

THE INFINITY LAPLACIAN WITH A TRANSPORT TERM

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ABSTRACT. We consider the following problem: given a bounded domain $\Omega \subset \mathbb{R}^n$ and a vector field $\zeta: \Omega \rightarrow \mathbb{R}^n$, find a solution to $-\Delta_\infty u - \langle Du, \zeta \rangle = 0$ in Ω , $u = f$ on $\partial\Omega$, where Δ_∞ is the 1-homogeneous infinity Laplace operator that is formally given by $\Delta_\infty u = \langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|} \rangle$ and f a Lipschitz boundary datum. If we assume that ζ is a continuous gradient vector field then we obtain existence and uniqueness of a viscosity solution by an L^p -approximation procedure. Also we prove the stability of the unique solution with respect to ζ . In addition when ζ is more regular (Lipschitz continuous) but not necessarily a gradient, using tug-of-war games we prove that this problem has a solution.

1. INTRODUCTION

Our aim is to study the following problem: given a bounded domain $\Omega \subset \mathbb{R}^n$, a Lipschitz continuous function $f: \partial\Omega \rightarrow \mathbb{R}$ and a vector field $\zeta: \Omega \rightarrow \mathbb{R}^n$, find a solution to

$$(1.1) \quad \begin{cases} -\Delta_\infty u - \langle Du, \zeta \rangle = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where the operator

$$\Delta_\infty u = \left\langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|} \right\rangle$$

is known as the 1-homogeneous infinity Laplacian, see the survey [6]. The infinity Laplacian $\Delta_\infty u$, was introduced by Aronsson [5] in 1960's and can be viewed as the "Laplacian of L^∞ -variational problems" in the sense that the equation $\Delta_\infty u(x) = 0$ is the Euler-Lagrange equation for the variational problem of finding absolute minimizers for the prototypical L^∞ -functional $I(u) = \|Du\|_{L^\infty(\Omega)}$ with given boundary values, see e.g. [12], [18]. One of the main difficulties when dealing with such operator is the lack of regularity results, see [14], [27] and references therein. The infinity Laplacian also arises from certain random turn games [25], [7], [22], and mass transportation problems [16], and it appears in several applications, such as image reconstruction and enhancement, [10].

The infinity Laplacian appears as the limit as $p \rightarrow \infty$ of the well known and widely studied p -Laplacian, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, in the sense that solutions to $\Delta_p u_p = 0$ with a Dirichlet data $u_p = f$ on $\partial\Omega$ converge as $p \rightarrow \infty$ to the solution to $\Delta_\infty u = 0$ with $u = f$ on $\partial\Omega$ in the viscosity sense (see [6], [9] and [15]). In our case, when ζ is a gradient vector field we can obtain solutions to our problem by taking the limit as $p \rightarrow \infty$ in certain p -Laplacian type problems that we describe below. Note that infinity harmonic functions are limits of p -harmonic functions, but this limit procedure does not work for solutions to equations with a right hand side, like for $-\Delta_\infty u = g$ that is not the limit of $-\Delta_p u = g$, see [9], [18], [6]. For

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our purposes let us assume that ζ is a gradient vector field and let

$$\zeta(x) = D\eta(x) \quad \text{and} \quad b_p(x) = e^{(p-2)\eta(x)}.$$

Associated with this function b_p as diffusion coefficient let us consider the following p -Laplacian type problem:

$$(1.2) \quad \begin{cases} -\operatorname{div} \left(b_p |Du|^{p-2} Du \right) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Existence and uniqueness for this problem of a continuous weak solution in the Sobolev space $W^{1,p}(\Omega)$ can be easily obtained from variational arguments. It turns out that this weak solution is also a viscosity solution.

Now we state our main results for a gradient vector field. We have existence and uniqueness of a viscosity solution and, in addition, stability with respect to ζ .

Theorem 1.1. *Let ζ be a continuous gradient vector field. Then there exists a viscosity solution to (1.1) that can be obtained as the uniform limit of solutions to the p -Laplacian type problems (1.2). In addition, the viscosity solution is unique.*

Theorem 1.2. *When we consider gradient vector fields solutions to (1.1) depend continuously on ζ . In fact, we have the following stability estimate: there exists a constant C such that*

$$\|u_1 - u_2\|_{L^\infty(\Omega)} \leq \frac{C}{-\ln \|\zeta_1 - \zeta_2\|_{L^\infty(\Omega)}}.$$

Here u_1, u_2 are solutions to (1.1) with ζ_1, ζ_2 respectively and the same boundary datum f .

In this way, we have proved the stability for solutions to the equation in the sense that

$$\lim_{\|\zeta_1 - \zeta_2\|_{L^\infty(\Omega)} \rightarrow 0} \|u_1 - u_2\|_{L^\infty(\Omega)} = 0.$$

Equation (1.1) arises naturally when one considers Tug-of-War games (as introduced recently in [25], see also [11], [22]). In fact, let us describe a game that has as continuous value a solution to (1.1). This is a zero sum game with two players in which the earnings of one of them are the losses of the other. Starting with a token at a vertex $x_0 \in \Omega$, the players flip a biased coin with probabilities $C(\varepsilon)$ and $1 - C(\varepsilon)$. If the result is a head (probability $C(\varepsilon)$), they toss a fair coin to decide who move the token. If the outcome of the second toss is heads, then Player I moves the token to any $x_1 \in \overline{B}(x_0)$, while in case of tails, Player II moves the token to any $x_1 \in \overline{B}(x_0)$. In the other case, that is, if they get tails in the first coin toss (probability $1 - C(\varepsilon)$), the game state moves to the point $x_0 + \zeta(x_0)\varepsilon$. The game continues until the first time the token arrives to $x_\tau \in \mathbb{R}^n \setminus \Omega$ and then Player I earns $F(x_\tau)$, and thus Player II earns $-F(x_\tau)$ (here F is a suitable extension of the boundary datum f). This game has a value u_ε that verifies a Dynamic Programming Principle formula, see the formula (4.1). Moreover, the value functions for this game u_ε converge uniformly along subsequences as $\varepsilon \rightarrow 0$ to a limit u that is called the continuous value of the game and is a viscosity solution to our problem (1.1). See Section 4 for more details concerning the game.

To carry on this probabilistic approach we have to assume more regularity on ζ and impose Lipschitz continuity, but it is not necessary to assume that it is a gradient vector field.

Theorem 1.3. *Let ζ be a Lipschitz vector field. Then there exists a viscosity solution to (1.1) that can be obtained as the continuous value of the game described above.*

Let us end the introduction with a brief discussion of the previous bibliography and a description of the main techniques used in the proofs. Concerning approximations using p -Laplacian type operators, we quote [9] and [17], from where the main idea to show the key bounds for the L^p -norm of the gradient is taken. We use ideas from [19] and [20] (some of them being original from [18]) to show uniqueness of solutions and the stability result. Note however that the equation considered here is different from the one in [19] and [20] that appears as a limit of $p(x)$ -Laplacian problems. Here we have a diffusion coefficient but no x -dependence in the exponent. This change affects some of the arguments that has to be carefully adapted to the present situation. Concerning games we mention [25] (see also [26]) from where we mimic the technical details to obtain uniform convergence of the values of the game. Let us point out that in reference [23] a related problem is studied (here the probability of winning the coin toss depends on ε). The resulting equation is similar to ours (they consider ζ depending on u as $\zeta = Du$) and the main results there are obtained via a clever comparison result with exponential cones. We don't use any comparison with cones argument here but rely on the more probabilistic ideas from [25]. We also quote the recent references [1], [2], [3] and [4] related with the interplay between tug-of-war games and the infinity Laplacian.

