

# THE SUBLINEAR PROBLEM FOR THE 1-HOMOGENEOUS $p$ -LAPLACIAN

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ABSTRACT. In this paper we prove existence and uniqueness of a positive viscosity solution of the 1-homogeneous  $p$ -Laplacian with a sublinear right-hand side, that is,  $-|Du|^{2-p}\operatorname{div}(|Du|^{p-2}Du) = \lambda u^q$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ , where  $\Omega$  is a bounded starshaped domain,  $\lambda > 0$ ,  $p > 2$  and  $0 < q < 1$ .

## 1. INTRODUCTION

In this paper we analyze the sublinear problem corresponding to the 1-homogeneous  $p$ -Laplacian,

$$(1.1) \quad \begin{cases} -|Du|^{2-p}\operatorname{div}(|Du|^{p-2}Du) = \lambda u^q, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded starshaped domain,  $\lambda > 0$ ,  $p > 2$  and  $0 < q < 1$ .

Note that the operator  $-|Du|^{2-p}\operatorname{div}(|Du|^{p-2}Du)$  (also denoted by  $\Delta_p^N u$ ) is nonlinear, elliptic but not in divergence form and homogeneous of degree one. It is also called strong  $p$ -Laplacian in the literature [1, 2]. This operator appears naturally in relation with game theory when one considers Tug-of-War games (like the ones considered in [12]), see also [13, 14, 16, 17]. Formally, if we expand the divergence, we get

$$-|Du|^{2-p}\operatorname{div}(|Du|^{p-2}Du) = -|Du|^{2-p}\Delta_p u = \left( (p-2)\Delta_\infty u + \Delta u \right),$$

where  $\Delta_\infty u := |Du|^{-2}\langle D^2u Du, Du \rangle$  is the 1-homogeneous infinity Laplacian. This expansion is the key to understand what we will define as a viscosity solution for this problem, see Section 2.

Our main result reads as follows:

**Theorem 1.1.** *Let  $\Omega$  be a starshaped domain,  $p > 2$  and  $0 < q < 1$ , then there exists a unique positive viscosity solution of (1.1) for every  $\lambda > 0$ .*

A fundamental tool for our analysis is a comparison result for positive subsolutions and supersolutions of our problem. To clarify the argument

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(that follows adapting a result from [9]) we first prove a comparison principle for positive subsolutions and supersolutions of

$$(1.2) \quad \begin{cases} -|Du|^{2-p} \operatorname{div} (|Du|^{p-2} Du) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Existence of viscosity solutions of (1.2) has been proved using different techniques, including game-theoretic arguments and finite difference methods, see [15, 16]. Uniqueness of solutions of the Dirichlet problem (1.2) is known if the righthand side  $f \in C(\Omega) \cap L^\infty(\Omega)$  is either 0, or has constant sign, however, when these conditions are not fulfilled, interesting non-uniqueness phenomena can happen [15]. Regularity issues for this problem were analyzed in [6] where the authors prove  $L^\infty$ -bounds and estimates for the modulus of continuity of (1.2). Since they also study the limit as  $p \rightarrow \infty$ , they take special care to get uniform estimates in  $p$ .

In view of these previous results, it seems natural to look for problems like  $-|Du|^{2-p} \operatorname{div} (|Du|^{p-2} Du) = F(u)$ . Here, we deal with the sublinear case,  $F(u) = \lambda u^q$  with  $q < 1$ . For the usual Dirichlet Laplacian ( $p = 2$  in our problem) with a sublinear term,  $-\Delta u = \lambda u^q$ , it is known that there exists a unique positive solution for every  $\lambda > 0$  [4]. Although for our problem we obtain the same result, there are significant differences in the proofs between the linear case and our problem due to the non-variational nature of  $-|Du|^{2-p} \operatorname{div} (|Du|^{p-2} Du)$ . Due to this lack of variational structure we will use the concept of viscosity solutions ([5, 7]) that reveals to be well suited for the problem under consideration.

The paper is organized as follows: in Section 2 we introduce the definition of what we understand by a viscosity solution of our problem and we prove a comparison principle for (1.2); in Section 3 we show existence of a solution using the classical sub-supersolution method; finally, in Section 4 we prove uniqueness of the positive solution.

