On The Asymptotic Distribution of Radial Eigenvalues

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Abstract

In this paper we find the asymptotic distribution of eigenvalues for the radial p-Laplacian in R^N, $-\Delta_p u = -div(|\nabla u|^{p-2}\nabla u) = (\lambda - q(|x|)|u|^{p-2}u$ when the potential q is increasing.

1 Introduction

In this work we answer a question posed in [1], concerning the asymptotic distribution of eigenvalues for the radial p-Laplacian in R^N for $1 < p < +\infty$,

$$-div(|\nabla v|^{p-2}\nabla v) = (\lambda - q(|x|)|v|^{p-2}v \qquad (1.1)$$

with a radially symmetric potential $q(|x|)$. The value $\lambda \in R$ is called a radial eigenvalue if a radially symmetric solution $v \neq 0$, $v \in L^p(R^N)$ of (1.1) exists. Let us observe that problem (1.1) is a one-dimensional eigenvalue problem,

$$\begin{gathered} -(r^{N-1}|u'|^{p-2}u')' = r^{N-1}(\lambda - q(r))|u|^{p-2}u, \\ u'(0) = 0, \quad u \in L^p(0, \infty; r^{N-1}), \end{gathered} \qquad (1.2)$$

The existence of a sequence of isolated eigenvalues $\lambda_1 < \lambda_2 < \ldots \to \infty$ was proved recently by Brown and Reichel, [1], for potentials $q(r) \in C^1(0, \infty)$ satisfying the following condition:

(a) There exist $\alpha > 0$ and $\beta > max\{(p-n)/(p-1), 0\}$ such that $q(r) \geq \alpha r^\beta$ for large r, and $q'(r)/q(r)^{1+1/p} \to 0$ as $r \to \infty$.

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Throughout the paper we will write $f \sim g$ to mean that $\lim_{n\to\infty} f/g = 1$. We will need the following condition on the potential $q(r)$:

(b) There exist $r_0 > 0$ such that $q(r)$ is increasing for $r \geq r_0$, and $q(0) = 0$.

Condition $q(0) = 0$ gives $\lambda_1 > 0$. This is no restriction, since the general case $q(0) = c \neq 0$ can be obtained from the case $q(0) = 0$, by defining $\hat{\lambda} = \lambda - c$ and $\hat{q}(x) = q(x) - c$, which gives

$$\hat{\lambda} - \hat{q}(x) = (\lambda - c) - (q(x) - c) = \lambda - q(x).$$

Our main result is the following theorem:

Theorem 1.1 *Let $\{\lambda_n\}_n$ be the sequence of eigenvalues of problem (1.2), and $r_{\lambda_n} \in R$ such that $q(r_{\lambda_n}) = \lambda_n$, where $q(r)$ satisfies conditions (a) and (b). Then,*

$$(n-1) \sim \frac{1}{\pi_p (p-1)^{1/p}} \int_0^{r_{\lambda_n}} (\lambda_n - q(r))^{1/p} dr.$$

From Theorem 1.1 we obtain the asymptotic distribution of eigenvalues:

Corollary 1.2 *Let $\{(\lambda_n, u_n)\}_n$ be the eigenvalues and eigenfunctions of problem (1.2), and $q(r)$ as in Theorem 1.1. Let $N(\lambda)$ be the eigenvalue counting function,*

$$N(\lambda) = \#\{n \in N : \lambda_n \leq \lambda\}.$$

Then,

$$N(\lambda) \sim \frac{1}{\pi_p (p-1)^{1/p}} \int_0^{r_\lambda} (\lambda - q(r))^{1/p} dr$$

as $\lambda \to +\infty$.

Theorem 1.1 and Corollary 1.2 generalizes the classical result of Milne [5] for $p = 2$, his result was proved assuming $q \in C^3$, convex, and $q''(x) = o(q'(x)^{4/3})$. A simplified proof is due to Titchmarsh [6], retaining all of Milne's assumptions except $q \in C^3$. Later, Hartman [4] obtained the same result under weaker assumptions, and deduce it from the asymptotic formula for the number of zeros of solutions of $u'' + Q(t)u = 0$, where no parameter λ occurs. We combine here his idea with the Prufer transformation techniques used in [3] instead of the ones in [1]. Let us mention that the results in [3, 4, 6] deals with a regular case, without singular coefficients like r^{N-1}.