The paper is organized as follows: In Section 2 we prove the existence part of Theorem 1.1 showing that there is a sequence of solutions to (1.2) that converges uniformly (this fact comes from uniform in p estimates of the gradients of such solutions); in Section 3 we deal with the uniqueness part of Theorem 1.1 and with the stability with respect to ζ ; finally, in Section 4 we perform the game theoretical approach.

2. A p -LAPLACIAN APPROXIMATION

In this section our aim is to obtain solutions to our problem (1.1) as limits of solutions to (1.2). Let us recall that we assume that ζ is a gradient vector field, $\zeta(x) = D\eta(x)$ and that we consider $b_p(x) = e^{(p-2)\eta(x)}$ as the diffusion coefficient in the following problem:

$$(2.1) \quad \begin{cases} -\operatorname{div} \left(b_p |Du|^{p-2} Du \right) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Adding a constant if necessary we can assume that $\eta \geq 0$ and hence $b_p \geq 1$.

First, we show existence and uniqueness of a continuous weak solution to (2.1). The proof is standard but we include the details for the sake of completeness.

Lemma 2.1. *Let $p > n$, then there it exists a unique continuous solution to the variational problem*

$$\min_S \int_{\Omega} b_p \frac{|Du|^p}{p}$$

where $S = \{u \in W^{1,p}(\Omega) : u|_{\partial\Omega} = f\}$. This minimum is a weak solution of the problem (2.1), that is, it verifies, $\int_{\Omega} b_p |Du|^{p-2} Du D\phi = 0$, for every $\phi \in C_0^\infty(\Omega)$ and $u = f$ on $\partial\Omega$.

Proof. As $b_p(x)$ is a bounded in $\bar{\Omega}$, we obtain that for every $u \in W^{1,p}(\Omega)$ there holds

$$\int_{\Omega} \frac{|Du|^p}{p} \leq \int_{\Omega} b_p \frac{|Du|^p}{p} \leq c_1 \int_{\Omega} \frac{|Du|^p}{p}$$

and hence the functional

$$\Theta(u) = \int_{\Omega} b_p \frac{|Du|^p}{p},$$

is well defined in the set S which is convex, weakly closed and non empty.

On the other hand, Θ is coercive, bounded below and lower semicontinuous in S , for this reason there is a minimizing sequence $u_n \in S \subset W^{1,p}(\Omega)$, such that $u_n \rightharpoonup u \in S$, and

$$\inf_S \Theta = \liminf_{n \rightarrow +\infty} \Theta(u_n) \geq \Theta(u).$$

Hence the minimum of Θ in S is attained. From the strict convexity of Θ we obtain that u_p is the unique minimum of Θ in S . Finally, u_p , the unique minimizer, is a weak solution of (2.1). The fact that u_p is continuous follows from the fact that $W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ for $p > n$, see [13]. \square

Notice that since $p > n$ and f is Lipschitz, the conditions $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and $u = f$ on $\partial\Omega$ are equivalent to the statement that $u - F \in W_0^{1,p}(\Omega)$ being F a Lipschitz extension of f to the whole Ω .

Now we have to introduce the definition of a viscosity solution, see [15]. For later use we state the definition with full generality using the upper and lower semicontinuous envelopes of a function $G(X, \xi, u, x)$ (that we call G^* and G_* respectively) defined for $X \in M_{n \times n}$, the set of symmetric matrices in $\mathbb{R}^{n \times n}$, $\xi \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $x \in \Omega$.

Definition 2.1. *Let us consider a general second order elliptic equation*

$$-G(D^2u, Du, u, x) = 0.$$

We say that a lower semicontinuous proper function $v : \Omega \rightarrow (-\infty, \infty]$ is a viscosity supersolution in Ω if, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $\phi(x_0) = v(x_0)$ and $\phi(x) < v(x)$ when $x \neq x_0$, we have,

$$-G_*(D^2\phi(x_0), D\phi(x_0), \phi(x_0), x_0) \geq 0.$$

The viscosity subsolutions have a similar definition: they are upper semicontinuous, the test functions touch from above and the differential inequality is reversed using G^ instead of G_* .*

Finally, a viscosity solution is a function that is both a viscosity supersolution and viscosity subsolution.

Observe that since we assume that p is large, the equation $\operatorname{div}(b_p |Du|^{p-2} Du) = 0$ is not singular at the points where the gradient vanishes, and thus $x \mapsto \operatorname{div}(b_p |Du|^{p-2} Du)(x)$ is well defined and continuous for any $\phi \in C^2(\Omega)$. Hence, when we consider viscosity solutions to equation (2.1), we take

$$G_p(X, \xi, u, x) = |\xi|^{p-2} \langle Db_p(x), \xi \rangle + (p-2)b_p(x) |\xi|^{p-4} \langle X\xi, \xi \rangle + b_p(x) |\xi|^{p-2} \operatorname{trace}(X).$$

Now we show that for this equation of p -Laplacian type a continuous weak solution is also a viscosity solution.

Lemma 2.2. *Let u_p be a continuous weak solution of (2.1) then u_p is a viscosity solution.*

Proof. Let $\phi(x) \in C^2$ be a test function such that $\phi(x_0) = u(x_0)$ and $(u - \phi)(x)$ has a strict minimum at $x_0 \in \Omega$. Assume that $D\phi(x_0) \neq 0$ (otherwise the conclusion is immediate). We want to show that

$$\begin{aligned} & -|D\phi(x_0)|^{p-2} Db_p(x_0) D\phi(x_0) - (p-2)b_p(x_0) |D\phi(x_0)|^{p-2} \left\langle D^2\phi(x_0) \frac{D\phi}{|D\phi|}(x_0), \frac{D\phi}{|D\phi|}(x_0) \right\rangle \\ & -b_p(x_0) |D\phi(x_0)|^{p-2} \Delta\phi(x_0) \geq 0. \end{aligned}$$

Assume that this is not the case, then there exists a radius $r > 0$ such that

$$\begin{aligned} & -|D\phi(x)|^{p-2} Db_p(x)D\phi(x) - (p-2)b_p(x)|D\phi(x)|^{p-2} \langle D^2\phi(x) \frac{D\phi}{|D\phi|}(x), \frac{D\phi}{|D\phi|}(x) \rangle \\ & -b_p(x)|D\phi(x)|^{p-2} \Delta\phi(x) < 0. \end{aligned}$$

for every $x \in B_r(x_0)$. Set $m = \inf_{|x-x_0|=r} (u - \phi)(x)$ and let $\psi(x) = \phi(x) + \frac{m}{2}$. This function verifies $\psi(x_0) > u(x_0)$ and $-\operatorname{div}(b_p |D\psi|^{p-2} D\psi) < 0$. Take $(\psi - u)_+$ and we extend it by zero outside $B_r(x_0)$, then $(\psi - u)_+ \in W_0^{1,p}(\Omega)$. Taking $(\psi - u)_+$ as test function in the weak form of the equation we get,

$$\int_{\psi > u} b_p \langle |Du|^{p-2} Du; D((\psi - u)_+) \rangle = 0.$$

So, we obtain

$$\begin{aligned} C(N, p) \int_{\psi > u} b_p |D\psi - Du|^p & \leq \int_{\psi > u} b_p \langle |D\psi|^{p-2} D\psi - |Du|^{p-2} Du; D\psi - Du \rangle \\ & = \int_{\psi > u} b_p \langle |D\psi|^{p-2} D\psi; D\psi - Du \rangle. \end{aligned}$$

But, by the divergence theorem,

$$\int_{\psi > u} b_p \langle |D\psi|^{p-2} D\psi; D\psi - Du \rangle = \int_{\psi > u} -\operatorname{div}(b_p |D\psi|^{p-2} D\psi)(\psi - u) < 0,$$

a contradiction. This proves that u is a viscosity supersolution. The proof that is a viscosity subsolution is analogous and we omit the details. \square

Our next step is to prove that from a sequence of solutions to (2.1) with $p \rightarrow \infty$ we can extract a subsequence that converges uniformly.

