MASS TRANSPORT PROBLEMS FOR THE EUCLIDEAN DISTANCE OBTAINED AS LIMITS OF $p-$LAPLACIAN TYPE PROBLEMS WITH OBSTACLES

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Abstract. In this paper we analyze a mass transportation problem that consists in moving optimally (paying a transport cost given by the Euclidean distance) an amount of a commodity larger or equal than a fixed one to fulfil a demand also larger or equal than a fixed one, with the obligation of paying an extra cost of $-g_1(x)$ for extra production of one unit at location x and an extra cost of $g_2(y)$ for creating one unit of demand at y. The extra amounts of mass (commodity/demand) are unknowns of the problem. Our approach to this problem is by taking the limit as $p \to \infty$ to a double obstacle problem (with obstacles g_1, g_2) for the $p-$Laplacian. In fact, under a certain natural constraint on the extra costs (that is equivalent to impose that the total optimal cost is bounded) we prove that this limit gives the extra material and extra demand needed for optimality and a Kantorovich potential for the mass transport problem involved. We also show that this problem can be interpreted as an optimal mass transport problem in which one can make the transport directly (paying a cost given by the Euclidean distance) or may hire a courier that cost $g_2(y) - g_1(x)$ to pick up a unit of mass at y and deliver it to x. For this different interpretation we provide examples and a decomposition of the optimal transport plan that shows when we have to use the courier.

1. Introduction.

Our main goal in this paper is to show that the limit as $p \to \infty$ for the double obstacle problem for the $p-$Laplacian gives a complete answer to an optimal mass transport problem with the Euclidean distance.

Consider the following variational problem where a double obstacle is considered,

$$\inf_{\substack{u \in W^{1,p}(\Omega):\\ g_1 \le u \le g_2 \ \text{ in } \Omega}} \int_\Omega \frac{|\nabla u(x)|^p}{p}\, dx - \int_\Omega f(x) u(x)\, dx. \tag{1.1}$$

We show that, provided that the restriction set is not empty, the obstacle problem has a solution for every fixed $p > N$ and, in addition, under a natural Lipschitz-type constraint

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on the obstacles, we prove that there is a uniform limit (along subsequences) as $p \to \infty$, u_∞, that is a solution of the variational problem

$$\max_{\substack{w \in W^{1,\infty}(\Omega):\\ \|\nabla w\|_{L^\infty(\Omega)} \le 1,\\ g_1 \le w \le g_2 \ \text{ in } \Omega}} \int_\Omega w(x) f(x)\, dx \,. \tag{1.2}$$

As we have mentioned, our main aim is to relate this optimization problem with an adequate optimal transport problem.

Limits as $p \to \infty$ of similar type problems are related to optimal mass transport problems for the Euclidean distance. In fact, this relation was the key to the first complete proof of the existence of an optimal transport map for the classical Monge problem given by Evans and Gangbo in [7]. See also [9] (in fact this work contains and extends some of the results proved there), where a problem with import/export taxes was studied, and [10], where an optimal matching problem is analyzed. Note that the usual Euclidean distance is not a strictly convex cost. This makes this optimal mass transport different from the strictly convex cost case in which there is existence of a convex function (solution to a Monge Ampere type problem) whose gradient provides an optimal transport map, see [13]. For notation and general results on Mass Transport Theory we refer to [1, 3, 6, 7, 13] and [14].

We are going to show that the limit variational problem (1.2) is related with the following optimal transport problem (see the next section for a precise mathematical formulation):

An optimal mass transport problem with taxes. Assume that we have some production in a domain Ω encoded in f_+ and some consumption encoded in f_-. We have the right to enlarge our previous production f_+, overall the domain included the boundary, paying an extra cost given by $-g_1(x)$ for each extra unit that we can produce at x, and we can create new demand paying an extra cost given by $g_2(y)$ for each extra unit of demand created at y (for example, this can come from advertising). Our main goal is to move the whole production and satisfy the whole demand minimizing the total cost of the operation. To transport one unit of material from x to y we pay as transport cost the Euclidean distance $|x-y|$. We will prove that solutions to the $p-$Laplacian type problem associated with (1.1) give an approximation to the extra production/demand necessary in the process and to a Kantorovich potential for the corresponding transport problem.

Let us now introduce a different interpretation (for precise details see Section 3).

An optimal mass transport problem with courier. Assume that we want to transport an amount of material f_+ to a location f_- (now, for simplicity, we can suppose with $\int f_+ = \int f_-$ for a better understanding of the interpretation). To do this task we have to possibilities: we can use our own vehicle and pay the distance $|x-y|$ for every unit of mass that we move from x to y or we can hire a courier from some location z to w, this courier charges us with $g_2(z)$ for taking a unit of mass at z and $-g_1(w)$ for delivering it at w, in this case, to send one unit of mass from x to y, we pay $|x-z|$ (the transport cost from the position x to the location z where we hire the courier), plus $g_2(z)-g_1(w)$ (the cost of the courier), plus $|w-y|$ (the transport cost from the delivery place w to the final destination y). Our goal is to transport the total amount of f_+ and cover the total amount to f_- minimizing the total cost. This problem is equivalent to the previously described. Note that for locations z and w with $g_2(z)-g_1(w)$ small respect to the distance $|z-w|$ it will be convenient to use the courier instead of taking the mass ourselves. On the other hand, when $g_2(z)-g_1(w)$ is large it will be better to do the task by ourselves. Of course, the total amount of mass that is carried by the courier and the locations at which/from it is delivered are unknown relevant quantities. In particular we want to know when we can find an optimal solution to this problem without using the courier and when we need to use it. As we will see here the $p-$Laplacian approximation provides an approximation to the whole set of unknowns.

The organization of the paper is the following: In Section 2 we state and prove our main results concerning the limit in the obstacle problem for the $p-$Laplacian and the solution to the optimal mass transport problem with taxes; in Section 3 we deal with the courier interpretation of the limit and finally we gather in the Appendix some max-min duality arguments that are not needed when we perform the $p-$Laplacian approach but are related to the optimal mass transport problems studied here.

2. Main results.

Let f_+, f_- be two bounded non-negative functions in $\mathbb{R}^N$ with compact disjoint supports. We set $f := f_+ - f_-$. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with smooth boundary such that $\operatorname{supp}(f_\pm) \subset\subset \Omega$. Given $g_i \in C(\overline{\Omega})$, with $g_1 \leq g_2$ in $\overline{\Omega}$ and $N < p < +\infty$, we set

$$W^{1,p}_{g_1,g_2}(\Omega) = \{u \in W^{1,p}(\Omega) \ : \ g_1 \leq u \leq g_2 \ \text{ in } \Omega\}$$

and consider the functional

$$\Psi_p(u) := \int_\Omega \frac{|\nabla u(x)|^p}{p}\, dx - \int_\Omega f(x) u(x)\, dx.$$

Assuming that $W^{1,p}_{g_1,g_2}(\Omega) \neq \emptyset$, since $W^{1,p}_{g_1,g_2}(\Omega)$ is a closed convex subset of $W^{1,p}(\Omega)$ and the functional Ψ_p is convex, lower semi-continuous and coercive, the variational problem

$$\min_{u \in W^{1,p}_{g_1,g_2}(\Omega)} \Psi_p(u) \tag{2.1}$$

has a minimizer u_p in $W^{1,p}_{g_1,g_2}(\Omega)$.

Note that, in general, this minimizer is not unique. However, it can be unique in some cases, a simple example is the following: if $\min g_2 = \max g_1 = k$ and $f_+ = f_- = 0$ then $u_p \equiv k$ is the unique minimizer.

Also it is well known that u_p is a solution of the variational inequality

$$\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla(w-u_p) \geq \int_\Omega f(w-u_p) \qquad \forall\, w\in W^{1,p}_{g_1,g_2}(\Omega). \tag{2.2}$$

Let us assume that g_1, g_2 satisfy the following condition:

$$g_1(x) - g_2(y) \leq |x-y| \qquad \forall\, x,y\in\Omega. \tag{2.3}$$

We use this condition in the next result to obtain that $W^{1,p}_{g_1,g_2}(\Omega)\neq\emptyset$.

Theorem 2.1. *Assume that g_1, g_2 verifies (2.3). Then, the minimizer u_p of problem (2.1) exists and, there exists a sequence $p_i\to+\infty$ such that $u_{p_i}\to u_\infty$ uniformly as $i\to\infty$, being u_∞ a maximizer of the variational problem (1.2), that is,*

$$\int_\Omega u_\infty(x)f(x)\,dx = \max\left\{\int_\Omega w(x)f(x)\,dx\colon w\in W^{1,\infty}_{g_1,g_2}(\Omega),\ \|\nabla w\|_{L^\infty(\Omega)}\leq 1\right\}, \tag{2.4}$$

where $W^{1,\infty}_{g_1,g_2}(\Omega) = \{u\in W^{1,\infty}(\Omega) \ :\ g_1\leq u\leq g_2 \ \text{ in } \Omega\}$.

Proof. Let us see that there exists $w\in W^{1,\infty}_{g_1,g_2}(\Omega)$ with $\|\nabla w\|_{L^\infty(\Omega)}\leq 1$. In fact, let us consider

$$w(x) := \max_{y\in\overline{\Omega}}\left\{g_1(y) - |x-y|\right\}.$$

We have that $|w(x)-w(y)|\leq|x-y|$ and

$$w(x)\geq g_1(x) \qquad \text{for } x\in\Omega.$$

Moreover, as (2.3) holds, we have

$$g_1(y) - |x-y| \leq g_2(x) \qquad \forall\, x,y\in\overline{\Omega},$$

and hence we obtain

$$w(x) = \max_{y\in\overline{\Omega}}\left\{g_1(y)-|x-y|\right\}\leq g_2(x) \qquad \forall\, x\in\Omega.$$

Therefore, $w\in W^{1,\infty}_{g_1,g_2}(\Omega)$ with $\|\nabla w\|_{L^\infty(\Omega)}\leq 1$. Consequently, $W^{1,p}_{g_1,g_2}(\Omega)\neq\emptyset$ and a minimizer u_p exists. Moreover, for any functions $w\in W^{1,\infty}_{g_1,g_2}(\Omega)$ with $\|\nabla w\|_{L^\infty(\Omega)}\leq 1$, we have

$$-\int_\Omega fu_p \leq \frac{1}{p}\int_\Omega|\nabla u_p|^p - \int_\Omega fu_p \leq \frac{1}{p}\int_\Omega|\nabla w|^p - \int_\Omega fw \leq \frac{|\Omega|}{p} - \int_\Omega fw. \tag{2.5}$$

The following Morrey's inequality holds

$$\|u\|_{L^\infty(\Omega)}\leq C_1\|\nabla u\|_{L^p(\Omega)} \quad \text{for any } \ u\in W^{1,p}_0(\Omega),\ p>N, \tag{2.6}$$

holds with constant C_1 not depending on p (this follows, for example, from [11, Theorem 2.E]). Since the functions $(u_p - \max_{\partial\Omega} g_2)^+, (u_p - \min_{\partial\Omega} g_1)^- \in W_0^{1,p}(\Omega)$, applying inequality (2.6), we get

$$\|u_p^+\|_{L^\infty(\Omega)} \leq C_1 \|\nabla u_p\|_{L^p(\Omega)} + |\max_{\partial\Omega} g_2|,$$

and

$$\|u_p^-\|_{L^\infty(\Omega)} \leq C_1 \|\nabla u_p\|_{L^p(\Omega)} + |\min_{\partial\Omega} g_1|.$$

Hence, we have

$$\|u_p\|_{L^\infty(\Omega)} \leq 2C_1 \|\nabla u_p\|_{L^p(\Omega)} + \|g_1\|_{L^\infty(\overline{\Omega})} + \|g_2\|_{L^\infty(\overline{\Omega})}.$$

From where it follows that

$$\|u_p\|_{L^p(\Omega)} \leq C_2 \|\nabla u_p\|_{L^p(\Omega)} + C_3, \tag{2.7}$$

where the constants C_i are independent of p. Moreover, from (2.5), using Hölder's inequality and having in mind (2.7), we get

$$\frac{1}{p} \int_\Omega |\nabla u_p|^p \leq C_4 (\|u_p\|_{L^p(\Omega)} + 1) \leq C_5 (\|\nabla u_p\|_{L^p(\Omega)} + 1),$$

from where we get

$$\|\nabla u_p\|_{L^p(\Omega)}^{p-1} \leq p\, C_6 \qquad \forall p > N, \tag{2.8}$$

with C_i independent of p. From (2.7) and (2.8), we obtain that $\{u_p\}_{p>N}$ is bounded in $W^{1,p}(\Omega)$. We have in fact (see [4]) that,

$$|u_p(x) - u_p(y)| \leq C_7 |x - y|^{1 - \frac{N}{p}},$$

with C_7 not depending on p. Then, by the Morrey-Sobolev's embedding and Arzela-Ascoli compactness criterion we can extract a sequence $p_i \to +\infty$ such that

$$u_{p_i} \rightrightarrows u_\infty \quad \text{uniformly in } \overline{\Omega}.$$

Moreover, by (2.8), we obtain that

$$\|\nabla u_\infty\|_\infty \leq 1.$$

Finally, passing to the limit in (2.5), we get

$$\int_\Omega u_\infty(x) f(x)\, dx = \max \left\{ \int_\Omega w(x) f(x)\, dx \colon w \in W^{1,\infty}_{g_1,g_2}(\Omega),\ \|\nabla w\|_{L^\infty(\Omega)} \leq 1 \right\},$$

as we wanted to prove. □

2.1. An optimal mass transport problem with taxes.

Let

$$\mathcal{A}(f_+, f_-) := \left\{ \mu \in \mathcal{M}^+(\overline{\Omega} \times \overline{\Omega}) \ : \ \pi_1\#\mu \geq f_+ \text{ and } \pi_2\#\mu \geq f_- \right\}$$

be the set of transport plans between masses larger or equal than f_+ and than f_-. π_1 and π_2 are the first and second projection on $\mathbb{R}^N \times \mathbb{R}^N$. By commodity, and for simplicity in the notation, we will write in all the paper $f_\pm$ instead of $f_\pm \mathcal{L}^N$ when we use such identification.

