game after i movements beginning at x and y respectively). Indeed, just reproduce the movements shifting points by $y - x$ when possible (if not, that is, if the jump is too large and ends outside the interval, just remain at the larger interior position $x = 1$).

Then,

$$\sum_i f(x_{i,x}) \le \sum_i f(x_{i,y}).$$

Taking expectations, infimum and supremum, it follows that

$$u_\epsilon(x) \le u_\epsilon(y).$$

Note that for two sequences of positions $x_{i,x} \le x_{i,y}$ (corresponding to the same sequence coin tosses) we have that the game with the first sequence ends before the second and the number of times that the position of the game enters into $(1-\epsilon, 1]$ is greater for the second.

Now, we just observe that there is a positive probability of obtaining a sequence of $1/\epsilon$ consecutive heads (exactly $2^{-1/\epsilon}$), hence the probability of reaching the interval where a positive running payoff is paid is strictly positive. Therefore,

$$u_\epsilon(x) > 0,$$

for every $x \neq 0$.

Step 3. In this one dimensional case it is easy to identify the optimal strategies for players I and II: to jump ϵ to the right for Player I and to jump ϵ to the left for Player II. That is, if we are at x, the optimal strategies lead to

$$x \to \min\{x+\epsilon, 1\}$$

for Player I, and to

$$x \to \max\{x-\epsilon, 0\}$$

for Player II.

This follows from step 2, where we have proved that the function u_ϵ is increasing in x. As a consequence, the optimal strategies follow: for instance, Player I will choose the point where the expected payoff is maximized and this is given by $\min\{x+\epsilon, 1\}$,

$$\sup_{z\in[x-\epsilon,x+\epsilon]\cap[0,1]} u_\epsilon(z) = \max_{z\in[x-\epsilon,x+\epsilon]\cap[0,1]} u_\epsilon(z) = u_\epsilon(\min\{x+\epsilon, 1\}),$$

since u_ϵ is increasing.

This is also clear from the following intuitive fact: player I wants to maximize the payoff (reaching the interval $(1-\epsilon, 1]$) and player II wants the game to end as soon as possible (hence pointing to 0).

Step 4. u_ϵ is constant in every interval of the form $(k\epsilon, (k+1)\epsilon)$ for $k = 1, ..., N$ (we denote by N the total number of such intervals in $(0, 1]$).

Indeed, from step 3 we know what are the optimal strategies for both players, and hence the result follows noticing that the number of steps that one has to advance to reach zero (or the payoff interval $(1-\epsilon, 1]$) is the same for every point in $(k\epsilon, (k+1)\epsilon)$.

Remark 1. Note that u_ϵ is necessarily discontinuos at every point of the form $y_k = k\epsilon \in (0, 1)$.

Step 5. Let us call $a_k := u_\epsilon \mid_{(k\epsilon,(k+1)\epsilon)}$. Then we have

$$a_0 = 0,$$
$$a_k = \frac{1}{2}(a_{k-1} + a_{k+1}),$$

for every $i = 2, ..., N-1$, and

$$a_N = \frac{1}{2}(a_{N-1} + a_N) + \frac{1}{2}\epsilon g(1)$$

Notice that these identities follow from the Dynamic Programming Principle as stated in the introduction.

Using again that from step 3 we know the optimal strategies, that from step 4 u_ϵ is constant in every subinterval of the form $(k\epsilon, (k+1)\epsilon)$ and that there is a final payoff in the last interval of amount $\epsilon g(1)/2$, we immediately get the conclusion.

Remark 2. Note the similarity with a finite difference scheme used to solve $u_{xx} = 0$ in $(0, 1)$ with boundary conditions $u(0) = 0$ and $u_x(1) = g(1)$. In fact, a discretization of this problem in a uniform mesh of size ϵ leads to the same formulas obtained in step 5.

Step 6. We have

$$u_\epsilon(x) = \epsilon g(1)k, \qquad x \in (k\epsilon, (k+1)\epsilon). \tag{2.1}$$

Indeed, the constants

$$a_k = \epsilon g(1)k$$

are the unique solution to the formulas obtained in step 5.

Remark 3. Since formula (2.1) is in fact valid for u_I^ϵ and u_{II}^ϵ, this proves that the game has a value.

Remark 4. Note that u_ϵ verifies that

$$0 \le u_\epsilon(x) - u_\epsilon(y) \le 2g(1)(x-y)$$

for every $x > y$ with $x - y > \epsilon$.

In this one dimensional case, we can pass to the limit directly, by using the explicit formula for u_ϵ (see Step 7 below). However, in the n-dimensional case there is no explicit formula, and then we will need a compactness result, which is proved in Theorem 3 (see the Appendix).

Step 7.

$$\lim_{\epsilon\to 0} u_\epsilon(x) = g(1)x,$$

uniformly in $[0,1]$.

Indeed, this follows from the explicit formula for u_ϵ in every interval of the form $(k\epsilon, (k+1)\epsilon)$ found in step 6 and from the monotonicity stated in step 2 (to take care of the values of u_ϵ at points of the form $k\epsilon$, we have $a_{k-1} \le u_\epsilon(k\epsilon) \le a_k$).

Remark 5. Note that from these results it follows that the functions u_ϵ are uniformly bounded for ϵ small, in fact,

$$0 \le u_\epsilon(x) \le 2g(1),$$

for every ϵ small enough.

Remark 6. Note that the limit function

$$u(x) = g(1)x$$

is the unique viscosity (and classical) solution to

$$\Delta_\infty u(x) = (u_{xx}(u_x)^2)(x) = 0 \qquad x \in (0,1),$$

with boundary conditions

$$u(0) = 0, \qquad u_x(1) = g(1).$$

Remark 7. Notice that an alternative approach to the previous analysis can be done by using the theory of Markov chains.

Running payoff given by (1.3)**.** In the case we have a running payoff given by

$$\epsilon\frac{g(1)}{2}\chi_{(1-\epsilon,1]} + \epsilon^3,$$

which is strictly positive in [0.1] and hence by [18] there exists the value of this game, u_ϵ.

Now we can argue almost exactly as before at steps 1 to 5, therefore we omit details. The only difference is that the running payoff changes the equations for the a_k and we have

$$a_0 = 0,$$

$$a_k = \frac{1}{2}(a_{k-1} + a_{k+1}) + \epsilon^3,$$

for every $i = 2, ..., N-1$, and

$$a_N = \frac{1}{2}(a_{N-1} + a_N) + \frac{1}{2}\epsilon g(1) + \epsilon^3.$$

Now we have to modify the previous **Step 6.** Let us look for a solution a_k of the form

$$a_k = -\epsilon^3 k^2 + \Theta \epsilon k.$$

Of course we have

$$a_0 = 0.$$

For every $k = 1, ..., N-1$ we need

$$a_k = \frac{1}{2}(a_{k-1} + a_{k+1}) + \epsilon^3,$$

that is,

$$-\epsilon^3 k^2 + \Theta\epsilon k = \frac{1}{2}\Big(- \epsilon^3 (k-1)^2 + \Theta\epsilon(k-1) - \epsilon^3(k+1)^2 + \Theta\epsilon(k+1)\Big) + \epsilon^3.$$

Simplifying this expression we get

$$-\epsilon^3 k^2 + \Theta\epsilon k = \frac{1}{2}\Big(- 2\epsilon^3 k^2 - 2\epsilon^3 + 2\Theta\epsilon k\Big) + \epsilon^3,$$

that holds true for any choice of Θ.