Lemma 2.3. *Let $\{u_p\}$ be a sequence of solutions to (2.1) with $p \rightarrow \infty$, then there exists a subsequence $p_j \rightarrow \infty$ such that $u_{p_j} \rightarrow u$ uniformly in $\bar{\Omega}$.*

Proof. We already proved in Lemma 2.1 that u_p is a minimizer of Θ in S . Let v a fixed Lipschitz function such that $|Dv| \leq L$ and $v = f$ on $\partial\Omega$, then we have that $v \in S$ and hence

$$\int_{\Omega} b_p \frac{|Du_p|^p}{p} \leq \int_{\Omega} b_p \frac{|Dv|^p}{p} \leq \int_{\Omega} b_p \frac{L^p}{p} \leq \frac{L^p}{p} |\Omega| \max_{\bar{\Omega}} |b_p|.$$

Let $k = \max_{x \in \bar{\Omega}} \eta(x)$ then $1 \leq |b_p(x)| \leq e^{(p-2)k}$, and we obtain,

$$\int_{\Omega} |Du_p|^p \leq C \frac{L^p}{p} |\Omega| e^{k(p-2)}.$$

Therefore we have,

$$\left(\int_{\Omega} |Du_p|^p \right)^{\frac{1}{p}} \leq C^{\frac{1}{p}} \frac{L}{p^{\frac{1}{p}}} |\Omega|^{\frac{1}{p}} e^{\frac{k(p-2)}{p}} \leq C_1,$$

being the constant C_1 independent of p . Now we take m such that $n < m \leq p$ and obtain the following bound

$$\|Du_p\|_{L^m(\Omega)} = \left(\int_{\Omega} |Du_p|^m \cdot 1 \right)^{\frac{1}{m}} \leq \left[\left(\int_{\Omega} |Du_p|^p \right)^{\frac{m}{p}} \left(\int_{\Omega} 1 \right)^{\frac{p-m}{p}} \right]^{\frac{1}{m}} \leq C_1 |\Omega|^{\frac{p-m}{pm}} \leq C_2,$$

the constant C_2 being independent of p . We have proved that u_p is a bounded sequence in $W^{1,m}(\Omega)$, and we know that $u_p = f$ in $\partial\Omega$, so we can obtain a subsequence $u_{p_j} \rightharpoonup u \in W^{1,m}(\Omega)$ with $p_j \rightarrow +\infty$. Since $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$ and $u_{p_j} \rightharpoonup u \in W^{1,p}(\Omega)$, we obtain $u_{p_j} \rightarrow u$ in $C^{0,\alpha}(\Omega)$, and in particular $u_{p_j} \rightarrow u$ uniformly in $\bar{\Omega}$. As $u_{p_j} \in C(\bar{\Omega})$, so $u \in C(\bar{\Omega})$. \square

Remark 2.1. *The function $u \in W^{1,m}(\Omega)$, given by Lemma 2.3, is Lipschitz. In fact, we proved that,*

$$\left(\int_{\Omega} |Du_p|^m \right)^{\frac{1}{m}} \leq \liminf_{p_j \rightarrow +\infty} \left(\int_{\Omega} |Du_{p_j}|^m \right)^{\frac{1}{m}} \leq C_1 |\Omega|^{\frac{1}{m}} \leq C_2.$$

Now, we take $p \rightarrow \infty$ and then $m \rightarrow \infty$ to obtain $\|Du\|_{L^\infty(\Omega)} \leq C_2$. So, we have proved $u \in W^{1,\infty}(\Omega)$, that is, u is a Lipschitz function.

Now we are ready to prove existence of a viscosity solution to our main problem, (1.1). Observe that, in order to define the 1-homogeneous infinity Laplacian, we have to give sense to the following function,

$$G_\infty(X, \xi, u, x) = \langle X \frac{\xi}{|\xi|}, \frac{\xi}{|\xi|} \rangle + \langle \xi, \zeta(x) \rangle, \quad \xi \in \mathbb{R}^n, X \in M_{n \times n},$$

when $\xi = 0$. Since this function is discontinuous at $\xi = 0$ we have to take into account the upper and lower envelopes. To this end, associated with a symmetric matrix let us denote by $M(X)$ and $m(X)$ the largest and the smallest eigenvalues of $X \in M_{n \times n}$, respectively, i.e.

$$M(X) = \max_{|\eta|=1} \langle X\eta; \eta \rangle, \quad \text{and} \quad m(X) = \min_{|\eta|=1} \langle X\eta; \eta \rangle.$$

Then the upper and lower envelopes of G_∞ are given by

$$(G_\infty)^*(X, \xi, u, x) = \begin{cases} \langle X \frac{\xi}{|\xi|}, \frac{\xi}{|\xi|} \rangle + \langle \xi, \zeta(x) \rangle & \xi \neq 0 \\ M(X) & \xi = 0, \end{cases}$$

and

$$(G_\infty)_*(X, \xi, u, x) = \begin{cases} \langle X \frac{\xi}{|\xi|}, \frac{\xi}{|\xi|} \rangle + \langle \xi, \zeta(x) \rangle & \xi \neq 0 \\ m(X) & \xi = 0. \end{cases}$$

With these semicontinuous envelopes we refer to Definition 2.1 for the concept of a continuous function u being a *viscosity solution* of (1.1).

Theorem 2.1. *Let u_p be a sequence of viscosity solutions of (2.1), such that*

$$\lim_{p \rightarrow +\infty} u_p = u$$

uniformly in $\bar{\Omega}$, then u is a viscosity solution of

$$\begin{cases} -\Delta_\infty u - \langle Du, \zeta \rangle = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Proof. Since $u_p = f$ for every p , the limit u verifies the boundary condition $u = f$ on $\partial\Omega$. Let $\phi \in C^2(\Omega)$ be a test function such that $\phi(x_0) = u(x_0)$ and $(u - \phi)(x)$ has a strict minimum at $x_0 \in \Omega$. We want to show that, $-\Delta_\infty \phi(x_0) - \langle D\phi(x_0), \zeta(x_0) \rangle \geq 0$. Since $\lim_{p \rightarrow +\infty} u_p = u$ uniformly in $\bar{\Omega}$, we have that $\phi - u_p$ has a strict minimum at $x_p \in \Omega$, with $x_p \rightarrow x_0$. Assume

$|D\phi|(x_0) \neq 0$, so we obtain $|D\phi|(x_p) \neq 0$, by the continuity of $|D\phi|$ and the fact that $x_p \rightarrow x_0$. As u_p is a viscosity solution of (2.1), we have,

$$(2.2) \quad \left\langle D\phi(x_p), \frac{Db_p(x_p)}{b_p(x_p)(p-2)} \right\rangle + \frac{\Delta\phi}{p-2}(x_p) + \left\langle D^2\phi(x_p) \frac{D\phi}{|D\phi|}(x_p), \frac{D\phi}{|D\phi|}(x_p) \right\rangle \leq 0.$$

Now, we want to compute the limits of the different terms that appear in (2.2). Taking into account that $x_p \rightarrow x_0$, by our choice of b_p as $b_p = e^{(p-2)\eta}$, it holds that

$$\zeta(x) = \lim_{p \rightarrow +\infty} \frac{Db_p(x)}{b_p(x)(p-2)}, \quad \text{uniformly in } \bar{\Omega}.$$

