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Homogenización de autovalores en operadores elípticos cuasilineales

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

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Para Mora & Robert.

Homogenización de autovalores en operadores elípticos cuasilineales

(Resumen)

Distintos problemas clásicos de vibraciones mecánicas son modelados con ecuaciones diferenciales, y las frecuencias de vibración corresponden a los autovalores de éstas. Estructuras tales como columnas, placas, membranas o cuerdas, obedecen distintas clases de problemas elípticos (el sistema de ecuaciones de la elasticidad, el laplaciano, el bilaplaciano, ecuaciones de Sturm Liouville). Estos operadores han sido muy estudiados y se conocen numerosas propiedades de sus autovalores, ver por ejemplo los trabajos clásicos de Courant, Hormander, Timoshenko, Titchmarsh, Weinstein [CoHi53, Hor68, Hor07, Ti46] entre otros.

Durante el siglo XX, la teoría no lineal generó nuevas herramientas y problemas, y los autovalores son interpretados en este contexto como un parámetro de bifurcación, correponden a valores críticos para los cuales una estructura puede deformarse, colapsar o salir de equilibrio (buckling, bending). Podemos citar como ejemplo los trabajos de Antman, Browder, Berger, y Amann [Am72, An83, Be68, Br65].

En los últimos años, los nuevos materiales han creado nuevos desafíos. En particular, cuando se consideran mezclas de dos o más materiales se van obteniendo mejores propiedades específicas, y gracias a estas mejores características los materiales heterogeneos reemplazan a los homogeneos. Particularmente, materiales compuestos como por ejemplo los polímeros reforzados con fibras de vidrio o fibras de carbono, presentan unas excelentes relaciones rigidez/peso y resistencia/peso que los hace idóneos para determinados sectores productivos, esto hace que vayan desplazando a materiales tradicionales como el acero, la madera o el aluminio. Desde el punto de vista matemático esto significa principalmente que las soluciones de un problema de valores de contorno, que dependen solo de un parametro pequeño, convergen a la solución de un problema límite de contorno que puede ser explícitamente descripto [Al02, CD99, OSY92, BCR06, SV93].

Un problema interesante, común a muchos problemas diferentes más, es obtener información sobre la existencia de transiciones de fases, situaciones en las cuales la variación del parámetro ε provoca diferentes comportamientos de las soluciones.

En este trabajo nos centramos en el estudio de la homogenización de problemas de autovalores

en ecuaciones elípticas con condiciones de contorno del tipo Dirichlet y Neumann.

Esta tesis se divide esencialmente en tres partes. Primero, recolectamos propiedades conocidas sobre el espectro del $p-$Laplaciano, y luego las generalizamos a una familia mas general de operadores. Hecho esto, definimos las nociones de $H-$ y $G-$convergencia para operadores elípticos. Luego nos centramos en el estudio del comportamiento de integrales oscilantes, esto es, integrales que involucran coeficientes rápidamente oscilantes. En una última parte aplicamos estos resultados al estudio de la homogenización de problemas de autovalores elípticos y la estimación las tasas de convergencia de los autovalores.

Palabras Claves: $p-$Laplaciano; operadores monotonos; homogenización; autovalores; tasas de convergencia; $G-$convergencia; integrales oscilantes.

Eigenvalue homogenization for quasilinear elliptic operators

(Abstract)

Different classical problems of mechanic vibration are modeled with differential equations, and the vibration frequencies correspond to the eigenvalues of these. Structures such as plates, membranes and strings, obey different class of elliptic problems (the laplacian, the bilaplacian, Sturm Liouville equations). Those operators have been extensively studied and are known many properties of their eigenvalues, see for instance the classical works of Hormander, Timoshenko, Titchmarsh, Weinstein [CoHi53, Hor68, Hor07, Ti46].

Along the XX century, the non-linear theory has generated new tools and problems, and in this context, eigenvalues are interpreted like a bifurcation parameter, corresponding to the critical values for which a structure can be deformed, collapse or lose the equilibrium (buckling, bending). We cite, for instance, works of Antman, Browder, Berger, y Amann [Am72, An83, Be68, Br65].

During the last years, new materials have created new challenges, Particularly, when are considered mixing of two or more materials, better specific properties are obtained. Due to this better characteristics, heterogeneous materials replace to homogeneous ones. Particularly, materials like polymers reinforced with glass fibers or carbon fibers, present excellent relations stiffness / weight and strength / weight. For these characteristics are ideal to certain sectors of production, and they are displacing to traditional materials like steel, wood or aluminum.

From a mathematical point of view, this means mainly that solutions of a boundary value problem, which only depend of a small parameter, converge to the solution of a limit boundary problem which can be explicitly described [Al02, CD99, OSY92, BCR06, SV93].

Homogenization describes the global behavior of the *composite materials*. They are heterogeneous but the heterogeneities are very small compared to its dimension. The aim of this theory is to give macroscopic properties of the composite by taking into account the properties of the microscopic structure.

In this work we focus in the study of the homogenization of elliptic eigenvalue problems either with homogeneous Dirichlet or Neumann boundary conditions.

This thesis is divided in three parts. First, we collect known properties about the spectrum of the $p-$Laplacian operator, and then, we extend them to a more general family of operators. Done this, we define the $H-$ and $G-$convergence for elliptic operators. Then, we focus in the study of the behavior of rapidly oscillating integrals, i.e., integrals involving rapidly oscillating coefficients. In the last part we apply these results to the study of the homogenization of elliptic eigenvalue problems and estimate the eigenvalue convergence rates.

Key words: $p-$Laplacian; monotone operators; homogenization; eigenvalues; rate convergence; $G-$convergence; oscillating integrals.

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A *Sabina* por su amor, apoyo y compañia, la vida es mucho más fácil estando a su lado.

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Contents

Notation

For convenience of the reader, we list some symbols used in the thesis.

$||u||_X$: The norm of $u \in X$, where X is a normed space.

$\{u_h\}$: A sequence of functions u_h.

$u_h \to u$: $\{u_h\}$ converges strongly to u.

$u_h \rightharpoonup u$: $\{u_h\}$ converges weakly to u.

$u_h \overset{*}{\rightharpoonup} u$: $\{u_h\}$ converges weakly* to u.

Ω : Any open bounded subset of $\mathbb{R}^N$.

$\mathcal{M}_n$: The linear space of square matrices of order N.

$\mathcal{M}_{\alpha,\beta}$: The subespace of $\mathcal{M}_N$ made of coercive matrices with coercive inverses.

$\mathcal{L}^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$: Space of admissible coefficients on Ω.

List of the asymptotic notation.

$\phi(x) = O(\psi(x))$ when $x \to x_0$: mean that $|\phi(x)| \le C\psi(x)$ when $x \to x_0$ for some $C > 0$.

$\phi(x) = o(\psi(x))$ when $x \to x_0$: mean that $\phi(x)/\psi(x) \to 0$ when $x \to x_0$.

$\phi(x) \sim \psi(x)$ when $x \to x_0$: mean that $\phi(x)/\psi(x) \to 1$ when $x \to x_0$.

$\phi(x) \asymp \psi(x)$ when $x \to x_0$: mean that $c\psi(x) \le \phi(x) \le C\psi(x)$ when $x \to x_0$ for some $c, C > 0$.

List of function spaces. All functions $u$are assumed to be measurable.

$L^p(\Omega)$: All functions $u : \Omega \to \mathbb{R}$ such that

$$||u||_{L^p(\Omega)} = \Big(\int_\Omega |u(x)|^p\Big)^{1/p} < \infty, p \ge 1.$$

$L^\infty(\Omega)$: All functions $u : \Omega \to \mathbb{R}$ such that

$$||u||_{L^\infty(\Omega)} = \text{ess sup}_{x\in\Omega}|u(x)| < \infty.$$

$W^{1,p}(\Omega)$: All functions $u \in L^p(\Omega)$ such that their first-order distributional derivates are in $L^p(\Omega)$

$$||u||_{W^{1,p}(\Omega)} = \Big(||u||^p_{L^p(\Omega)} + ||\nabla u||^p_{L^p(\Omega)}\Big) < \infty.$$

$W^{1,p}_0(\Omega)$: All functions $u \in W^{1,p}(\Omega)$ such that $u = 0$ on $\partial\Omega$

$$||u||_{W^{1,p}_0(\Omega)} = ||\nabla u||_{L^p(\Omega)}.$$

$W^{-1,p'}(\Omega)$: The dual space of $W^{1,p}_0(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

1

Introduction

The mathematical theory of homogenization try to describe the behavior of composite materials. This kind of materials are characterized by having two or more finely mixed constituents, for instance, the fibred or layered structures are widely used. Some composites built are reinforced concrete, plastic reinforced by glass or carbon fiber, but there exists some heterogeneous materials with a fine microstructure that occur naturally, such as in porous rocks. Although they are heterogeneous, the heterogeneities are very small compared to its dimension. This fact allows us to differentiate two scales that characterize the material: the microscopic one, describing the heterogeneities, and the macroscopic one, describing the global behavior of the composite. From a macroscopic point of view, the composite looks like a homogeneous material. The aim of the homogenization is to give macroscopic properties of the composite by taking into account the properties of the microscopic structure.