The last equation reads as

$$a_N = \frac{1}{2}(a_{N-1} + a_N) + \frac{1}{2}\epsilon g(1) + \epsilon^3.$$

That is,

$$-\epsilon^3 N^2 + \Theta\epsilon N = \frac{1}{2}\Big(- \epsilon^3 (N-1)^2 + \Theta\epsilon(N-1) - \epsilon^3 N^2 + \Theta\epsilon N\Big) + \frac{1}{2}\epsilon g(1) + \epsilon^3,$$

which is equivalent to,

$$-\epsilon^3 N^2 + \Theta\epsilon N = \frac{1}{2}\Big(- 2\epsilon^3 N^2 + 2N\epsilon^3 - \epsilon^3 + 2\Theta\epsilon N - \Theta\epsilon\Big) + \frac{1}{2}\epsilon g(1) + \epsilon^3.$$

Hence we need,

$$0 = 2N\epsilon^3 - \epsilon^3 - \Theta\epsilon + \epsilon g(1) + 2\epsilon^3.$$

From this we find the value of Θ, using that $\epsilon N = 1$,

$$\Theta = g(1) + 2N\epsilon^2 + \epsilon^2 = g(1) + 2\epsilon + \epsilon^2.$$

Therefore, we conclude that it holds

$$u_\epsilon(x) = -\epsilon^3 k^2 + \Theta\epsilon k, \qquad x \in (k\epsilon, (k+1)\epsilon),$$

with

$$\Theta = g(1) + 2\epsilon + \epsilon^2,$$

and hence, from the explicit formula for u_ϵ in every interval of the form $(k\epsilon, (k+1)\epsilon)$ and using that Θ verifies that $\lim_{\epsilon\to 0}\Theta = g(1)$, we conclude

$$\lim_{\epsilon\to 0} u_\epsilon(x) = g(1)x,$$

uniformly in $[0, 1]$.

3. The n-dimensional case: estimates for u_ϵ

In this section we prove some estimates, based in arguments from game theory, that allow us to show that u_ϵ converge uniformly along subsequences $\epsilon_j \to 0$ to a Lipschitz continuous limit u. We remark again that proving uniform convergence is not an easy task since u_ϵ are in general discontinuous functions, c.f. the previous section.

Notation and description of the setting:

To get an estimate for the sequence $\{u_\epsilon\}$, we will use that these functions are defined as an *sup-inf* (or equivalently as *inf-sup*). Let us fix a point x_0 in the domain Ω. Let us also denote by K the maximum of $\frac{g(x)}{2}$ and, in a first stage of the game, assume that the running payoff is much more favourable to Player I, and it is given by

$$f(x) = \begin{cases} \epsilon K & \text{for } d(x, \partial\Omega) \le \epsilon, \\ \epsilon^3 & \text{for } d(x, \partial\Omega) > \epsilon. \end{cases} \tag{3.1}$$

With this new running payoff, we play a first stage of the game which ends when the position reaches the point x_0, with final payoff 0. Once the position reaches x_0, and this happens almost surely, since for this modified game this point is the natural target for Player I, the game follows as usual, without any restriction on the strategies and with the original running payoff and final states (on Γ_D).

Main task: Assuming that the modified game starts from a point y_0, to find a bound for the value of the first part of the game, we call it v, in terms of the distance $|x_0 - y_0|$, K and the diameter of Ω but we want it to be independent of ϵ for ϵ small enough.

For points x in Ω that do not lie in the strip, dynamic programming principle holds. That is, the value of the game in Ω, v, (we omit the subscript ϵ to simplify the notation) verifies

$$v(x) = \frac{1}{2}\sup_{B_\epsilon(x)} v(z) + \frac{1}{2}\inf_{B_\epsilon(x)} v(z) + \epsilon^3. \tag{3.2}$$

For points $x \in \Omega$ that are in the strip of width ϵ around Γ_N the dynamic programming principle is different, since the running payoff is much larger, and the ball of radius ϵ may not be contained in the domain Ω. Therefore, in this case the dynamic programming principle gives

$$v(x) = \frac{1}{2}\sup_{B_\epsilon(x)\cap\Omega} v(z) + \frac{1}{2}\inf_{B_\epsilon(x)\cap\Omega} v(z) + \epsilon K. \tag{3.3}$$

To estimate v we will look for a suitable supersolution. Consider

$$\overline{v}(x) = \epsilon + C_1|x-x_0| - C_2\epsilon|x-x_0|^2.$$

We want to choose C_1 and C_2 in such a way that $\overline{v}$ is a supersolution to the dynamic programming principle, (3.2)–(3.3), that is, we want to obtain a " $\geq$ " in both equations.

Case 1. Concerning (3.2), we have

$$\begin{aligned}
&\overline{v}(x) - \frac{1}{2}\overline{v}\left(x + \epsilon\frac{(x-x_0)}{|x-x_0|}\right) - \frac{1}{2}\overline{v}\left(x - \epsilon\frac{(x-x_0)}{|x-x_0|}\right)\\
&= -C_2\epsilon|x-x_0|^2 + \frac{1}{2}C_2\epsilon(|x-x_0|+\epsilon)^2 + \frac{1}{2}C_2\epsilon(|x-x_0|-\epsilon)^2\\
&= C_2\epsilon^3 \geq \varepsilon^3,
\end{aligned}$$

taking $C_2 \geq 1$.

Case 2. We have to check (3.3), and at this point, the geometry of the domain has to be taken into account. We assume that there exists a constant c (that only depends on Ω), that bounds the maximum distance from x_0 that can be reached by Player I.

Definition 1. Given a point $x_0 \in \overline{\Omega}$ we say that it satisfies the transversality condition with constant $c > 0$, if and only if for any point $y \in \partial\Omega$, $y \neq x_0$, it holds

$$\langle \frac{y-x_0}{|y-x_0|}, \vec{n}(y)\rangle \geq c \tag{3.4}$$

where $\vec{n}(y)$ denotes the unit outwards normal at the point y.

Remark 8. Note that if Ω is strictly convex, there exists a uniform transversality constant c. However, if $\partial\Omega$ contains flat pieces, the transversality constant becomes 0 at some points, see Figure 1.