## 2. PRELIMINARIES

First, we fix what we understand by solutions of the problem. Note that, if  $Du = 0$  then the operator  $|Du|^{2-p} \operatorname{div} (|Du|^{p-2} Du)$  is not well defined and we need to take care of this fact. As we mentioned in the introduction, if  $u$  is smooth and  $Du \neq 0$  we have

$$(2.3) \quad |Du|^{2-p} \operatorname{div} (|Du|^{p-2} Du) = (p-2)\Delta_\infty u + \Delta u,$$

where  $\Delta_\infty u = |Du|^{-2} \langle D^2 u Du, Du \rangle$  is the 1-homogeneous infinity Laplacian. Now, to define what is a solution to the 1-homogeneous infinity Laplacian, observe that we have to give sense to the following function,

$$F(\xi, X) = \left\langle X \frac{\xi}{|\xi|}, \frac{\xi}{|\xi|} \right\rangle, \quad \xi \in \mathbb{R}^N, X \in M_{N \times N},$$

when  $\xi \neq 0$ . Here we have used the notation  $M_{N \times N}$  to denote the set of symmetric matrices in  $\mathbb{R}^{N \times N}$ . Let us denote by  $M(A)$  and  $m(A)$  the largest

and the smallest eigenvalue of  $A \in M_{N \times N}$ , respectively, i.e.

$$M(A) = \max_{|\eta|=1} \langle A\eta, \eta \rangle, \quad m(A) = \min_{|\eta|=1} \langle A\eta, \eta \rangle.$$

Taking into account (2.3) we rewrite the problem (1.2) as follows

$$(2.4) \quad \begin{cases} -(p-2)\Delta_\infty u - \Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and consider the following standard definitions of viscosity sub and supersolutions of (2.4).

**Definition 2.1.** *We say that an upper semicontinuous function  $u : \Omega \rightarrow \mathbb{R}$  is a viscosity subsolution of (2.4) if,  $u|_{\partial\Omega} \leq 0$  and, whenever  $x_0 \in \Omega$  and  $\psi \in C^2(\Omega)$  are such that  $u(x_0) = \psi(x_0)$  and  $u(x) < \psi(x)$ , if  $x \neq x_0$ , then*

$$\begin{cases} -(p-2)\Delta_\infty \psi(x_0) - \Delta \psi(x_0) \leq f(x_0), & \text{if } D\psi(x_0) \neq 0, \\ -(p-2)M(D^2\psi(x_0)) - \Delta \psi(x_0) \leq f(x_0), & \text{if } D\psi(x_0) = 0. \end{cases}$$

*We say that a lower semicontinuous function  $u : \Omega \rightarrow \mathbb{R}$  is a viscosity supersolution of (2.4) if,  $u|_{\partial\Omega} \geq 0$  and, whenever  $x_0 \in \Omega$  and  $\varphi \in C^2(\Omega)$  are such that  $u(x_0) = \varphi(x_0)$  and  $u(x) > \varphi(x)$ , if  $x \neq x_0$ , then*

$$\begin{cases} -(p-2)\Delta_\infty \varphi(x_0) - \Delta \varphi(x_0) \geq f(x_0), & \text{if } D\varphi(x_0) \neq 0, \\ -(p-2)m(D^2\varphi(x_0)) - \Delta \varphi(x_0) \geq f(x_0), & \text{if } D\varphi(x_0) = 0. \end{cases}$$

*Finally, a continuous function  $u : \Omega \rightarrow \mathbb{R}$  is a viscosity solution if it is both, a viscosity supersolution and a viscosity subsolution.*

Analogous definitions hold for solutions to our sublinear problem (1.1), replacing  $f$  with  $\lambda u^q$ .

Note that in both of the above definitions the strict inequality can be relaxed, since the second condition is required just in a neighbourhood of  $x_0$ . We refer to [7] for more details about general theory of viscosity solutions, and [10, 11] for viscosity solutions related to the  $\infty$ -Laplacian and the  $p$ -Laplacian operators.

We are ready to prove a comparison principle for problem (2.4) following the ideas in [9].