The paper is organized as follows: in Section 2 we introduce the Prufer transformation and we reformulate Theorem 1.1 in terms of the zeros of solutions. In Section 3 we prove Theorem 1.1 and we obtain the asymptotic distribution of eigenvalues.

2 The Prufer Transformation

Let us introduce the generalized sine function $S_p(x)$, the unique solution of

$$\begin{array}{c} -(|u'(r)|^{p-2}u'(r))' = (p-1)|u(r)|^{p-2}u(r) \\ u(0) = 0 \quad u'(0) = 1 \end{array} \tag{2.3}$$

(see [2]). This function has a zero if and only if $r = k\pi_p$, where

$$\pi_p = 2(p-1)^{1/p} \int_0^1 \frac{ds}{(1-s^p)^{1/p}}.$$

Let us call $C_p(r) = S_p'(r)$. Both functions are well defined, and the following identity could be easily derived from equation (2.3):

$$|C_p(r)|^p + |S_p(r)|^p = 1, \tag{2.4}$$

Also, if $C_p(x) \neq 0$,

$$|C_p(r)|^{p-2}C_p'(r) + |S_p(r)|^{p-2}S_p(r) = 0. \tag{2.5}$$

We define the following Prufer transform:

$$\begin{array}{rcl} u(r) & = & f(r)\rho(r)S_p(\varphi(r)), \\ u'(x) & = & g(r)\rho(r)C_p(\varphi(r)). \end{array} \tag{2.6}$$

where

$$f(r) = \left(\frac{r^{N-1}(\lambda - q(r))}{p-1}\right)^{-1/p} \qquad \text{and} \qquad g(r) = (r^{N-1})^{-1/p} \tag{2.7}$$

Remark 2.1 *Let us observe that our Prufer transformation is different to the one in [1]. Moreover, it is valid only for* $0 \leq r \leq r_\lambda$.

Differentiating the first equation in (2.6), and multiplying by $gf^{-1}|C_p|^{p-2}C_p$, we obtain from the second equation,

$$gf^{-1}f'\rho|C_p|^{p-2}C_pS_p + g\rho'|C_p|^{p-2}C_pS_p + g\rho|C_p|^p\varphi' = g^2f^{-1}\rho|C_p|^p \tag{2.8}$$

On the other hand, replacing u and u' in equation (1.2) we obtain

$$-(r^{N-1}g^{p-1}\rho^{p-1}|C_p|^{p-2}C_p)' = r^{N-1}(\lambda - q(r))f^{p-1}\rho^{p-1}|S_p|^{p-2}S_p.$$

Hence, from (2.7) we have

$$-(g^{-1}\rho^{p-1}|C_p|^{p-2}C_p)' = (p-1)f^{-1}\rho^{p-1}|S_p|^{p-2}S_p$$

Now, we differentiate the first term and replace C_p' from equation (2.5). After multiplying by $g^2\rho^{2-p}S_p/(p-1)$, we have

$$g'\rho|C_p|^{p-2}C_pS_p/(p-1) - g\rho'|C_p|^{p-2}C_pS_p + g\rho|S_p|^p\varphi' = g^2 f^{-1}\rho|S_p|^p$$

Adding the last equation to equation (2.8), and multiplying by $(\rho g)^{-1}$, we obtain

$$(g'g^{-1}/(p-1) + f'f^{-1})|C_p|^{p-2}C_pS_p + (|C_p|^p + |S_p|^p)\varphi' = gf^{-1}(|C_p|^p + |S_p|^p)$$

By using the identity (2.4) and rearranging the terms, we obtain

$$\varphi' = -(g'g^{-1}/(p-1) + f'f^{-1})|C_p|^{p-2}C_pS_p + gf^{-1}.$$

Finally, replacing f and g from equations (2.7) we have

$$\varphi' = -\left(\frac{1-N}{p-1}r^{-1} + \frac{q'}{p(\lambda-q)}\right)|C_p|^{p-2}C_pS_p + \left(\frac{\lambda-q}{p-1}\right)^{1/p}. \tag{2.9}$$

Remark 2.2 *In much the same way, it is possible to obtain a first order differential equation for ρ, which could be used to compute the corresponding eigenfunctions. However, we will need only the phase function φ, and let us observe that equation (2.9) is independent of ρ.*