Hence, if we are at a point x in the strip, when we look at the positions in the ball $B_\epsilon(x) \cap \Omega$, the minimum of the distance to x_0 is $|x_0 - x| - \epsilon$, while the maximum of this distance is bounded by $|x_0 - x| + c\epsilon$. Therefore, to get an inequality in (3.3), we need to compute

$$\begin{aligned}
&\overline{v}(x) - \frac{1}{2}\overline{v}\left(x + c\epsilon\frac{(x-x_0)}{|x-x_0|}\right) - \frac{1}{2}\overline{v}\left(x - \epsilon\frac{(x-x_0)}{|x-x_0|}\right)\\
&= C_1|x-x_0| - C_2\epsilon|x-x_0|^2 - \frac{1}{2}C_1(|x-x_0|+c\epsilon) + \frac{1}{2}C_2\epsilon(|x-x_0|+c\epsilon)^2\\
&\qquad -\frac{1}{2}C_1(|x-x_0|-\epsilon) + \frac{1}{2}C_2\epsilon(|x-x_0|-\epsilon)^2\\
&\geq \epsilon\frac{1}{2}C_1(1-c) - O(\epsilon^2)\\
&\geq \epsilon K.
\end{aligned}$$

This indeed holds for ϵ small if we choose $C_1 = \frac{4}{1-c}K$.

Using a comparison argument, which follows from the proof of Theorem 2.4 in [18], taking $C_1 = \frac{4}{1-c}K$ and $C_2 = 1$, this function $\overline{v}$ gives a bound for v at points y_0 such that $\epsilon < |y_0 - x_0|$,

$$
\begin{aligned}
v(y_0) \le \overline{v}(y_0) &= \epsilon + \frac{4}{1-c} K|y_0 - x_0| - \epsilon|y_0 - x_0|^2 \\
&\le \left(\frac{4}{1-c}K + 1\right)|y_0 - x_0|.
\end{aligned} \tag{3.5}
$$

Remark 9. This estimate is valid as long as $|y_0 - x_0| > \epsilon$. Moreover, this is natural since, as was shown in the previous section, in the one-dimensional case, $v(y_0)$ is bounded from below by a strictly positive constant (depending on ϵ).

An estimate for u: Let $u_\epsilon(x)$ be a game as described in the introduction. We have

$$
u_\epsilon(y_0) \le v(y_0) + u_\epsilon(x_0) \tag{3.6}
$$

where v is as before. This estimate follows by prescribing that the second player plays with a strategy in the family S^*_{II} of strategies which consists on pointing to x_0 until the game reaches this position, and then continue the game from the starting position x_0. By simplicity of writing, we will use the same symbol S^*_{II} to denote the family of such strategies, and any particular element in this family. For any $\delta > 0$ there exists a particular strategy S^δ_I for Player I, such that

$$
\begin{aligned}
u_\epsilon(y_0) &= \sup_{S_I} \inf_{S_{II}} V_{y_0,I}(S_I, S_{II}) \le \sup_{S_I} \inf_{S^*_{II}} V_{y_0,I}(S_I, S^*_{II}) \\
&\le \inf_{S^*_{II}} \mathbb{E}^{y_0}_{S^\delta_I, S^*_{II}}[\sum_i f(x_i)] + \delta.
\end{aligned}
$$

Notice that the fact that $F = 0$ and $g > 0$ implies that, in this case, the "good strategies" (in the sense that they are close to realize the supremum) for Player I do not allow to end the game jumping to a point in Γ_D. In any case, if the game ends during this first stage, we trivially obtain that

$$
\mathbb{E}^{y_0}_{S^\delta_I, S^*_{II}}[\sum_{i \le \tau^*} f(x_i)] \le v(y_0) + u_\epsilon(x_0).
$$

For the second part of the game, once we arrive to x_0, we can fix a quasi-optimal strategy for Player II, in the set of strategies S^*_{II}, that we denote by S^δ_{II}. Calling τ^* the first time where the game reaches x_0,

$$
\begin{aligned}
u_\epsilon(y_0) &\le \inf_{S^*_{II}} \mathbb{E}^{y_0}_{S^\delta_I, S^*_{II}}[\sum_{i \le \tau^*} f(x_i) + \sum_{i > \tau^*} f(x_i)] + \delta \\
&\le \mathbb{E}^{y_0}_{S^\delta_I, S^\delta_{II}}[\sum_{i \le \tau^*} f(x_i)] + \mathbb{E}^{x_0}_{S^\delta_I, S^\delta_{II}}[\sum_{i > \tau^*} f(x_i)] + 2\delta \\
&\le v(y_0) + \sup_{S_I} \inf_{S_{II}} V_{x_0,I}(S_I, S_{II}) + 2\delta \\
&= v(y_0) + u_\epsilon(x_0) + 2\delta.
\end{aligned}
$$

Here we have selected S^δ_I and S^δ_{II} such that

$$
\sup_{S_I} \inf_{S^*_{II}} V_{y_0,I}(S_I, S^*_{II}) \le \inf_{S^*_{II}} \mathbb{E}^{y_0}_{S^\delta_I, S^*_{II}}[\sum_i f(x_i)] + \delta
$$

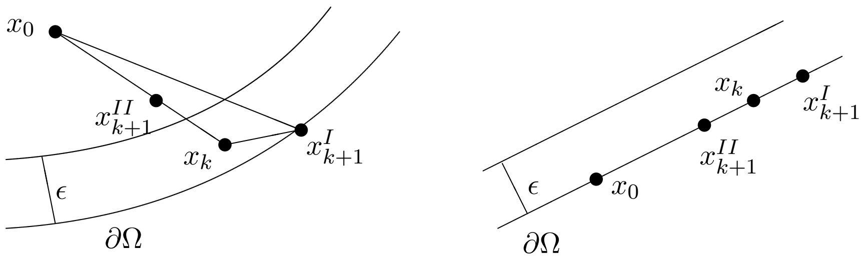

FIGURE 1. Possible states of the modified game.

and

$$\mathbb{E}^{x_0}_{S^\delta_I,S^\delta_{II}}[\sum_{i>\tau^*} f(x_i)] \leq \sup_{S_I}\inf_{S_{II}} V_{x_0,I}(S_I,S_{II}) + \delta.$$

Since δ is arbitrary we conclude (3.6).

When the transversality condition holds, using our previous estimate for v, we get

$$u_\epsilon(y_0) \leq u_\epsilon(x_0) + \left(\frac{4}{1-c}K + 1\right)|y_0 - x_0|.$$

By a symmetric argument we obtain

$$|u_\epsilon(y_0) - u_\epsilon(x_0)| \leq \left(\frac{4}{1-c}K + 1\right)|y_0 - x_0|, \tag{3.7}$$

for every y_0, x_0 such that $|y_0 - x_0| > \epsilon$, and such that the transversality condition in Ω holds. Note that when Ω is strictly convex the transversality condition holds for every x_0, y_0 in Ω, assuming $|y_0 - x_0| > \epsilon$. Therefore, our enemy at this point is that the boundary can have flat pieces (recall that we have assumed only that Ω is convex). See Remark 8.

