NONLOCAL DIFFUSION PROBLEMS THAT APPROXIMATE THE HEAT EQUATION WITH DIRICHLET BOUNDARY CONDITIONS

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Abstract. We present a model for nonlocal diffusion with Dirichlet boundary conditions in a bounded smooth domain. We prove that solutions of properly re-scaled non local problems approximate uniformly the solution of the corresponding Dirichlet problem for the classical heat equation.

1. Introduction

Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radial, continuous function with $\int_{\mathbb{R}^N} J(z)\,dz = 1$. Assume also that J is strictly positive in $B(0,d)$ and vanishes in $\mathbb{R}^N \setminus B(0,d)$. Nonlocal evolution equations of the form

$$u_t(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t), \tag{1.1}$$

and variations of it, have been widely used to model diffusion processes. As stated in [11] equation (1.1) models a, continuous in time, "random walk" where the probability distribution of jumping from location y to location x is given by $J(x-y)$. For recent references on nonlocal diffusion see, [1], [2], [3], [4], [5], [6], [10], [11], [13], [14] and references therein.

In this article we propose the following nonlocal "Dirichlet" boundary value problem: Given $g(x,t)$ defined for $x \in \mathbb{R}^N \setminus \Omega$ and $u_0(x)$ defined for $x \in \Omega$, find $u(x,t)$ such that

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy, & x \in \Omega,\ t>0, \\ u(x,t) = g(x,t), & x \notin \Omega,\ t>0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

In this model we prescribe the values of u outside Ω which is the analogous of prescribing the so called Dirichlet boundary conditions for the classical heat equation. However, the boundary data is not understood in the usual sense as we will see in Remark 2.1 below. As explained before in this model the right hand side models the diffusion, the integral $\int J(x-y)(u(y,t)-$

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$u(x,t))\,dy$ takes into account the individuals arriving or leaving position $x \in \Omega$ from or to other places while we are prescribing the values of u outside the domain Ω by imposing $u = g$ for $x \notin \Omega$. When $g = 0$ we get that any individuals that leave Ω die, this is the case when Ω is surrounded by a hostile enviroment.

Existence and uniqueness of solutions of (1.2) is proved by a fixed point argument in Section 2 where a comparison principle is also obtained.

Let us consider the classical Dirichlet problem for the heat equation,

$$\begin{cases} v_t(x,t) - \Delta v(x,t) = 0, & x \in \Omega,\ t > 0, \\ v(x,t) = g(x,t), & x \in \partial\Omega,\ t > 0, \\ v(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.3}$$

The nonlocal Dirichlet model (1.2) and the classical Dirichlet problem (1.3) share many properties, among them the asymptotic behavior of their solutions as $t \to \infty$ is similar as was proved in [7].

The main goal of this article is to show that the Dirichlet problem for the heat equation (1.3) can be approximated by suitable nonlocal problems of the form of (1.2).

More precisely, for a given J and a given $\varepsilon > 0$ we consider the rescaled kernel

$$J_\varepsilon(\xi) = C_1 \frac{1}{\varepsilon^N} J\left(\frac{\xi}{\varepsilon}\right), \qquad \text{with} \qquad C_1^{-1} = \frac{1}{2}\int_{B(0,d)} J(z) z_N^2\, dz. \tag{1.4}$$

Here C_1 is a normalizing constant in order to obtain the Laplacian in the limit instead of a multiple of it. Let $u^\varepsilon(x,t)$ be the solution of

$$\begin{cases} u_t^\varepsilon(x,t) = \displaystyle\int_\Omega \frac{J_\varepsilon(x-y)}{\varepsilon^2}(u^\varepsilon(y,t) - u^\varepsilon(x,t))dy, & x \in \Omega, t > 0, \\ u(x,t) = g(x,t), & x \notin \Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.5}$$

Our main result now reads as follows.