When we are studying some physical phenomenon like heat conduction, elasticity or fluid dynamics, differential equations are good tools to describe the process and its behavior. The main difficulty when we try to solve the equations arises from the characteristics of the material. In the case of a composite material due to the fine microstructure, the physical parameters describing it are rapidly oscillating. For this reason, to handle with the corresponding differential equation can be very hard.

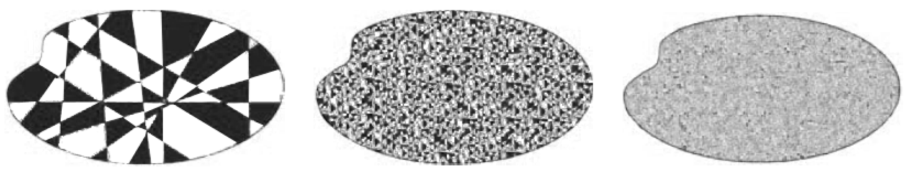

Figure 1.1: The process of homogenization of a microestructure.

1.1 The simplest model problem

The idea of the method of homogenization is to describe how a material behave at the macroscopic level from its microscopic structure.

To illustrate we study a simple model problem. Suppose we want to know the stationary temperature in a *homogeneous* body occupying a bounded open subset $\Omega \subset \mathbb{R}^N$ with constant heat conductivity A, with a heat source given by f and zero temperature on the surface $\partial\Omega$ of the body. Then the temperature can be modelated by the following boundary value problem:

$$\begin{cases} -div(A\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1.1}$$

where f is a given function on Ω and $A : \mathbb{R}^N \to \mathbb{R}^N$ satisfies certain suitable conditions that guaranties the existence and uniqueness of the solution of (1.1.1).

Now, suppose that the material is *heterogeneous*, i.e., A is not constant on Ω, $A = A(x)$, here we obtain:

$$\begin{cases} -div(A(x)\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1.2}$$

The dependence of (1.1.2) on x does more difficult to handle.

An interesting special case is the case of periodic homogenization. We will assume that the body Ω is a heterogeneous material which is built by of identical cubes with side length ε, where ε is a small positive number.

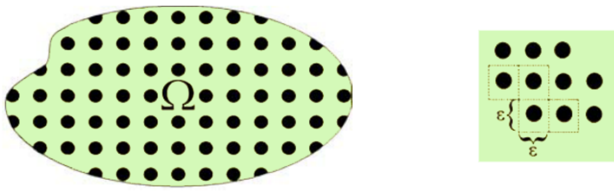

Figure 1.2: A periodic heterogeneous material.

The heat conductivity A now is a periodic function which represents how the heat varies over a reference cell Y. For simplicity we can choose Y to be the unit cube. Substituting y by $\frac{x}{\varepsilon}$, we obtain that the function $A(\frac{x}{\varepsilon})$ oscillates periodically with period ε as x go over Ω, i.e.,

$$A^{\varepsilon}(x) := A(\tfrac{x}{\varepsilon}), \quad x \in \Omega.$$

The variable x is called the *macroscopic variable*, and $\frac{x}{\varepsilon}$ the *microscopic variable*. In this case the distribution of temperature u^{ε} will be the solution of the problem

$$\begin{cases} -div(A^{\varepsilon}\nabla u^{\varepsilon}) = f & \text{in } \Omega \\ u^{\varepsilon} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1.3}$$

For each value of the parameter ε there is a corresponding equation like (1.1.3), and as ε tends to zero we obtain a sequence $\{u^\varepsilon\}$ of solutions.

From a numerical point of view, solving equation (1.1.3) by any method will require too much effort if ε is small since the number of elements (of degree of freedom) for a fixed level of accuracy grows like $1/\varepsilon^N$. It is this preferable to average of homogenize the properties of Ω and compute an approximation of u^ε on a coarse mesh.

Many natural questions arise:

Q1: Does the temperature u^ε converge to some limit function u? Is u a good approximation of u^ε?

Let us observe the following example: let $\Omega = (0,1)$, $f(x) = x^2$ and $A(x) = 1/(2 + sin(2\pi x))$. In figures 1.3, 1.4 and 1.5 we have plotted the limit solution u of (1.1.3) (which can be obtained explicitly) and the solution u^ε calculated by a numerical method for different values of ε. Moreover, it was plotted the difference between both solutions to appreciate how the approximation improves as we let ε get smaller.

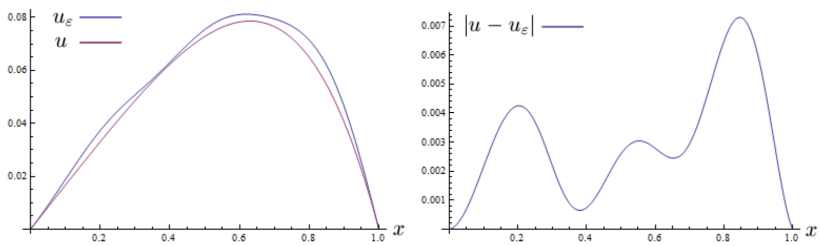

Figure 1.3: u and u^ε for $\varepsilon = 0.4$.

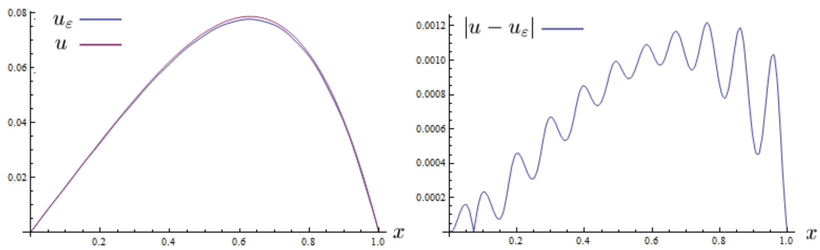

Figure 1.4: u and u^ε for $\varepsilon = 0.1$.

Q2: If u^ε converges to a limit function u, does u solve some limit boundary value problem? Are then the coefficients of the limit problem constant?

When we study the convergence of the solutions u^ε as ε go to zero we would expect that the material behaves like a homogeneous one. From a macroscopic point of view, it would be

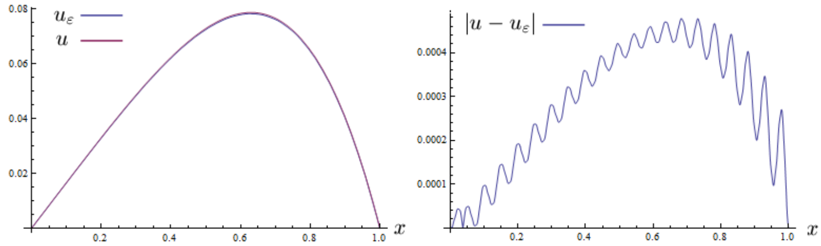

Figure 1.5: u and u^ε for $\varepsilon = 0.01$.

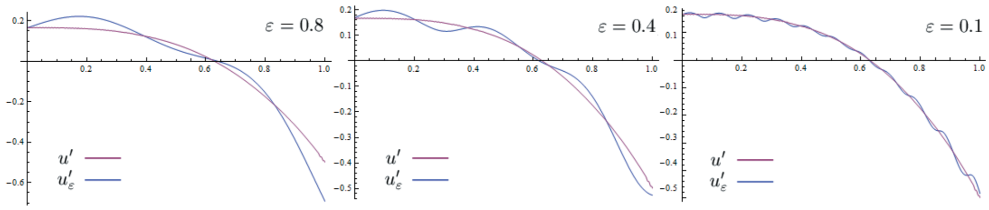

Figure 1.6: u' and $(u^\varepsilon)'$ for some values of ε.

reasonable that the limit u be described by an equation of the form

$$\begin{cases} -div(A^*\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1.4}$$

with A^* a constant matrix. Since this limit problem does not contain any oscillation, it is easier to solve than the original one. Thus, if A^* was known we could find u, which give us a very good approximation of the temperature distribution in the limit material. But, how can we find A^*?

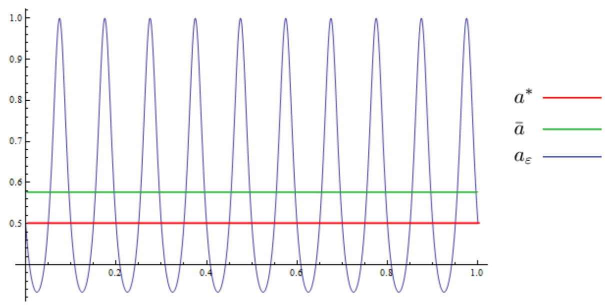

Figure 1.7: $a_\varepsilon(x) = 1/(2 + \sin(2\pi\frac{x}{\varepsilon}))$, $\bar{a} = \int_0^1 a(x)\,dx$ and a^*.