In Figure 1 we show possible movements of the first stage of the game according to Player I wins, x^I_{k+1}, or Player II wins, x^{II}_{k+1}. Player I tries to move far away from x_0 while Player II aims to reach x_0. Notice that when the domain is strictly convex then Player I can not move a distance ϵ away from x_0 when x_k lies in the strip; while this is possible when the boundary $\partial\Omega$ has flat pieces, the risk being the fact that when Player II uses the strategy of pointing to x_0 at every position then the states of the game can be traped in the strip. Our next task is to extend the estimate to convex domains that may have flat pieces on its boundary.

If the transversality condition does not hold we argue as follows: we choose a point x_1 in the interior of the domain with the required transversality condition for y_0 and x_1 and for x_0 and x_1. It suffices to take $x_1 \in \Omega$ such that

$$|x_0 - x_1| \leq C|x_0 - y_0|, \qquad |y_0 - x_1| \leq C|x_0 - y_0|,$$

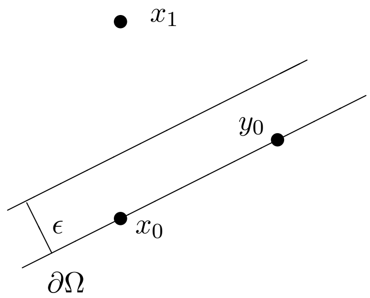

FIGURE 2. A possible position of x_1

and

$$\operatorname{dist}(x_1, \partial\Omega) \geq c|x_0 - y_0|.$$

In Figure 2, we draw a possible location of x_0 and y_0 for which the transversality condition is not satisfied. Note that when pointing to x_0 Player II remains in the strip a large number of times. We also depicted a possible location of the point x_1 mentioned above. Note that when playing with the strategy of pointing to x_1 we leave the strip immediately after at most a finite number (independent of ϵ) of consecutive winnings of Player II.

Then we obtain that there exists a constant C (depending on K and the geometry of Ω trough the constants in the transversality condition but not on ϵ) such that

$$\begin{aligned} |u_\epsilon(y_0) - u_\epsilon(x_0)| &\leq |u_\epsilon(x_1) - u_\epsilon(y_0)| + |u_\epsilon(x_1) - u_\epsilon(x_0)| \\ &\leq C|y_0 - x_0|, \end{aligned} \tag{3.8}$$

for all y_0, x_0 such that $|y_0 - x_0| > \epsilon$.

This estimate allows us to conclude two results:

Lemma 1. *Let u_ϵ be the value of the game described in the introduction, then there exists a constant C_1 (depending on g and Ω but not on ϵ) such that*

$$0 \leq u_\epsilon(x) \leq C_1. \tag{3.9}$$

Proof. It is enough to use (3.8) twice (if necessary) with y_0 an arbitrary point in Ω and two points $x_0^1, x_0^2 \in \Gamma_D$ with $|x_0^1 - x_0^2| >> \epsilon$ and recall that $u_\epsilon(x_0^i) = 0$ to obtain the desired bound. □

Lemma 2. *Let u_ϵ be the value of the game described in the introduction, then there exists a constant C_2 (depending on g and Ω but not on ϵ) such that*

$$|u_\epsilon(y_0) - u_\epsilon(x_0)| \leq C_2|y_0 - x_0|, \tag{3.10}$$

for all x, x_0 such that $|y_0 - x_0| > \epsilon$.

Proof. Immediate from (3.8). □

Remark 10. This result is analogous to Lemma 3.2 in [18]. However, due to the fact that our running payoff is of order ϵ (much bigger than ϵ^2) in the strip near Γ_N we need a different argument for the proof. Also note that since our running payoff is of order ϵ^3 (much smaller than ϵ^2) outside the strip we find as supersolution a cone plus a correction of order ϵ (see (3.5)), hence Lemma 3.2 in [18] is not a consequence of our results.

Therefore we fall into the hypotheses of the variant of the Ascoli-Arzela's compactness lemma (see the Appendix) and we conclude that u_ϵ converge uniformly taking a subsequence if necessary. This proves the first part of Theorem 1.

Theorem 2. *Let u_ϵ denote the value of the game described in the introduction, then there exists a subsequence ϵ_j and a Lipschitz continuous function u, such that*

$$u_{\epsilon_j} \to u$$

uniformly in $\overline{\Omega}$.

Proof. Thanks to Lemmas 1 and 2 we can apply Theorem 3 in the Appendix with modulus of continuity $\omega(s) = C_2|s|$. Hence there is a subsequence u_{ϵ_j} that converges uniformly in $\overline{\Omega}$ to a limit with the same modulus of continuity. ☐

4. The continuous value of the game is a viscosity solution to the mixed problem

As we have already mentioned in the Introduction, it is shown in [18] that the continuous value of the game u is infinity harmonic within Ω and, in the case that $\Gamma_D = \partial\Omega$, it satisfies a Dirichlet boundary condition $u = 0$ on $\partial\Omega$.

In this paper, we are concerned with the case in which $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_N \neq \emptyset$. Our aim in the present section is to prove that u satisfies an homogeneous Neumann boundary condition on Γ_N, namely

$$\frac{\partial u}{\partial n}(x) = g(x) \qquad \text{on } \Gamma_N.$$

For completeness we will include here the full proof of the fact that u is a solution to

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = g(x) & \text{on } \Gamma_N, \\ u(x) = 0 & \text{on } \Gamma_D, \end{cases} \tag{4.1}$$

in the viscosity sense. As stated in the introduction the precise definition of the 1−homogeneous infinity Laplacian needs some care. We will consider

$$\Delta_\infty u(x) = \begin{cases} \Big\langle D^2u(x)\dfrac{\nabla u(x)}{|\nabla u(x)|}, \dfrac{\nabla u(x)}{|\nabla u(x)|}\Big\rangle, & \text{if } \nabla u(x) \neq 0, \\ \lim\limits_{y\to x} \dfrac{2\big(u(y)-u(x)\big)}{|y-x|^2}, & \text{otherwise.} \end{cases} \tag{4.2}$$

In defining $\Delta_\infty u$ we have followed [18]. Let us point out that it is possible to define the infinity laplacian at points with zero gradient in an alternative way, as in [13]. However, it is easy to see that both definitions are equivalent.