The first mass transport problem described in the introduction can be stated as the following minimization problem:

$$\min_{\mu \in \mathcal{A}(f_+, f_-)} \mathcal{F}(\mu), \tag{2.9}$$

where

$$\mathcal{F}(\mu) = \int_{\overline{\Omega} \times \overline{\Omega}} |x - y| \, d\mu - \int_{\overline{\Omega}} g_1 d(\pi_1\#\mu - f_+) + \int_{\overline{\Omega}} g_2 d(\pi_2\#\mu - f_-).$$

Remark 2.2. Observe that this problem makes sense if we impose condition (2.3), otherwise the above minimum can be $-\infty$.

We have the following result, in which we give the relation between the limit variational problem (1.2) and the above mass transport problem.

Theorem 2.3. *If g_1 and g_2 satisfy* (2.3)*, then*

$$\max \left\{ \int_\Omega w(x) f(x) \, dx \colon w \in W^{1,\infty}_{g_1, g_2}(\Omega), \ \|\nabla w\|_{L^\infty(\Omega)} \leq 1 \right\} = \min_{\mu \in \mathcal{A}(f_+, f_-)} \mathcal{F}(\mu). \tag{2.10}$$

Before proving this result let us pay attention to the following remark.

Remark 2.4. Fix $\mu \in \mathcal{A}(f_+, f_-)$ a measure where the minimum in (2.10) is taken. If $\mu_i := \pi_i\#\mu$, $i = 1, 2$, by the Kantorovich-Rubinstein Theorem (see [13]), we have

$$\min \left\{ \int_{\overline{\Omega} \times \overline{\Omega}} |x - y| \, d\nu \ : \ \nu \in \Pi(\mu_1, \mu_2) \right\} = \max \left\{ \int_{\overline{\Omega}} u d(\mu_1 - \mu_2) \ : \ u \in K_1(\overline{\Omega}) \right\}, \tag{2.11}$$

where $\Pi(\mu_1, \mu_2)$ denotes the set of transport plans between μ_1 and μ_2, that is,

$$\Pi(\mu_1, \mu_2) := \left\{ \nu \in \mathcal{M}^+(\overline{\Omega} \times \overline{\Omega}) \ : \ \pi_1\#\nu = \mu_1 \text{ and } \pi_2\#\nu = \mu_2 \right\},$$

and $K_1(\overline{\Omega})$ is the set of 1-Lipschitz continuous functions in $\overline{\Omega}$.

Let us see that μ is an optimal transport plan for (2.11), that is, a minimizer for Problem (2.11). Indeed, if $\nu_\mu \in \Pi(\mu_1, \mu_2)$ is an optimal transport plan for (2.11), then, as $\mu \in \Pi(\mu_1, \mu_2)$,

$$\int_{\overline{\Omega} \times \overline{\Omega}} |x - y| d\nu_\mu \leq \int_{\overline{\Omega} \times \overline{\Omega}} |x - y| d\mu.$$

Now, since

$$\int_{\overline{\Omega}} g_1 d(\pi_1\#\nu_\mu - f^+) - \int_{\overline{\Omega}} g_2 d(\pi_2\#\nu_\mu - f_-) = \int_{\overline{\Omega}} g_1 d(\mu_1 - f_+) - \int_{\overline{\Omega}} g_2 d(\mu_2 - f_-),$$

we have

$$\int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\nu_\mu - \int_{\overline{\Omega}} g_1 d(\pi_1\#\nu_\mu - f_+) + \int_{\overline{\Omega}} g_2 d(\pi_2\#\nu_\mu - f_-)$$
$$\leq \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\mu - \int_{\overline{\Omega}} g_1 d(\mu_1 - f_+) + \int_{\overline{\Omega}} g_2 d(\mu_2 - f_-).$$

On the other hand, since $\nu_\mu \in \mathcal{A}(f_+, f_-)$,

$$\int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\mu - \int_{\overline{\Omega}} g_1 d(\mu_1 - f_+) + \int_{\overline{\Omega}} g_2 d(\mu_2 - f_-)$$
$$\leq \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\nu_\mu - \int_{\overline{\Omega}} g_1 d(\pi_1\#\nu_\mu - f_+) + \int_{\overline{\Omega}} g_2 d(\pi_2\#\nu_\mu - f_-).$$

Therefore, the above inequality is an equality and then

$$\int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\mu = \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\nu_\mu,$$

and consequently μ is an optimal transport plan for (2.11).

Let u^* be a Kantorovich potential in (2.11), then

$$\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu = \int_{\overline{\Omega}} u^* d(\mu_1 - \mu_2).$$

Hence, for u_∞ the maximizer in (2.4),

$$\int_{\Omega} u_\infty (f_+ - f_-) = \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu - \int_{\overline{\Omega}} g_1\, d(\mu_1 - f_+) + \int_{\overline{\Omega}} g_2\, d(\mu_2 - f_-)$$
$$= \int_{\overline{\Omega}} u^* d(\mu_1 - \mu_2) - \int_{\overline{\Omega}} g_1\, d(\mu_1 - f_+) + \int_{\overline{\Omega}} g_2\, d(\mu_2 - f_-),$$

and, then, since $u_\infty \geq g_1$ and $\mu_1 - f_+ \geq 0$, and $u_\infty \leq g_2$ and $\mu_2 - f_- \geq 0$, we have

$$\int_{\overline{\Omega}} u_\infty\, d(\mu_1 - \mu_2) = \int_{\overline{\Omega}} u_\infty\, d\left(f_+ + \mu_1 - f_+ - (f_- + \mu_2 - f_-)\right)$$
$$\geq \int_{\Omega} u_\infty (f_+ - f_-) + \int_{\overline{\Omega}} g_1\, d(\mu_1 - f_+) - \int_{\overline{\Omega}} g_2\, d(\mu_2 - f_-) = \int_{\overline{\Omega}} u^*\, d(\mu_1 - \mu_2),$$

that is, u_∞ is also a Kantorovich potential for (2.11). From this last expression we also deduce that

$$u_\infty = g_1 \quad (\mu_1 - f_+) - \text{a.e.}, \qquad \text{and} \qquad u_\infty = g_2 \quad (\mu_2 - f_-) - \text{a.e.} \tag{2.12}$$

We will prove Theorem 2.3 by using the following key result. The importance of this result not only resides on the above question but on the fact that, as we will interpret afterwards, it gives a direct approximation to the unknowns of the mass transport problem.

Theorem 2.5. *Assume that* g_1, g_2 *verifies*

$$g_1(x) - g_2(y) < |x-y| \quad \forall\, x, y \in \overline{\Omega}. \tag{2.13}$$

For $p > N$*, let* u_p *be minimizers to Problem* (2.1)*, and set* $\mathcal{X}_p := |Du_p|^{p-2} Du_p$*. Then,*

1. *There exist Radon measures* η_p *supported on* $\overline{\Omega}$ *such that*

$$\int_\Omega \mathcal{X}_p \cdot \nabla\varphi = \int_\Omega f\varphi + \int_{\overline{\Omega}} \varphi d\eta_p \qquad \forall \varphi \in W^{1,p}(\Omega). \tag{2.14}$$

2. *There exists a sequence* $p_i \to +\infty$*, converging uniformly to* u_∞*, stated in Theorem 2.1, such that*

$$\mathcal{X}_{p_i} \to \mathcal{X} \quad \text{weakly* in the sense of measures,}$$

with $-div(\mathcal{X}) = f$ *in the sense of distributions in* $\{x \in \Omega \ : \ g_1(x) < u_\infty(x) < g_2(x)\}$*, and*

$$\eta_{p_i} \to \mathcal{V} \quad \text{weakly* in the sense of measures,}$$

with

$$\int_\Omega \nabla\varphi \, d\mathcal{X} = \int_\Omega f\varphi + \int_{\overline{\Omega}} \varphi \, d\mathcal{V} \quad \forall\, \varphi \in W^{1,\infty}(\Omega). \tag{2.15}$$

3. u_∞ *is a Kantorovich potential for the classical transport problem for the measures* $f_+\mathcal{L}^N \llcorner \Omega + \mathcal{V}^+$ *and* $f_-\mathcal{L}^N \llcorner \Omega + \mathcal{V}^-$.

Proof. Since g_i and u_p are continuous functions and having in mind (2.13), we have the set, which depends on p, $O := \{x \in \Omega \ : \ g_1(x) < u_p(x) < g_2(x)\}$ is a nonempty open set. We claim that

$$-\mathrm{div}(\mathcal{X}_p) = f \quad \text{in the sense of distributions in } O. \tag{2.16}$$

To see this, fix any test function $\varphi \in C_0^\infty(O)$. Then if $|t|$ is sufficiently small, $w := u_p + t\varphi \in W^{1,\infty}_{g_1,g_2}(\Omega)$. Thus (2.2) implies

$$t \int_O |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla\varphi \geq t \int_O f\varphi.$$

This inequality is valid for all sufficiently small t, both positive and negative, and so in fact

$$\int_O |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla\varphi = \int_O f\varphi,$$

and consequently (2.16) holds.

By (2.16), the distribution η_p defined defined in $\mathbb{R}^N$ as

$$\langle \eta_p, \varphi \rangle := \int_\Omega \mathcal{X}_p \cdot \nabla \varphi - \int_\Omega f\varphi \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^N), \tag{2.17}$$

vanishes in O, consequently

$$\text{supp}(\eta_p) \subset \{x \in \overline{\Omega} : u_p(x) = g_1(x)\} \cup \{x \in \overline{\Omega} : u_p(x) = g_2(x)\}. \tag{2.18}$$

On the other hand, if φ is a positive smooth function whose support does not touch $\{x \in \overline{\Omega} : u_p(x) = g_2(x)\}$ (which is separated from $\{x \in \overline{\Omega} : u_p(x) = g_1(x)\}$ by the continuity of u_p and the strict inequality in (2.13)) then there exists $\delta > 0$ such that $u_p + t\varphi \in W^{1,p}_{g_1,g_2}(\Omega)$ for all $0 \le t < \delta$. Thus (2.2) implies

$$t \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \varphi \ge t \int_\Omega f\varphi,$$

and consequently

$$\langle \eta_p, \varphi \rangle \ge 0.$$

Similarly, if φ is a positive smooth function whose support does not touch $\{x \in \overline{\Omega} : u_p(x) = g_1(x)\}$,

$$\langle \eta_p, \varphi \rangle \le 0.$$

Fix two functions $\varphi_i \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that

$$\varphi_1(x) = \begin{cases} 1, & u_p(x) = g_1(x), \\ 0, & u_p(x) = g_2(x), \end{cases}$$

and

$$\varphi_2(x) = \begin{cases} 1, & u_p(x) = g_2(x), \\ 0, & u_p(x) = g_1(x). \end{cases}$$

By (2.18), we can write $\eta_p = T_1 + T_2$ with $\langle T_i, \varphi \rangle = \langle \eta_p, \varphi\varphi_i \rangle$. Now, the above arguments show that T_1 and $-T_2$ are nonnegative distributions and so Radon measures. Consequently, η_p is a Radon measure. Moreover,

$$\text{supp}((\eta_p)^+) \subset \{x \in \overline{\Omega} : u_p(x) = g_1(x)\}, \tag{2.19}$$

and

$$\text{supp}((\eta_p)^-) \subset \{x \in \overline{\Omega} : u_p(x) = g_2(x)\}. \tag{2.20}$$

In addition, by density ([4, Corollary 9.8]) and Rellich-Kondrachov's Theorem ([4, Corollary 9.16]), we obtain (2.14).