Answering these questions is the aim of the mathematical *theory of homogenization*.

1.2 $G-$ and $H-$convergence. Homogenization

Related to the convergence of the solution of elliptic problems of the type (1.1.3) are the notion of $H-$ and $G-$convergence. The main difference between these two notions of convergence is that $G-$convergence deals with symmetric matrices while $H-$convergence is defined for general sequences (not necessarily symmetric). Moreover, $G-$convergence supposes the convergence of the solutions u^ε only while $H-$convergence supposes not only the convergence of the solutions u^ε but also of $A^\varepsilon \nabla u^\varepsilon$.

Let $\mathcal{M}_N$ be the linear space of square real matrices of order N with bounded coefficients. Given α, β two positive constants, we define a space of $\mathcal{M}_N$ made of coercive matrices with coercive inverses

$$\mathcal{M}_{\alpha,\beta} = \{M \in \mathcal{M}_N : M\xi \cdot \xi \geq \alpha|\xi|^2, \quad M^{-1}\xi \cdot \xi \geq \beta|\xi|^2 \quad \forall \xi \in \mathbb{R}^N\}.$$

Given a sequence of matrices $\{A^\varepsilon\} \in \mathcal{M}_{\alpha,\beta}$ we say that A^ε *$H-$converges* to A^* if and only if for every function $f \in W^{-1,2}(\Omega)$ the solution u^ε of (1.1.3) is such that

$$\begin{cases} i) & u^\varepsilon \rightharpoonup u \quad \text{weakly in } W_0^{1,2}(\Omega), \\ ii) & A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^* \nabla u \quad \text{weakly in } (L^2(\Omega))^N, \end{cases}$$

where u is the unique solution of the problem (1.1.4).

In the particular case of symmetric matrices in $\mathcal{M}_{\alpha,\beta}$ we say that $\{A^\varepsilon\}$ *$G-$converges* to A^* if and only if for every function $f \in W^{-1,2}(\Omega)$ the solution u^ε of (1.1.3) is such that

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } W_0^{1,2}(\Omega),$$

where u is the unique solution of the problem (1.1.4). Let us observe that in the case $N = 1$, $H-$convergence always implies $G-$ convergence. When $N > 1$ this implication is true in the case of symmetric matrices.

1.2.1 A one-dimensional example

In the following classical example we will see difficulties that arises when we try to obtain the homogenized equation as ε tends to zero in (1.1.3). Here, in the one-dimensional case, the diffusion matrix $A(x)$ it is reduced to a real function $a(x)$ which we will assume be $1-$periodic for simplicity.

We will see that the main difficulty presents when we need to pass to the limit in products of only weak convergent sequences. To overcome this problem it is used the notion of the called *compensated compactness*. Particularly, in the one-dimensional case, we will be able to obtain an explicit formulation of the limit coefficient a^* through algebraic manipulation of $a(x)$.

We consider Ω be a bounded interval in $\mathbb{R}$, $\Omega = (0, 1)$ for simplicity. Let f be a function belonging to $L^2(\Omega)$ and let a be a positive $1-$periodic function in $L^\infty(\Omega)$ such that for some constants α, β

$$0 < \alpha \leq a(x) \leq \beta < +\infty, \qquad \text{for a.e. } x \in \mathbb{R}. \tag{1.2.1}$$

We define $a_\varepsilon(x) := a(\frac{x}{\varepsilon})$ and consider the following sequence of equations

$$\begin{cases} -(a_\varepsilon(u^\varepsilon)')' = f & \text{in } \Omega \\ u^\varepsilon(0) = u^\varepsilon(1) = 0, \end{cases} \tag{1.2.2}$$

where $' := \frac{d}{dx}$.

The weak form of (1.2.2) is

$$\begin{cases} \int_0^1 a_\varepsilon(u^\varepsilon)'\varphi' = \int_0^1 f\varphi & \text{for every } \varphi \in W_0^{1,2}(\Omega) \\ u^\varepsilon \in W_0^{1,2}(\Omega). \end{cases} \tag{1.2.3}$$

By a standard result in the existence theory of partial differential equations, using Lax-Milgram Lemma (see for instance [Ev10]), there exists a unique solution of these problems for each ε.

Let us observe that by duality

$$\alpha\|u^\varepsilon\|^2_{L^2(\Omega)} \leq \int_0^1 a_\varepsilon|u^\varepsilon|^2 = \int_0^1 fu^\varepsilon \leq \|f\|_{W^{-1,2}(\Omega)}\|u^\varepsilon\|_{W_0^{1,2}(\Omega)}. \tag{1.2.4}$$

By Poincaré inequality we have that

$$\|u^\varepsilon\|_{L^2(\Omega)} \leq \|(u^\varepsilon)'\|_{L^2(\Omega)}$$

which implies that

$$\|u^\varepsilon\|_{W_0^{1,2}(\Omega)} \leq \frac{1}{\alpha}\|f\|_{L^2(\Omega)}. \tag{1.2.5}$$

Since $W_0^{1,2}(\Omega)$ is a reflexive space, there exists a subsequence still denoted by ε such that

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } W_0^{1,2}(\Omega) \tag{1.2.6}$$

and since $W_0^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$ we have by Rellich Embedding Theorem (for instance, see [Ev10]) that

$$u^\varepsilon \to u \quad \text{strongly in } L^2(\Omega).$$

In general, however, we only have that

$$(u^\varepsilon)' \rightharpoonup u' \quad \text{weakly in } L^2(\Omega). \tag{1.2.7}$$

Since a is a $1-$periodic function we have that the sequence $\{a_\varepsilon\}$ converges weakly* in $L^\infty(\Omega)$ (and hence weakly in $L^2(\Omega)$) to its arithmetic mean, i.e.,

$$a_\varepsilon \overset{*}{\rightharpoonup} \bar{a} = \int_0^1 a \quad \text{weakly* in } L^\infty(\Omega). \tag{1.2.8}$$

From (1.2.3),(1.2.6) and (1.2.8) it could be reasonable that in the limit we have that u is solution of:

$$\begin{cases} \int_0^1 \bar{a}u'\varphi' = \int_0^1 f\varphi & \text{for every } \varphi \in W_0^{1,2}(\Omega) \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{1.2.9}$$

However this **is not true** in general, since $a_\varepsilon(u^\varepsilon)'$ is the product of two weakly converging sequences. This is the **main difficulty** in the limit process. To obtain the correct answer we proceed as follows.

Let $\xi_\varepsilon = a_\varepsilon(u^\varepsilon)'$. According to (1.2.8) and (1.2.5), $\{\xi_\varepsilon\}$ is bounded in $L^2(\Omega)$ and (1.2.3) implies that $-\xi'_\varepsilon = f$ in Ω. Therefore, there exists a constant C independent of k such that

$$||\xi'_\varepsilon||_{L^2(\Omega)} \leq C.$$

Again, since $W^{1,2}(\Omega)$ is reflexive, there exists a subsequence still denoted by ε, such that

$$\xi_\varepsilon \to \xi \quad \text{strongly in } L^2(\Omega).$$

Since $\{\frac{1}{a_\varepsilon}\}$ converges to $\overline{\frac{1}{a}}$ weakly* in $L^\infty(\Omega)$ (and hence weakly in $L^2(\Omega)$), we can pass to the limit in the *weak-strong* product

$$(u^\varepsilon)' = \frac{1}{a_\varepsilon}\xi_\varepsilon \rightharpoonup \overline{\frac{1}{a}}\xi \quad \text{weakly in } L^2(\Omega). \tag{1.2.10}$$

Thus, by (1.2.6) and (1.2.10), we obtain that

$$\xi = (\overline{a^{-1}})^{-1}u'. \tag{1.2.11}$$

Now we can pass to the limit in (1.2.3) obtaining

$$\begin{cases} \int_0^1 a^* u'\varphi' = \int_0^1 f\varphi & \text{for every } \varphi \in W_0^{1,2}(\Omega) \\ u^0 \in W_0^{1,2}(\Omega) \end{cases}$$

where $a^* = (\overline{a^{-1}})^{-1}$. Being $\beta^{-1} \leq a^{-1} \leq \alpha^{-1}$ we conclude that the homogenized equation has a unique solution and thus that the whole sequence $\{u^\varepsilon\}$ converges. Finally u is solution of the equation

$$\begin{cases} -(a^*u')' = f & \text{in } \Omega \\ u(0) = u(1) = 0. \end{cases}$$

Here, through algebraic manipulation, we have obtained the value of the G−limit a^* explicitly. However, when $N > 1$ or the problem is non-linear the procedure can be much more difficult.