Hence, we have

$$(2.3) \quad \lim_{p \rightarrow +\infty} \left\langle D\phi(x_p), \frac{Db_p(x_p)}{b_p(x_p)(p-2)} \right\rangle + \frac{\Delta\phi}{p-2}(x_p) = \langle D\phi(x_0), \zeta(x_0) \rangle$$

and, on the other hand, it holds that

$$(2.4) \quad \lim_{p \rightarrow +\infty} \left\langle D^2\phi(x_p) \frac{D\phi}{|D\phi|}(x_p), \frac{D\phi}{|D\phi|}(x_p) \right\rangle = \left\langle D^2\phi(x_0) \frac{D\phi}{|D\phi|}(x_0), \frac{D\phi}{|D\phi|}(x_0) \right\rangle.$$

Moreover, as $\Delta\phi$ is bounded for a fixed C^2 -function we obtain,

$$(2.5) \quad \lim_{p \rightarrow +\infty} \frac{\Delta\phi(x_p)}{p-2} = 0.$$

If we collect these results, we obtain,

$$\Delta_\infty\phi(x_0) + \langle D\phi(x_0), \zeta(x_0) \rangle \leq 0.$$

as we wanted to prove.

Now, assume that $D\phi(x_0) = 0$. As before we get that (2.3) and (2.5) hold. Concerning (2.4) we get

$$\liminf_{p \rightarrow +\infty} \left\langle D^2\phi(x_p) \frac{D\phi}{|D\phi|}(x_p), \frac{D\phi}{|D\phi|}(x_p) \right\rangle \geq m(D^2\phi(x_0)).$$

Putting all together we obtain that u is a viscosity supersolution.

An analogous argument considering $\psi \in C^2$ test function such that $\psi(x_0) = u(x_0)$ and $(u - \psi)(x)$ has a strict maximum at $x_0 \in \Omega$ shows that u is a viscosity subsolution to the equation. We omit the details. \square

Remark 2.2. Consider the modified p -Laplacian equation

$$-\Delta_p u - (p-2)|Du|^{p-2} \langle Du, \zeta \rangle = 0.$$

It can be checked that if there exists a sequence of solutions u_p with $u_p = f$ on $\partial\Omega$ that converges uniformly then the limit is a solution to (1.1). However, this equation is not variational and hence the required estimates cannot be obtained as before.

3. UNIQUENESS AND STABILITY

The goal of this section is to study uniqueness and stability with respect to the vector field ζ of solutions to the equation (1.1). Along this section we assume that ζ is a gradient vector field. Following [19], [20], we consider two auxiliary problems with a positive parameter ε ,

$$(3.1) \quad \begin{aligned} \min \{ & -\Delta_\infty u^+ - \langle \zeta, Du^+ \rangle; -\varepsilon + e^\eta |Du^+| \} = 0 & \text{upper equation} \\ & -\Delta_\infty u - \langle \zeta, Du \rangle = 0 & \text{equation} \end{aligned}$$

$$(3.2) \quad \max \{ -\Delta_\infty u^- - \langle \zeta, Du^- \rangle; \varepsilon - e^\eta |Du^-| \} = 0 \quad \text{lower equation.}$$

Let u_p^+ and u_p^- be the unique weak solutions of the problems

$$(3.3) \quad \begin{cases} -\operatorname{div}(b_p |Du| Du) = \varepsilon^{p-1} & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\operatorname{div}(b_p |Du| Du) = -\varepsilon^{p-1} & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases}$$

respectively. Existence and uniqueness for these problems can be obtained as in Lemma 2.1. For example u_p^+ is the unique solution to the following minimization problem,

$$\min_{u \in W^{1,p}(\Omega), u|_{\partial\Omega} = f} \left(\int_\Omega b_p \frac{|Du|^p}{p} - \varepsilon^{p-1} \int_\Omega u \right).$$

The weak solutions are viscosity solutions of their respective equations, for example, u_p^+ is a solution to $G_p(u) + \varepsilon^{p-1} = 0$. This fact can be proved as in Lemma 2.2. In addition, there are subsequences such that $u_p^- \rightarrow u^-$, $u_p \rightarrow u$, and $u_p^+ \rightarrow u^+$ uniformly in $\bar{\Omega}$. We also obtain that there exists a constant K such that

$$(3.4) \quad \max\{\|Du^+\|_{L^\infty(\Omega)}, \|Du^-\|_{L^\infty(\Omega)}\} \leq K.$$

By a comparison argument we have $u_p^- \leq u_p \leq u_p^+$, hence $u^- \leq u \leq u^+$.

Let us see that the limits u^+ , u^- are viscosity solutions to (3.1) and (3.2) respectively. We provide the proof for u^+ and leave the details for u^- to the reader.

Lemma 3.1. *Every uniform limit of u_p^+ as $p \rightarrow \infty$ is a viscosity solution to (3.1).*

Proof. Let $\phi \in C^2(\Omega)$ be a test function such that $\phi(x_0) = u^+(x_0)$ and $(u^+ - \phi)$ has a strict minimum at $x_0 \in \Omega$. By the uniform convergence, there exists $x_p \rightarrow x_0$, such that $\phi(x_p) = u_p^+(x_p)$ and $(u_p^+ - \phi)$ has a minimum at x_p . Thus, using that u_p^+ is a viscosity solution to (3.3),

$$\begin{aligned} & -|D\phi(x_p)|^{p-2} Db_p(x_p) D\phi(x_p) - (p-2)b_p(x_p) |D\phi(x_p)|^{p-2} \left\langle D^2\phi(x_p) \frac{D\phi}{|D\phi|}(x_p), \frac{D\phi}{|D\phi|}(x_p) \right\rangle \\ & - b_p(x_p) |D\phi|^{p-2}(x_p) \Delta\phi(x_p) \geq \varepsilon^{p-1}. \end{aligned}$$

Hence $D\phi(x_p) \neq 0$ and we obtain,

$$-\frac{Db_p(x_p)}{b_p(x_p)} \frac{D\phi(x_p)}{p-2} - \left\langle D^2\phi(x_p) \frac{D\phi}{|D\phi|}(x_p), \frac{D\phi}{|D\phi|}(x_p) \right\rangle - \frac{\Delta\phi(x_p)}{p-2} \geq \frac{\varepsilon^{p-1}}{b_p(x_p)(p-2) |D\phi(x_p)|^{p-2}}$$

We have to compute the limits of the different terms. As we did before, in (2.3), (2.4), (2.5), we can compute the limits in the left hand side. Now we write the left hand side as

$$\frac{\varepsilon^{p-1}}{b_p(x_p)(p-2) |D\phi(x_p)|^{p-2}} = \left[\frac{\varepsilon^{\frac{p-1}{p-2}}}{e^{\eta(x_p)}(p-2)^{\frac{1}{p-2}} |D\phi(x_p)|} \right]^{p-2}$$

and we observe that

$$\lim_{p \rightarrow +\infty} \frac{\varepsilon^{\frac{p-1}{p-2}}}{e^{\eta(x_p)} (p-2)^{\frac{1}{p-2}} |D\phi(x_p)|} = \frac{\varepsilon}{e^{\eta(x_0)} |D\phi(x_0)|}.$$

Hence, as the limit of the left hand side is finite we get

$$\frac{\varepsilon}{e^{\eta(x_0)} |D\phi(x_0)|} \leq 1,$$

and

$$-\Delta_\infty \phi(x_0) - \langle D\phi(x_0), \zeta(x_0) \rangle \geq 0.$$

Moreover, if $-\varepsilon + e^{\eta(x_0)} |D\phi(x_0)| > 0$, then $-\Delta_\infty \phi(x_0) - \langle D\phi(x_0), \zeta(x_0) \rangle = 0$. Then, we have obtained $\min \{-\Delta_\infty \phi - \langle \zeta, D\phi \rangle; -\varepsilon + e^\eta |D\phi|\}(x_0) \geq 0$, as we wanted to prove.