Using (2.13), there is $0 < L < 1$ such that

$$g_1(x) - g_2(y) < L|x - y| \quad \forall\, x, y \in \overline{\Omega}.$$

Therefore, if we define

$$w(x) := \inf_{y \in \overline{\Omega}} (g_2(y) + L|x - y|),$$

we have w is a L-Lipschitz function in $\overline{\Omega}$ satisfying

$$g_1(x) < w(x) \leq g_2(x) \quad \forall\, x \in \overline{\Omega}.$$

By (2.14), since $u_p - w \in W^{1,p}(\Omega)$, and having in mind (2.19) and (2.20), we get

$$\begin{aligned} \int_\Omega (u_p - w) f &= \int_\Omega \mathcal{X}_p \cdot \nabla (u_p - w) - \int_{\overline{\Omega}} (u_p - w) d\eta_p \\ &= \int_\Omega \mathcal{X}_p \cdot \nabla (u_p - w) - \int_{\{g_1 = u_p\}} (g_1 - w)\, d\eta_p^+ + \int_{\{g_2 = u_p\}} (g_2 - w)\, d\eta_p^-. \end{aligned}$$

Then, since $g_1 - w \leq -c$, with $c > 0$, and $g_2 - w \geq 0$, by Hölder's and Young inequalities, it follows that

$$\begin{aligned} \int_\Omega |\nabla u_p|^p + c \int_{\overline{\Omega}} d\,\eta_p^+ &\leq \int_\Omega (u_p - w) f + \int_\Omega \mathcal{X}_p \cdot \nabla w \\ &\leq C + \left(\int_\Omega |\nabla u_p|^p \right)^{\frac{1}{p'}} L |\Omega|^{\frac{1}{p'}} \leq C + \frac{L}{p'} \int_\Omega |\nabla u_p|^p + \frac{1}{p} |\Omega|. \end{aligned}$$

Hence,

$$\left(1 - \frac{L^{p'}}{p'} \right) \int_\Omega |\nabla u_p|^p + c \int_{\overline{\Omega}} d\eta_p^+ \leq C + \frac{1}{p} |\Omega|.$$

Therefore, since $0 < L < 1$ and $c > 0$, we obtain that there exist positive constants A_1, A_2, not depending on p, such that

$$\int_\Omega |\nabla u_p|^p \leq A_1, \qquad \forall\, p \geq N + 1, \tag{2.21}$$

and

$$\int_{\overline{\Omega}} d\,\eta_p^+ \leq A_2, \qquad \forall\, p \geq N + 1. \tag{2.22}$$

Moreover, working similarly, changing the function w by the function

$$\tilde{w}(x) = \sup_{y \in \overline{\Omega}} (g_1(y) - L|x - y|),$$

we get

$$\int_{\overline{\Omega}} d\,\eta_p^- \leq A_3, \qquad \forall\, p \geq N + 1, \tag{2.23}$$

with A_3 a constant not depending on p.

As consequence of (2.21), we have that

$$\text{the measures } \ \mathcal{X}_p \mathcal{L}^N \llcorner \Omega \ \text{ are equi-bounded in } \Omega, \tag{2.24}$$

and from (2.22) and (2.23), we have that

$$\text{the measures } \ \eta_p \ \text{ are equi-bounded in } \overline{\Omega}. \tag{2.25}$$

Moreover, by Theorem 2.1, there exists a sequence $p_i \to +\infty$ such that

$$u_{p_i} \rightrightarrows u_\infty \quad \text{uniformly in } \overline{\Omega}, \quad \text{with } \|\nabla u_\infty\|_\infty \le 1,$$

From (2.24) and(2.25), there exists a subsequence of p_i, denoted equal, such that

$$\mathcal{X}_{p_i} \rightharpoonup \mathcal{X} \quad \text{weakly}^* \text{ as measures in } \Omega, \tag{2.26}$$

and

$$\eta_p^{\pm} \rightharpoonup \mathcal{V}_{\pm} \quad \text{weakly}^* \text{ as measures on } \overline{\Omega}. \tag{2.27}$$

Let us write $\mathcal{V} = \mathcal{V}_+ - \mathcal{V}_-$. Since the sets $\{u_\infty = g_1\}$ and $\{u_\infty = g_2\}$ are separated, we have, $\mathcal{V}^{\pm} = \mathcal{V}_{\pm}$,

$$\operatorname{supp}(\mathcal{V}^+) \subset \{x \in \overline{\Omega} : u_\infty(x) = g_1(x)\} \tag{2.28}$$

and

$$\operatorname{supp}(\mathcal{V}^-) \subset \{x \in \overline{\Omega} : u_\infty(x) = g_2(x)\}. \tag{2.29}$$

From (2.16) , (2.14), (2.26) and (2.27), we obtain that $-\text{div}(\mathcal{X}) = f$ in the sense of distributions in $\{x \in \Omega : g_1(x) < u_\infty(x) < g_2(x)\}$, and (2.15).

Set $\varphi = u_\infty$ in (2.14) for $p = p_i$. Then taking limit as $i \to \infty$ and having in mind (2.27), we get

$$\lim_{i\to\infty} \int_\Omega \mathcal{X}_{p_i} \cdot \nabla u_\infty = \int_\Omega f u_\infty + \int_{\overline{\Omega}} u_\infty \, d\mathcal{V}. \tag{2.30}$$

Let $v_\epsilon \in C_0^\infty(\mathbb{R}^N)$ uniformly converging to u_∞ as $\epsilon \searrow 0$, with $\|\nabla v_\epsilon\|_\infty \le 1$. By (2.14), we have

$$\int_\Omega \mathcal{X}_{p_i} \cdot \nabla u_\infty = \int_\Omega f(u_\infty - v_\epsilon) + \int_{\overline{\Omega}} (u_\infty - v_\epsilon) \, d\eta_{p_i} + \int_\Omega \mathcal{X}_{p_i} \cdot \nabla v_\epsilon.$$

Then, by (2.26), (2.27) and (2.30), taking limit in the above equality as $i \to \infty$, we obtain

$$\int_\Omega f u_\infty + \int_{\overline{\Omega}} u_\infty \, d\mathcal{V} = \int_\Omega f(u_\infty - v_\epsilon) + \int_{\overline{\Omega}} (u_\infty - v_\epsilon) \, d\mathcal{V} + \int_\Omega \nabla v_\epsilon \, d\mathcal{X}. \tag{2.31}$$

Now we are going to show that, as $\epsilon \searrow 0$,

$$\nabla v_\epsilon \quad \text{converges in } L^2(|\mathcal{X}|) \text{ to the Radon-Nikodym derivative } \frac{\mathcal{X}}{|\mathcal{X}|}. \tag{2.32}$$

To do that we use the technique used in [1, Theorem 5.2]. We first notice that the functional $\Psi : [C(\overline{\Omega}, \mathbb{R}^N)]^* \to \mathbb{R}$ defined by

$$\Psi(\nu) := \int_\Omega \left| \frac{\nu}{|\nu|} - w \right|^2 d|\nu|$$

is lower semicontinuous with respect to the the weak convergence of measures for any $w \in C(\overline{\Omega}, \mathbb{R}^N)$. Next, we observe that

$$\lim_{\epsilon \to 0^+} \limsup_{i \to \infty} \int_\Omega \left| \frac{\mathcal{X}_{p_i}}{|\mathcal{X}_{p_i}|} - \nabla v_\epsilon \right|^2 d|\mathcal{X}_{p_i}| = 0, \tag{2.33}$$

where $v_\epsilon \in C_0^\infty(\mathbb{R}^N)$ are functions uniformly converging to u_∞ with $\|\nabla v_\epsilon\|_\infty \leq 1$. Indeed,

$$\begin{aligned} \int_\Omega \left| \frac{\mathcal{X}_{p_i}}{|\mathcal{X}_{p_i}|} - \nabla v_\epsilon \right|^2 d|\mathcal{X}_{p_i}| &\leq 2 \int_\Omega |\nabla u_{p_i}|^{p_i - 1} \left(1 - \frac{\nabla v_\epsilon \cdot \nabla u_{p_i}}{|\nabla u_{p_i}|} \right) \\ &\leq 2 \int_\Omega |\nabla u_{p_i}|^{p_i - 2} \left(|\nabla u_{p_i}|^2 - \nabla v_\epsilon \cdot \nabla u_{p_i} \right) + \omega_{p_i} \\ &= 2 \int_\Omega f(u_{p_i} - v_\epsilon) + \int_{\overline{\Omega}} (u_{p_i} - v_\epsilon) d\, \eta_{p_i} + \omega_{p_i}, \end{aligned}$$

where $\omega_{p_i} := \sup_{t \geq 0} t^{p_i - 1} - t^{p_i}$ tends to 0 as $i \to +\infty$. Then, having in mind (2.27) and the uniform convergence of u_{p_i} and v_ϵ to u_∞, we obtain (2.33). Now, from (2.33), taking into account the lower semicontinuity of Ψ, passing to the limit as $i \to +\infty$, we obtain

$$\lim_{\epsilon \to 0^+} \int_\Omega \left| \frac{\mathcal{X}}{|\mathcal{X}|} - \nabla v_\epsilon \right|^2 d|\mathcal{X}| = 0.$$

Consequently, (2.32) holds true.

Now, having in mind (2.32), if we take the limit in (2.31) as $\epsilon \searrow 0$, we get

$$\int_\Omega f u_\infty + \int_{\overline{\Omega}} u_\infty \, d\mathcal{V} = \lim_{\epsilon \downarrow 0} \int_\Omega \nabla v_\epsilon \cdot \frac{\mathcal{X}}{|\mathcal{X}|} \, d|\mathcal{X}| = \int_\Omega d|\mathcal{X}|. \tag{2.34}$$

Giving a function $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\|\nabla \varphi\|_\infty \leq 1$, by (2.15) and (2.34), we have

$$\int_\Omega u_\infty f + \int_{\overline{\Omega}} u_\infty d\mathcal{V} = \int_\Omega d|\mathcal{X}| \geq \int_\Omega \frac{\mathcal{X}}{|\mathcal{X}|} \cdot \nabla \varphi \, d|\mathcal{X}| = \int_\Omega \nabla \varphi \, d\mathcal{X} = \int_\Omega \varphi f + \int_{\overline{\Omega}} \varphi \, d\mathcal{V}.$$

Then, by approximation, given a Lipschitz continuous function w with $\|\nabla w\|_\infty \leq 1$, we obtain

$$\int_\Omega u_\infty f + \int_{\overline{\Omega}} u_\infty d\mathcal{V} \geq \int_\Omega w f + \int_{\overline{\Omega}} w d\mathcal{V}.$$

On the other hand, taking $\varphi = \chi_\Omega$ in (2.15), it follows that the total masses of the measures $f_+ \mathcal{L}^N \llcorner \Omega + \mathcal{V}^+$ and $f_- \mathcal{L}^N \llcorner \Omega + \mathcal{V}^-$ are the same. Therefore u_∞ is a Kantorovich potential for the classical transport problem associated to the measures $f_+ \mathcal{L}^N \llcorner \Omega + \mathcal{V}^+$ and $f_- \mathcal{L}^N \llcorner \Omega + \mathcal{V}^-$. □

To prove Theorem 2.3 we will also use the following result, which, in addition, will be relevant in the next section.

Theorem 2.6. *Given an optimal measure μ for Problem* (2.9)*, there exist non-negative Radon measures μ_i, $i = I, II, III, IV$, satisfying:*

1.
$$\begin{aligned}
&\mu = \mu_I + \mu_{II} + \mu_{III} + \mu_{IV},\\
&\pi_1\#(\mu_I + \mu_{II}) = f_+,\\
&\pi_1\#(\mu_{III} + \mu_{IV}) = \pi_1\#\mu - f_+,\\
&\pi_2\#(\mu_I + \mu_{III}) = f_-,\\
&\pi_2\#(\mu_{II} + \mu_{IV}) = \pi_2\#\mu - f_-,
\end{aligned}$$

2. *μ_{IV} is supported in the set $\{(x,y) : g_1(x) - g_2(y) = |x-y|\}$.*
3. *$\mu_I + \mu_{II} + \mu_{III}$ is also an optimal measure for Problem* (2.9)*.*
4. *$u_\infty(x) = g_1(x)$ for $\pi_1\#\mu_{III} -$ a.e. x and $u_\infty(x) = g_2(x)$ for $\pi_2\#\mu_{II} -$ a.e. x.*

Proof. To construct measures μ_i satisfying the above properties, we first disintegrate μ as

$$\int_{\overline{\Omega}\times\overline{\Omega}} \psi(x,y)d\mu(x,y) = \int_{\overline{\Omega}} \left(\int_{\overline{\Omega}} \psi(x,y)d\mu_x(y) \right) d\pi_1\#\mu(x), \quad \forall\, \psi \in C(\overline{\Omega}\times\overline{\Omega}),$$

being μ_x probability measures in $\overline{\Omega}$ such that $x \mapsto \mu_x$ is $\pi_1\#\mu$-measurable (see [5] or [2]).