1.3 Eigenvalue problems

Having defined the notion of convergence of problem (1.1.3), we are devoted to the study of the eigenvalue problem and its behavior as $\varepsilon \to 0$. Let us consider a sequence of symmetric matrices in $\mathcal{M}_{\alpha,\beta}(\Omega)$ for a bounded domain Ω in $\mathbb{R}^N$. Fixed a positive value of ε, the constant λ^ε is an *eigenvalue* of the operator $\mathcal{A}_\varepsilon = -div(A^\varepsilon\nabla)$ with Dirichlet boundary conditions, if there exists $u^\varepsilon \not\equiv 0$ solution of

$$\begin{cases} -div(A^\varepsilon\nabla u^\varepsilon) = \lambda^\varepsilon u^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3.1}$$

The function u^ε is called an *eigenfunction* of $\mathcal{A}_\varepsilon$, associated with the eigenvalue λ^ε. The set of the eigenvalues is called the *spectrum* of $\mathcal{A}_\varepsilon$. The symmetry assumption implies that the spectrum of $\mathcal{A}_\varepsilon$ is a countable subset of $\mathbb{R}_0^+$ whose unique accumulation point is $+\infty$, i.e., the spectrum is a increasing sequence $\{\lambda_k^\varepsilon\}$ with

$$0 < \lambda_1^\varepsilon \le \lambda_2^\varepsilon \le \cdots \to +\infty.$$

Given the matrices $\{A^\varepsilon\}$, let A^* be the corresponding homogenized matrix in the sense of the $G-$convergence. Obviously, from the symmetry of A^ε, the matrix A^* is symmetric too. Consequently, there exists a sequence of eigenvalues $\{\lambda_k\}$ corresponding to the operator $\mathcal{A}_* = -div(A^*\nabla)$ such that

$$0 < \lambda_1 \le \lambda_2 \le \cdots \to +\infty.$$

Some natural question arise:

Q1: Is $\{\lambda_k\}$ the limit of $\{\lambda_k^\varepsilon\}$ as $\varepsilon \to 0$?

Q2: If the answer is positive, can the rate of the convergence be estimated?

When we mention the order of convergence of the eigenvalues, we refer to find explicit bounds on ε and k for the difference $|\lambda_k^\varepsilon - \lambda_k|$.

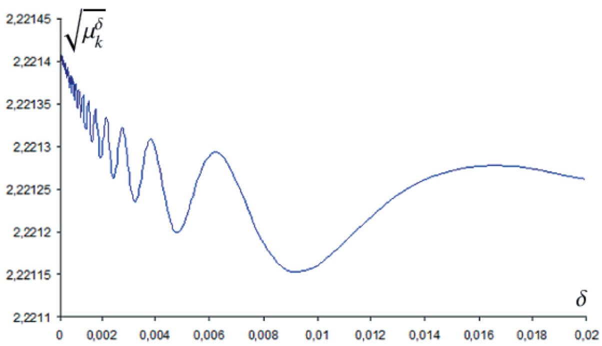

Figure 1.8: Behavior of μ_k^δ, eigenvalues of (1.3.2).

As we will see in Section 7.3.2, in the one-dimensional case $N = 1$ when a_ε is a $1-$periodic function, through a change of variables, problem

$$\begin{cases} -(a_\varepsilon (u^\varepsilon)')' = \lambda^\varepsilon u^\varepsilon & \text{in } I := (0,1) \\ u^\varepsilon(0) = u^\varepsilon(1) = 0, \end{cases}$$

can be converted in one of the form

$$\begin{cases} -w_\delta'' = \mu^\delta \rho_\delta w & \text{in } I := (0,1) \\ w_\delta(0) = w_\delta(1) = 0 \end{cases} \tag{1.3.2}$$

where ρ is a $1-$periodic function defined by

$$\rho(y) = a(Ly), \quad \rho_\delta(y) := \rho(\tfrac{y}{\delta}),$$

with

$$L_\varepsilon = \int_0^1 \frac{1}{a_\varepsilon(s)} ds \to L = \overline{a^{-1}}.$$

The new parameter is $\delta = \varepsilon L/L_\varepsilon$ and the eigenvalue is $\mu^\delta = L_\varepsilon^2 \lambda^\varepsilon$. From a computational point of view, estimate of the eigenvalues is easier in equations involving only a weight function.

For example, let us consider $\rho(x) = 2 + \sin(2\pi x)$. In this case, we obtain that $\bar{\rho} = \int_I 2 + \sin(2\pi x) dx = 2$, and the eigenvalues of the limit problem associated to (1.3.2) are given by $\mu_k = \frac{k^2\pi^2}{2}$. When δ tends to zero the value of $\sqrt{\mu_1^\delta}$ tends to the limit value $\sqrt{\mu_1} = \pi/\sqrt{2} \sim 2.2214$ displaying oscillations, as we see in Figure 1.8.

1.4 The Fučik spectrum

Consider the Laplacian eigenvalue problem with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = \lambda m(x) u & \text{in } \Omega \subset \mathbb{R}^N \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4.1}$$

As have been mentioned, (1.4.1) admits a sequence of eigenvalues $\{\lambda_k\}_k$ such that

$$0 < \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \cdots \to +\infty. \tag{1.4.2}$$

Given a function u, it can be written as $u = u^+ - u^-$, being $u^\pm$ the positive and negative part of u respectively. Now, instead consider λu in the right term of (1.4.1), we are interested in consider a more general case. Let α and β be two real parameters such that

$$\begin{cases} -\Delta u = m(x)(\alpha u^+ - \beta u^-) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4.3}$$

We denote by $\tilde{\Sigma}$ to the *spectrum* of (1.4.3), i.e., the set of points $(\alpha,\beta) \in \mathbb{R}^2$ such that (1.4.3) has non-trivial solution.

Observe that if $\alpha = \beta$ we recover (1.4.1) then, eigenvalues (1.4.2) will be contained in the spectrum of (1.4.3).

One can ask:

Q1: What happens with the spectrum of (1.4.3)? Is the spectrum a discrete sequence?

Taking a look to (1.4.3) we immediately observe that the spectrum is not a discrete sequence: Σ contain the lines $\lambda_1(m)\times\mathbb{R}$ and $\mathbb{R}\times\lambda_1(m)$, which are called the *trivial lines* of the spectrum. Here, the sequence $\{\lambda_k(m)\}$ denotes the eigenvalues of the weighted linear problem (1.4.1). See figure 1.9.

Q2: Can the spectrum be characterized?

In the case $N = 1$ (for instance, see [Ry00], [Dr92]) $\tilde{\Sigma}$ is made of the two trivial lines $\mathbb{R} \times \lambda_1(m)$ and $\lambda_1(m) \times \mathbb{R}$ together with a sequence of hyperbolic like curve in $\mathbb{R}^+ \times \mathbb{R}^+$ passing through $(\lambda_k(m), \lambda_k(m))$, $k \geq 2$; one or two such curves emanate from $(\lambda_k(m), \lambda_k(m))$, and the corresponding solutions of (1.4.3) along these curves have exactly $k-1$ zeros in $(0, 1)$.

When $N > 1$, the situation is different and a characterization of the full spectrum is not known .

Understanding the behavior of problem (1.4.3) it is useful for the study of the *Fučik spectrum* with weights, that is, the following asymmetric problem:

$$\begin{cases} -\Delta u = \alpha m(x) u^+ - \beta n(x) u^- & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4.4}$$

where m and n are two positive functions.

What happens with the spectrum Σ of (1.4.4)?

In the one-dimensional case $N = 1$ with constant coefficients (let $m = n = 1$ and $\Omega = (0, 1)$ for simplicity) the spectrum of (1.4.4) can be characterized explicitly, for instance see [FH80]. Moreover, it is composed by the following curves:

$$\begin{aligned} \Sigma_{2i} : \qquad & \frac{i\pi}{\sqrt{\alpha}} + \frac{i\pi}{\sqrt{\beta}} = 1, \\ \Sigma_{2i-1}^+ : \qquad & \frac{i\pi}{\sqrt{\alpha}} + \frac{(i-1)\pi}{\sqrt{\beta}} = 1, \\ \Sigma_{2i-1}^- : \qquad & \frac{(i-1)\pi}{\sqrt{\alpha}} + \frac{i\pi}{\sqrt{\beta}} = 1, \end{aligned}$$

In Fig 1.9 we plot this spectrum, where the axes have been moved to $\sqrt{\alpha}/\pi$ and $\sqrt{\beta}/\pi$, respectively.

In the case in which m and n are non-constants weights, in [AG01] a characterization of the spectrum is proved in terms of the so-called zeroes-functions. Unfortunately, such construction does not provides an explicit characterization of the curves.

When $N > 1$, is only known a full description of the first nontrivial curve of Σ, which we will denote by $C_1 := C_1(m, n)$, see [ACCG02, ACCG08].

1.4.1 Homogenization of the spectrum

Let us consider two sequences of functions $\{m_\varepsilon(x)\}$ and $\{n_\varepsilon(x)\}$ depending on a real parameter ε, where m, n are uniformly bounded away from zero.