To motivate the above definition, the starting point is that $\Delta_\infty u$ is the second derivative of u in the direction of the gradient, as it is said in the introduction. In points where $\nabla u(x) = 0$, no direction is preferred to take second derivatives, and then expression (4.2) arises from the second-order Taylor's expansion of u at the point x,

$$\frac{2(u(y)-u(x))}{|y-x|^2} = \left\langle D^2u(x)\frac{y-x}{|y-x|}, \frac{y-x}{|y-x|}\right\rangle + o(1).$$

We say that, at these points, $\Delta_\infty u(x)$ is defined if $D^2u(x)$ is the same in every direction, that is, if the limit $\dfrac{(u(y)-u(x))}{|y-x|^2}$ exists as $y \to x$.

Because of the singular nature of (4.2) in points where $\nabla u(x) = 0$, we have to restrict our class of test functions. We will denote

$$S(x) = \left\{\phi \in \mathcal{C}^2 \text{ near } x \text{ for which } \Delta_\infty \phi(x) \text{ has been defined}\right\},$$

this is, $\phi \in S(x)$ if $\phi \in \mathcal{C}^2$ in a neighborhood of x and either $\nabla\phi(x) \neq 0$ or $\nabla\phi(x) = 0$ and the limit

$$\lim_{y\to x} \frac{2\big(\phi(y)-\phi(x)\big)}{|y-x|^2},$$

exists.

Now, using the above discussion of the infinity laplacian, we give the precise definition of viscosity solution to (4.1) following [5].

Definition 2. Consider the boundary value problem (4.1). Then,

(1) A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in S(x_0)$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \Gamma_N$, the inequality

$$\max\left\{\langle n(x_0), \nabla\phi(x_0)\rangle - g(x_0),\ -\Delta_\infty\phi(x_0)\right\} \geq 0$$

holds, if $x_0 \in \Omega$ then we require

$$-\Delta_\infty\phi(x_0) \geq 0,$$

with $\Delta_\infty\phi(x_0)$ given by (4.2), and if $x_0 \in \Gamma_D$ then $\phi(x_0) \geq 0$.

(2) An upper semi-continuous function u is a subsolution if for every $\phi \in S(x_0)$ such that $u-\phi$ has a strict maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \Gamma_N$, the inequality

$$\min\left\{\langle n(x_0), \nabla\phi(x_0)\rangle - g(x_0),\ -\Delta_\infty\phi(x_0)\right\} \leq 0$$

holds, if $x_0 \in \Omega$ then we require

$$-\Delta_\infty\phi(x_0) \leq 0,$$

with $\Delta_\infty\phi(x_0)$ given by (4.2), and if $x_0 \in \Gamma_D$ then $\psi(x_0) \leq 0$.

(3) Finally, u is a viscosity solution if it is both a super- and a subsolution.

Let us recall the result that we have at this point: the sequence u_ϵ contains a uniformly convergent subsequence (that by simplicity we will denote by u_ϵ) with limit a Lipschitz function u. To end the proof of Theorem 1 we have to identify the problem verified by u.

End of the proof of Theorem 1. The starting point is the Dynamic Programming Principle, which is satisfied by the value of the $\epsilon-$game:

$$2u_\epsilon(x) = \sup_{y\in \overline{B_\epsilon(x)}\cap\bar\Omega} u_\epsilon(y) + \inf_{y\in \overline{B_\epsilon(x)}\cap\bar\Omega} u_\epsilon(y) + \epsilon g(x)\chi_{\omega_\epsilon}(x) + 2\epsilon^3\chi_{\Omega\setminus\omega_\epsilon}(x) \tag{4.3}$$

for every $x \in \bar\Omega \setminus \Gamma_D$, where $B_\epsilon(x)$ denotes the open ball of radius ϵ centered at x and ω_ϵ is the small strip near Γ_N given by

$$\omega_\epsilon = \{x \in \Omega \,:\, d(x,\Gamma_N) \le \epsilon\}.$$

Let us check that u (a uniform limit of the subsequence that we still denote by u_ϵ) is a viscosity supersolution to (4.1). To this end, consider a function $\phi \in S(x_0)$ such that $u-\phi$ has a strict local minimum at x_0, this is,

$$u(x) - \phi(x) > u(x_0) - \phi(x_0), \quad x \ne x_0.$$

Without loss of generality, we can suppose that $\phi(x_0) = u(x_0)$. Let us see the inequality that these test functions satisfy, as a consequence of the Dynamic Programming Principle. For a sequence $\epsilon \to 0$ we can choose a sequence of positive numbers $\eta = \eta(\epsilon) = o(\epsilon^2)$. Then, by the uniform convergence of u_ϵ to u, there exist a sequence $x_\epsilon \to x_0$ such that

$$u_\epsilon(x) - \phi(x) \ge u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \eta(\epsilon), \tag{4.4}$$

for every x in a fixed neighborhood of x_0. From (4.4), we deduce

$$\sup_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} u_\epsilon(y) \ge \max_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) + u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \eta(\epsilon)$$

and

$$\inf_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} u_\epsilon(y) \ge \min_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) + u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \eta(\epsilon).$$

Then, we have from (4.3)

$$\begin{aligned} 2\phi(x_\epsilon) \ge & \max_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) + \min_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) \\ & + \epsilon g(x_\epsilon)\chi_{\omega_\epsilon}(x_\epsilon) + 2\epsilon^3\chi_{\Omega\setminus\omega_\epsilon}(x_\epsilon) - 2\eta(\epsilon). \end{aligned} \tag{4.5}$$

The above expression can be read as an aproximated *Dynamic Programming Principle in the viscosity sense.*

It is clear that the uniform limit of u_ϵ, u, verifies

$$u(x) = 0, \qquad x \in \Gamma_D.$$

Hence

$$\phi(x_0) = 0, \qquad \text{if } x_0 \in \Gamma_D.$$

In $\overline{\Omega} \setminus \Gamma_D$ there are two possibilities: $x_0 \in \Omega$ and $x_0 \in \Gamma_N$. In the former case we have to check that

$$-\Delta_\infty \phi(x_0) \ge 0, \tag{4.6}$$

while in the latter, what we have to prove is

$$\max\left\{\frac{\partial\phi}{\partial n}(x_0)-g(x_0),-\Delta_\infty\phi(x_0)\right\}\geq 0. \tag{4.7}$$

CASE A First, assume that $x_0\in\Omega$.

In this case, we just observe that for ϵ small the points x_0 and x_ϵ belong to $\Omega\setminus\omega_\epsilon$. In addition we can assume that $B_\epsilon(x_\epsilon)\cap\omega_\epsilon=\emptyset$ (considering ϵ smaller if necessary).

Now we use ideas from [8].

If $\nabla\phi(x_0)\neq 0$ we proceed as follows. Since $\nabla\phi(x_0)\neq 0$ we also have $\nabla\phi(x_\epsilon)\neq 0$ for ϵ small enough.