Now, we define the non-negative Radon measure μ^a in $\overline{\Omega}\times\overline{\Omega}$ as:

$$\int_{\overline{\Omega}\times\overline{\Omega}} \psi(x,y)d\mu^a(x,y) := \int_{\overline{\Omega}} \left(\int_{\overline{\Omega}} \psi(x,y)d\mu_x(y) \right) df_+(x), \quad \forall\, \psi \in C(\overline{\Omega}\times\overline{\Omega}).$$

Since $\int d\mu^a_x(y) = 1$ for $\pi_1\#\mu^a$-a.e. x, and, so, for f_+-a.e. x, for any $\varphi \in C(\overline{\Omega})$, we have

$$\begin{aligned}
\int_{\overline{\Omega}} \varphi(x)d\pi_1\#\mu^a(x) &= \int_{\overline{\Omega}\times\overline{\Omega}} \varphi(\pi_1(x,y))d\mu^a(x,y) = \int_{\overline{\Omega}} \left(\int_{\overline{\Omega}} \varphi(x)d\mu^a_x(y) \right) df_+(x)\\
&= \int_{\overline{\Omega}} \varphi(x) \left(\int_{\overline{\Omega}} d\mu^a_x(y) \right) df_+(x) = \int_{\overline{\Omega}} \varphi(x)df_+(x).
\end{aligned}$$

Thus,

$$\pi_1\#\mu^a = f_+. \tag{2.35}$$

Let us also define $\mu^b := \mu - \mu^a$, which is also non-negative. Disintegrating the auxiliary measures μ^a and μ^b with respect the second projection we have, for all $\psi \in C(\overline{\Omega}\times\overline{\Omega})$,

$$\int_{\overline{\Omega}\times\overline{\Omega}} \psi(x,y)d\mu^a(x,y) = \int_{\overline{\Omega}} \left(\int_{\overline{\Omega}} \psi(x,y)d\mu^a_y(x) \right) d\pi_2\#\mu^a(y)$$

and

$$\int_{\overline{\Omega}\times\overline{\Omega}} \psi(x,y)d\mu^b(x,y) = \int_{\overline{\Omega}} \left(\int_{\overline{\Omega}} \psi(x,y)d\mu^b_y(x) \right) d\pi_2\#\mu^b(y).$$

Then, we define, for $\psi \in C(\overline{\Omega}\times\overline{\Omega})$, the non-negative Radon measures μ_i as:
(2.36)

$$\int_{\overline{\Omega}\times\overline{\Omega}} \psi(x,y)d\mu_I(x,y) := \int_{\overline{\Omega}} \left(\int_{\overline{\Omega}} \psi(x,y)d\mu_y^a(x)\right) d(f_- \wedge \pi_2\#\mu^a)(y),$$

$$\int_{\overline{\Omega}\times\overline{\Omega}} \psi(x,y)d\mu_{II}(x,y) := \int_{\overline{\Omega}} \left(\int_{\overline{\Omega}} \psi(x,y)d\mu_y^a(x)\right) d(\pi_2\#\mu^a - f_-)^+(y),$$

$$\int_{\overline{\Omega}\times\overline{\Omega}} \psi(x,y)d\mu_{III}(x,y) := \int_{\overline{\Omega}} \left(\int_{\overline{\Omega}} \psi(x,y)d\mu_y^b(x)\right) d(f_- - f_- \wedge \pi_2\#\mu^a)(y),$$

$$\int_{\overline{\Omega}\times\overline{\Omega}} \psi(x,y)d\mu_{IV}(x,y) := \int_{\overline{\Omega}} \left(\int_{\overline{\Omega}} \psi(x,y)d\mu_y^b(x)\right) d\big(\pi_2\#\mu^b - (f_- - f_- \wedge \pi_2\#\mu^a)\big)(y).$$

Since, $0 \le f_- - f_- \wedge \pi_2\#\mu^a \le \pi_2\#\mu^b$, we have that all the measures μ_i are non-negative Radon measures in $\overline{\Omega}$.

Since $\pi_2\#\mu^a = f_- \wedge \pi_2\#\mu^a + (\pi_2\#\mu^a - f_-)^+$, we have $\mu^a = \mu_I + \mu_{II}$ and consequently by (2.35), we get

$$\pi_1\#(\mu_I + \mu_{II}) = f_+.$$

On the other hand, $\mu^b = \mu_{III} + \mu_{IV}$, then

$$\pi_1\#(\mu_{III} + \mu_{IV}) = \pi_1\#\mu^b = \pi_1\#\mu - \pi_1\#\mu^a = \pi_1\mu - f_+.$$

Moreover, for any $\varphi \in C(\overline{\Omega})$, we have

$$\begin{aligned}
\int_{\overline{\Omega}} \varphi(y)d\pi_2\#(\mu_I + \mu_{III})(y) &= \int_{\overline{\Omega}\times\overline{\Omega}} \varphi(\pi_2(x,y))d(\mu_I + \mu_{III})(x,y) \\
&= \int_{\overline{\Omega}} \varphi(y) \left(\int_{\overline{\Omega}} d\mu_y^a(x)\right) d(f_- \wedge \pi_2\#\mu^a)(y) \\
&\quad + \int_{\overline{\Omega}} \varphi(y) \left(\int_{\overline{\Omega}} d\mu_y^b(x)\right) d(f_- - f_- \wedge \pi_2\#\mu^a)(y) \\
&= \int_{\overline{\Omega}} \varphi(y)df_-(y).
\end{aligned}$$

Hence,

$$\pi_2\#(\mu_I + \mu_{III}) = f_-.$$

Finally,

$$\pi_2\#(\mu_{II} + \mu_{IV}) = \pi_2\#(\mu - (\mu_I + \mu_{III})) = \pi_2\#\mu - f_-.$$

Therefore the whole contents of Item 1 are proved.

Having in mind the decomposition of μ obtained in the previous step, we can then write the total cost functional as

$$\mathcal{F}(\mu) = \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu_I(x,y) \tag{2.37}$$

$$+ \int_{\overline{\Omega}\times\overline{\Omega}} |x-z|\, d\mu_{II}(x,z) \tag{2.38}$$

$$+ \int_{\overline{\Omega}} g_2(z) d\pi_2\#\mu_{II}(z) \tag{2.39}$$

$$- \int_{\overline{\Omega}} g_1(w) d\pi_1\#\mu_{III}(w) \tag{2.40}$$

$$+ \int_{\overline{\Omega}\times\overline{\Omega}} |w-y|\, d\mu_{III}(w,y) \tag{2.41}$$

$$+ \int_{\overline{\Omega}\times\overline{\Omega}} |w-z|\, d\mu_{IV}(w,z) - \int_{\overline{\Omega}} g_1(w) d\pi_1\#\mu_{IV}(w) \quad + \int_{\overline{\Omega}} g_2(z) d\pi_1\#\mu_{IV}(z). \tag{2.42}$$

Now, it is clear that $\mu_I + \mu_{II} + \mu_{III} \in \mathcal{A}(f_+, f_-)$ and that

$$(2.42) = \int_{\overline{\Omega}\times\overline{\Omega}} (|x-y| - g_1(x) + g_2(y)) d\mu_{IV} \geq 0.$$

Therefore

$$\mathcal{F}(\mu) = \mathcal{F}(\mu_I + \mu_{II} + \mu_{III}) + (2.42) \geq \mathcal{F}(\mu_I + \mu_{II} + \mu_{III}),$$

and the assertions of Item 2 and 3 the proved.

Finally, as consequence of (2.12) and Item 1, we get Item 4. □

Remark 2.7. Observe that Item 1 in the above theorem is also true for any $\mu \in A(f_+, f_-)$, and that for any such μ, the transport cost $\mathcal{F}(\mu)$ is also given by (2.38)-(2.42).

Proof of Theorem 2.3. Given $\mu \in \mathcal{A}(f_+, f_-)$ and u_∞ as in Theorem 2.1,

$$\begin{aligned}
&\int_\Omega u_\infty(x)(f_+(x) - f_-(x))\, dx \\
&= \int_{\overline{\Omega}} u_\infty d\pi_1\#\mu - \int_{\overline{\Omega}} u_\infty d(\pi_1\#\mu - f^+) - \int_{\overline{\Omega}} u_\infty d\pi_2\#\mu + \int_{\overline{\Omega}} u_\infty d(\pi_2\#\mu - f^-) \\
&\leq \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu - \int_{\overline{\Omega}} g_1 d(\pi_1\#\mu - f^+) + \int_{\overline{\Omega}} g_2 d(\pi_2\#\mu - f^-) \\
&\leq \inf_{\mu\in\mathcal{A}(f_+,f_-)} \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu - \int_{\overline{\Omega}} g_1 d(\pi_1\#\mu - f^+) + \int_{\overline{\Omega}} g_2 d(\pi_2\#\mu - f^-) \right\}.
\end{aligned}$$

Then,

$$\max\left\{\int_\Omega w(x)f(x)\,dx\colon w\in W^{1,\infty}_{g_1,g_2}(\Omega),\ \|\nabla w\|_{L^\infty(\Omega)}\le 1\right\}\le \min_{\mu\in\mathcal{A}(f_+,f_-)}\mathcal{F}(\mu).$$

Let us prove now the reverse inequality. Let us first assume that we are under assumption (2.13). Take $\tilde f_1 := f_+\mathcal{L}^N\llcorner\Omega+\mathcal{V}^+$ and $\tilde f_2 := f_-\mathcal{L}^N\llcorner\Omega+\mathcal{V}^-$. We have that

$$\int_{\overline\Omega} u_\infty d(\tilde f_1-\tilde f_2)=\min_{\nu\in\Pi(\tilde f_1,\tilde f_2)}\int_{\overline\Omega\times\overline\Omega}|x-y|d\nu=\int_{\overline\Omega\times\overline\Omega}|x-y|d\nu_0,$$

for some $\nu_0\in\Pi(\tilde f_1,\tilde f_2)$. Since $\pi_1\#\nu_0=f_+\mathcal{L}^N\llcorner\Omega+\mathcal{V}^+$, $\pi_2\#\nu_0=f_-\mathcal{L}^N\llcorner\Omega+\mathcal{V}^-$, we have $\nu_0\in\mathcal{A}(f_+,f_-)$. Then, having in mind (2.28) and (2.29), we get

$$\begin{aligned}\int_\Omega u_\infty(f_+-f_-)&=\int_{\overline\Omega\times\overline\Omega}|x-y|d\nu_0-\int_{\overline\Omega}g_1\,d\mathcal{V}^+ +\int_{\overline\Omega}g_2\,d\mathcal{V}^-\\&=\int_{\overline\Omega\times\overline\Omega}|x-y|d\nu_0-\int_{\overline\Omega}g_1\,d(\pi_1\#\nu_0-f^+)+\int_{\overline\Omega}g_2\,d(\pi_2\#\nu_0-f^-),\end{aligned}$$

and consequently

$$\int_\Omega u_\infty(x)(f_+(x)-f_-(x))dx\ge\min_{\mu\in\mathcal{A}(f_+,f_-)}\mathcal{F}(\mu).$$

The result under condition (2.3) now follows by approximation. Indeed, let $g_{1,n},g_{2,n}\in C(\overline\Omega)$, satisfying

$$g_{1,n}(x)-g_{2,n}(y)<|x-y|\quad\forall\,x,y\in\overline\Omega,$$

and

$$g_{i,n}\rightrightarrows g_i\quad\text{uniformly on }\ \overline\Omega,\quad i=1,2.$$

By the previous argument, there exist $u_{\infty,n}\in W^{1,\infty}(\Omega)$, with $\|\nabla u_{\infty,n}\|_\infty\le 1$ and $g_{1,n}\le u_{\infty,n}\le g_{2,n}$ on Ω, and there exist measures $\mu_n\in\mathcal{A}(f_+,f_-)$ satisfying

$$\begin{aligned}&\int_\Omega u_{\infty,n}(x)(f_+(x)-f_-(x))\,dx\\&=\int_{\overline\Omega\times\overline\Omega}|x-y|\,d\mu_n-\int_{\overline\Omega}g_{1,n}d(\pi_1\#\mu_n-f^+)+\int_{\overline\Omega}g_{2,n}d(\pi_2\#\mu_n-f^-).\end{aligned}\tag{2.43}$$

By the Morrey-Sobolev's embedding and Arzela-Ascoli compactness criterion we can suppose that, for a subsequence if necessary,

$$u_{\infty,n}\rightrightarrows u_\infty\quad\text{uniformly in }\overline\Omega.$$

Moreover,

$$\|\nabla u_\infty\|_\infty\le 1,\quad\text{and}\quad g_1\le u_\infty\le g_2\ \text{ on }\Omega.$$

On the other hand, since $g_{1,n}(x) - g_{2,n}(y) < |x-y|$, by Theorem 2.6 $\mu_n = (\mu_n)_I + (\mu_n)_{II} + (\mu_n)_{III}$, and we have

$$\mu_n(\overline{\Omega} \times \overline{\Omega}) \leq ((\mu_n)_I + (\mu_n)_{II})(\overline{\Omega} \times \overline{\Omega}) + ((\mu_n)_I + (\mu_n)_{III})(\overline{\Omega} \times \overline{\Omega})$$
$$= \pi_1\#((\mu_n)_I + (\mu_n)_{II})(\overline{\Omega}) + \pi_2\#((\mu_n)_I + (\mu_n)_{III})(\overline{\Omega}) = \int_\Omega f_+ + \int_\Omega f_-,$$

for every $n \in \mathbb{N}$. Therefore, we can assume that $\mu_n \rightharpoonup \mu_0$ weakly* as measures in $\overline{\Omega} \times \overline{\Omega}$, with $\mu_0 \in \mathcal{A}(f_+, f_-)$. Then, passing to the limit in (2.43) we finish the proof. □

3. An optimal mass transport problem with courier.

Let us see that Theorem 2.6 is decomposing a minimizer of Problem (2.9) (which is not unique in general) in the sum of measures that describe the mass transport problem with courier stated in the introduction.

1. Since $\pi_1\#(\mu_I + \mu_{II}) = f_+$ and $\pi_2\#(\mu_I + \mu_{III}) = f_-$, μ_I is the part of the plan μ that takes a part of f_+ to f_- using our own vehicle; and for this part of the transport plan we are paying (2.37).

2. Since $\pi_1\#(\mu_I + \mu_{II}) = f_+$ and $\pi_2\#(\mu_{II} + \mu_{IV}) = \pi_2\#\mu - f_-$, μ_{II} is the part of the plan that takes (with our vehicle) what is left of f_+ to places where we will use the courier; the costs of this transport are represented in (2.38).

3. Now the courier takes $\pi_2\#\mu_{II}$, we pay g_2 for each unit of mass that we deliver, and leave it at $\pi_1\#\mu_{III}$ where we pick it up, paying $-g_1$ for each unit of mass; these costs are the sum of (2.39)+(2.40). Note that $\pi_2\#\mu_{II}$ and $\pi_1\#\mu_{III}$ have the same total mass if it is so for f_+ and f_-.

4. Since $\pi_1\#(\mu_{III} + \mu_{IV}) = \pi_1\#\mu - f_+$ and $\pi_2\#(\mu_I + \mu_{III}) = f_-$, μ_{III} is the measure that tells us how to transport what is in $\pi_1\#\mu_{III}$ to what is needed to complete f_-; the cost of this operation is given by (2.41).