**Proposition 2.1.** *Let  $v \in C(\bar{\Omega})$  such that  $v > 0$  in  $\bar{\Omega}$  and  $-(p-2)\Delta_\infty v - \Delta v \geq f$ . If  $u \in C(\bar{\Omega})$  verifies  $-(p-2)\Delta_\infty u - \Delta u < f$  and  $u < v$  on  $\partial\Omega$ , then  $u \leq v$  in  $\Omega$ .*

*Proof.* Arguing by contradiction, suppose that there exists an interior point at which  $u - v$  is strictly positive. Since  $u < v$  on  $\partial\Omega$ , this implies that there exists  $x_0 \in \Omega$  where  $u(x) - v(x)$  attains a positive maximum. We have

$$(2.5) \quad \begin{aligned} -(p-2)\Delta_\infty v - \Delta v &\geq f, \\ -(p-2)\Delta_\infty u - \Delta u &< f. \end{aligned}$$

Then, using arguments of [9], we consider

$$\psi_j(x, y) = u(x) - v(y) - \theta_j(x, y), \quad j \in \mathbb{N}, \quad \theta_j(x, y) = \frac{j}{4}|x - y|^4,$$

and let  $(x_j, y_j) \in \Omega_0 \times \Omega_0$  such that  $\psi_j(x_j, y_j) = \sup_{(x,y) \in \Omega_0 \times \Omega_0} \psi_j(x, y)$ . Then we have that

$$(2.6) \quad x_j \rightarrow x_0, \quad y_j \rightarrow x_0, \quad j|x_j - y_j|^4 \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

Thus we can assume that  $\psi_j(x, y)$  attains positive maxima at  $(x_j, y_j)$  for  $j$  large. Applying the maximum principle for semicontinuous functions we obtain that, using the standard notations of [9], there exist symmetric matrices  $X_j, Y_j \in \mathbb{S}_N$ , such that

$$(2.7) \quad (\eta_j, X_j) \in \bar{J}^{2,+}u(x_j), \quad (\eta_j, Y_j) \in \bar{J}^{2,+}v(y_j),$$

being  $\eta_j = j|x_j - y_j|^2(x_j - y_j)$ , and

$$\begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \leq D^2\theta_j(x_j, y_j) + \frac{1}{j} \left( D^2\theta_j(x_j, y_j) \right)^2.$$

After some computations and denoting  $z_j = x_j - y_j$ , the previous inequality reads as follows

$$(2.8) \quad \begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \leq j(|z_j|^2 + 2|z_j|^4) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 16j|z_j|^2 \begin{pmatrix} z_j \otimes z_j & -z_j \otimes z_j \\ -z_j \otimes z_j & z_j \otimes z_j \end{pmatrix}.$$

Evaluating these quadratic forms at  $(\xi, \xi) \in \mathbb{R}^{2N}$  it leads to

$$(2.9) \quad \langle X_j \xi, \xi \rangle \leq \langle Y_j \xi, \xi \rangle, \quad \text{for all } \xi \in \mathbb{R}^N,$$

that is,  $Y_j - X_j$  is positive semidefinite. From (2.5) and (2.7) together with (2.9), if  $x_j \neq y_j$  we can conclude that

$$\begin{aligned} f &\leq -(p-2) \langle Y_j \frac{\eta_j}{|\eta_j|}, \frac{\eta_j}{|\eta_j|} \rangle - \text{trace}(Y_j) \\ &\leq -(p-2) \langle X_j \frac{\eta_j}{|\eta_j|}, \frac{\eta_j}{|\eta_j|} \rangle - \text{trace}(X_j) < f, \end{aligned}$$

which is a contradiction.

If  $x_j = y_j$  then  $\eta_j = z_j = 0$  and by (2.8) we get

$$\begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \leq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

thus  $X_j \leq 0 \leq Y_j$ . Hence,  $-m(Y_j) \leq 0 \leq -M(X_j)$ . Taking into account (2.5) and (2.7), this implies that

$$f \leq -(p-2)m(Y_j) - \text{trace}(Y_j) \leq -(p-2)M(X_j) - \text{trace}(X_j) < f,$$

getting again a contradiction.  $\square$

Now, using a change of variables that allows us to apply the same ideas used in the previous result, we prove a comparison result for our problem.