In the sequel, $x_1^\epsilon, x_2^\epsilon\in\bar\Omega$ will be the points such that

$$\phi(x_1^\epsilon)=\max_{y\in\overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega}\phi(y) \qquad\text{and}\quad \phi(x_2^\epsilon)=\min_{y\in\overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega}\phi(y).$$

We remark that $x_1^\epsilon, x_2^\epsilon\in\partial B_\epsilon(x_\epsilon)$. Suppose to the contrary that there exists a subsequence $x_1^{\epsilon_j}\in B_{\epsilon_j}(x_{\epsilon_j})$ of maximum points of ϕ. Then, $\nabla\phi(x_1^{\epsilon_j})=0$ and, since $x_1^{\epsilon_j}\to x_0$ as $\epsilon_j\to 0$, we have by continuity that $\nabla\phi(x_0)=0$, a contradiction. The argument for x_2^ϵ is similar.

Hence, since $\overline{B_\epsilon(x_\epsilon)}\cap\partial\Omega=\emptyset$, we have

$$x_1^\epsilon=x_\epsilon+\epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|}+o(1)\right] \quad\text{and}\quad x_2^\epsilon=x_\epsilon-\epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|}+o(1)\right] \tag{4.8}$$

as $\epsilon\to 0$. This can be deduced from the fact that, for ϵ small enough ϕ is approximately the same as its tangent plane.

In fact, if we write $x_1^\epsilon=x_\epsilon+\epsilon v^\epsilon$ with $|v^\epsilon|=1$, and we fix any direction w, then the Taylor expansion of ϕ gives

$$\phi(x_\epsilon)+\langle\nabla\phi(x_\epsilon),\epsilon v^\epsilon\rangle+o(\epsilon)=\phi(x_1^\epsilon)\geq\phi(x_\epsilon+\epsilon w)$$

and hence

$$\langle\nabla\phi(x_\epsilon),v^\epsilon\rangle+o(1)\geq\frac{\phi(x_\epsilon+\epsilon w)-\phi(x_\epsilon)}{\epsilon}=\langle\nabla\phi(x_\epsilon),w\rangle+o(1)$$

for any direction w. This implies

$$v^\epsilon=\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|}+o(1).$$

Now, consider the Taylor expansion of second order of ϕ

$$\phi(y)=\phi(x_\epsilon)+\nabla\phi(x_\epsilon)\cdot(y-x_\epsilon)+\frac12\langle D^2\phi(x_\epsilon)(y-x_\epsilon),(y-x_\epsilon)\rangle+o(|y-x_\epsilon|^2)$$

as $|y-x_\epsilon|\to 0$. Evaluating the above expansion at the point at which ϕ attains its minimum in $\overline{B_\epsilon(x_\epsilon)}$, x_2^ϵ, we get

$$\phi(x_2^\epsilon)=\phi(x_\epsilon)+\nabla\phi(x_\epsilon)(x_2^\epsilon-x_\epsilon)+\frac12\langle D^2\phi(x_\epsilon)(x_2^\epsilon-x_\epsilon),(x_2^\epsilon-x_\epsilon)\rangle+o(\epsilon^2),$$

as $\epsilon\to 0$.

Evaluating at its symmetric point in the ball $\overline{B_\epsilon(x_\epsilon)}$, that is given by

$$\tilde x_2^\epsilon=2x_\epsilon-x_2^\epsilon \tag{4.9}$$

we get

$$\phi(\tilde{x}_2^\epsilon) = \phi(x_\epsilon) - \nabla\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon) + \frac{1}{2}\langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2).$$

Adding both expressions we obtain

$$\phi(\tilde{x}_2^\epsilon) + \phi(x_2^\epsilon) - 2\phi(x_\epsilon) = \langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2).$$

We observe that, by our choice of x_2^ϵ as the point where the minimum is attained, using (4.5), and estimating $\phi(\tilde{x}_2^\epsilon)$ by $\max_{y\in\overline{B_\epsilon(x)}\cap\bar{\Omega}} \phi(y)$,

$$\begin{aligned}\phi(\tilde{x}_2^\epsilon) &+ \phi(x_2^\epsilon) - 2\phi(x_\epsilon) - 2\epsilon^3 \\ &\le \max_{y\in\overline{B_\epsilon(x)}\cap\bar{\Omega}} \phi(y) + \min_{y\in\overline{B_\epsilon(x)}\cap\bar{\Omega}} \phi(y) - 2\phi(x_\epsilon) + 2\epsilon^3 \le 2\eta(\epsilon).\end{aligned}$$

Therefore, since $\eta(\epsilon) = o(\epsilon^2)$,

$$0 \ge \langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2).$$

Note that from (4.8) we get

$$\lim_{\epsilon\to 0} \frac{x_2^\epsilon - x_\epsilon}{\epsilon} = -\frac{\nabla\phi}{|\nabla\phi|}(x_0).$$

Then we get, dividing by ϵ^2 and passing to the limit,

$$0 \le -\Delta_\infty\phi(x_0).$$

Now, if $\nabla\phi(x_0) = 0$ we can argue exactly as above and moreover, we can suppose (considering a subsequence) that

$$\frac{(x_2^\epsilon - x_\epsilon)}{\epsilon} \to v_2 \qquad \text{as } \epsilon \to 0,$$

for some $v_2 \in \mathbb{R}^n$. Thus

$$0 \le -\left\langle D^2\phi(x_0)v_2, v_2\right\rangle = -\Delta_\infty\phi(x_0)$$

by definition, since $\phi \in S(x_0)$.

CASE B Suppose that $x_0 \in \Gamma_N$. There are four sub-cases to be considered depending on the direction of the gradient $\nabla\phi(x_0)$ and the distance of the points x_ϵ to the boundary.

CASE 1: If either $\frac{\partial\phi}{\partial n}(x_0) \ge g(x_0)$, then

$$\frac{\partial\phi}{\partial n}(x_0) \ge g(x_0) \quad \Rightarrow \quad \max\left\{\frac{\partial\phi}{\partial n}(x_0), -\Delta_\infty\phi(x_0)\right\} \ge 0, \tag{4.10}$$

where

$$\Delta_\infty\phi(x_0) = \lim_{y\to x_0} \frac{2\big(\phi(y) - \phi(x_0)\big)}{|y - x_0|^2}$$

is well defined since $\phi \in S(x_0)$.

Therefore, we conclude that we can always assume in the sequel (cases 2,3,4) that

$$\frac{\partial\phi}{\partial n}(x_0) < g(x_0)$$

and hence we get

(4.11)
$$
\begin{aligned}
2\phi(x_\epsilon) &\geq \max_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) + \min_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) + \epsilon g(x_\epsilon)\chi_{\omega_\epsilon}(x_\epsilon) - 2\eta(\epsilon) \\
&\geq \max_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) + \min_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) + \epsilon \frac{\partial\phi}{\partial n}(x_\epsilon)\chi_{\omega_\epsilon}(x_\epsilon) - 2\eta(\epsilon).
\end{aligned}
$$

CASE 2: $\liminf_{\epsilon\to 0} \dfrac{\operatorname{dist}(x_\epsilon,\partial\Omega)}{\epsilon} > 1$, and $\nabla\phi(x_0)\neq 0$.