An analogous argument considering $\psi \in C^2(\Omega)$ a test function such that $\psi(x_0) = u_p^+(x_0)$ and $(u_p^+ - \psi)$ has a strict maximum at $x_0 \in \Omega$ shows a reverse inequality. \square

Using the weak form of the equations with $u_p^+ - u_p^-$ as test function we have,

$$\int_{\Omega} b_p |Du_p^+|^{p-2} Du_p^+ D(u_p^+ - u_p^-) = \int_{\Omega} \varepsilon^{p-1} (u_p^+ - u_p^-)$$

and

$$\int_{\Omega} b_p |Du_p^-|^{p-2} Du_p^- D(u_p^+ - u_p^-) = - \int_{\Omega} \varepsilon^{p-1} (u_p^+ - u_p^-)$$

and if we subtract, we obtain

$$\int_{\Omega} b_p \langle |Du_p^+|^{p-2} Du_p^+ - |Du_p^-|^{p-2} Du_p^-; D(u_p^+ - u_p^-) \rangle = 2 \int_{\Omega} \varepsilon^{p-1} (u_p^+ - u_p^-).$$

With the aid of the elementary inequality: $\langle |b|^{q-2}b - |a|^{q-2}a, b-a \rangle \geq 2^{2-q}|b-a|^q$ valid for vectors $a, b \in \mathbb{R}^n$ and $q \geq 2$, we get

$$4 \int_{\Omega} b_p \frac{|Du_p^+ - Du_p^-|^p}{2} \leq \frac{1}{\varepsilon} \int_{\Omega} \varepsilon^p |u_p^+ - u_p^-|$$

Extracting the p th root, and taking $p \rightarrow \infty$ we conclude that $\|Du^+ - Du^-\|_{L^\infty(\Omega)} \leq C\varepsilon$ and then $\|u^+ - u^-\|_{L^\infty(\Omega)} \leq C\varepsilon$. Hence we have proved,

$$(3.5) \quad u^+ \leq u^- + C\varepsilon \quad \text{and hence} \quad u^- \leq u \leq u^+ \leq u^- + C\varepsilon \quad \text{c.t.p. } x \in \Omega.$$

Now, using these functions u^+ , u^- we prove a comparison result valid for any viscosity solution to (1.1).

Lemma 3.2. *Let $u \in C(\bar{\Omega})$ be a viscosity solution to (1.1), then*

$$u^- \leq u \leq u^+.$$

Proof. We prove that $u \leq u^+$. The proof of $u^- \leq u$ is analogous and we omit the details.

By adding a constant if necessary we can assume that $u^+ > 0$. Arguing by contradiction we assume that

$$\max_{\bar{\Omega}} (u - u^+) > 0 = \max_{\partial\Omega} (u - u^+).$$

Now we introduce a function g which is an approximation of the identity,

$$g(t) = \frac{1}{\alpha} \ln(1 + A(e^{\alpha t} - 1)), \quad \text{where } A > 1, \alpha > 0.$$

Let $w = g(u^+)$. Analogously as in [19], see also [20], we have that there exists $\mu > 0$ such that w verifies $-\Delta_\infty w - \langle Dw, \zeta \rangle \geq \mu$ in the viscosity sense. Since g is an approximation of the identity we have

$$\max_{\Omega}(u - w) > \max_{\partial\Omega}(u - w).$$

Now, we double the variables and consider

$$\sup_{x, y \in \Omega} \left\{ u(x) - w(y) - \frac{j}{2}|x - y|^2 \right\}.$$

For large j the supremum is attained at interior points x_j, y_j such that $x_j \rightarrow \hat{x}$, $y_j \rightarrow \hat{x}$, where \hat{x} is an interior point (that \hat{x} cannot be on the boundary can be obtained as in [19]).

Now, we have to prove that there exists a constant C such that $j|x_j - y_j| \leq C$. To show this fact we just observe that $u(x_j) - w(y_j) - \frac{j}{2}|x_j - y_j|^2 \geq u(x_j) - w(x_j)$. Hence, by (3.4) and the explicit form of g we get (here we are using that u^+ is obtained taking the limit as $p \rightarrow \infty$ of a sequence of solutions to variational p -Laplacian type problems),

$$\frac{j}{2}|x_j - y_j|^2 \leq w(x_j) - w(y_j) \leq \|g'(u^+)Du^+\|_{L^\infty(\Omega)}|x_j - y_j| \leq AK|x_j - y_j|$$

from where the claim follows.

Now, the theorem of sums implies that there are symmetric matrices $\mathbb{X}_j, \mathbb{Y}_j$, with $\mathbb{X}_j \leq \mathbb{Y}_j$ such that $(j|x_j - y_j|, \mathbb{X}_j) \in \overline{J^{2,+}}(u)(x_j)$ and $(j|x_j - y_j|, \mathbb{Y}_j) \in \overline{J^{2,-}}(w)(y_j)$, where $\overline{J^{2,+}}(u)(x_j)$ and $\overline{J^{2,-}}(w)(y_j)$ are the closures of the super and subjets of u and w respectively. Using the equations, assuming that $x_j \neq y_j$, we have

$$\left\langle \mathbb{Y}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle + j \langle x_j - y_j, \zeta(y_j) \rangle \leq -\mu$$

and

$$\left\langle \mathbb{X}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle + j \langle x_j - y_j, \zeta(x_j) \rangle \geq 0.$$

Substracting these equations we obtain

$$\begin{aligned} 0 &\leq \left\langle (\mathbb{Y}_j - \mathbb{X}_j) \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle \leq -\mu + j \langle x_j - y_j, \zeta(x_j) - \zeta(y_j) \rangle \\ &\leq -\mu + j|x_j - y_j| |\zeta(x_j) - \zeta(y_j)| \leq -\mu + C|\zeta(x_j) - \zeta(y_j)|. \end{aligned}$$

This gives a contradiction taking the limit $j \rightarrow \infty$ using the continuity of ζ .

When $x_j = y_j$ we obtain $M(\mathbb{Y}_j) \leq -\mu$ and $m(\mathbb{X}_j) \geq 0$, that also gives a contradiction since $0 \leq \mathbb{Y}_j - \mathbb{X}_j$.

Hence we have obtained that $u \leq u^+$, as we wanted to prove. \square

Now we can prove uniqueness of viscosity solutions to (1.1).