Theorem 1.1. *Let Ω be a bounded $C^{2+\alpha}$ domain for some $0 < \alpha < 1$. Let $v \in C^{2+\alpha, 1+\alpha/2}(\overline{\Omega} \times [0,T])$ be the solution to (1.3) and let u^ε be the solution to (1.5) with J_ε as above. Then, there exists $C = C(T)$ such that*

$$\sup_{t \in [0,T]} \|v - u^\varepsilon\|_{L^\infty(\Omega)} \le C\varepsilon^\alpha \to 0, \qquad \text{as } \varepsilon \to 0. \tag{1.6}$$

Related results for the Neumann problem where recently obtained in [9].

Note that the assumed regularity of v is a consequence of regularity assumptions on the boundary data g, the domain Ω and the initial condition u_0, see [12].

The rest of the paper is organized as follows: in Section 2 we prove existence, uniqueness and a comparison principle for our nonlocal equation and in Section 3 we prove the convergence result.

2. Existence, uniqueness and a comparison principle

Existence and uniqueness of solutions is a consequence of Banach's fixed point theorem. We look for $u \in C([0,\infty); L^1(\Omega))$ satisfying (1.2). Fix $t_0 > 0$ and consider the Banach space $X_{t_0} = \{w \in C([0,t_0]; L^1(\Omega))\}$ with the norm

$$|||w||| = \max_{0\le t\le t_0} \|w(\cdot,t)\|_{L^1(\Omega)}.$$

We will obtain the solution as a fixed point of the operator $\mathcal{T} : X_{t_0} \to X_{t_0}$ defined by

$$\mathcal{T}_{w_0}(w)(x,t) = w_0(x) + \int_0^t \int_{\mathbb{R}^N} J(x-y)\,(w(y,s) - w(x,s))\,dy\,ds,$$

where we impose

$$w(x,t) = g(x,t), \qquad \text{for } x \notin \Omega.$$

Lemma 2.1. *Let $w_0, z_0 \in L^1(\Omega)$, then there exists a constant C depending on J and Ω such that*

$$|||\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)||| \le Ct_0|||w-z||| + ||w_0 - z_0||_{L^1(\Omega)}$$

for all $w, z \in X_{t_0}$.

Proof. We have

$$\begin{aligned}\int_\Omega |\mathcal{T}_{w_0}(w)(x,t) - \mathcal{T}_{z_0}(z)(x,t)|\,dx &\le \int_\Omega |w_0 - z_0|(x)\,dx \\ &+ \int_\Omega \left| \int_0^t \int_{\mathbb{R}^N} J(x-y) \Big[(w(y,s) - z(y,s)) \right. \\ &\qquad \left. -(w(x,s) - z(x,s)) \Big]\,dy\,ds \right| dx.\end{aligned}$$

Hence, taking into account that w and z vanish outside Ω,

$$|||\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)||| \le ||w_0 - z_0||_{L^1(\Omega)} + Ct_0|||w - z|||,$$

as we wanted to prove. □

Theorem 2.1. *For every $u_0 \in L^1(\Omega)$ there exists a unique solution u, such that $u \in C([0,\infty); L^1(\Omega))$.*

Proof. We check first that $\mathcal{T}_{u_0}$ maps X_{t_0} into X_{t_0}. Taking $z_0 \equiv 0$ and $z \equiv 0$ in Lemma 2.1 we get that $\mathcal{T}_{u_0}(w) \in C([0,t_0]; L^1(\Omega))$ for any $w \in X_{t_0}$.

Choose t_0 such that $Ct_0 < 1$. Now taking $z_0 \equiv w_0 \equiv u_0$ in Lemma 2.1 we get that $\mathcal{T}_{u_0}$ is a strict contraction in X_{t_0} and the existence and uniqueness part of the theorem follows from Banach's fixed point theorem in the interval $[0,t_0]$. To extend the solution to $[0,\infty)$ we may take as initial data $u(x,t_0) \in L^1(\Omega)$ and obtain a solution up to $[0,2t_0]$. Iterating this procedure we get a solution defined in $[0,\infty)$. □

Remark 2.1. Note that in general a solution u with $u_0 > 0$ and $g = 0$ is strictly positive in $\overline{\Omega}$ (with a positive continuous extension to $\overline{\Omega}$) and vanishes outside $\overline{\Omega}$. Therefore a discontinuity occurs on $\partial\Omega$ and the boundary value is not taken in the usual "classical" sense, see [7].