**Proposition 2.2.** *Let  $v \in C(\bar{\Omega})$  such that  $v > 0$  in  $\bar{\Omega}$  and  $-(p-2)\Delta_\infty v - \Delta v \geq \lambda v^q$ . If  $u \in C(\bar{\Omega})$  verifies  $-(p-2)\Delta_\infty u - \Delta u < \lambda u^q$  and  $u < v$  on  $\partial\Omega$ , then  $u \leq v$  in  $\Omega$ .*

*Proof.* Our proof is by contradiction, and hence we suppose that the function  $\frac{u(x)}{v(x)}$  attains a positive maximum greater than one at an interior point  $\hat{x} \in \Omega$ . Denote by  $\Omega_0$  an open set containing  $\hat{x}$ , where  $u$  is positive. Let

$$w(x) = \ln(u(x)) \quad \text{and} \quad z(x) = \ln(v(x)).$$

Note that  $w$  is defined only where  $u$  is positive, but this includes  $\Omega_0$ . These functions verify

$$(2.10) \quad \begin{aligned} -(p-2)\Delta_\infty w - \Delta w - (p-1)|Dw|^2 &\leq \lambda e^{(q-1)w}, & \text{in } \Omega_0, \\ -(p-2)\Delta_\infty z - \Delta z - (p-1)|Dz|^2 &\geq \lambda e^{(q-1)z} & \text{in } \Omega. \end{aligned}$$

Here we interpret the infinity Laplacian at the points where the gradient vanishes as in our previous definition. We also have that  $\hat{x}$  is a maximum point for  $w - z = \ln\left(\frac{u(x)}{v(x)}\right)$ .

Now, as in the previous comparison result, we consider

$$\psi_j(x, y) = w(x) - z(y) - \theta_j(x, y), \quad j \in \mathbb{N}, \quad \theta_j(x, y) = \frac{j}{4}|x - y|^4,$$

and let  $(x_j, y_j) \in \Omega_0 \times \Omega_0$  such that  $\psi_j(x_j, y_j) = \sup_{(x,y) \in \Omega_0 \times \Omega_0} \psi_j(x, y)$ . Then we have that (2.6) holds and then we can assume that  $\psi_j(x, y)$  attains positive maxima at  $(x_j, y_j)$  for  $j$  large. Applying again the maximum principle for semicontinuous functions we obtain that, there exist symmetric matrices  $X_j, Y_j \in \mathbb{S}_N$ , such that

$$(2.11) \quad (\eta_j, X_j) \in \bar{J}^{2,+} w(x_j), \quad (\eta_j, Y_j) \in \bar{J}^{2,+} z(y_j),$$

and verifying (2.8) and (2.9). That is,  $Y_j - X_j$  is positive semidefinite. From (2.10) and (2.11) together with (2.9), if  $x_j \neq y_j$  we can conclude that

$$\begin{aligned} \lambda e^{(q-1)z} &\leq -(p-2)\langle Y_j \frac{\eta_j}{|\eta_j|}, \frac{\eta_j}{|\eta_j|} \rangle - \text{trace}(Y_j) - (p-1)|\eta_j|^2 \\ &\leq -(p-2)\langle X_j \frac{\eta_j}{|\eta_j|}, \frac{\eta_j}{|\eta_j|} \rangle - \text{trace}(X_j) - (p-1)|\eta_j|^2 \leq \lambda e^{(q-1)w}, \end{aligned}$$

which is a contradiction since  $q < 1$ .

If  $x_j = y_j$  then  $\eta_j = z_j = 0$  and by (2.8) we get that  $X_j \leq Y_j$ . As before, taking into account (2.8), (2.10) and (2.11), it implies that

$$\begin{aligned} \lambda e^{(q-1)z} &\leq -(p-2)m(Y_j) - \text{trace}(Y_j) \\ &\leq -(p-2)M(X_j) - \text{trace}(X_j) \leq \lambda e^{(q-1)w}, \end{aligned}$$

getting again a contradiction.  $\square$

### 3. EXISTENCE

We begin this section by including some preliminary known results, see [3].

**Lemma 3.1.** *Let  $u \in C(\Omega)$  such that  $u \geq 0$  verifying  $-(p-2)\Delta_\infty u - \Delta u \geq 0$  in  $\Omega$  in the viscosity sense. If  $x_0 \in \Omega$  and  $0 < r < R \leq \text{dist}(x_0, \partial\Omega)$ , then*

$$u(y) \leq u(z) e^{\frac{|y-z|}{R-r}}, \quad \text{for all } y, z \in B_r(x_0).$$

Moreover,

$$|Du(x)| \leq \frac{u(x)}{\text{dist}(x, \partial\Omega)}, \quad \text{a.e. } x \in \Omega.$$

Now, we state an existence result valid for any bounded domain.