Theorem 3.1. *There is a unique viscosity solution to (1.1).*

Proof. From Lemma 3.2 we get that any two solutions u_1, u_2 to (1.1) satisfy $u^- \leq u_1, u_2 \leq u^+$. This fact, together with (3.5) gives $|u_1 - u_2| \leq C\varepsilon$. The result follows letting $\varepsilon \rightarrow 0$. \square

Our next task is to prove the stability of solutions to our equation with respect to ζ . First, let us prove an estimate when ζ_1 is small. Let u_1 be a viscosity solution of,

$$\begin{cases} -\Delta_\infty u - \langle Du, \zeta_1 \rangle = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases}$$

and u_2 be a viscosity solution of

$$\begin{cases} -\Delta_\infty u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Take u_2^+ which is viscosity solution of (3.1) and verifies $u_2^+ \geq u_2 > 0$. Let $w_2 = g(u_2^+)$. As in [20] one can show that

$$-\Delta_\infty w_2 \geq \mu,$$

where μ is given by,

$$(3.6) \quad \mu = \frac{(A-1)\varepsilon^4}{Ae \|u_2^+\|_{L^\infty(\Omega)}}.$$

Using the properties of g and [20], we obtain the following estimate,

$$(3.7) \quad u_1 - u_2 = (u_1 - w_2) + (w_2 - u_2^+) + (u_2^+ - u_2) \leq (u_1 - g(u_2^+)) - \frac{A-1}{\alpha} + \varepsilon \operatorname{diam}(\Omega).$$

Lemma 3.3. *We have the following bound,*

$$u_1 - w_2 \leq C\varepsilon^{-4} \|u_2^+\|_{L^\infty(\Omega)}^2 \|\zeta\|_{L^\infty(\Omega)}.$$

Proof. Let $\sigma = \max_{\bar{\Omega}}(u_1 - w_2)$. If we suppose $\sigma \leq 0$, the conclusion is immediate, so we assume $\sigma > 0$. Let us consider

$$M_j = \sup_{x,y \in \Omega} \left\{ u_1(x) - w_2(y) - \frac{j}{2} |x - y|^2 \right\}.$$

We know $M_j \geq \sigma$, and the supremum is attained at some points x_j, y_j . Now $|x_j - y_j| \rightarrow 0$ as $j \rightarrow +\infty$, and $x_j \rightarrow \hat{x}$, $y_j \rightarrow \hat{x}$ at least for a subsequence. As before, we have that \hat{x} is an interior point of Ω . We have the following bounds: $\varepsilon \leq j|x_j - y_j| \leq C$. The upper bound follows as before, while the lower bound can be obtained as in [19].

According to the Theorem on Sums there exist matrices \mathbb{X}_j and \mathbb{Y}_j such that $\mathbb{X}_j \leq \mathbb{Y}_j$ and, assuming that $x_j \neq y_j$ (if $x_j = y_j$ we can proceed as in the previous proof, we omit the details),

$$\left\langle \mathbb{Y}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle \leq -\mu$$

and

$$\left\langle \mathbb{X}_j \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle + j \langle x_j - y_j, \zeta(x_j) \rangle \geq 0.$$

If we subtract them

$$0 \leq \left\langle (\mathbb{Y}_j - \mathbb{X}_j) \frac{(x_j - y_j)}{|x_j - y_j|}, \frac{(x_j - y_j)}{|x_j - y_j|} \right\rangle \leq -\mu + \langle j(x_j - y_j); \zeta(x_j) \rangle \leq -\mu + C \|\zeta\|_{L^\infty(\Omega)}$$

so, we have

$$\mu \leq C \|\zeta\|_{L^\infty(\Omega)}$$

and from the expression of μ in (3.6), the above estimate can be written as follows:

$$\frac{(A-1)\varepsilon^4}{Ae \|u^+\|_{L^\infty(\Omega)}} \leq C \|\zeta\|_{L^\infty(\Omega)}.$$

Now, we fix $A > 1$ such that $\frac{A-1}{\alpha} = \sigma$. So we obtain, $\sigma \leq C\varepsilon^{-4}Ae \|u_2^+\|_{L^\infty(\Omega)}^2 \|\zeta\|_{L^\infty(\Omega)}$. As $A < 2$, Ae can be absorbed into the constant C , and since $u_1 - w_2 \leq \sigma$, we obtain

$$u_1 - w_2 \leq C\varepsilon^{-4} \|u_2^+\|_{L^\infty(\Omega)}^2 \|\zeta\|_{L^\infty(\Omega)}$$

as we wanted to prove. \square

Theorem 3.2. *It holds the following stability estimate,*

$$|u_1 - u_2| \leq C \|\zeta\|_{L^\infty(\Omega)} + C \|\zeta\|_{L^\infty(\Omega)}^{\frac{1}{5}}.$$

Proof. If we return to (3.7), using the last result we have

$$u_1 - u_2 \leq 2C\varepsilon^{-4} \|u_2^+\|_{L^\infty(\Omega)}^2 \|\zeta\|_{L^\infty(\Omega)} + \varepsilon \operatorname{diam}(\Omega)$$

It remains to choose ε nearly optimal. To simplify, we use (3.5), $u_2^+ \leq \|f\|_{L^\infty(\partial\Omega)} + \varepsilon \operatorname{diam}(\Omega)$ to obtain

$$u_1 - u_2 \leq 2\frac{C}{\varepsilon^4} \left(\|f\|_{L^\infty(\Omega)} + \varepsilon \operatorname{diam}(\Omega) \right)^2 \|\zeta\|_{L^\infty(\Omega)} + \varepsilon \operatorname{diam}(\Omega).$$

That is, renaming constants,

$$u_1 - u_2 \leq \frac{C_1}{\varepsilon^5} \|\zeta\|_{L^\infty(\Omega)} \varepsilon + C_2 \varepsilon.$$

We consider two cases. If $C_2 \leq 2C_1 \|\zeta\|_{L^\infty(\Omega)}$, we take $\varepsilon = 1$ and obtain a bound of the form $C \|\zeta\|_{L^\infty(\Omega)}$, then, $u_1 - u_2 \leq C \|\zeta\|_{L^\infty(\Omega)}$. If $C_2 > 2C_1 \|\zeta\|_{L^\infty(\Omega)}$, we choose ε as:

$$\varepsilon = \left[\frac{C_1 \|\zeta\|_{L^\infty(\Omega)}}{C_2} \right]^{1/5},$$

and we obtain $u_1 - u_2 \leq C \|\zeta\|_{L^\infty(\Omega)}^{\frac{1}{5}}$. We conclude that

$$u_1 - u_2 \leq C \|\zeta\|_{L^\infty(\Omega)} + C \|\zeta\|_{L^\infty(\Omega)}^{\frac{1}{5}},$$

and considering u_2^- instead of u_2^+ we get,

$$|u_1 - u_2| \leq C \|\zeta\|_{L^\infty(\Omega)} + C \|\zeta\|_{L^\infty(\Omega)}^{\frac{1}{5}},$$

as we wanted to prove. \square

Now we obtain a general stability estimate. Let $u_1, u_2 > 0$ be viscosity solutions of

$$\begin{cases} \Delta_\infty u + \langle Du; \zeta_1 \rangle = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \Delta_\infty u + \langle Du; \zeta_2 \rangle = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases}$$

respectively. As before we let $w_2 = g(u_2^+)$ that verifies

$$\Delta_\infty w + \langle Dw, \zeta_2 \rangle \leq -\mu$$

with

$$\mu = \frac{\alpha(A-1)}{A} e^{-\alpha \|u_2^+\|_{L^\infty(\Omega)} - 2\|\eta_2\|_{L^\infty(\Omega)}} \varepsilon^2$$

and we obtain

$$u_1 - u_2 = (u_1 - w_2) + (w_2 - v_2) + (v_2 - u_2) \leq (u_1 - g(u_2^+)) - \frac{A-1}{\alpha} + C\varepsilon.$$

Lemma 3.4. *It holds that*

$$u_1 - w_2 \leq CAe^{\frac{1}{\varepsilon}\|u_2\|_{L^\infty(\Omega)}} \|\zeta_1 - \zeta_2\|_{L^\infty(\Omega)}.$$