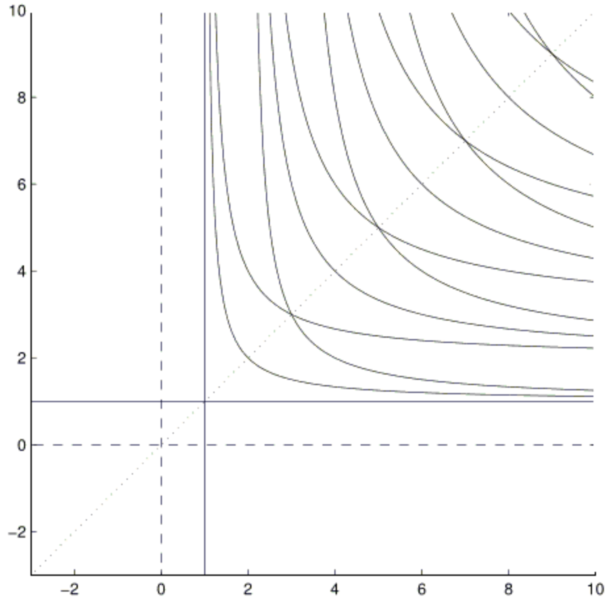

Figure 1.9: The Fučik spectrum

We are interested in studying the spectrum $\Sigma_\varepsilon(m_\varepsilon, n_\varepsilon)$ of problem (1.4.4) with weights m_ε and n_ε, i.e.,

$$\begin{cases} -\Delta u^\varepsilon = \alpha_\varepsilon m_\varepsilon(x) u_\varepsilon^+ - \beta_\varepsilon n_\varepsilon(x) u_\varepsilon^- & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4.5}$$

Particularly, for each value of ε there exists a curve $C_1^\varepsilon := \{\alpha_\varepsilon, \beta_\varepsilon\}$ in the spectrum $\Sigma_\varepsilon(m_\varepsilon, n_\varepsilon)$ associated to problem (1.4.5).

Having in mind these problems, we wonder:

Q1: There exists a limit curve $C_1 = \{(\alpha_0, \beta_0)\}$ such that $C_1^\varepsilon \to C_1$ as $\varepsilon \to 0$?

Q2: Can this limit curve be characterized like a curve of a limit problem?

Q3: If the answer is positive, can be estimated a rate of convergence of C_1^ε?

When $C_1^\varepsilon \to C_1$ as $\varepsilon \to 0$ we would like to obtain a estimate of the remainders $|\alpha_\varepsilon - \alpha_0|$ and $|\beta_\varepsilon - \beta_0|$, that is, if C_1^ε can be described as $\{(\alpha_\varepsilon(s), \beta_\varepsilon(s)), s \in \mathbb{R}^+\}$ and C_1 as $\{(\alpha_0(s), \beta_0(s)), s \in \mathbb{R}^+\}$, we want a estimate of the kind

$$|\alpha_\varepsilon - \alpha_0| \leq c\tau(s)\varepsilon$$

with c a constant independent of ε and s, and τ a functions depending only on s.

Q4: What happens with other boundary conditions? Can a similar results be obtained ?

1.5 Outline of the thesis

In **Chapter §2**, we deal with the eigenvalue problem of the weighted *p−Laplacian* operator $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ with Dirichlet boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, i.e.,

$$\begin{cases} -\Delta_p u = \lambda\rho|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.5.1}$$

where ρ is a weight bounded uniformly away from zero and infinity. Here, we define the concept of eigenvalue and eigenfunction associated to (1.5.1) as well as its variational spectrum. Then, we remember some useful properties about the (variational) eigenvalues: the first eigenvalue λ_1 is positive, simple (the eigenfunctions associated to it are one multiple of the other one) and isolated (there is no eigenvalue between λ_1 and $\lambda_1 + \delta$ for a small δ). Moreover, eigenfunctions associated to the first eigenvalue do not change sign in Ω. The second eigenvalue is variational and an eigenfunction associated to it has two nodal domains. In the one-dimensional case $N = 1$ it is well-known that the k-th eigenvalue are simple and its associated eigenfunctions have $k-1$ zeroes and the variational eigenvalues exhaust the full spectrum.

Also, we define the concept of monotone operators, which extends (1.5.1) to a more general family of eigenvalue problems of the form

$$\begin{cases} -div(a(x,\nabla u)) = \lambda\rho|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.5.2}$$

where $a(\cdot,\cdot)$ satisfies certain properties that we made precise later (c.f Section 2.4). Here, we generalize all the properties known for the eigenvalues of the $p-$Laplacian for the case $N = 1$ and $N > 1$.

In **Chapter §3** we study the asymptotic distribution of eigenvalues in one-dimensional open sets. We consider a set $\Omega \subset \mathbb{R}$ which is a disjoint union of bounded intervals, $\Omega = \bigcup_{j\in\mathbb{N}} I_j$ such that

$$|I_1| \geq |I_2| \geq \cdots \geq |I_j| \geq \cdots \searrow 0,$$

and we assume that there exists some nonincreasing function $g : (0,\infty) \to (0,\infty)$ such that

$$|I_j| = g(j).$$

We consider the two equivalent following problems:

- A Lattice Point Problem: to estimate, for $x \nearrow \infty$, the number of lattice points below the curve $xg(t)$,

$$N(x) = \#\{(j,k) \in \mathbb{N}\times\mathbb{N} : k \leq xg(j)\} = \sum_{j=1}^{\infty}[xg(j)]. \tag{1.5.3}$$

- An Eigenvalue Counting Problem: to estimate, for $\lambda \nearrow \infty$, the number of eigenvalues less than or equal to λ of $-(|u'|^{p-1}u')' = \lambda|u|^{p-2}u$ in Ω with zero Dirichlet boundary conditions on $\partial\Omega$,
$$N(\lambda) = \#\{j \in \mathbb{N} \colon \lambda_j \leq \lambda\},$$

Indeed, both problems are the same: $N(\lambda) = \frac{1}{\pi_p}\sum_{j=1}^{\infty}\left[g(j)\lambda^{1/p}\right]$.

We are interested in the asymptotic number of eigenvalues of the following problem in Ω:

$$-(|u'|^{p-2}u')' = \lambda|u|^{p-2}u, \tag{1.5.4}$$

with zero Dirichlet boundary conditions on $\partial\Omega$, and $1 < p < +\infty$.

In the linear case ($p = 2$) and when the measure of Ω is finite, He, Lapidus and Pomerance [HL97, La93] obtain that

$$N(\lambda) = \#\{j \in \mathbb{N} \colon \lambda_j \leq \lambda\} = \frac{|\Omega|}{\pi}\lambda^{1/2} + \frac{\zeta(d)}{\pi^d}f(\lambda^{1/2}) + o(f(\lambda^{1/2}))$$

where $0 < d < 1$, $f(x) = g^{-1}(1/x)$ and ζ is the Riemann Zeta function.

In this Chapter we characterize the growth of the number of eigenvalues $N(\lambda)$ in terms of the decay of the lengths of the intervals when the measure of Ω is finite. When $\Omega \subset \mathbb{R}$ is bounded and $p > 1$, we obtain

$$N(\lambda,\Omega) = \frac{|\Omega|}{\pi_p}\lambda^{1/p} + \frac{\zeta(d)}{\pi_p^d}f(\lambda^{1/p}) + o(f(\lambda^{1/p})) \quad \text{as} \quad \lambda \to \infty$$

where $f(x) = g^{-1}(1/x)$ and $0 < d < 1$. Moreover, when the measure of Ω is not finite we obtain the following non-standard asymptotic formula

$$N(\lambda) = \#\{j \in \mathbb{N} \colon \lambda_j \leq \lambda\} = \frac{\zeta(d)}{\pi_p^d}f(\lambda^{1/p}) + o(f(\lambda^{1/p})),$$

where now $d > 1$.

In **Chapter §4** basically we discuss the definitions of $H-$ and $G-$*convergence* for elliptic operators. Here, we deal with classical examples of homogenization in the one-dimensional case and difficulties that arise. Furthermore, we define $G-$convergence for elliptic monotone operators and we review some essential results for the important case of periodic homogenization.

In **Chapter §5** we prove some results concerning to the estimate of integrals involving rapidly oscillating functions. They allows us to replace an integral involving a rapidly oscillating function with one that involves its average in the unit cube. This kind of results will be very useful to estimate the rate of convergence of eigenvalues in homogenization problems. Indeed, following the ideas of Oleĭnik [OSY92] we prove the following results. For every $u \in W^{1,p}(\Omega)$ there exists a constant C independent of ε such that

$$\left|\int_\Omega (g(\tfrac{x}{\varepsilon}) - \bar{g})u\right| \leq C\varepsilon\|u\|_{W^{1,p}(\Omega)}. \tag{1.5.5}$$

where g is a Q-periodic function and $\bar{g}$ denotes the average of g over Q

In the case of functions in $u \in W_0^{1,p}(\Omega)$ we prove that

$$\Big|\int_\Omega (g(\tfrac{x}{\varepsilon}) - \bar{g})|u|^p\Big| \le C_1 \varepsilon \|\nabla u\|^p_{L^p(\Omega)}. \tag{1.5.6}$$

In both cases constants C and C_1 are unknown. The fact of enlarge the set of test functions is reflected in the regularity of the domain Ω. In (1.5.5) we need little regularity, let us say Lipschitz boundary or less. Instead, in (1.5.6) is necessary a more regularity, for instance a domain with C^1 boundary.