5. Finally, μ_{IV} is a not a relevant part of the plan μ since (2.42)= 0. The fact that μ_{IV} is supported in the set $\{(x, y) : g_1(x) - g_2(y) = |x - y|\}$ does not exclude the possibility that part of the other measures that appear in the decomposition can be also supported there.

Remark 3.1. We want to remark that the variational approach to the problem gives an approximation to the extra masses involved in the process, which are unknowns of the problem, besides the approximation to the Kantorovich potential u_∞.

We want also observe that this potential u_∞ is a Kantorovich potential, not only for the transport of $\pi_1\#\mu$ to $\pi_2\#\mu$, but also for each transport detailed in the above description. Indeed, since $\pi_1\#\mu_I$, $\pi_1\#\mu_{II}$ and $\pi_2\#\mu_{III}$ are absolutely continuous with respect to the Lebesgue measure, by the Sudakov Theorem (see Theorem 6.2 in [1]), there exist

$t_1 : \mathrm{supp}(\pi_1\#\mu_I) \to \Omega$ an optimal transport map pushing $\pi_1\#\mu_I$ forward $\pi_2\#\mu_I$;

$t_2 : \mathrm{supp}(\pi_1\#\mu_{II}) \to \overline{\Omega}$ an optimal transport map pushing $\pi_1\#\mu_{II}$ forward $\pi_2\#\mu_{II}$; and

$t_3 : \mathrm{supp}(\pi_2\#\mu_{III}) \to \Omega$ an optimal map transport pushing $\pi_2\#\mu_{III}$ forward $\pi_1\#\mu_{III}$.

Consider now the measure μ^* defined, for $\varphi \in C(\overline{\Omega}\times\overline{\Omega})$, as

$$\int_{\overline{\Omega}\times\overline{\Omega}} \varphi(x,y)d\mu^*(x,y) = \int_{\overline{\Omega}} \varphi(x,t_1(x))d\pi_1\#\mu_I(x) + \int_{\overline{\Omega}} \varphi(x,t_2(x))d\pi_1\#\mu_{II}(x) + \int_{\overline{\Omega}} \varphi(t_3(y),y)d\pi_2\#\mu_{III}(y),$$

that we write formally as (we will use this formal notation afterwards)

$$\mu^*(x,y) = \pi_1\#\mu_I(x)\otimes\delta_{y=t_1(x)} + \pi_1\#\mu_{II}(x)\otimes\delta_{y=t_2(x)} + \pi_2\#\mu_{III}(y)\otimes\delta_{x=t_3(y)}.$$

Let us see that μ^* is an optimal transport plan of our problem with courier. In fact, $\mu^* \in \mathcal{A}(f_+,f_-)$ and, according to Theorem 2.6 and Remark 2.7, $\mu_I^*(x,y) = \pi_1\#\mu_I(x)\otimes\delta_{y=t_1(x)}$, $\mu_{II}^*(x,y) = \pi_1\#\mu_{II}(x)\otimes\delta_{y=t_2(x)}$, $\mu_{III}^*(x,y) = \pi_2\#\mu_{III}(y)\otimes\delta_{x=t_3(y)}$ and

$$\begin{aligned}
\mathcal{F}(\mu^*) &= \int_{\overline{\Omega}} |x-t_1(x)|d\pi_1\#\mu_I(x) + \int_{\overline{\Omega}} |x-t_2(x)|d\pi_1\#\mu_{II}(x) \\
&\quad + \int_{\overline{\Omega}} |y-t_3(y)|d\pi_2\#\mu_{III}(y) - \int_{\overline{\Omega}} g_1(x)d\pi_1\#\mu_{III} + \int_{\overline{\Omega}} g_2(x)d\pi_2\#\mu_{II} \\
&\le \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|d(\mu_I+\mu_{II}+\mu_{III}) - \int_{\overline{\Omega}} g_1(x)d\pi_1\#\mu_{III} + \int_{\overline{\Omega}} g_2(x)d\pi_2\#\mu_{II} \\
&= \mathcal{F}(\mu_I+\mu_{II}+\mu_{III}),
\end{aligned}$$

by the optimality of the transport maps t_i. Then, by Theorem 2.6, $\mathcal{F}(\mu^*) = \mathcal{F}(\mu_I + \mu_{II}+\mu_{III})$ which gives the optimality of μ^*. Therefore, by Theorems 2.1 and 2.3,

$$\int_\Omega u_\infty(x)(f_+(x)-f_-(x))dx = \mathcal{F}(\mu^*).$$

Now, from the above expression, using Item 4 of Theorem 2.6, we have that

$$\begin{aligned}
&\int_{\overline{\Omega}} u_\infty d(\pi_1\#\mu_I - \pi_2\#\mu_I) + \int_{\overline{\Omega}} u_\infty d(\pi_1\#\mu_{II} - \pi_2\#\mu_{II}) + \int_{\overline{\Omega}} u_\infty d(\pi_1\#\mu_{III} - \pi_2\#\mu_{III}) \\
&= \int_{\overline{\Omega}\times\overline{\Omega}} |x-t_1(x)|d\pi_1\#\mu_I(x) + \int_{\overline{\Omega}\times\overline{\Omega}} |x-t_2(x)|d\pi_1\#\mu_{II}(x) \\
&\quad + \int_{\overline{\Omega}\times\overline{\Omega}} |t_3(y)-y|d\pi_2\#\mu_{III}(y).
\end{aligned}$$

Taking into account that $|\nabla u_\infty| \le 1$, we obtain that u_∞ is a Kantorovich potential for the transport of $\pi_1\#\mu_I$ to $\pi_2\#\mu_I$, of $\pi_1\#\mu_{II}$ to $\pi_2\#\mu_{II}$, and of $\pi_1\#\mu_{III}$ to $\pi_2\#\mu_{III}$. Also it is easy to see that u_∞ is a Kantorovich potential for the transport of $\pi_1\#\mu_{IV}$ to $\pi_2\#\mu_{IV}$.

By Theorem 2.6, we know that $u_\infty(x) = g_1(x)$ for $\pi_1\#\mu_{III}-$ a.e. x and $u_\infty(x) = g_2(x)$ for $\pi_2\#\mu_{II}-$ a.e. x. Then, by the Dual Criteria for Optimality, we have

$$u_\infty(x) = u_\infty(t_1(x)) + |x - t_1(x)| \quad \text{for } a.e.\ x \in \operatorname{supp}(\pi_1\#\mu_I),$$
$$u_\infty(x) = g_2(t_2(x)) + |x - t_2(x)| \quad \text{for } a.e.\ x \in \operatorname{supp}(\pi_1\#\mu_{II}),$$
$$u_\infty(y) = g_1(t_3(y)) - |y - t_3(y)| \quad \text{for } a.e.\ y \in \operatorname{supp}(\pi_2\#\mu_{III}).$$

Moreover,

$$u_\infty(x) = \min_{z\in\overline{\Omega}}(g_2(z) + |x - z|) \quad \text{for } a.e.\ x \in \operatorname{supp}(\pi_1\#\mu_{II}),$$
$$u_\infty(y) = \max_{z\in\overline{\Omega}}(g_1(z) - |y - z|) \quad \text{for } a.e.\ y \in \operatorname{supp}(\pi_2\#\mu_{III}),$$

and therefore, u_∞ is only determined by g_1 and g_2 in such domains. Indeed, for each $x \in \operatorname{supp}(\pi_1\#\mu_{II})$ there exist $\hat{t}_2(x)$ such that

$$\min_{z\in\overline{\Omega}}(g_2(z) + |x - z|) = g_2(\hat{t}_2(x)) + |x - \hat{t}_2(x)| \le g_2(t_2(x)) + |x - t_2(x)|,$$

and $\hat{t}_2$ is Borel measurable. Also, for each $y \in \operatorname{supp}(\pi_2\#\mu_{III})$ there exists $\hat{t}_3(y)$ such that

$$\max_{z\in\overline{\Omega}}(g_1(z) - |y - z|) = g_1(\hat{t}_3(y)) - |y - \hat{t}_3(y)| \ge g_1(t_3(y)) - |y - t_3(y)|,$$

$\hat{t}_3$ Borel measurable. Now,

$$\hat{\mu}(x,y) = \pi_1\#\mu_I(x) \otimes \delta_{y=t_1(x)} + \pi_1\#\mu_{II}(x) \otimes \delta_{y=\hat{t}_2(x)} + \pi_2\#\mu_{III}(y) \otimes \delta_{x=\hat{t}_3(y)}$$

is also an optimal transport plan, and consequently, the above inequalities are equalities a.e.

Observe also that the supports of $\pi_1\#\mu$ and $\pi_2\#\mu$ could be not disjoint, even if we start (as we do) with the supports of f_+ and f_- disjoints.

Example 3.2. Consider $\Omega = (0,2)$, $f^+ = \chi_{(0,1)}$, $f^- = \chi_{(1,2)}$ and

$$g_1(x) = g_2(x) = \begin{cases} -x + \frac{1}{2}, & 0 < x < \frac{1}{2}, \\ x - \frac{1}{2}, & \frac{1}{2} < x < \frac{3}{2}, \\ -x + \frac{5}{2}, & \frac{3}{2} < x < 2. \end{cases}$$

Then, $u_\infty = g_1 = g_2$ and we have

$$\int_0^2 u_\infty(f^+ - f_-) = -\frac{1}{2}.$$

Now, let us describe different optimal measures for this problem.

1. Let $\mu(x,y) = \chi_{(0,1)}(x) \otimes \delta_{y=x} + \chi_{(1,2)}(y) \otimes \delta_{x=y}$. This is an optimal measure for which $\mu_I = 0$, $\mu_{II} = \chi_{(0,1)}(x) \otimes \delta_{y=x}$, $\mu_{III} = \chi_{(1,2)}(y) \otimes \delta_{x=y}$ and and $\mu_{IV} = 0$; so we are not moving any part of the masses, and the courier will do all the work.

Observe that, since $g_2(x)-g_1(y)<|x-y|$ for $x\in(0,1)$ and $y\in(1,2)$, taking a courier is always a better solution, but this can also be done in a different way:

2. $\mu(x,y)=\mathcal{X}_{(0,1)}(x)\otimes\delta_{y=\frac{1}{2}}+\mathcal{X}_{(1,2)}(y)\otimes\delta_{x=\frac{3}{2}}$ is an optimal measure for which $\mu_I=0$, but now $\mu_{II}=\mathcal{X}_{(0,1)}(x)\otimes\delta_{y=\frac{1}{2}}$ takes f_+ to $\delta_{\frac{1}{2}}$, leaving that mass there to the courier, the courier takes what we have at $x=\frac{1}{2}$ to $x=\frac{3}{2}$ and with $\mu_{III}=\mathcal{X}_{(1,2)}(y)\otimes\delta_{x=\frac{3}{2}}$ we distribute it to cover f_-. Here again $\mu_{IV}=0$.

3. $\mu(x,y)=\mathcal{X}_{(0,1)}(x)\otimes\delta_{y=\frac{1}{2}}+\mathcal{X}_{(1,2)}(y)\otimes\delta_{x=\frac{3}{2}}+\delta_{x=0}\otimes\delta_{y=\frac{1}{2}}+\delta_{x=\frac{3}{2}}\otimes\delta_{y=2}$ is also an optimal measure. It is the same *courier-plan* given above except that this measure has non-negative $\mu_{IV}=\delta_{x=0}\otimes\delta_{y=\frac{1}{2}}+\delta_{x=\frac{3}{2}}\otimes\delta_{y=2}$.

An small modification of the tax-functions g_i shows how a μ_I not null appears: take $g_{1,n}=g_1$ and $g_{2,n}=g_2+\frac{1}{n}$, then we have that

$$u_{n,\infty}=\begin{cases} -x+\frac{1}{2}, & 0<x<\frac{1}{2},\\ x-\frac{1}{2}, & \frac{1}{2}<x<1-\frac{1}{4n},\\ -x+\frac{3}{2}+\frac{1}{2n}, & 1-\frac{1}{4n}<x<1+\frac{1}{4n},\\ x-\frac{1}{2}+\frac{1}{n}, & 1+\frac{1}{4n}<x<\frac{3}{2},\\ -x+\frac{5}{2}+\frac{1}{n}, & \frac{3}{2}<x<2,\end{cases}$$

for which

$$\int_0^2 u_{n,\infty}(f^+-f_-)=-\frac{1}{2}+\frac{1}{n}-\frac{1}{8n^2}.$$

Now $\mu_n=\mathcal{X}_{(0,1-\frac{1}{4n})}(x)\otimes\delta_{y=x}+\mathcal{X}_{(1-\frac{1}{4n},1)}\otimes\delta_{y=x+\frac{1}{4n}}+\mathcal{X}_{(1+\frac{1}{4n},2)}(y)\otimes\delta_{x=y}$ is an optimal measure for the problem and $(\mu_n)_I=\mathcal{X}_{(1-\frac{1}{4n},1)}\otimes\delta_{y=x+\frac{1}{4n}}$. Observe that, as $n\to+\infty$, this measure μ_n weakly converges to the one described in the above point **1.**

Example 3.3. As a different example, with u_∞ not fixed a priori, let us consider $\Omega=(0,2)$, $f_+=\chi_{(0,1)}$, $f_-=\chi_{(1,2)}$ and the obstacles

$$g_1(x)=\begin{cases} 0, & 0\le x\le\frac{3}{2},\\ 2x-3, & \frac{3}{2}\le x\le 2,\end{cases}$$

and

$$g_2(y)=\begin{cases} -2y+1, & 0\le y\le\frac{1}{2},\\ y-\frac{1}{2}, & \frac{1}{2}\le y\le 2.\end{cases}$$

In this example we have that a Kantorovich potential is given by

$$u_\infty(x) = \begin{cases} -x+\frac{1}{2}, & 0\le x\le \frac{1}{2}, \\ x-\frac{1}{2}, & \frac{1}{2}\le x\le \frac{7}{8}, \\ -x+\frac{5}{4}, & \frac{7}{8}\le x\le \frac{9}{8}, \\ x-1, & \frac{9}{8}\le x\le 2, \end{cases}$$

and an optimal measure by

$$\mu = \chi_{(0,\frac{1}{2})}(x)\otimes\delta_{y=\frac{1}{2}} + \chi_{(\frac{1}{2},\frac{7}{8})}\otimes\delta_{y=\frac{1}{2}} + \chi_{(\frac{7}{8},1)}(x)\otimes\delta_{y=x+\frac{1}{8}} + \chi_{(\frac{9}{8},2)}(y)\otimes\delta_{x=2}.$$

We leave to the reader the description of the different transport plans that appear here.