Radially symmetric solutions of equation (1.2) satisfy $u'(0) = 0$, which gives an initial condition at $r = 0$,

$$\varphi(0) = \pi_p/2.$$

Let u_n be the n^{th}-eigenfunction, with zeros $r_1 < r_2 < \ldots < r_{n-1}$, (where $0 < r_1$). Then,

$$u_n(r) = f(r)\rho_n(r)S_p(\varphi_n(r)),$$

where

$$\varphi_n(r_j) = j\pi_p, \qquad 1 \le j \le n-1.$$

We may restate Theorem 1.1 in terms of the number of zeros of eigenfunctions. We introduce the function

$$N(u_n, (a,b)) = \#\{j \in N : r_j \in [a,b)\},$$

which counts the number of zeros r_j of u_n in $[a,b)$. Since $N(u_n,(0,\infty)) = n-1$, we have:

Theorem 2.3 *Let $\{(\lambda_n, u_n)\}_n$ be the eigenvalues and eigenfunctions of problem (1.2), and $r_{\lambda_n} \in R$ such that $q(r_{\lambda_n}) = \lambda_n$, where $q(r)$ satisfies conditions (a) and (b). Then,*

$$N(u_n,(0,\infty)) \sim \frac{1}{\pi_p(p-1)^{1/p}}\int_0^{r_{\lambda_n}} (\lambda_n - q(r))^{1/p}dr.$$

Thus, in order to prove Theorem 1.1, in the following section we will obtain the asymptotic expansion of $N(u_n,(0,\infty))$.

3 Proof of Theorems 1.1 and 2.3

Let us state and proof several useful Lemmas.

Lemma 3.1 *Let* $\{(\lambda_n, u_n)\}_n$ *and* $q(r)$ *as in Theorem 2.3. Then,*

$$N(u_n, (0,\infty)) \sim N(u_n, (0, r_{\lambda_n})).$$

Proof: Suppose that u_n has two consecutive zeros $z_1, z_2 \in (r_{\lambda_n}, \infty)$. Then, u_n is a solution in (z_1, z_2) of

$$-(r^{N-1}|u_n'|^{p-2}u_n')' = r^{N-1}(\lambda_n - q(r))|u_n|^{p-2}u_n.$$

Multiplying by u_n the previous equation and integrating by parts the left hand side, we obtain

$$\int_{z_1}^{z_2} r^{N-1}|u_n'|^p dr = \int_{z_1}^{z_2} r^{N-1}(\lambda_n - q(r))|u_n|^p dr,$$

which is impossible because $\lambda_n - q(r) < 0$ in (r_{λ_n}, ∞), and the right hand side is negative, unless $u_n \equiv 0$.

Since u_n has $n-1$ zeros, at least $n-2$ belongs to the interval $(0, r_{\lambda_n})$, which gives

$$\limsup_{n\to\infty} \frac{N(u_n, (0,\infty))}{N(u_n, (0, r_{\lambda_n}))} \le \lim_{n\to\infty} \frac{n-1}{n-2} = 1$$

On the other hand, $N(u_n, (0,\infty)) \ge N(u_n, (0, r_{\lambda_n}))$. Hence,

$$1 \le \liminf_{n\to\infty} \frac{N(u_n, (0,\infty))}{N(u_n, (0, r_{\lambda_n}))},$$

and the Lemma is proved. □

Remark 3.2 *Let us observe that*

$$N(u_n, (0, r_{\lambda_n})) = N(u_n, (0, a)) + N(u_n, [a, r_{\lambda_n})),$$

for every $a \in (0, r_{\lambda_n})$*. For the rest of the paper, we choose* a *such that*

$$q(a) = \lambda_n - r_{\lambda_n}^{1-p-N}. \tag{3.10}$$

Lemma 3.3 *Let* $\{(\lambda_n, u_n)\}_n$ *and* $q(r)$ *as in Theorem 2.3. Then,*

$$N(u_n, (a, r_{\lambda_n})) \sim O(1).$$

Proof: Let r_0 be fixed such that $q(r)$ is increasing for $r \geq r_0$. We can assume that λ_n is big enough to have $a > max\{1, r_0\}$. Now, we apply the Sturmian oscillation theory. We have $\lambda_n - q(r) \leq \lambda_n - q(a) = r_{\lambda_n}^{1-p-N}$, since q is increasing for $r \geq r_0$. By using the inequalities $1 \leq r^{N-1} \leq r_{\lambda_n}^{N-1}$, we have the following equations:

$$-(r^{N-1}|u_n'|^{p-2}u_n')' = r^{N-1}(\lambda_n - q(r))|u_n|^{p-2}u_n,$$

and

$$-(|v'|^{p-2}v')' = r_{\lambda_n}^{N-1}(r_{\lambda_n}^{1-p-N})|v|^{p-2}v.$$

Now, between two consecutive zeros of u_n, we have at least a zero of v. Hence, an upper bound for the number of zeros of u is given by $N(v, [a, r_{\lambda_n}))+1$, where $N(v, [a, r_{\lambda_n}))$ is the number of zeros of v . Let us observe that the number of zeros of two different solutions of the same equation can differ only by one. Hence, from equation (2.3), a solution can be computed explicitly and we have

$$v(r) = S_p\left(\frac{r}{r_{\lambda_n}(p-1)^{1/p}}\right)$$

A direct computation shows that the number of zeros of v is bounded by

$$\frac{r_{\lambda_n} - a}{\pi_p r_{\lambda_n}(p-1)^{1/p}} + 1 = O(1),$$

independent of n. □

Lemma 3.4 *Let $\{(\lambda_n, u_n)\}_n$ and $q(r)$ as in Theorem 2.3. Then,*

$$N(u_n, (0, a)) \sim \frac{1}{\pi_p(p-1)^{1/p}} \int_0^a (\lambda_n - q(r))^{1/p} dr.$$

Proof: We apply the Prufer transformation, and the phase function φ_n corresponding to u_n satisfies

$$\varphi_n(0) = \pi_p/2, \qquad \varphi_n(a) = \theta_a$$

By integrating equation (2.9), we get

$$\begin{aligned} \theta_a - \tfrac{\pi_p}{2} &= \varphi_n(a) - \varphi_n(0) \\ &= \textstyle\int_0^a \left(\frac{1-N}{(p-1)r} + \frac{q'}{p(\lambda_n - q)}\right)|C_p|^{p-2}C_pS_p + \left(\frac{\lambda_n - q}{p-1}\right)^{1/p} dr \\ &= \textstyle\int_0^a \left(\frac{\lambda_n - q}{p-1}\right)^{1/p} dr + R_1 + R_2, \end{aligned} \tag{3.11}$$

where

$$R_1 = -\frac{1-N}{p-1}\int_0^a r^{-1}|C_p|^{p-2}C_pS_p dr \tag{3.12}$$

$$R_2 = -\frac{1}{p}\int_0^a \frac{q'}{\lambda_n - q}|C_p|^{p-2}C_pS_p dr. \tag{3.13}$$

Our next task is to obtain upper bounds for $|R_1|$ and $|R_2|$.

Let us observe that the term R_1 is not present when $N = 1$. We may assume here that $N \geq 2$. From equation (2.4) we have $|C_p(r)| \leq 1$ and $|S_p(r)| \leq 1$. Then,

$$\left|\frac{p-1}{1-N}R_1\right| \leq \int_0^1 |r^{-1}S_p|dr + \int_1^a r^{-1}dr.$$

Since $S_p(0) = 0$, $S_p'(0) = 1$, by applying the L'Hopital rule, the first integral is bounded by a constant independent of n. The second integral is bounded by $\log(a)$. Now, condition (a) gives

$$\lambda_n > \lambda_n - r_{\lambda_n}^{1-p-N} = q(a) \geq \alpha a^\beta.$$

Thus,

$$\log(\lambda_n) > \beta\log(a) + \log(\alpha),$$

which gives

$$|R_1| = O(\log(\lambda_n)). \tag{3.14}$$

Let us consider now R_2. We have

$$\begin{array}{ll} |pR_2| & \leq \int_0^a \frac{q'}{\lambda_n - q}dr \\ & = -\log(\lambda_n - q(a)) + \log(\lambda_n - q(0)) \\ & \leq \log(\lambda_n) - \log(r_{\lambda_n}^{1-p-N}) \\ & \leq \log(\lambda_n) + (N+p-1)\log(r_{\lambda_n}) \end{array}$$