**Theorem 3.1.** *Let  $p > 2$ ,  $0 < q < 1$ , then there exists a nonnegative nontrivial viscosity solution of problem (1.1) for every  $\lambda > 0$ .*

*Proof.* First, we construct a supersolution and a subsolution for the problem (1.1). In order to construct the supersolution, we argue as follows: let us consider the problem

$$(3.12) \quad \begin{cases} -(p-2)\Delta_\infty w(x) - \Delta w(x) = 1, & \text{in } \Omega, \\ w(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

whose continuous viscosity solution exists (and it is positive in  $\Omega$ ) by the results in [6]. We claim that, for  $K > 0$  large enough and  $\delta$  small,  $\bar{u}(x) = Kw(x) + \delta$  is a supersolution of the problem (1.1). We compute and we get

$$\begin{aligned} -(p-2)\Delta_\infty \bar{u} - \Delta \bar{u} &= K(-(p-2)\Delta_\infty w - \Delta w) = K \\ &\geq \lambda \bar{u}^q = \lambda(Kw + \delta)^q. \end{aligned}$$

We obtain the desired supersolution if  $K^{1-q} \geq \lambda(w + \delta/K)^q$ , which holds for  $K$  large since  $q < 1$ . The parameter  $K$  can be selected independently of  $\delta$  as long as we consider  $0 < \delta \leq 1$ . In fact, it suffices to take  $K^{1-q} \geq \lambda \max_\Omega (w + 1/K)^q$ .

Now, we construct a subsolution of our problem. To this end, we assume that  $0 \in \Omega$ , we take  $r > 0$  such that  $B(0, r) \subset\subset \Omega$  and we consider the problem

$$(3.13) \quad \begin{cases} -(p-2)\Delta_\infty \varphi - \Delta \varphi = \lambda_1 \varphi, & \text{in } B(0, r), \\ \varphi = 0, & \text{on } \partial B(0, r). \end{cases}$$

If  $\varphi(|x|)$  is a radial solution of (3.13), it satisfies

$$(3.14) \quad \begin{cases} -(p-1)\varphi''(|x|) - \frac{(N-1)}{r}\varphi'(|x|) = \lambda_1 \varphi(|x|), \\ \varphi(r) = 0, \end{cases}$$

with  $\lambda_1$  depending on  $r, N, p$ . It is well known that there is a simple positive eigenfunction  $\varphi$  (with positive eigenvalue  $\lambda_1$ ) for this problem, see, for example, [8]. We claim that for  $\delta > 0$  small enough,  $\underline{u} = \delta\varphi$  is a subsolution. Indeed, we get

$$-(p-2)\Delta_\infty \underline{u} - \Delta \underline{u} = \delta(-(p-2)\Delta_\infty \varphi - \Delta \varphi) = \lambda_1 \delta \varphi \leq \lambda \underline{u}^q = \lambda \delta^q \varphi^q.$$

It is satisfied if  $\delta^{1-q} \leq \lambda C(\varphi)$ , which holds for  $\delta$  small (here we are using again that  $q < 1$ ).

Once we have found a super and a subsolution that are ordered, the existence of the asserted solution  $w$  follows from the standard Perron method,

see [7], Section 4. In fact, let us consider the supremum of subsolutions that are between  $\underline{u}$  and  $\bar{u}$ , that is

$$\hat{u}(x) = \sup \{u \text{ subsolutions such that } \underline{u} \leq u \leq \bar{u}\}.$$

Since we have a uniform bound for every function involved in this supremum, taking into account Lemma 3.1, we get that these functions verify

$$|Du(x)| \leq \frac{u(x)}{\text{dist}(x, \partial\Omega)}, \quad a.e. \ x \in \Omega.$$