Proof. The proof follows the same steps as the previous one until we arrive to

$$\varepsilon^2 \alpha^2 A^{-1} \frac{\sigma}{2} e^{-\alpha\|u_2^+\|_{L^\infty(\Omega)} - 2\|\eta_2\|_{L^\infty(\Omega)}} \leq C \|\zeta_1 - \zeta_2\|_{L^\infty(\Omega)}.$$

Then we choose $\alpha \approx \frac{1}{\varepsilon}$ and we get

$$\sigma \leq CAe^{-\frac{1}{\varepsilon}\|u_2^+\|_{L^\infty(\Omega)} - 2\|\eta_2\|_{L^\infty(\Omega)}} \|\zeta_1 - \zeta_2\|_{L^\infty(\Omega)}$$

as we wanted to prove. \square

Theorem 3.3. *It holds the general stability estimate,*

$$\|u_1 - u_2\|_{L^\infty(\Omega)} \leq \frac{C}{-ln \|\zeta_1 - \zeta_2\|_{L^\infty(\Omega)}}$$

Proof. We have

$$u_1 - u_2 \leq CAe^{-\frac{1}{\varepsilon}\|u_2\|_{L^\infty(\Omega)}} \|\zeta_1 - \zeta_2\|_{L^\infty(\Omega)} + \frac{A-1}{\alpha} + C\varepsilon$$

so that, by a symmetric argument and using that $\alpha \approx \frac{1}{\varepsilon}$,

$$\|u_1 - u_2\|_{L^\infty(\Omega)} \leq CAe^{-\frac{1}{\varepsilon}\|u_2\|_{L^\infty(\Omega)}} \|\zeta_1 - \zeta_2\|_{L^\infty(\Omega)} + C\varepsilon.$$

Now we choose ε as

$$\varepsilon \approx \frac{1}{-ln \|\zeta_1 - \zeta_2\|_{L^\infty(\Omega)}}$$

and we conclude the desired estimate. \square

4. A GAME THEORETICAL APPROACH

In this section our aim is to show that for a Lipschitz continuous vector field ζ solutions to our problem (1.1) can be obtained as the continuous value of a modification of the tug-of-war game introduced in [25]. Next, we briefly describe the tug-of-war game of [25] and refer to that reference for details.

4.1. The tug-of-war game. A tug-of-war game is a two person zero-sum game, that is, two players play knowing that the earnings of the first one are the losses of the second one. Player I chooses a strategy in order to maximize the expected outcome, and Player II chooses another in order to minimize the outcome.

Take a bounded smooth domain $\Omega \subset \mathbb{R}^n$. Let $f : \partial\Omega \rightarrow \mathbb{R}$ be a Lipschitz continuous function and extend it to a small strip of width ε around $\partial\Omega$ in $\mathbb{R}^n \setminus \Omega$, $\Gamma_\varepsilon = \{x \in \mathbb{R}^n \setminus \Omega : d(x, \partial\Omega) < \varepsilon\}$. This extension, that we call F , gives the final payoff of the game (that is, the earnings of Player I and the losses of Player II). At the beginning, a token is placed at a point $x_0 \in \Omega$. Then, a fair coin is tossed and the player who wins moves the token to any $x_1 \in \overline{B}_\varepsilon(x_0)$, being $\varepsilon > 0$ a parameter of the game. At the next turn, the coin is tossed again and the winner chooses to move the token to any $x_2 \in \overline{B}_\varepsilon(x_1)$. When the token arrives to any $x_\tau \in \mathbb{R}^n \setminus \Omega$, Player I earns $F(x_\tau)$, and thus Player II earns $-F(x_\tau)$.

Denote S_I and S_{II} the profile of strategies of Player I and Player II respectively, see [21] and [25], we define the expected payoff for Player I as

$$V_{x_0,I}(S_I, S_{II}) = \begin{cases} \mathbb{E}_{S_I, S_{II}}^{x_0} [F(x_\tau)], & \text{if the game terminate a.s.} \\ -\infty & \text{otherwise.} \end{cases}$$

Analogously, we define the expected payoff for Player II as

$$V_{x_0,II}(S_I, S_{II}) = \begin{cases} \mathbb{E}_{S_I, S_{II}}^{x_0} [F(x_\tau)], & \text{if the game terminate a.s.} \\ +\infty & \text{otherwise.} \end{cases}$$

Now, we define the ε - *value* of the game for Player I as

$$u_I^\varepsilon(x_0) = \sup_{S_I} \inf_{S_{II}} V_{x_0,II}(S_I, S_{II})$$

and the ε - *value* of the game for Player II as

$$u_{II}^\varepsilon(x_0) = \inf_{S_{II}} \sup_{S_I} V_{x_0,I}(S_I, S_{II}).$$

We have that $u_I^\varepsilon = u_{II}^\varepsilon := u_\varepsilon$, that is, the game has a value. Now comes a key fact, by the Dynamic Programming Principle, see [21], [24] and [25], the value of the game verifies

$$u_\varepsilon(x) = \frac{1}{2} \sup_{y \in \overline{B}_\varepsilon(x)} u_\varepsilon(y) + \frac{1}{2} \inf_{y \in \overline{B}_\varepsilon(x)} u_\varepsilon(y).$$

In [25] it is proved that u_ε converges uniformly when $\varepsilon \rightarrow 0$. This uniform limit is called the *continuous value* of the game that we denote by u and it can be proved (see also [25]) that u is a viscosity solution to the problem

$$\begin{cases} -\Delta_\infty u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

4.2. A modification of the game. We consider as before a smooth domain $\Omega \subset \mathbb{R}^n$ and F the final payoff function defined in a narrow strip around the boundary. The principal difference in this modified tug-of-war game is that we play a game with two stages. First we toss an unfair coin, which has head probability $0 < C(\varepsilon) < 1$, and tail probability $1 - C(\varepsilon)$. If we have obtained a head, we play to the game described before, i.e., we toss a new (fair) coin and the winner moves the token to any new position $x_1 \in \overline{B}_\varepsilon(x_0)$. But if in the first (unfair) coin toss we obtained a tail, the token is moved to $x_0 + \zeta(x_0)\varepsilon$, where $\zeta(x) : \Omega \rightarrow \mathbb{R}^n$ is the vector field that appears in (1.1) (that is assumed to be Lipschitz). Note that there is no strategies of the players involved if we get a tail in the first coin toss. The game continues until the first time the token arrives to $x_\tau \in \mathbb{R}^n \setminus \Omega$ and then Player I earns $F(x_\tau)$, and thus Player II earns $-F(x_\tau)$.

We choose the probability $C(\varepsilon)$ according to

$$C(\varepsilon) = 1 - \varepsilon.$$

This choice is motivated by the different scaling properties of the different terms that appear in (1.1).

Again we have that this game has a value (defined in an analogous way as before), and, using the Dynamic Programming Principle, we obtain that the value function for this game

verifies

$$(4.1) \quad u_\varepsilon(x) = (1 - \varepsilon) \left\{ \frac{1}{2} \sup_{y \in \overline{B}_\varepsilon(x)} u_\varepsilon(y) + \frac{1}{2} \inf_{y \in \overline{B}_\varepsilon(x)} u_\varepsilon(y) \right\} + \varepsilon u_\varepsilon(x + \zeta(x)\varepsilon).$$

Our next goal is to show that we can pass to the limit as $\varepsilon \rightarrow 0$ and that this limit, the continuous value of the game, is a viscosity solution to (1.1).

4.3. Uniform Convergence of the ε -value of the game as $\varepsilon \rightarrow 0$. Our first result shows uniform convergence using an Arzela-Ascoli type lemma from [22].