Since $\nabla\phi(x_0)\neq 0$ we also have $\nabla\phi(x_\epsilon)\neq 0$ for ϵ small enough. Hence, since $B_\epsilon(x_\epsilon)\cap\partial\Omega=\emptyset$, we have, as before,

$$
x_1^\epsilon = x_\epsilon + \epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|} + o(1)\right], \quad \text{and} \quad x_2^\epsilon = x_\epsilon - \epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|} + o(1)\right]
$$

as $\epsilon\to 0$. Notice that both $x_1^\epsilon, x_2^\epsilon \to \partial B_\epsilon(x_\epsilon)$. This can be deduced from the fact that, for ϵ small enough ϕ is approximately the same as its tangent plane.

Then we can argue exactly as before (when $x_0\in\Omega$) to obtain that

$$
0 \leq -\Delta_\infty \phi(x_0).
$$

CASE 3: $\limsup_{\epsilon\to 0} \dfrac{\operatorname{dist}(x_\epsilon,\partial\Omega)}{\epsilon} \leq 1$, and $\nabla\phi(x_0)\neq 0$ points outwards Ω.

Now we observe that if the normal derivative of ϕ is positive then the minimum in the ball of radius ϵ is contained in Ω since it is located in the direction of $-\nabla\phi$ and hence we can argue as before. In fact, we have,

$$
x_2^\epsilon = x_\epsilon - \epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|} + o(1)\right].
$$

Now, consider $\tilde x_2^\epsilon = 2x_\epsilon - x_2^\epsilon$ the symmetric point of x_2^ϵ with respect to x_ϵ. We go back to (4.5) and use the Taylor expansions of second order,

$$
\phi(x_2^\epsilon) = \phi(x_\epsilon) + \nabla\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon) + \frac12\langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon),(x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2),
$$

and

$$
\phi(\tilde x_2^\epsilon) = \phi(x_\epsilon) + \nabla\phi(x_\epsilon)(\tilde x_2^\epsilon - x_\epsilon) + \frac12\langle D^2\phi(x_\epsilon)(\tilde x_2^\epsilon - x_\epsilon),(\tilde x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2),
$$

to get

$$
\begin{aligned}
2\eta(\epsilon) &\geq \min_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) + \max_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) - 2\phi(x_\epsilon) \\
&\geq \phi(x_2^\epsilon) + \phi(\tilde x_2^\epsilon) - 2\phi(x_\epsilon) \\
&= \nabla\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon) + \nabla\phi(x_\epsilon)(\tilde x_2^\epsilon - x_\epsilon) \\
&\quad + \frac12\langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon),(x_2^\epsilon - x_\epsilon)\rangle \\
&\quad + \frac12\langle D^2\phi(x_\epsilon)(\tilde x_2^\epsilon - x_\epsilon),(\tilde x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2), \\
&= \langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon),(x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2),
\end{aligned}
$$

by the definition of $\tilde x_2^\epsilon$. Then, we can divide by ϵ^2 and use (4.13) to obtain

$$
-\Delta_\infty\phi(x_0) \geq 0.
$$

CASE 4: $\limsup_{\epsilon\to 0} \dfrac{\mathrm{dist}(x_\epsilon, \partial\Omega)}{\epsilon} \le 1$, and $\nabla\phi(x_0) \neq 0$ points inwards Ω.

In this case we have,

$$\frac{\partial\phi}{\partial n}(x_\epsilon) < 0.$$

and moreover, since $g \ge 0$ (note that the sign condition on g also appears here),

(4.12)

$$\begin{aligned} 2\phi(x_\epsilon) \ge & \max_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) + \min_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) + \epsilon g(x_\epsilon)\chi_{\omega_\epsilon}(x_\epsilon) - 2\eta(\epsilon) \\ \ge & \max_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) + \min_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) - 2\eta(\epsilon). \end{aligned}$$

In this case, for ϵ small enough we have that $\nabla\phi(x_\epsilon) \neq 0$ points inwards as well. Thus,

$$x_1^\epsilon = x_\epsilon + \epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|} + o(1)\right] \in \Omega,$$

while $x_2 \in \overline{\Omega} \cap \overline{B_\epsilon}(x_\epsilon)$. Indeed,

$$\frac{|x_2^\epsilon - x_\epsilon|}{\epsilon} = \delta_\epsilon \le 1.$$

We have the following first-order Taylor's expansions,

$$\phi(x_1^\epsilon) = \phi(x_\epsilon) + \epsilon|\nabla\phi(x_\epsilon)| + o(\epsilon),$$

and

$$\phi(x_2^\epsilon) = \phi(x_\epsilon) + \nabla\phi(x_\epsilon)\cdot(x_2^\epsilon - x_\epsilon) + o(\epsilon),$$

as $\epsilon \to 0$. Adding both expressions, we arrive at

$$\phi(x_1^\epsilon) + \phi(x_2^\epsilon) - 2\phi(x_\epsilon) = \epsilon|\nabla\phi(x_\epsilon)| + \nabla\phi(x_\epsilon)\cdot(x_2^\epsilon - x_\epsilon) + o(\epsilon).$$

Using (4.5), the fact that $\eta(\epsilon) = o(\epsilon^2)$ and dividing by $\epsilon > 0$,

$$0 \ge |\nabla\phi(x_\epsilon)| + \nabla\phi(x_\epsilon)\cdot\frac{(x_2^\epsilon - x_\epsilon)}{\epsilon} + o(1)$$

as $\epsilon \to 0$. We can write

$$0 \ge |\nabla\phi(x_\epsilon)|\cdot(1 + \delta_\epsilon\cos\theta_\epsilon) + o(1)$$

where

$$\theta_\epsilon = \mathrm{angle}\left(\nabla\phi(x_\epsilon), \frac{(x_2^\epsilon - x_\epsilon)}{\epsilon}\right).$$

For a subsequence $\epsilon_j \to 0$, we can assume that we can pass to the limit and we get

$$0 \ge |\nabla\phi(x_0)|\cdot(1 + \delta_0\cos\theta_0),$$

where $\delta_0 \le 1$, and

$$\theta_0 = \lim_{\epsilon_j\to 0}\theta_{\epsilon_j} = \mathrm{angle}\left(\nabla\phi(x_0), v(x_0)\right),$$

with

$$v(x_0) = \lim_{\epsilon_j\to 0}\frac{x_2^{\epsilon_j} - x_{\epsilon_j}}{\epsilon_j}.$$