In the one-dimensional case we can be more precise and we found an explicit value of the constants C and C_1: for every function $u \in W_0^{1,p}(I)$, $I := (0,1)$ we have

$$\Big|\int_I (g(\tfrac{x}{\varepsilon}) - \bar{g})|u|^p\Big| \le \|g - \bar{g}\|_{L^\infty(\mathbb{R})}\varepsilon \|u'\|^p_{L^p(I)}\Big[\frac{p}{\pi_p^{p-1}} + \frac{\varepsilon^{p-1}}{p}\Big]$$

where $\bar{g} = \int_0^1 g$. In the case of functions $u \in W^{1,p}(I)$ we obtain that

$$\Big|\int_I (g(\tfrac{x}{\varepsilon}) - \bar{g})u\Big| \le \|g - \bar{g}\|_{L^\infty(\mathbb{R})}\Big((2^p + p)^{\frac{1}{p}}(2^{\frac{p}{p-1}} + \tfrac{p-1}{p})^{\frac{p-1}{p}} + \tfrac{1}{(p-1)\pi_p}\Big)\varepsilon\|u\|_{W^{1,p}(I)}.$$

In **Chapter §6** we are devoted to study the asymptotic behavior (as $\varepsilon \to 0$) of the eigenvalues of the following problems

$$\begin{cases} -div(a_\varepsilon(x, \nabla u^\varepsilon)) = \lambda^\varepsilon \rho_\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5.7}$$

where ε is a positive real number, and λ^ε is the eigenvalue parameter.

The weight functions ρ_ε are assumed to be positive and uniformly bounded away from zero and infinity and the family of operators $a_\varepsilon(x, \xi)$ have precise hypotheses, but the prototypical example is

$$-\,div(a_\varepsilon(x, \nabla u^\varepsilon)) = -div(A^\varepsilon(x)|\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon), \tag{1.5.8}$$

with $1 < p < +\infty$, and $A^\varepsilon(x)$ is a family of uniformly elliptic matrices (both in $x \in \Omega$ and in $\varepsilon > 0$).

As ε tends to zero, eigenvalues of (1.5.7) tends to those of a limit problem of the kind

$$\begin{cases} -div(a(x, \nabla u)) = \lambda\bar{\rho}|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5.9}$$

where $a(x, \xi)$ is the $G-limit$ of $a_\varepsilon(x, \xi)$.

In this Chapter, we analyze the order of convergence of eigenvalues of (1.5.7) to the ones of its limit problem and prove that

$$\lambda_1^\varepsilon \to \lambda_1, \qquad \lambda_2^\varepsilon \to \lambda_2 \quad \text{as } \varepsilon \to 0.$$

In the periodic framework the first result in this problem, for the linear case, can be found in the work of Oleĭnik, Shamaev and Yosifian [OSY92]. In the case in which the diffusion matrix does not depends on ε, using tools from functional analysis in Hilbert spaces, they deduce that

$$|\lambda_k^\varepsilon - \lambda_k| \leq \frac{C\lambda_k^\varepsilon(\lambda_k)^2}{1-\lambda_k\beta_k^\varepsilon}\,\varepsilon^{\frac{1}{2}}.$$

Here, C is a positive constant, and β_ε^k satisfies

$$0 \leq \beta_\varepsilon^k < \lambda_k^{-1}, \qquad \lim_{\varepsilon\to 0}\beta_\varepsilon^k = 0$$

for each $k \geq 1$.

More recently, Kenig, Lin and Shen [KLS11] studied the linear problem (allowing an ε dependance in the diffusion matrix of the elliptic operator) and proved that for Lipschitz domains Ω one has

$$|\lambda_k^\varepsilon - \lambda_k| \leq C\varepsilon|\log(\varepsilon)|^{\frac{1}{2}+\sigma}$$

for any $\sigma > 0$, C depending on k and σ.

Moreover, the authors show that if the domain Ω is more regular ($C^{1,1}$ is enough) they can get rid of the logarithmic term in the above estimate. However, no explicit dependance of C on k is obtained in that work.

When the dependance on ε only appears in an oscillating weight ρ_ε we prove that the kth-variational eigenvalue of problem (1.5.7) converges to the kth-variational eigenvalue of the limit problem (1.5.9). In this case we estimate the rate of convergence as

$$|\lambda_k^\varepsilon - \lambda_k| \leq Ck^{\frac{2p}{N}}\varepsilon$$

with C independent of k and ε. By λ_k^ε and λ_k we refer to the variational eigenvalues of problems (1.5.7) and (1.5.9) respectively. This result generalizes the mentioned bounds for the linear case.

Also, we prove that this estimates still holds for the Neumann boundary condition:

$$|\lambda_k^\varepsilon - \lambda_k| \leq \tilde{C}k^{\frac{2p}{N}}\varepsilon$$

for some $\tilde{C}$ independent of k and ε, where λ_k^ε and λ_k are the variational eigenvalues of problems (1.5.7) and (1.5.9) with Neumann boundary condition, respectively.

In **Chapter §7** we study problem (1.5.7) in the one-dimensional case $N = 1$. Now, the function $a_\varepsilon(x,\xi^\varepsilon)$ in (1.5.7) can be explicitly expressed as $a_\varepsilon(x,\xi^\varepsilon) = a_\varepsilon(x)|\xi^\varepsilon|^{p-2}\xi^\varepsilon$ and its $G-$limit (1.5.9) is given by $a_h(x,\xi) = a_h(x)|\xi|^{p-2}\xi$. Moreover, in the periodic framework, $a_h(x)$ is constant and is given by

$$a_p^* = \Big(\int_I a(x)^{-\frac{1}{p-1}}\Big)^{-(p-1)}.$$

In this Chapter, we analyze the convergence of eigenvalues of (1.5.7) to the ones of its limit problem and prove that for each $k \in N$,

$$\lambda_k^\varepsilon \to \lambda_k, \qquad \text{as } \varepsilon \to 0$$

where λ_k^ε and λ_k are the variational eigenvalues of problems (1.5.7) and (1.5.9) with $N = 1$.

The problem, in the linear and periodic setting, and in dimension $N = 1$, with $a = 1$, was recently studied by Castro and Zuazua in [CZ00, CZ00b]. In those articles the authors, using the so-called WKB method which relays on asymptotic expansions of the solutions of the problem, and the explicit knowledge of the eigenfunctions and eigenvalues of the constant coefficient limit problem, proved

$$|\lambda_k^\varepsilon - \lambda_k| \leq Ck^4\varepsilon$$

Let us mention that their method needs higher regularity on the weight ρ and on the diffusion a, which must belong at least to C^2 and that the bound holds for $k \sim \varepsilon^{-1}$. Also, the value of the constant C entering in the estimate is unknown.

Our main result in this chapter is the following: in periodic settings, i.e., $a_\varepsilon = a(x/\varepsilon)$ and $\rho_\varepsilon = \rho(x/\varepsilon)$ are 1−periodic functions, there exists a constant C depending only on p, a and ρ such that

$$|\lambda_k^\varepsilon - \lambda_k| \leq Ck^{2p}\varepsilon$$

where λ_k^ε and λ_k are the variational eigenvalues of problems (1.5.7) and (1.5.9) with $N = 1$. Moreover, C can be estimated explicitly in terms of the functions a and ρ.

Also, we study problem (1.5.7) in the one-dimensional case $N = 1$ with Neumann boundary conditions. We prove that for each $k \in \mathbb{N}$, $\lambda_k^\varepsilon \to \lambda_k$ as $\varepsilon \to 0$. In the periodic framework we find an explicit expression of the constant C in the inequality

$$|\lambda_k^\varepsilon - \lambda_k| \leq \tilde{C}(k-1)^{2p}\varepsilon$$

where λ_k^ε and λ_k are the eigenvalues of problems (1.5.7) and (1.5.9) with Neumann boundary conditions and $N = 1$, respectively.