We now define what we understand by sub and supersolutions.

Definition 2.1. *A function $u \in C([0,T);L^1((\Omega))$ is a supersolution of* (1.2) *if*

$$\begin{cases} u_t(x,t) \geq \displaystyle\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy, & x\in\Omega,\ t>0,\\ u(x,t) \geq g(x,t), & x\notin\Omega,\ t>0,\\ u(x,0)\geq u_0(x), & x\in\Omega. \end{cases} \tag{2.1}$$

As usual, subsolutions are defined analogously by reversing the inequalities.

Lemma 2.2. *Let $u_0 \in C(\overline{\Omega})$, $u_0 \geq 0$, and $u \in C(\overline{\Omega}\times[0,T])$ a supersolution to* (1.2) *with $g \geq 0$. Then, $u \geq 0$.*

Proof. Assume for contradiction that $u(x,t)$ is negative somewhere. Let $v(x,t) = u(x,t) + \varepsilon t$ with ε so small such that v is still negative somewhere. Then, if (x_0,t_0) is a point where v attains its negative minimum, there holds that $t_0 > 0$ and

$$\begin{aligned} v_t(x_0,t_0) = u_t(x_0,t_0) + \varepsilon &> \int_{\mathbb{R}^N} J(x-y)(u(y,t_0)-u(x_0,t_0))\,dy \\ &= \int_{\mathbb{R}^N} J(x-y)(v(y,t_0)-v(x_0,t_0))\,dy \geq 0 \end{aligned}$$

which is a contradiction. Thus, $u \geq 0$. □

Corollary 2.1. *Let $J \in L^\infty(\mathbb{R}^N)$. Let u_0 and v_0 in $L^1(\Omega)$ with $u_0 \geq v_0$ and g, $h \in L^\infty((0,T);L^1(\mathbb{R}^N\setminus\Omega))$ with $g \geq h$. Let u be a solution of* (1.2) *with $u(x,0)=u_0$ and Dirichlet datum g and v be a solution of* (1.2) *with $v(x,0)=v_0$ and datum h. Then, $u \geq v$ a.e.*

Proof. Let $w = u - v$. Then, w is a supersolution with initial datum $u_0 - v_0 \geq 0$ and datum $g - h \geq 0$. Using the continuity of solutions with respect to the data and the fact that $J \in L^\infty(\mathbb{R}^N)$, we may assume that u, $v \in C(\overline{\Omega}\times[0,T])$. By Lemma 2.2 we obtain that $w = u - v \geq 0$. So the corollary is proved. □

Corollary 2.2. *Let $u \in C(\overline{\Omega}\times[0,T])$ (resp. v) be a supersolution (resp. subsolution) of* (1.2). *Then, $u \geq v$.*

Proof. It follows the lines of the proof of the previous corollary. □

3. Convergence to the heat equation

In order to prove Theorem 1.1 let $\tilde{v}$ be a $C^{2+\alpha,1+\alpha/2}$ extension of v to $\mathbb{R}^N \times [0,T]$.

Let us define the operator

$$\tilde{L}_\varepsilon(z) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} J_\varepsilon(x-y)\big(z(y,t)-z(x,t)\big)dy.$$

Then $\tilde{v}$ verifies

$$(3.1)\qquad \begin{cases} \tilde{v}_t(x,t) = \tilde{L}_\varepsilon(\tilde{v})(x,t) + F_\varepsilon(x,t) & x\in\Omega,\ (0,T],\\ \tilde{v}(x,t) = g(x,t)+G(x,t), & x\notin\Omega,\ (0,T],\\ \tilde{v}(x,0) = u_0(x), & x\in\Omega. \end{cases}$$

where, since $\Delta v = \Delta\tilde{v}$ in Ω,

$$F_\varepsilon(x,t) = -\tilde{L}_\varepsilon(\tilde{v})(x,t) + \Delta\tilde{v}(x,t).$$