If we modify g_1 by taking

$$g_1(x) = \begin{cases} 0, & 0\le x\le \frac{11}{6}, \\ 2x-\frac{11}{3}, & \frac{11}{6}\le x\le 2, \end{cases}$$

we obtain

$$u_\infty(x) = \begin{cases} -x+\frac{1}{2}, & 0\le x\le \frac{1}{2}, \\ x-\frac{1}{2}, & \frac{1}{2}\le x\le \frac{5}{6}, \\ -x+\frac{7}{6}, & \frac{5}{6}\le x\le \frac{7}{6}, \\ 0, & \frac{7}{6}\le x\le \frac{5}{3}, \\ x-\frac{5}{3}, & \frac{5}{3}\le x\le 2, \end{cases}$$

and

$$\mu = \chi_{(0,\frac{1}{2})}(x)\otimes\delta_{y=\frac{1}{2}} + \chi_{(\frac{1}{2},\frac{5}{6})}\otimes\delta_{y=\frac{1}{2}} + \chi_{(\frac{5}{6},1)}(x)\otimes\delta_{y=x+\frac{1}{6}}$$
$$+\chi_{(\frac{7}{6},\frac{5}{3})}(y)\otimes\delta_{x=y} + \chi_{(\frac{5}{3},2)}(y)\otimes\delta_{x=2}$$

as optimal measure.

3.1. **Some properties of the optimal transport.**

3.1.1. *Continuity of the total cost with respect to the obstacles.*

Theorem 3.4. *The total cost depends continuously on the obstacles. In fact, if $g_{1,n}$ and $g_{2,n}$ are two sequences that converge uniformly as $n\to\infty$ to g_1 and g_2 respectively, then the total cost with $g_{1,n}$ and $g_{2,n}$ as obstacles converge to the total cost with g_1 and g_2.*

Proof. Let $\mu_n\in\mathcal{A}(f_+,f_-)$ optimal measures for the functionals

$$\mathcal{F}_n(\mu) = \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,d\mu - \int_{\overline{\Omega}} g_{1,n} d(\pi_1\#\mu - f^+) + \int_{\overline{\Omega}} g_{2,n} d(\pi_2\#\mu - f^-),$$

and let $\mathcal{F}$ be the corresponding functional associated with g_1, g_2. By Theorem 2.6, if $\tilde{\mu}_n := (\mu_n)_I + (\mu_n)_{II} + (\mu_n)_{III}$, we have $\mathcal{F}_n(\mu_n) = \mathcal{F}_n(\tilde{\mu}_n)$. Moreover, as in the proof of Theorem 2.3, we obtain

$$\tilde{\mu}_n(\overline{\Omega} \times \overline{\Omega}) \leq \int_\Omega f_+ + \int_\Omega f_- \qquad \forall\, n \in \mathbb{N}.$$

Therefore, we can assume that $\tilde{\mu}_n \rightharpoonup \mu$ weakly as measures.

Then, given $\nu \in \mathcal{A}(f_+, f_-)$, since $\mathcal{F}_n(\tilde{\mu}_n) \leq \mathcal{F}_n(\nu)$, $\tilde{\mu}_n \rightharpoonup \mu$ weakly as measures and $g_{i,n}$ converges uniformly to g_i, $i = 1, 2$, we have

$$\mathcal{F}(\mu) = \lim_{n\to\infty} \mathcal{F}_n(\tilde{\mu}_n) \leq \lim_n \mathcal{F}_n(\nu) = \mathcal{F}(\nu).$$

Consequently, μ is an optimal measure for the functional $\mathcal{F}$ and

$$\mathcal{F}(\mu) = \lim_{n\to\infty} \mathcal{F}_n(\mu_n).$$

□

Remark 3.5. We can provide a more precise estimate. Let us call $\mathrm{c}(g_1, g_2)$ to the optimal total cost of the courier problem for the obstacles g_1 and g_2, then we have

$$|\mathrm{c}(g_1, g_2) - \mathrm{c}(g_{1,n}, g_{2,n})| \leq K\Big(\|g_1 - g_{1,n}\|_\infty + \|g_2 - g_{2,n}\|_\infty \Big) \tag{3.1}$$

with K a constant that only depends on $\|f_\pm\|_1$. The proof of such a bound runs as follows: for μ_n an optimal transport plan for $g_{i,n}$, with $(\mu_n)_{IV} = 0$, and μ an optimal transport plan for g_i, with $(\mu)_{IV} = 0$, we have that

$$\mathcal{F}(\mu) - \mathcal{F}_n(\mu) \leq \mathcal{F}(\mu) - \mathcal{F}_n(\mu_n) \leq \mathcal{F}(\mu_n) - \mathcal{F}_n(\mu_n). \tag{3.2}$$

Observe that $\mathrm{c}(g_1, g_2) - \mathrm{c}(g_{1,n}, g_{2,n}) = \mathcal{F}(\mu) - \mathcal{F}_n(\mu_n)$. Now,

$$\begin{aligned} \Big|\mathcal{F}(\mu_n) - \mathcal{F}_n(\mu_n)\Big| &= \left| -\int_{\overline{\Omega}} (g_1 - g_{1,n}) d(\pi_1 \# \mu_n - f_+) + \int_{\overline{\Omega}} (g_2 - g_{2,n}) d(\pi_2 \# \mu_n - f_-) \right| \\ &\leq \left(\int_\Omega f_+ + \int_\Omega f_- \right) (\|g_1 - g_{1,n}\|_\infty + \|g_2 - g_{2,n}\|_\infty), \end{aligned}$$

and similarly,

$$\Big|\mathcal{F}(\mu) - \mathcal{F}_n(\mu)\Big| \leq \left(\int_\Omega f_+ + \int_\Omega f_- \right) (\|g_1 - g_{1,n}\|_\infty + \|g_2 - g_{2,n}\|_\infty).$$

From this estimates combined with (3.2) we obtain (3.1).

Moreover, with the same ideas we can get the following estimate, depending also on the masses,

$$\begin{aligned} &\Big| \mathrm{c}(g_1, g_2, f_+, f_-) - \mathrm{c}(g_{1,n}, g_{2,n}, f_{+,n}, f_{-,n}) \Big| \\ &\qquad \leq K\Big(\|g_1 - g_{1,n}\|_\infty + \|g_2 - g_{2,n}\|_\infty + \|f_+ - f_{+,n}\|_1 + \|f_- - f_{-,n}\|_1 \Big), \end{aligned}$$

where K is a uniform bound for $\|g_{i,n}\|_\infty$, $\|f_{\pm,n}\|_1$, $\|g_i\|_\infty$ and $\|f_\pm\|_1$. We conclude the continuity of the total cost with respect to the obstacles (in uniform topology) and the involved masses (in L^1 topology).

3.1.2. *When do we need to add extra mass ?*

Let us begin with an example to determine in terms of a parameter that controls the size of the upper obstacle when extra masses appear or not.

Example 3.6. Let us consider $\Omega = (0,3)$, $f_+ = \chi_{[0,1]}$, $f_- = \chi_{[2,3]}$ and $g_2(x) = k$, $g_1(x) = 0$ as obstacles.

First, let us compute the total cost without obstacles. A Kantorovich potential is given by $u(x) = 3 - x$ and the total cost is 2 with transport map $T(x) = x + 2$.

Case 1: $\mathbf{k \geq 3}$. In this case we do not want to use the courier, since to move a unit of mass from x to y we will be charged by $g_2(x) - g_1(y) = k \geq 3$ (note that 3 is the maximum distance between points in $supp(f_+)$ and points in $supp(f_-)$). In this case we have that the total cost is still 2 (like in the unconstrained situation) and a Kantorovich potential is $u(x) = 3 - x$. Note that it is exactly for $k \geq 3$ that we have room for a Kantorovich potential without obstacles to belong to the set $\{u : g_1 \leq u \leq g_2\}$.

In this case we do not use any extra mass.

Case 2: $\mathbf{k \leq 1}$. In this case we will always use the courier, since the cost to move a unit of mass from x to y we will be charged by $g_2(x) - g_1(y) = k \leq 1$ (note that 1 is the minimum distance between $supp(f_+)$ and $supp(f_-)$). In this case a Kantorovich potential is given by

$$u(x) = \begin{cases} k, & x \in [0,1], \\ k(2-x), & x \in (1,2), \\ 0, & x \in [2,3]. \end{cases}$$

The total cost is then k.

Note that the same u is the solution obtained with the $p-$Laplacian approximation. This follows since $u_\infty = \lim_{p\to\infty} u_p$ maximizes $\int u(f_+ - f_-)$ and then $u_\infty(x) = k$ for $0 \leq x \leq 1$ and $u_\infty(x) = 0$ for $2 \leq x \leq 3$. Moreover u_p is a strait line in the interval $(1,2)$ and hence its uniform limit u_∞ is also a strait line there.

For u_p we have

$$\begin{aligned} \int_0^3 |u_p'(x)|^{p-2} u_p'(x) \varphi'(x) dx &= \int_1^2 k^{p-1} \varphi'(x) dx = k^{p-1}(\varphi(2) - \varphi(1)) \\ &= \int_0^3 \varphi(f_+ - f_-) + \int_0^3 \varphi d\eta_p^+ - \int_0^3 \varphi d\eta_p^-, \end{aligned}$$

where (using the notation of Theorem 2.5)

$$\eta_p^+ = f_-\mathcal{L}^N + k^{p-1}\delta_2 \geq 0$$

and

$$\eta_p^- = f_+\mathcal{L}^N + k^{p-1}\delta_1 \geq 0.$$

Also u_p is a minimizer of Ψ_p. In fact, we have,

$$\begin{aligned} \frac{1}{p}\int_0^3 |u_p'|^p - \frac{1}{p}\int_0^3 |\varphi'|^p &\leq \int_0^3 |u_p'|^{p-2}u_p'(u_p' - \varphi') \\ = \int_0^3 f(u_p - \varphi) &+ \int_0^3 (u_p - \varphi)d\eta_p \leq \int_0^3 f(u_p - \varphi) \end{aligned}$$

for every $g_1 \leq \varphi \leq g_2$, since for such a φ it holds that

$$\int_0^3 (u_p - \varphi)d\eta_p = \int_0^3 (g_1 - \varphi)d\eta_p^+ - \int_0^3 (g_2 - \varphi)d\eta_p^- \leq 0.$$

Now, for $k < 1$, we can pass to the limit in η_p^+ and η_p^- as $p \to \infty$ and obtain as limits $f_-\mathcal{L}^N$ and $f_+\mathcal{L}^N$ respectively.

Remark that in the case $0 < k < 1$ there are infinitely many Kantorovich potentials. In fact, for any $a \in (1, 2-k)$,

$$u(x) = \begin{cases} k, & 0 \leq x \leq a, \\ (k+a) - x, & a \leq x \leq k+a, \\ 0, & k+a \leq x \leq 3, \end{cases}$$

is also a Kantorovich potential.

In this case two extra masses appear: $\mathcal{V}^- = \chi_{[0,1]}$ and $\mathcal{V}^+ = \chi_{[2,3]}$.

Case 3: $3 > k > 1$. This is the intermediate case. For some points it is convenient to use the courier while for others it is more convenient to carry the mass by ourselves. The cost of using the courier to move a unit of mass from x to y is exactly $g_2(x) - g_1(y) = k$ hence we want to use it for points that are at a distance larger than k. We have the following Kantorovich potential

$$u(x) = \begin{cases} k, & 0 \leq x \leq \frac{3-k}{2}, \\ \frac{3+k}{2} - x, & \frac{3-k}{2} \leq x \leq \frac{3+k}{2}, \\ 0, & \frac{3+k}{2} \leq x \leq 3. \end{cases}$$

We have that

$$\int_0^3 u(f_+ - f_-) = \frac{3}{2}k - \frac{1}{4}k^2 - \frac{1}{4}.$$

In this case we use the courier to take the mass in $[0, \frac{3-k}{2}]$ to the mass in $[\frac{3+k}{2}, 3]$ and we pay $\frac{k(3-k)}{2}$ and we move with our vehicle the mass in $(\frac{3-k}{2}, 1)$ to the mass in $(2, \frac{3+k}{2})$ (note that both intervals have the same length), with cost $\frac{k^2-1}{4}$.