In **Chapter §8** we deal with the following asymmetric problem

$$-\Delta_p u_\varepsilon = \alpha_\varepsilon m_\varepsilon (u_\varepsilon^+)^{p-1} - \beta_\varepsilon n_\varepsilon (u_\varepsilon^-)^{p-1} \quad \text{in } \Omega \subset \mathbb{R}^N \tag{1.5.10}$$

either with homogeneous Dirichlet or Neumann boundary conditions. For each $\varepsilon > 0$, consider the Fučik spectrum defined as the set

$$\Sigma(m_\varepsilon, n_\varepsilon) := \{(\alpha_\varepsilon, \beta_\varepsilon) \in \mathbb{R}^2 : \text{(1.5.10) has nontrivial solution}\}.$$

It is known that Σ contain the trivial lines $\lambda_1^\varepsilon(m) \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1^\varepsilon(n)$. Also, only a characterization of the first non-trivial curve in the spectrum, say C_1, is known:

$$C_1^\varepsilon = \{(\alpha_\varepsilon(s), \beta_\varepsilon(s)), s \in \mathbb{R}^+\} \tag{1.5.11}$$

where $\alpha(s)$ and $\beta(s)$ are continuous functions defined by in terms of a min-max quotient. Assuming that $m_\varepsilon(x) \rightharpoonup m(x)$ and $n_\varepsilon(x) \rightharpoonup n(x)$ weakly* in $L^\infty(\Omega)$, the natural limit of (1.5.10) as $\varepsilon \to 0$ is

$$-\Delta_p u_0 = \alpha_0 m(x)(u_0^+)^{p-1} - \beta_0 n(x)(u_0^-)^{p-1} \quad \text{in } \Omega \tag{1.5.12}$$

either with homogeneous Dirichlet or Neumann boundary. The first non-trivial curve in the spectrum of (1.5.12) is given by

$$C_1 = \{(\alpha_0(s), \beta_0(s)), s \in \mathbb{R}^+\}. \tag{1.5.13}$$

Under these considerations we prove that

$$C_1^\varepsilon(m_\varepsilon, n_\varepsilon) \to C_1(m,n) \quad \text{as } \varepsilon \to 0$$

in the sense that $\alpha_\varepsilon(s) \to \alpha(s)$ and $\beta_\varepsilon(s) \to \beta(s)$ $\forall s \in \mathbb{R}^+$. Moreover, when the weights m_ε and n_ε are given in terms of Q−periodic functions m, n in the form $m_\varepsilon(x) = m(\frac{x}{\varepsilon})$ and $n_\varepsilon(x) = n(\frac{x}{\varepsilon})$, being Q the unit cube in $\mathbb{R}^N$, for each $s \in \mathbb{R}^+$ we have the following estimates

$$|\alpha_\varepsilon(s) - \alpha_0(s)| \le c(1+s)\tau(s)\varepsilon, \quad |\beta_\varepsilon(s) - \beta_0(s)| \le cs(1+s)\tau(s)\varepsilon \tag{1.5.14}$$

where c is a constant independent of ε and s and τ is a function depending only on s.

Included publications

The results in have appeared published as research articles. These results are readable as individuals contributions linked by a common theme and all of them are either published, accepted for publication or submitted for publication in refereed journals. The chapters contain the following papers:

[FBPS10] Julián Fernández Bonder, Juan Pablo Pinasco, Ariel M. Salort. *Refined asymptotics for eigenvalues on domains of infinite measure.* J. Math. Anal. Appl., 371 (2010), no. 1, 41–56.
http://arxiv.org/abs/0906.2198.

[FBPS12] Julián Fernández Bonder, Juan Pablo Pinasco, Ariel M. Salort. *Eigenvalue homogenization for quasilinear elliptic operators.* Submitted for publication.
http://arxiv.org/abs/1201.1219.

[FBPS12b] Julián Fernández Bonder, Juan Pablo Pinasco, Ariel M. Salort. *Eigenvalue homogenization for quasilinear elliptic operators in one space dimension.* Submitted for publication.
http://arxiv.org/abs/1203.2091.

[Sa12] Ariel M. Salort. *Convergence rates in a weighted Fučik problem.* Submitted for publication.
http://arxiv.org/abs/1205.2075.

2

Eigenvalues

2.1 Eigenvalues of the weighted $p-$Laplacian in $\mathbb{R}^N$

For $1 < p < \infty$ the $p-$Laplacian operator is defined as

$$\Delta_p u := div(|\nabla u|^{p-2}\nabla u) \tag{2.1.1}$$

Obviously, $\Delta_2 = \Delta$ is the usual Laplace Operator. Note that for $p \neq 2$ the operator (2.1.1) is $(p-1)-$homogeneous but not additive. For this reason, some authors call equations involving the $p-$Lalacian *half-linear* equation.

Eigenvalue problems for the $p-$Laplacian operator subject to zero Dirichlet boundary conditions on a bounded domain have been studied extensively during the past two decades and many interesting results have been obtained. In this section we collect some of those one more important to our purpouse.

We consider the following weighted eigenvalue problem with Dirichlet boundary conditions

$$\begin{cases} -\Delta_p u = \lambda\rho(x)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and λ is a parameter. The weight ρ is such that for two positive constants $\rho^- < \rho^+$

$$0 < \rho^- \leq \rho(x) \leq \rho^+ < \infty \qquad \text{a.e. in } \Omega. \tag{2.1.3}$$

The solution of problem (2.1.2) is understood in the weak sense; we say that λ is a *eigenvalue* if there exists a function $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$, such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\xi = \lambda \int_\Omega \rho|u|^{p-2}u\xi$$

for every $\xi \in W_0^{1,p}(\Omega)$. The function u is called an *eigenfunction*.

The first eigenvalue $\lambda_1 = \lambda_1(\Omega)$ is obtained as the minimum of the Rayleigh quotient

$$\lambda_1 = \inf_u \frac{\int_\Omega |\nabla u|^p}{\int_\Omega \rho |u|^p} \tag{2.1.4}$$

where the infimum is taken over all $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$. If u realizes the infimum in (2.1.4), so does $|u|$, and this leads immediately to the following statement.

The following *strong maximum principle* holds:

Theorem 2.1. *If $u \in W_0^{1,p}(\Omega)$ is non-negative such that $-\Delta_p u \geq 0$ then either $u \not\equiv 0$ or $u(x) > 0$ for all $x \in \Omega$.*

Proof. It follows from the Harnack's Inequality, see [SePu84]. □

Theorem 2.2. *The eigenfunction u associated with the first eigenvalue λ_1 does not changes its sign in Ω. We assume $u \geq 0$, then $u > 0$ in Ω.*

Proof. The function $v = |u|$ minimizes (2.1.4) then $v \geq 0$ verifies $-\Delta_p v = \lambda \rho |v|^{p-2} v \geq 0$. Then from the Strong Maximum Principle given in Theorem (2.1) it follows that $v > 0$ in Ω and so $u > 0$ in Ω. □

The first eigenvalue λ_1 satisfies two important properties: it is *simple* (i.e., if u and v are two eigenfunctions corresponding to λ_1 then $u = \alpha v$ for some $\alpha \in \mathbb{R}$), and it is *isolated* (i.e., there exists $\delta > 0$ such that in the interval $(\lambda_1, \lambda_1 + \delta)$ there are no other eigenvalues of (2.1.2)). These results are proved in the following theorem.

Theorem 2.3. *The first eigenvalue of (2.1.2) is simple and isolated for any bounded domain $\Omega \subset \mathbb{R}^N$.*

We omit the proof, which can be found in [Cu01], Proposition 4.1 and Proposition 4.2, where the more general case in which the weight ρ may change sign in Ω and satisfies

$$\rho \in L^q(\Omega) \text{ where } \begin{cases} q > \frac{N}{p} \text{ if } 1 < p \leq N \\ q = 1 \text{ if } p > N. \end{cases} \tag{2.1.5}$$

is considered.

We recall that a *nodal domain* of an eigenfunction u is a connected component of $\Omega \setminus \{x \in \Omega : u(x) = 0\}$. In the following result is given an estimate of the measure of the nodal domains of the eigenfunctions for the general case in which ρ may changes of sign.

Theorem 2.4. *Let ρ satisfying (2.1.5). Then any eigenfunction u associated to a positive eigenvalue $0 < \lambda \neq \lambda_1$ changes signs. Moreover if $\mathcal{N}$ is a nodal domain of u then*

$$|\mathcal{N}| \geq (C\lambda \|\rho\|_{L^q(\Omega)})^{-\gamma}$$

where $\gamma = \frac{qN}{qp-N}$ and C is some constant depending only on N and p if $p \neq N$ and on N and q' if $p = N$.

The result is proved in Theorem 3.2 of [Cu01]. In the case of positive weights, see [AP96]. As corollary, it is obtained that each eigenfunction of (2.1.2) has a finite number of nodal domains. Moreover, Theorems 2.2 and 2.9 say that an eigenfunction associated to the first eigenvalue of (2.1.2) has only one nodal domain; and any eigenfunction associated to the second eigenvalue of (2.1.2) has exactly two nodal domains.

The following definition will be useful to define the variational eigenvalues of (2.1.2).

Definition 2.5. Let A be a symmetric subset in a Banach space, i.e., $A = -A$, we define the *Kranoselskii genus* of A as

$\gamma(A) = \{$the minimal integer m such that there exists a continuous odd mapping of $C \to \mathbb{R}^m \setminus \{0\}\}$.

If no such integer exists we set $\gamma(A) = \infty$, and for the empty set, we define $\gamma(\emptyset) = 0$, see [Ra74, DSP03] for properties.