Moreover as G is smooth and $G(x,t)=0$ if $x\in\partial\Omega$ we have

$$G(x,t) = O(\varepsilon), \qquad \text{for } x \text{ such that } \operatorname{dist}(x,\partial\Omega)\le \varepsilon d.$$

We set $w^\varepsilon = \tilde{v} - u^\varepsilon$ and we note that

$$(3.2)\qquad \begin{cases} w^\varepsilon_t(x,t) = \tilde{L}_\varepsilon(w^\varepsilon)(x,t) + F_\varepsilon(x,t) & x\in\Omega,\ (0,T],\\ w^\varepsilon(x,t) = G(x,t), & x\notin\Omega,\ (0,T],\\ w^\varepsilon(x,0) = 0, & x\in\Omega. \end{cases}$$

First, we claim that, by the choice of C_1, the fact that J is radially symmetric and $\tilde{u}\in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N\times[0,T])$, we have that

$$(3.3)\qquad \sup_{t\in[0,T]}\|F_\varepsilon\|_{L^\infty(\Omega)} = \sup_{t\in[0,T]}\|\Delta\tilde{v} - \tilde{L}_\varepsilon(\tilde{v})\|_{L^\infty(\Omega)} = O(\varepsilon^\alpha)\,.$$

In fact,

$$\Delta\tilde{v}(x,t) - \frac{C_1}{\varepsilon^{N+2}}\int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right)(\tilde{v}(y,t)-\tilde{v}(x,t))\,dy$$

becomes, under the change variables $z=(x-y)/\varepsilon$,

$$\Delta\tilde{v}(x,t) - \frac{C_1}{\varepsilon^2}\int_{\mathbb{R}^N} J(z)\,(\tilde{v}(x-\varepsilon z,t)-\tilde{v}(x,t))\,dz$$

and hence (3.3) follows by a simple Taylor expansion. This proves the claim.

We proceed now to prove Theorem 1.1.

Proof of Theorem 1.1. In order to prove the theorem by a comparisonwe first look for a supersolution. Let $\overline{w}$ be given by

$$(3.4)\qquad \overline{w}(x,t) = K_1\varepsilon^\alpha t + K_2\varepsilon.$$

For $x \in \Omega$ we have, if K_1 is large,

$$\overline{w}_t(x,t) - \tilde{L}(\overline{w})(x,t) = K_1\varepsilon^\alpha \geq F_\varepsilon(x,t) = w_t^\varepsilon(x,t) - \tilde{L}_\varepsilon(w^\varepsilon)(x,t). \tag{3.5}$$

Since

$$G_\varepsilon(x,t) = O(\varepsilon) \qquad \text{for } x \text{ such that } \operatorname{dist}(x,\partial\Omega) \leq \varepsilon$$

choosing K_2 large, we obtain

$$\overline{w}(x,t) \geq w^\varepsilon(x,t) \tag{3.6}$$

for $x \notin \Omega$ such that $\operatorname{dist}(x,\partial\Omega) \leq \varepsilon d$ and $t \in [0,T]$. Moreover it is clear that

$$\overline{w}(x,0) = K_2\varepsilon > w^\varepsilon(x,0) = 0. \tag{3.7}$$

Thanks to (3.5), (3.6) and (3.7) we can apply the comparison result and conclude that

$$w^\varepsilon(x,t) \leq \overline{w}(x,t) = K_1\varepsilon^\alpha t + K_2\varepsilon. \tag{3.8}$$

In a similar fashion we prove that $\underline{w}(x,t) = -K_1\varepsilon^\alpha t - K_2\varepsilon$ is a subsolution and hence

$$w^\varepsilon(x,t) \geq \underline{w}(x,t) = -K_1\varepsilon^\alpha t - K_2\varepsilon. \tag{3.9}$$

Therefore

$$\sup_{t\in[0,T]} \|u - u^\varepsilon\|_{L^\infty(\Omega)} \leq C(T)\varepsilon^\alpha, \tag{3.10}$$

as we wanted to prove. □

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