Since $|\nabla\phi(x_0)| \neq 0$, we find out $(1+\delta_0\cos\theta_0) \leq 0$, and then $\theta_0 = \pi$ and $\delta_0 = 1$. Hence all the sequence converges and

$$\lim_{\epsilon\to 0} \frac{x_2^\epsilon - x_\epsilon}{\epsilon} = -\frac{\nabla\phi}{|\nabla\phi|}(x_0), \tag{4.13}$$

or what is equivalent,

$$x_2^\epsilon = x_\epsilon - \epsilon\left[\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|} + o(1)\right].$$

Now, consider $\tilde{x}_2^\epsilon = 2x_\epsilon - x_2^\epsilon$ the symmetric point of x_2^ϵ with respect to x_ϵ. We go back to (4.5) and use the Taylor expansions of second order,

$$\phi(x_2^\epsilon) = \phi(x_\epsilon) + \nabla\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon) + \frac{1}{2}\langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2),$$

and

$$\phi(\tilde{x}_2^\epsilon) = \phi(x_\epsilon) + \nabla\phi(x_\epsilon)(\tilde{x}_2^\epsilon - x_\epsilon) + \frac{1}{2}\langle D^2\phi(x_\epsilon)(\tilde{x}_2^\epsilon - x_\epsilon), (\tilde{x}_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2),$$

to get again

$$\begin{aligned} 2\eta(\epsilon) &\geq \min_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) + \max_{y\in \overline{B_\epsilon(x_\epsilon)}\cap\bar\Omega} \phi(y) - 2\phi(x_\epsilon) \\ &\geq \phi(x_2^\epsilon) + \phi(\tilde{x}_2^\epsilon) - 2\phi(x_\epsilon) \\ &= \langle D^2\phi(x_\epsilon)(x_2^\epsilon - x_\epsilon), (x_2^\epsilon - x_\epsilon)\rangle + o(\epsilon^2), \end{aligned}$$

by the definition of $\tilde{x}_2^\epsilon$. Then, we can divide by ϵ^2 and use (4.13) to obtain

$$-\Delta_\infty\phi(x_0) \geq 0.$$

It remains to check that u is a viscosity subsolution of (4.1). This fact can be proved in an analogous way, taking some care in the choice of the points where we perform Taylor expansions. In fact, instead of taking (4.9) we have to choose

$$\tilde{x}_1^\epsilon = 2x_\epsilon - x_1^\epsilon,$$

that is, the reflection of the point where the maximum in the ball $\overline{B_\epsilon(x_\epsilon)}$ of the test function is attained.

This ends the proof. □

Remark 11. The previous argument fix an inaccuracy in the proof of Theorem 1 in [8] where it was omitted the error term $\eta(\epsilon)$. This omission was also pointed out in [2] where the authors develop a regularization technique which also could fill the gap.

5. Appendix

In this appendix we prove a compactness result, which is a variant of the classical Ascoli-Arzela's Lemma. Note that the involved functions need not be continuous, but they have a uniform modulus of continuity for points that are not too close.

Theorem 3. *Let $\{u_\epsilon\}$ be a sequence, $u_\epsilon : \overline{\Omega} \to R$, such that,*

(1) *u_ϵ is uniformly bounded, $|u_\epsilon(x)| \leq K$ for all $x \in \overline{\Omega}$.*

(2) *There exists a sequence $a_\epsilon \to 0$, and a uniform modulus of continuity $\omega(s)$ such that, if $|x-y| > a_\epsilon$, then $|u_\epsilon(x) - u_\epsilon(y)| \le \omega(|x-y|)$.*

Then, there exists a subsequence $\{u_{\epsilon_j}\}$ that converges uniformly in $\overline{\Omega}$ to a continuous function $u(x)$, that verifies $|u(x) - u(y)| \le \omega(|x-y|)$.

Proof. First, we find a candidate to be the uniform limit u. Let $X \subset \overline{\Omega}$ be a dense numerable set. By a diagonal procedure we can extract a subsequence of u_ϵ, $u_{\epsilon_j}(x)$, that converges for all $x \in X$. Let $u(x)$ denote this limit (note that at this point $u(x)$ is defined for $x \in X$.

Moreover, we have that, given $r, s \in X$, there exists an index j_0 such that, if $j > j_0$ then $|r-s| > a_{\epsilon_j}$, and therefore $|u_{\epsilon_j}(r) - u_{\epsilon_j}(s)| \le \omega(|r-s|)$.

Taking limits as $\epsilon \to 0$ we get

$$|u(r) - u(s)| \le \omega(|r-s|).$$

Hence, we can extend u to the whole $\overline{\Omega}$, continuously (keeping the same modulus of continuity ω).

Our next step is to prove that u_ϵ converges to u pointwise. Given $z \in \overline{\Omega}$, fix, $x \in X$ such that $|x-z| < \delta_0$, with $\omega(\delta_0) < \delta/3$. We have

$$|u_{\epsilon_j}(z) - u(z)| \le |u_{\epsilon_j}(z) - u_{\epsilon_j}(x)| + |u_{\epsilon_j}(x) - u(x)| + |u(x) - u(z)|.$$

For j large enough, $|x-z| > a_{\epsilon_j}$, and then the first term in the right hand side is less or equal to $\omega(|x-z|)$. The same estimate is valid for the third term and hence,

$$|u_{\epsilon_j}(z) - u(z)| \le 2\omega(|x-z|) + |u_{\epsilon_j}(x) - u(x)| \le 2\delta/3 + |u_{\epsilon_j}(x) - u(x)| \le \delta,$$

for j large enough.

Now, we prove uniform convergence. We argue by contradiction. Assume that there exists a constant $M > 0$ such that we can find two sequences $\{\epsilon_j\}$ and $\{z_j\} \subset \overline{\Omega}$ such that

$$M < |u_{\epsilon_j}(z_j) - u(z_j)|.$$

Since $\overline{\Omega}$ is compact, we can assume $z_j \to z$, and then we obtain,

$$M < |u_{\epsilon_j}(z_j) - u(z_j)| \le |u_{\epsilon_j}(z_j) - u_{\epsilon_j}(z)| + |u_{\epsilon_j}(z) - u(z)| + |u(z) - u(z_j)|.$$

The second and third terms go to zero as $j \to \infty$, hence to get a contradiction it is enough to estimate the first term.

Fix $s \in \overline{\Omega}$ such that $|s-z| < M/4$, but $|s-z| > a_{\epsilon_j}$ for $j > j_0$. Then we have,

$$\begin{aligned} |u_{\epsilon_j}(z_j) - u_{\epsilon_j}(z)| &\le |u_{\epsilon_j}(z_j) - u_{\epsilon_j}(s)| + |u_{\epsilon_j}(s) - u_{\epsilon_j}(z)| \\ &\le \omega(|z_j - s|) + \omega(|s-z|) < M/2, \end{aligned}$$

from where we obtain the desired contradiction. □

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