Note that also in this case this Kantorovich potential is the limit of the $p-$Laplacian approximation (this follows as in the previous example, using the fact that u_p is invariant under symmetry around $x = 3/2$). In addition, this is the unique possible Kantorovich potential.

In this case two extra masses appear: $\mathcal{V}^- = \chi_{[0, \frac{3-k}{2}]}$ and $\mathcal{V}^+ = \chi_{[\frac{3+k}{2}, 3]}$.

Let us now characterize, in general, when some extra mass appears. First, let us observe that when

$$\int_\Omega f_- \neq \int_\Omega f_+$$

then necessarily an extra mass is needed (since it is not possible to perform a usual mass transport without obstacles between f_+ and f_-). Therefore, let us assume that

$$\int_\Omega f_- = \int_\Omega f_+.$$

In this case, let us consider the total transport cost without obstacles, that is given by (we use the Kantorovich-Rubinstein Theorem here)

$$A = \max_{u \in W^{1,\infty}(\Omega)} \left\{ \int_\Omega w(x)(f_+(x) - f_-(x))\, dx \colon \ \|\nabla w\|_{L^\infty(\Omega)} \leq 1 \right\},$$

and compare it with the optimal cost with obstacles,

$$B_{g_1,g_2} = \max_{\{u \in W^{1,\infty}(\Omega) \ : \ g_1 \leq u \leq g_2 \quad \text{in } \Omega\}} \left\{ \int_\Omega w(x) f(x)\, dx \colon \ \|\nabla w\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

Note that we always have that

$$A \geq B_{g_1,g_2}.$$

Now we can state the following result.

Theorem 3.7. *Assume that $\int f_- = \int f_+$. No extra mass is needed if and only if*

$$A = B_{g_1,g_2},$$

or equivalently, if and only if there exists a Kantorovich potential for the usual transport of f_+ to f_- (without obstacles) that belongs to the set

$$\{u \in W^{1,\infty}(\Omega) \ : \ g_1 \leq u \leq g_2 \ \ \text{in } \Omega\}.$$

Proof. It is clear that if no extra mass appears then the optimal transport cost is the same as the one without obstacles. Viceversa, if the costs are the same then an optimal transport plan for the cost of transporting f_+ to f_- with obstacles is an optimal transport plan for the problem without obstacles (since we have equality of optimal costs) and for this plan no extra masses appear (the marginals of this plan are $f_\pm$). Finally, we just have to observe that there exists such a Kantorovich potential if and only if the two maxima A and B_{g_1,g_2} are equal. □

As a consequence, we get that no extra mass is needed when the distance between the obstacles, compared with the distance at which $f_\pm$ are located, is large.

Corollary 3.8. *Assume that* $\int f_- = \int f_+$. *If*

$$\inf_\Omega g_2 - \sup_\Omega g_1 \geq \max_{x\in supp(f_+);y\in supp(f_-)} |x-y|$$

no extra mass is needed.

Proof. This follows from the fact that a Kantorovich potential for the usual transport without obstacles verifies $|Du| \leq 1$ and therefore

$$|u(x) - u(y)| \leq |x-y|$$

for $x \in supp(f_+)$ and $y \in supp(f_-)$. Then

$$|u(x) - u(y)| \leq \inf_\Omega g_2 - \sup_\Omega g_1 \tag{3.3}$$

for $x \in supp(f_+)$ and $y \in supp(f_-)$. Now we observe that the maximum of u is located in $supp(f_+)$ and the minimum in $supp(f_-)$ (this can be deduced from the fact that this property holds for the $p-$Laplacian approximation problems without obstacles, that is, the maximum of u_p is located in $supp(f_+)$ and the minimum in $supp(f_-)$).

Take x_0 a point in $supp(f_+)$ where the maximum of u is attained and a point y_0 in $supp(f_-)$ where the minimum is attained. From (3.3), we have

$$\max_{\overline{\Omega}} u - \min_{\overline{\Omega}} u = u(x_0) - u(y_0) \leq \inf_\Omega g_2 - \sup_\Omega g_1$$

and hence there is room for a Kantorovich potential to belong to the set

$$\{u \in W^{1,\infty}(\Omega) \ : \ g_1 \leq u \leq g_2 \ \text{ in } \Omega\}.$$

Indeed, there is a constant l such that $u + l$ is between the obstacles. □

Conversely, some extra mass appears when the distance between the obstacles is small (again compared with the distance at which $f_\pm$ are located).

Corollary 3.9. *Assume that* $\int f_- = \int f_+$. *If*

$$g_2(x) - g_1(y) < |x-y|$$

for every $x \in supp(f_+)$ *and every* $y \in supp(f_-)$, *then some extra mass appears.*

Proof. It follows from the fact that a Kantorovich potential for the usual transport without obstacles verifies that

$$u(x) - u(y) = |x - y|$$

for any two points $x \in supp(f_+)$ and $y \in supp(f_-)$ joined by a transport ray. Hence our hypothesis excludes the possibility of u to belong to the set

$$\{u \in W^{1,\infty}(\Omega) \ : \ g_1 \le u \le g_2 \ \text{ in } \Omega\}.$$

In fact, assume that u belong to the set

$$\{u \in W^{1,\infty}(\Omega) \ : \ g_1 \le u \le g_2 \ \text{ in } \Omega\}$$

then for two points $x \in supp(f_+)$ and $y \in supp(f_-)$ that are joined by a transport ray we have

$$g_2(x) - g_1(y) \ge u(x) - u(y) = |x - y|$$

a contradiction with our hypothesis. □

3.2. **Effective obstacles.**

Let us define the *effective obstacles* as follows: first, for g_2 let us consider the set where we will never use the courier:

$$B_2 := \{x \in \overline{\Omega} \ : \ \exists z \in \overline{\Omega} \text{ with } g_2(x) - g_2(z) > |x - z|\}.$$

In fact, if $x \in B_2$ it is better to transport the mass to a z such that $g_2(x) - g_2(z) > |x - z|$ and then use the courier from z. Therefore, the effective obstacle of g_2 can be defined as

$$\tilde{g}_2(x) := \begin{cases} g_2(x) & x \in \overline{\Omega} \setminus B_2, \\ +\infty & x \in B_2. \end{cases}$$

Observe that

$$\tilde{g}_2(x) = \begin{cases} g_2(x) & \text{if } \ g_2(x) = \min\limits_{z \in \overline{\Omega}} \{g_2(z) + |x - z|\}, \\ +\infty & \text{else.} \end{cases}$$

Note that if we relax the strict inequality in the definition of the set B_2 and consider

$$\tilde{B}_2 := \{x \in \overline{\Omega} \ : \ \exists z \neq x \text{ with } g_2(x) - g_2(z) \ge |x - z|\}.$$

we do not need to use the courier from a point $x \in \tilde{B}_2$ to absorb mass since we are not paying more if we transport the mass to a z such that $g_2(x) - g_2(z) \ge |x - z|$ and then use the courier from z compared with the cost that we pay if we use the courier directly from x. At points in $\tilde{B}_2 \setminus B_2$ if we decide to use the courier from x we can also make the choice of transport to z and then use the courier from there with the same cost.

Analogously, for g_1 let us consider

$$B_1 := \{x \ : \ \exists z \neq x \text{ with } g_1(z) - g_1(x) > |x - z|\}.$$

By the same arguments used before we never use the courier in B_1 to create mass, so we have the effective obstacle

$$\tilde{g}_1(x) := \begin{cases} g_1(x) & x \in \overline{\Omega} \setminus B_1, \\ -\infty & x \in B_1. \end{cases}$$

Observe also that

$$\tilde{g}_1(x) := \begin{cases} g_1(x) & \text{if } \ g_1(x) = \max_{z \in \overline{\Omega}} \{g_1(z) - |x - z|\}, \\ -\infty & \text{else.} \end{cases}$$

These obstacles $\tilde{g}_1$ and $\tilde{g}_2$ are called effective obstacles since for the limit problem they define the same set of $1-$Lipschitz functions. In fact, we have

$$\{u \ : \ |Du| \le 1 \text{ and } g_1 \le u \le g_2\} = \{u \ : \ |Du| \le 1 \text{ and } \tilde{g}_1 \le u \le \tilde{g}_2\}.$$

Indeed, since $\tilde{g}_1 \le g_1$ and $g_2 \le \tilde{g}_2$ we get

$$\{u \ : \ |Du| \le 1 \text{ and } g_1 \le u \le g_2\} \subseteq \{u \ : \ |Du| \le 1 \text{ and } \tilde{g}_1 \le u \le \tilde{g}_2\}.$$

Now assume that there exists a function with $|Du| \le 1$ such that $\tilde{g}_1 \le u \le \tilde{g}_2$ but there is a point x_0 with $u(x_0) > g_2(x_0)$ (the case in which the lower obstacle restriction is not satisfied is analogous). As $\tilde{g}_2 = g_2$ in $\overline{\Omega} \setminus B_2$ we must have $x_0 \in B_2$, then there exists z such that

$$g_2(x_0) - g_2(z) > |x_0 - z|.$$

Therefore

$$g_2(x_0) > g_2(z) + |x_0 - z| \ge \min_{w \in \overline{\Omega}} \{g_2(w) + |x_0 - w|\} = g_2(w_\infty) + |x_0 - w_\infty|.$$

Therefore if $w_\infty \notin B_2$ we have $u(w_\infty) \le g_2(w_\infty)$ and then

$$u(x_0) - u(w_\infty) > g_2(x_0) - g_2(w_\infty) > |x_0 - w_\infty|,$$

a contradiction with $|Du| \le 1$. But if $w_\infty \in B_2$, there exists z_∞ such that

$$g_2(w_\infty) - g_2(z_\infty) > |w_\infty - z_\infty|$$

and then

$$g_2(w_\infty) + |x_0 - w_\infty| > g_2(z_\infty) + |w_\infty - z_\infty| + |x_0 - w_\infty| \ge g_2(z_\infty) + |x_0 - z_\infty|$$

a contradiction with the choice of w_∞ as the minimum in $\min_{w \in \overline{\Omega}} \{g_2(w) + |x_0 - w|\}$.

Example 3.10. Now, as an example, take $\Omega = [0, 1]$ and assume that we only have positive mass, that is, assume that $f_- = 0$. Take as the upper obstacle (the lower obstacle does not enters into play since we want to maximize $\int_0^1 u f_+$ so we want u as big as possible everywhere),

$$g_2(x) = kx.$$

Case 1. $k \le 1$. In this case we have

$$u(x) = kx = g_2(x)$$

everywhere in $[0,1]$ and we will use the courier from every point in $supp(f_+)$ in $(0,1)$. In this case the total cost with the potential is $\int_0^1 kxf_+(x)dx$ and an optimal measure is given by $\mu(x,y) = f_+(x) \otimes \delta_{y=x}$. The effective obstacle coincides with the obstacle and the effective set $[0,1] \setminus B_2$ is the whole $[0,1]$.

Case 2. $k > 1$**.** In this case we have

$$u(x) = x < g_2(x)$$

everywhere in $[0,1]$ and we will use the usual transport in our vehicle to send everything to $x = 0$ and then use the courier from the point $x = 0$. The effective obstacle coincides with the obstacle only at $x = 0$ and the effective set $[0,1] \setminus B_2$ is the set $\{0\}$. Now an optimal measure is given by $\mu(x,y) = f_+(x) \otimes \delta_{y=0}$.

This case is related to Example 3.13 given in the next section.

3.2.1. *Localizing the courier.*

In this subsection we consider the case in which we want to restrict the possibility of using the courier to a subset $K \subset \overline{\Omega}$.

Let K a compact subset of $\overline{\Omega}$ and let us consider $\hat{g}_1$, $\hat{g}_2$ the costs of using the courier there. As before, these costs have to satisfy the following compatibility condition:

$$\hat{g}_1(x) - \hat{g}_2(y) \leq |x-y| \qquad \forall\, x, y \in K.$$

Now, let us consider

$$g_1(x) = \max_{z \in K} (\hat{g}_1(z) - c|x-z|), \qquad x \in \overline{\Omega}, \tag{3.4}$$

and

$$g_2(y) = \min_{w \in K} (\hat{g}_2(w) + c|y-w|), \qquad y \in \overline{\Omega}, \tag{3.5}$$

with $c > 1$, and consider the transport problem with courier studied above for this pair g_1, g_2. One can see that the effective obstacles (the ones obtained in the previous subsection 3.2) are equal to $\hat{g}_1(x)$ for $x \in \overline{\Omega} \setminus B_1$, with $\overline{\Omega} \setminus B_1 \subset K$, and to $\hat{g}_2(x)$ for $x \in \overline{\Omega} \setminus B_2$, with $\overline{\Omega} \setminus B_2 \subset K$ (see details afterwards). Then, from our previous results, see Subsection 3.2, we have that there is a transport plan that does not use the courier in $\overline{\Omega} \setminus K$.

We can even localize the possible action of the courier in two different sets (one in which we are allowed to leave mass and a different one where we can pick up mass). Let us assume that, given two compact sets K_1 and K_2 (that can be disjoint or not), we have two functions $\hat{g}_1(x)$ and $\hat{g}_2(y)$ that verify the compatibility condition,

$$\hat{g}_1(x) - \hat{g}_2(y) \leq |x-y| \qquad \forall\, x \in K_1, \ \forall\, y \in K_2.$$

Then we consider

$$g_1(x) = \max_{z \in K_1} (\hat{g}_1(z) - c|x - z|), \qquad x \in \overline{\Omega},$$

and

$$g_2(y) = \min_{w \in K_2} (\hat{g}_2(w) + c|y - w|), \qquad y \in \overline{\Omega},$$

with $c > 1$, as obstacles in the whole $\overline{\Omega}$. Note that these obstacles also verify

$$g_1(x) - g_2(y) \le |x - y|, \qquad x, y \in \overline{\Omega},$$

hence we may use our previous general results. But we have that the effective obstacles that corresponds to these extensions g_1, g_2, are the same as the effective obstacles of $\hat{g}_1$ and $\hat{g}_2$ inside K_1 and K_2 respectively. In this way we can deal with a transport problem in which the possibility of using the courier is restricted to two different sets.