By using again condition (a), we get

$$\lambda_n = q(r_{\lambda_n}) \geq \alpha r_{\lambda_n}^\beta.$$

Hence,

$$\log(\lambda_n) \geq \beta\log(r_{\lambda_n}) + \log(\alpha)$$

which gives

$$|R_2| = O(\log(\lambda_n)). \tag{3.15}$$

Finally, since

$$N(u_n, (0,a)) = \left[\frac{\theta_a - \pi_p/2}{\pi_p}\right] = \frac{\theta_a - \pi_p/2}{\pi_p} + O(1),$$

from equations (3.14) and (3.15) we have:

$$N(u_n, (0,a)) = \frac{1}{\pi_p(p-1)^{1/p}}\int_0^a (\lambda_n - q(r))^{1/p}dr + O(log(\lambda_n))$$

and the proof is finished. □

Lemma 3.5 *Let* $\{(\lambda_n, u_n)\}_n$ *and* $q(r)$ *as in Theorem 2.3. Then,*

$$\frac{1}{\pi_p}\int_a^{r_{\lambda_n}} \left(\frac{\lambda_n - q(r)}{p-1}\right)^{1/p} dr = O(1)$$

Proof: Let us observe that

$$\begin{aligned}\int_a^{r_{\lambda_n}} (\lambda_n - q(r))^{1/p} dr &\leq (r_{\lambda_n} - a)(\lambda_n - q(a))^{1/p} \\ &\leq r_{\lambda_n}(\lambda_n - \lambda_n + r_{\lambda_n}^{1-p-N})^{1/p} \\ &= r_{\lambda_n}^{(1-N)/p},\end{aligned}$$

which goes to zero when $N > 1$, since $r_{\lambda_n} \to \infty$ as $n \to \infty$. For $N = 1$, we obtain

$$\int_a^{r_{\lambda_n}} (\lambda_n - q(r))^{1/p} dr \leq 1,$$

and the lemma is proved. □

We can prove now Theorem 2.3:

Proof of Theorem 2.3: From Lemma 3.1 and Remark 3.2 we have

$$N(u_n, (0, \infty)) \sim N(u_n, (0, a)) + N(u_n, (a, r_{\lambda_n})).$$

Now, Lemma 3.3 gives

$$N(u_n, (0, a)) = \frac{1}{\pi_p (p-1)^{1/p}} \int_0^a (\lambda_n - q(r))^{1/p} dr + O(log(\lambda_n)),$$

and Lemmas 3.4 and 3.5 gives

$$N(u_n, (a, r_{\lambda_n})) = \frac{1}{\pi_p (p-1)^{1/p}} \int_a^{r_{\lambda_n}} (\lambda_n - q(r))^{1/p} dr + O(1).$$

Hence,

$$N(u_n, (0, \infty)) = \frac{1}{\pi_p (p-1)^{1/p}} \int_0^{r_{\lambda_n}} (\lambda_n - q(r))^{1/p} dr + O(log(\lambda_n))$$

and Theorem 2.3 is proved. □

Remark 3.6 *Since* $N(u_n, (0, \infty)) = n - 1$, *the proof of Theorem 1.1 follows immediately:*

$$n - 1 = N(u_n, (0, \infty)) \sim \frac{1}{\pi_p (p-1)^{1/p}} \int_0^{r_{\lambda_n}} (\lambda_n - q(r))^{1/p} dr.$$

We close the paper with the proof of Corollary 1.2, which gives the asymptotic distribution of eigenvalues:

Proof of Corollary 1.2: Given λ, there exists n such that

$$\lambda_n \leq \lambda < \lambda_{n+1}.$$

Now, by using that $\nu < \mu$ implies

$$\int_0^{r_\nu} (\nu - q(r))^{1/p} dr < \int_0^{r_\mu} (\mu - q(r))^{1/p} dr,$$

the result follows from Theorem 1.1 since

$$\frac{1}{\pi_p (p-1)^{1/p}} \int_0^{r_{\lambda_n}} (\lambda_n - q(r))^{1/p} dr \sim n - 1$$

$$\frac{1}{\pi_p (p-1)^{1/p}} \int_0^{r_{\lambda_{n+1}}} (\lambda_{n+1} - q(r))^{1/p} dr \sim n. \qquad \square$$

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