Therefore,  $|Du|$  is locally bounded. Hence, we have a sequence of functions being uniformly bounded and locally equicontinuous, thus the supremum can be approximated locally uniformly in  $\Omega$ . Therefore,  $\hat{u}$  is continuous in  $\Omega$ . In addition, we have that  $\hat{u}$  verifies  $\underline{u} \leq \hat{u} \leq \bar{u}$  in  $\bar{\Omega}$ , hence,  $\hat{u}$  is positive in  $\Omega$  (the ball considered for constructing the subsolution can be any ball contained in  $\Omega$ ). To obtain that  $\hat{u}$  vanishes on  $\partial\Omega$  we just observe that we have  $0 \leq \hat{u} \leq \bar{u} = Kw + \delta$  in  $\bar{\Omega}$ , and we just let  $\delta \rightarrow 0$  to obtain that  $\hat{u} = 0$  on  $\partial\Omega$  and moreover, we get that  $\hat{u}$  is continuous in the whole  $\bar{\Omega}$ .  $\square$

#### 4. UNIQUENESS

In this section we deal with the issue of uniqueness of a positive solution to (1.1). Note that  $u = 0$  is a solution to (1.1) hence we analyze the uniqueness for positive solutions.

**Theorem 4.1.** *Let  $\Omega$  be a starshaped domain,  $p > 2$  and  $0 < q < 1$ , then there exists a unique positive solution of (1.1) for every  $\lambda > 0$ .*

*Proof.* Existence was proved in the previous section, see Theorem 3.1. Here we use the scaling invariance of the equation together with the comparison principle Proposition 2.2. Let us assume that there are two solutions  $u_1, u_2$  of (1.1). We rescale one of them, let us say  $u_1$  according to

$$u_\gamma(x) = \gamma^a u_1(\gamma x), \quad \gamma > 0, \quad a = \frac{-2}{1-q}.$$

Observe that  $u_\gamma$  is defined for  $x \in \Omega_\gamma = \{y/\gamma : y \in \Omega\}$  and  $a < 0$  since  $q < 1$ . Since we assume that  $\Omega$  is starshaped, we have  $\Omega \subset \Omega_\gamma$  for  $\gamma < 1$ . Simple computations show that  $u_\gamma$  is also a solution to (1.1) in  $\Omega_\gamma$ . We consider the following set  $\Gamma = \{\gamma < 1 : u_\gamma(x) \geq u_2(x), x \in \Omega\}$ . Taking  $\gamma$  sufficiently small we assure that  $\Gamma \neq \emptyset$  (recall  $a < 0$ ). Denote  $\gamma^* = \sup_\Gamma \gamma$ .

Our goal is to conclude that  $\gamma^* = 1$ , then  $u_1 \geq u_2$  and a symmetric argument shows that  $u_1 = u_2$ . Arguing by contradiction we assume that  $\gamma^* < 1$ . By continuity of  $u_1, u_2$  we have  $u_{\gamma^*}(x) \geq u_2(x)$  for  $x \in \Omega$  (and moreover, there exists  $x_0 \in \Omega$  such that  $u_{\gamma^*}(x_0) = u_2(x_0)$ ) and  $u_{\gamma^*}(x) \geq k > 0$ , for  $x \in \partial\Omega$ . Concerning the equation verified by  $u_{\gamma^*}$  in  $\Omega$  we have

$$-(p-2)\Delta_\infty u_{\gamma^*}(x) - \Delta u_{\gamma^*}(x) = \lambda(u_{\gamma^*}(x))^q \geq \lambda u_2^q(x).$$

Take

$$z(x) = u_{\gamma^*}(x) - \frac{k}{2}.$$

This function  $z$  is strictly positive on  $\partial\Omega$  (in fact it is greater or equal to  $k/2$ ) and verifies  $-(p-2)\Delta_\infty z(x) - \Delta z(x) \geq \lambda u_2^q(x)$  for  $x \in \Omega$ . By the comparison principle Proposition 2.2 we obtain  $z(x) \geq u_2(x)$ ,  $x \in \Omega$ , but at  $x_0$  we have

$$z(x_0) = u_{\gamma^*}(x_0) - \frac{k}{2} < u_2(x_0)$$

which is a contradiction that proves our claim.  $\square$

As a consequence of this uniqueness result, when  $\Omega = B_R(0)$  is a ball we get that the unique positive solution is radial,  $u(x) = g(r)$ , with  $g$  the positive solution to  $-(p-1)g''(r) - \frac{N-1}{r}g'(r) = \lambda g^q(r)$ , with  $g(R) = 0$ ,  $g'(0) = 0$ .

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