Let us now give some details on how to prove the above observations. We can see, in fact, that the way we define g_2 in (3.5) (a similar discussion also holds for g_1 given in (3.4)), localizes *exactly* the effective domain for a transport involving only the obstacles in the compact K: Define the effective obstacle defined in K (relative to K) as

$$\bar{g}_2(x) := \begin{cases} \hat{g}_2(x) & \text{if } \ \hat{g}_2(x) = \min\limits_{z \in K} \{\hat{g}_2(z) + |x - z|\}, \\ +\infty & \text{else.} \end{cases}$$

Proposition 3.11. *Given g_2 defined in* (3.5)*, for the effective obstacle $\tilde{g}_2$ defined in $\overline{\Omega}$ we have that*

$$\tilde{g}_2(x) := \begin{cases} \hat{g}_2(x) & \text{if } x \in K \text{ and } \hat{g}_2(x) = \min\limits_{z \in K} \{\hat{g}_2(z) + |x - z|\}, \\ +\infty & \text{else.} \end{cases}$$

Proof. The thesis holds if we prove that

$$\Big\{ x \in \overline{\Omega} : g_2(x) = \min_{z \in \overline{\Omega}} \{g_2(z) + |x - z|\} \Big\} = \Big\{ x \in K : \hat{g}_2(x) = \min_{z \in K} \{\hat{g}_2(z) + |x - z|\} \Big\}$$

and in this set $g_2(x) = \hat{g}_2(x)$. So, let us begin by taking $x \in \overline{\Omega}$ such that

$$g_2(x) = \min_{z \in \overline{\Omega}} \{g_2(z) + |x - z|\}.$$

Then

$$\begin{aligned} g_2(x) &= \min_{z \in K} \{\hat{g}_2(z) + c|x - z|\} = \min_{z \in \overline{\Omega}} \{g_2(z) + |x - z|\} \\ &\le \min_{z \in K} \{g_2(z) + |x - z|\} \le \min_{z \in K} \{\hat{g}_2(z) + |x - z|\}. \end{aligned}$$

Therefore, for a minimizer $z_x \in K$ in the first minimum of the above expression we have,

$$g_2(x) = \hat{g}_2(z_x) + c|x - z_x| \le \min_{z \in K} \{\hat{g}_2(z) + |x - z|\} \le \hat{g}_2(z_x) + |x - z_x|.$$

This implies $z_x = x$ (since $c > 1$), and hence $x \in K$, $\hat{g}_2(x) = \min_{z \in K} \{\hat{g}_2(z) + |x - z|\}$ and $g_2(x) = \hat{g}_2(x)$.

Let us now take $x \in K$ such that $\hat{g}_2(x) = \min_{z\in K}\{\hat{g}_2(z) + |x-z|\}$. Then

$$\hat{g}_2(x) \leq g_2(x) \leq \hat{g}_2(x),$$

that is, $\hat{g}_2(x) = g_2(x)$. Suppose that $g_2(x) \neq \min_{z\in\overline{\Omega}}\{g_2(z) + |x-z|\}$, that is,

$$\min_{z\in\overline{\Omega}}\{g_2(z) + |x-z|\} < g_2(x),$$

then there exists $z_x \in \overline{\Omega}$ such that

$$g_2(z_x) + |x - z_x| < g_2(x) = \hat{g}_2(x).$$

Now, there exists $w_x \in K$ such that $g_2(z_x) = \hat{g}_2(w_x) + c|z_x - w_x|$, hence

$$\hat{g}_2(w_x) + |x-w_x| \leq g_2(w_x) + c|z_x - w_x| + |x - z_x| = g_2(z_x) + |x-z_x| < \hat{g}_2(x),$$

a contradiction with the choice of x. □

Remark 3.12. In particular, if we choose $K = \partial\Omega$ we are recovering and extending some of the results given in [9].

Example 3.13. If $K = \{x_0\}$ with $\hat{g}_2(x_0) = b_0$, we take a cone of the form

$$g_2(x) = c|x - x_0| + b$$

with $c > 1$ then the effective obstacle is just

$$\tilde{g}_2(x) = \begin{cases} b, & x = x_0, \\ +\infty & x \neq x_0. \end{cases}$$

That is, we can only create negative extra mass at the single point x_0 (with a cost given by b). Analogously, if $K = \{x_1, x_2, ..., x_N\}$ with $\hat{g}_2(x_i) = b_i$ we take

$$g_2(x) = \min_i\{c|x - x_i| + b_i\}$$

with $c > 1$, then we can obtain a transport plan that only creates negative extra mass at the points x_i (with costs given by b_i).

Assume now that $f_- = 0$ (we only have positive mass) and that the lower obstacle is $g_1 = -m$ with $m > 0$ large enough. In this case a Kantorovich potential is given by

$$u_\infty(x) = \min_i\{|x - x_i| + b_i\}.$$

This situation can be used to describe the use of *localized resources*. Assume that we only have positive mass (that is, $f_- = 0$) that we want to store at a finite number of prescribed locations, $x_1, ..., x_N$, with a cost given by b_i to store a unit of mass at x_i, then the amount of mass that each location receives (and from where it comes) is exactly the solution to our mass transport problem with obstacle $g_2(x) = \min_i\{c|x - x_i| + b_i\}$ and $g_1 = -m$, $m > 0$ large enough. In fact, the point x_j will receive all the mass of f_+ located in the set

$$S_j = \Big\{x \,:\, \min_i\{|x - x_i| + b_i\} = |x - x_j| + b_j\Big\}.$$

This fact is rather intuitive since if we want to store a unit of mass located at x we will choose the location x_j that minimizes the cost. Then the optimal measure for the transport problem is given by $\mu(x,y) = \sum_j \chi_{S_j}(x) f_+(x) \otimes \delta_{y=x_j}$.

4. Appendix

In this appendix we include a duality argument for our optimal mass transport problem. We remark that this argument was not needed in the previous sections, since the $p-$Laplacian approximation gives a complete approximation of all the relevant quantities.

Let $J : C(\overline{\Omega}) \times C(\overline{\Omega}) \to \mathbb{R}$ be defined by

$$J(\varphi,\psi) := \int_\Omega \varphi(x) f_+(x)\, dx + \int_\Omega \psi(y) f_-(y)\, dy,$$

and set

$$\mathcal{B}(g_1,g_2) := \left\{ \begin{array}{l} (\varphi,\psi) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : \varphi(x) + \psi(y) \le |x-y|, \\ \hfill g_1 \le \varphi, \ -g_2 \le \psi \text{ on } \overline{\Omega} \end{array} \right\}.$$

Given $(\varphi,\psi) \in \mathcal{B}(g_1,g_2)$ and $\mu \in \mathcal{A}(f_+,f_-)$ we have

$$\begin{aligned} J(\varphi,\psi) &= \int_\Omega \varphi(x) f_+(x)\, dx + \int_\Omega \psi(y) f_-(y)\, dy \\ &= \int_{\overline{\Omega}} \varphi(x) d\pi_1\#\mu - \int_{\overline{\Omega}} \varphi d(\pi_1\#\mu - f^+) + \int_{\overline{\Omega}} \psi(y) d\pi_2\#\mu - \int_{\overline{\Omega}} \psi d(\pi_2\#\mu - f^-) \\ &\le \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu - \int_{\overline{\Omega}} g_1 d(\pi_1\#\mu - f^+) + \int_{\overline{\Omega}} g_2 d(\pi_2\#\mu - f^-). \end{aligned}$$

Then,

$$\begin{aligned} &\sup\{J(\varphi,\psi) \ : \ (\varphi,\psi) \in \mathcal{B}(g_1,g_2)\} \\ &\le \inf_{\mu \in \mathcal{A}(f_+,f_-)} \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\mu - \int_{\overline{\Omega}} g_1 d(\pi_1\#\mu - f^+) + \int_{\overline{\Omega}} g_2 d(\pi_2\#\mu - f^-) \right\}. \end{aligned}$$

Moreover, if $w \in W^{1,\infty}(\Omega)$, with $g_1 \le w \le g_2$ in Ω and $\|\nabla w\|_{L^\infty(\Omega)} \le 1$, we have $(w,-w) \in \mathcal{B}(g_1,g_2)$, and hence

$$\begin{aligned} \max\left\{ \int_\Omega w(x) f(x)\, dx \colon \ w \in W^{1,\infty}_{g_1,g_2}(\Omega), \ \|\nabla w\|_{L^\infty(\Omega)} \le 1 \right\} \\ \le \sup\left\{ J(\varphi,\psi) : (\varphi,\psi) \in \mathcal{B}(g_1,g_2) \right\}. \end{aligned}$$

Therefore, by Theorem 2.3 we have that

(4.1)

$$\max_{\substack{\varphi,\psi\in C(\overline{\Omega}):\\ \varphi\ge g_1,\psi\ge -g_2,\\ \varphi(x)+\psi(y)\le |x-y|}} \int_\Omega \varphi(x)f_+(x)\,dx+\int_\Omega \psi(y)f_-(y)\,dy$$
$$=\min_{\substack{\mu\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}):\\ \pi_1\#\mu\ge f_+ \text{ and } \pi_2\#\mu\ge f_-}} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,d\mu-\int_{\overline{\Omega}} g_1 d(\pi_1\#\mu-f^+)+\int_{\overline{\Omega}} g_2 d(\pi_2\#\mu-f^-)\,,$$

Now, (4.1) can be written as

$$\max_{\substack{\varphi,\psi\in C(\overline{\Omega}):\\ \varphi\ge g_1,\psi\ge -g_2,\\ \varphi(x)+\psi(y)\le |x-y|}} \int_{\overline{\Omega}} (\varphi(x)-g_1(x))f_+(x)\,dx+\int_\Omega (\psi(y)+g_2(x))f_-(y)\,dy$$
$$=\min_{\substack{\mu\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}):\\ \pi_1\#\mu\ge f_+ \text{ and } \pi_2\#\mu\ge f_-}} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,d\mu-\int_{\overline{\Omega}} g_1 d\pi_1\#\mu+\int_{\overline{\Omega}} g_2 d\pi_2\#\mu\,,$$

that is,

$$\max_{\substack{\varphi,\psi\in C(\overline{\Omega}):\\ \varphi-g_1\ge 0,\psi+g_2\ge 0,\\ \varphi(x)-g_1(x)+\psi(y)+g_2(y)\\ \le |x-y|-g_1(x)+g_2(y)}} \int_\Omega (\varphi(x)-g_1(x))f_+(x)\,dx+\int_\Omega (\psi(y)+g_2(x))f_-(y)\,dy$$
$$=\min_{\substack{\mu\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}):\\ \pi_1\#\mu\ge f_+ \text{ and } \pi_2\#\mu\ge f_-}} \int_{\overline{\Omega}\times\overline{\Omega}} (|x-y|-g_1(x)+g_2(y))\,d\mu\,.$$

So, we can rewrite (4.1) as follows

$$\max_{\substack{\varphi,\psi\in C(\overline{\Omega}):\\ \varphi\ge 0,\psi\ge 0,\\ \varphi(x)+\psi(y)\le c(x,y)}} \int_\Omega \varphi(x)f_+(x)\,dx+\int_\Omega \psi(y)f_-(y)\,dy$$
$$=\min_{\substack{\mu\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}):\\ \pi_1\#\mu\ge f_+ \text{ and } \pi_2\#\mu\ge f_-}} \int_{\overline{\Omega}\times\overline{\Omega}} c(x,y)\,d\mu\,,$$

where $c(x,y)=|x-y|-g_1(x)+g_2(y)$.

We want to remark that the above min-max result can be proven using Fenchel-Rockafellar's Theorem under the condition $c(x,y)>0$, $\forall\, x,y\in\overline{\Omega}$. In fact, we can follow the same steps of the proof of [13, Theorem 1.3], the main change appearing in the definition of the functional Ξ, that we define as

$$\Xi : u \in C_b(\overline{\Omega} \times \overline{\Omega}) \mapsto \begin{cases} \displaystyle\int_{\overline{\Omega}} \varphi(x) f_+(x)\, dx + \int_{\overline{\Omega}} \psi(y) f_-(y)\, dy \\ \qquad\qquad \text{if } u(x,y) = \varphi(x) + \psi(y), \varphi \le 0, \psi \le 0, \\ 0 \quad \text{else,} \end{cases}$$

and for which, the calculus of the Legendre-Fenchel transform gives

$$\Xi^*(\pi) = \begin{cases} 0, & \text{if } \pi_1 \# \pi - \mu \ge 0 \text{ and } \pi_2 \# \pi - \nu \ge 0, \\ +\infty, & \text{in any other case,} \end{cases}$$

for $\pi \in \mathcal{M}(\overline{\Omega} \times \overline{\Omega})$.

Nevertheless, as we pointed out before, our approach using the $p-$Laplacian approximation gives also a Kantorovich potential for this problem, and, in addition, provides a method to approximate that potential and the extra masses needed in the mass transport process.

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