Lemma 4.1. *Let $\{u_\varepsilon : \overline{\Omega} \rightarrow \mathbb{R}, \varepsilon > 0\}$ be a set of functions such that*

- (1) *there exists $C > 0$ so that $|u_\varepsilon(x)| < C$ for every $\varepsilon > 0$ and every $x \in \overline{\Omega}$,*
- (2) *given $\eta > 0$ there are constants r_0 and ε_0 such that for every $\varepsilon < \varepsilon_0$ and any $x_0, y_0 \in \overline{\Omega}$ with $|x_0 - y_0| < r_0$ it holds $|u_\varepsilon(x) - u_\varepsilon(y)| < \eta$.*

Then, there exists a uniformly continuous function $u : \overline{\Omega} \rightarrow \mathbb{R}$ and a subsequence still denoted by $\{u_\varepsilon\}$ such that $u_\varepsilon \rightarrow u$ uniformly in $\overline{\Omega}$, as $\varepsilon \rightarrow 0$.

Now, we proceed with the proof of the uniform convergence.

Lemma 4.2. *There exists a subsequence $\varepsilon_j \rightarrow 0$ such that $u_{\varepsilon_j} \rightarrow u$ uniformly in $\overline{\Omega}$.*

Proof. First, let us point out that since F is bounded then u_ε is uniformly bounded. In fact, it holds,

$$\min_{y \in \Gamma_\varepsilon} F(y) \leq u_\varepsilon(x) \leq \max_{y \in \Gamma_\varepsilon} F(y).$$

Now, we want to prove that condition (2) in Lemma 4.1 holds. If $x_0 \in \Gamma_\varepsilon$ and $y_0 \in \Gamma_\varepsilon$ then, due to the fact that F is Lipschitz we have,

$$|u_\varepsilon(x_0) - u_\varepsilon(y_0)| = |F(x_0) - F(y_0)| \leq L|x_0 - y_0|,$$

and this shows (2) in this case. If $x_0 \in \Omega$ and $y_0 \in \Gamma_\varepsilon$, then, using the same arguments as in [25], taking the strategy of pointing to y starting at x one can show that there exists a constant K such that $|u_\varepsilon(x_0) - u_\varepsilon(y_0)| \leq K d_\varepsilon(x_0, y_0)$, where d_ε is the discrete distance given by $d_\varepsilon(x, y) = \lceil \frac{|x-y|}{\varepsilon} \rceil + 1$ and we also get (2) in this case. Finally, if $x_0 \in \Omega$ and $y_0 \in \Omega$ we can mimic the strategies of the players starting at x with those starting at y . That is, when we fix $S_I(x) \in \overline{B}_\varepsilon(x)$, we choose $S_I(y) = S_I(x) - x + y \in \overline{B}_\varepsilon(y)$ and analogously for S_{II} . In this way, each time that the tug-of-war game is played, we have $|x_k - y_k| = |x_{k-1} - y_{k-1}|$. In case the movement is given by the vector field ζ we have (here we use that ζ is Lipschitz)

$$|x_k - y_k| = |x_{k-1} - y_{k-1} + \varepsilon(\zeta(x_{k-1}) - \zeta(y_{k-1}))| \leq (1 + L\varepsilon)|x_{k-1} - y_{k-1}|.$$

Now, we observe that $\mathbb{E}[\#\text{ plays with } \zeta] = \mathbb{E}[\#\text{ total number of plays}]\varepsilon$, and the expected number of total plays can be bounded by K/ε^2 , see [25], [22], hence we get $\mathbb{E}[\#\text{ plays with } \zeta] \leq K/\varepsilon$. Therefore, if we let τ be the first time such that $x_\tau \in \Gamma_\varepsilon$ or $y_\tau \in \Gamma_\varepsilon$, we have

$$\mathbb{E}[|x_\tau - y_\tau|] \leq (1 + L\varepsilon)^{K/\varepsilon} |x_0 - y_0| \leq C|x_0 - y_0|.$$

And by the same arguments that we used for the case $x \in \Omega$ and $y \in \Gamma_\varepsilon$ we conclude that

$$|u_\varepsilon(x_0) - u_\varepsilon(y_0)| \leq C d_\varepsilon(x_0, y_0),$$

and also in this case we get (2).

Therefore we are under the hypotheses of Lemma 4.1 and we get the desired uniform convergence, extracting a subsequence if it is necessary. \square

Now our aim is to show that the uniform limit as $\varepsilon \rightarrow 0$ of u_ε , called the *continuous value* of the game, that we denote by u is a viscosity solution to the problem (1.1).

Theorem 4.1. *Any uniform limit of u_ε as $\varepsilon \rightarrow 0$ is a viscosity solution to (1.1).*

Proof. First, let us observe that from the fact that $u_\varepsilon = F$ in Γ_ε and the uniform convergence to u , we obtain that the boundary condition $u|_{\partial\Omega} = f$ is satisfied.

Now, let us check the equation in (1.1). To this end, let us first consider a smooth test function ϕ that touches u from above, that is, $\phi(y) - \phi(x_0) \geq u(y) - u(x_0)$, for every $y \neq x_0 \in \bar{\Omega}$. As u_ε converge uniformly to u as $\varepsilon \rightarrow 0$ there are points x_ε converging to x_0 such that

$$\phi(y) - \phi(x_\varepsilon) \geq u_\varepsilon(y) - u_\varepsilon(x_\varepsilon) - \varepsilon^3.$$

Using that u_ε verifies the Dynamic Programming Principle, (4.1), at the point x_ε , we obtain that

$$0 \leq (1 - \varepsilon) \left\{ \frac{1}{2} \sup_{y \in \bar{B}_\varepsilon(x_\varepsilon)} \phi(y) + \frac{1}{2} \inf_{y \in \bar{B}_\varepsilon(x_\varepsilon)} \phi(y) - \phi(x_\varepsilon) \right\} + \varepsilon \left\{ \phi(x_\varepsilon + \zeta(x_\varepsilon)\varepsilon) - \phi(x_\varepsilon) \right\} + O(\varepsilon^3).$$

Now we divide by ε^2 and pass to the limit as $\varepsilon \rightarrow 0$. The first term

$$A(\varepsilon, \phi) = \frac{(1 - \varepsilon)}{\varepsilon^2} \left\{ \frac{1}{2} \sup_{y \in \bar{B}_\varepsilon(x_\varepsilon)} \phi(y) + \frac{1}{2} \inf_{y \in \bar{B}_\varepsilon(x_\varepsilon)} \phi(y) - \phi(x_\varepsilon) \right\}$$

can be handled as in [22], see also [11], and gives as limit when $\varepsilon \rightarrow 0$ the infinity Laplacian of ϕ at x_0 ,

$$\lim_{\varepsilon \rightarrow 0} A(\varepsilon, \phi) = \Delta_\infty \phi(x_0).$$

While the second term

$$B(\varepsilon, \phi) = \frac{1}{\varepsilon} \left\{ \phi(x_\varepsilon + \zeta(x_\varepsilon)\varepsilon) - \phi(x_\varepsilon) \right\}$$

gives as limit

$$\lim_{\varepsilon \rightarrow 0} B(\varepsilon, \phi) = \langle D\phi(x_0), \zeta(x_0) \rangle.$$

Therefore we get

$$0 \leq \Delta_\infty \phi(x_0) + \langle D\phi(x_0), \zeta(x_0) \rangle$$

and we have obtained that u is a viscosity supersolution according to Definition 2.1.

The fact that u is a viscosity subsolution is analogous and we omit the details. \square

Remark 4.1. *To prove that the a uniform limit of the values of the game is a viscosity solution to (1.1) we only used that ζ is continuous (but we don't need Lipschitz continuity here).*

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