Let us denote

$$\Gamma_k = \{C \subset W_0^{1,p}(\Omega) : C \text{ compact}, C = -C, \ \gamma(C) \geq k\}.$$

By means of the critical point theory of Ljusternik–Schnirelmann (see [Sz88],[An06]) it is straightforward to obtain a discrete sequence of variational eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ tending to $+\infty$. The $k-$th variational eigenvalue is given by (see Theorem 6.1.2 in [DP05], for $\rho \equiv 1$ see [GAP88])

$$\lambda_k = \inf_{C\in\Gamma_k} \sup_{v\in C} \frac{\int_\Omega |\nabla v|^p}{\int_\Omega \rho |v|^p} \tag{2.1.6}$$

We denote the sequence of variational eigenvalues by

$$\Sigma_{var} := \{\lambda_k \text{ given by (2.1.6)}, k \in \mathbb{N}\}. \tag{2.1.7}$$

Note that it is an open problem whether (2.1.6) described all eigenvalues of (2.1.2) (in contrast to the scalar case $N = 1$, see Theorem 2.13).

Remark 2.6. Intuitively, the Kranoselskii genus γ provides a measure of the dimension of a symmetric set. For example, if Ω is a bounded symmetric neighborhood of the origin in $\mathbb{R}^m$, then $\gamma(\partial\Omega) = m$.

Remark 2.7. One can also define another sequence of critical values minimaxing along a smaller family of symmetric subsets of $W_0^{1,p}(\Omega)$. The following result can be proved using the minimax principle of [Cu03]. Let us denote by S^k the unit sphere of $\mathbb{R}^{k+1}$ and

$$O(S^k, W_0^{1,p}(\Omega)) := \{h \in C(S^k, W_0^{1,p}(\Omega)) : \text{h is odd}\}.$$

Then for any $k \in \mathbb{N}$ the value

$$\tilde{\lambda}_k := \inf_{h\in O(S^{k-1}, W_0^{1,p}(\Omega))} \max_{z\in S^{k-1}} \frac{\int_\Omega |\nabla h(z)|^p}{\int_\Omega \rho |z|^p} \tag{2.1.8}$$

is a eigenvalue of (2.1.2). Moreover $\lambda_k \leq \tilde{\lambda}_k$ and it is a trivial fact that $\lambda_1 = \tilde{\lambda}_1$ is the infimum given in (2.1.4). In Theorem 2.8 we also see that $\lambda_2 = \tilde{\lambda}_2$. Whether or not $\lambda_k = \tilde{\lambda}_k$ for other values of k is still an open question when $p \neq 2$. For $p = 2$ the proof that $\lambda_k = \tilde{\lambda}_k$ is simple: when $N = 1$, $\rho = 1$ and $p > 1$ it is proved (for instance in [Cu98]) that $\lambda_k = \tilde{\lambda}_k$ for all $k \geq 1$ but this last equality remains an open question when $N > 1$.

Since λ_1 is isolated in the spectrum and there exists eigenvalues different from λ_1, it makes sense to define the *second eigenvalue* of (2.1.2) as

$$\Lambda_2 = \inf\{\lambda\colon \lambda \text{ is eigenvalue of (2.1.2), and } \lambda > \lambda_1\}.$$

There exist several variational characterizations of λ_2 through minimax formulas.

The following result it is obtained as a consequence of the construction of the first *Fučik's curve* in the paper [ACCG02] of Arias, Campos, Cuesta and Gossez (see Chapter §8).

Theorem 2.8. *Assume that ρ satisfies (2.1.3). Then*

$$\Lambda_2 = \inf_{h\in\mathcal{F}} \max_{u\in h([-1,1])} \int_\Omega |\nabla u|^p$$

where $\mathcal{F} := \{\gamma \in C([-1,1], W_0^{1,p}(\Omega) : \gamma(\pm 1) = \pm\varphi_1\}$ and φ_1 is the positive eigenfunction associated to λ_1. Moreover,

$$\Lambda_2 = \lambda_2 = \mu_2$$

where λ_2 is given by (2.1.6) and μ_2 by (2.1.8).

From Theorem 2.2 it follows that an eigenfunction associated with λ_1 has an only one nodal domain. With respect to the number of nodal domains of eigenfunctions associated to λ_2 we have the following result.

Theorem 2.9. *An eigenfunction associated to the second eigenvalue of problem (2.1.2) admits exactly two nodal domains.*

This result was proved by Cuesta,De Figueiredo and Gossez in the case $\rho = 1$. For positive weights, see Theorem 3.1 in [ACFK07].

2.2 Eigenvalues of the $p-$Laplacian in $\mathbb{R}$

2.2.1 The one-dimensional $p-$Laplace operator

For the one dimensional $p-$Laplace operator in Ω

$$-(|u'|^{p-2}u')' = \lambda|u|^{p-2}u \tag{2.2.1}$$

with zero Dirichlet boundary conditions all the eigenvalues and eigenfunctions can be found explicitly.

To give such characterization, first we remember the definition of the generalized trigonometric functions.

The function $\sin_p(x)$ is the solution of the initial value problem

$$\begin{cases} -(|u'|^{p-2}u')' = |u|^{p-2}u \\ u(0) = 0, \qquad u'(0) = 1, \end{cases}$$

and is defined implicitly as

$$x = \int_0^{\sin_p(x)} \Big(\frac{p-1}{1-t^p}\Big)^{1/p} dt.$$

Moreover, its first zero is π_p, given by

$$\pi_p = 2\int_0^1 \Big(\frac{p-1}{1-t^p}\Big)^{1/p} dt.$$

Note that

$$p\pi_p = 2\Gamma(1/p')\Gamma(1/p) = p'\pi_{p'}. \tag{2.2.2}$$

Where Γ is the Gamma function.

It is well known from the basic calculus that

$$\int_0^1 \frac{1}{\sqrt{1-t^2}} dt = \frac{\pi}{2}$$

and that

$$\arcsin(x) = \int_0^x \frac{1}{\sqrt{1-t^2}} dt$$

define a differentiable function on $[0,1]$. Since $\frac{1}{\sqrt{1-t^2}}$ is positive on $(0,1)$, the function is increasing and one-to-one from $[0,1]$ to $[0,\pi/2]$. This function is $arcsin(x)$ and can be used to define the function sin on $[0,\pi/2]$. By standard symmetry arguments we can extend the sin function to the whole $\mathbb{R}$. We extend this to $1 < p < \infty$. We define for $1 < p < \infty$ the function

$$F_p(x) = \int_0^x \frac{1}{\sqrt[p]{1-t^p}} dt, \ x \in [0,1].$$

Then $F_2(x) = \sin^{-1}(x)$. As F_p is strictly increasing it is a one-to-one function on $[0,1]$ with range $[0,\pi_p/2]$. Then it has an inverse, which we denote by $\sin_p$ to emphasize the confection with the usual sine function. This is defined in the interval $[0,\pi_p/2]$, where

$$\frac{\pi_p}{2} = \sin_p^{-1}(1) = \int_0^1 \frac{1}{\sqrt[p]{1-t^p}} dt = \frac{1}{p}\int_0^1 \frac{1}{\sqrt[p]{1-s}} s^{-\frac{1}{p'}} ds = \frac{1}{p} B\left(\frac{1}{p'}, \frac{1}{p}\right)$$

where B is the Beta function. Hence

$$\pi_p = \frac{2\pi}{p\sin(\pi/p)} \tag{2.2.3}$$

Note that $\pi_2 = \pi$. Moreover, π_p decreases as p increases, and

$$\lim_{p\to 1} \pi_p = \infty, \ \lim_{p\to\infty} \pi_p = 2, \ \lim_{p\to 1}(p-1)\pi_p = \lim_{p\to 1} \pi_{p'} = 2$$

We see that $\sin_p$ is strictly increasing on $[0, \pi_p/2]$, $\sin_p(0) = 0$ and $\sin_p(\pi_p/2) = 1$. It may be extended to $[0, \pi_p]$ by defining $\sin_p(x) = \sin_p(\pi_p - x)$ for $x \in [\pi_p/2, \pi_p]$; further extension to $[-\pi_p, \pi_p]$ is made by oddness, and finally $\sin_p$ is extended to the whole of $\mathbb{R}$ by $2\pi_p$-periodicity.

Let us call $\sin_p(x)$ to the generalized sine function, the unique solution of

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = (p-1)|u(x)|^{p-2}u(x) & \text{in } (0,1) \\ u(0) = 0 \\ u'(0) = 1. \end{cases} \tag{2.2.4}$$

The function $\sin_p(x)$ has a zero if and only if $x = k\pi_p$, where

$$\pi_p = \frac{2\pi/p}{\sin(\pi/p)}. \tag{2.2.5}$$

We define the function $\cos_p$ by the rule

$$\cos_p(x) = \frac{d}{dx}\sin_p(x), x \in \mathbb{R}.$$

Clearly $\cos_p$ is even, $2\pi_p$-periodic and odd about π_p; $\cos_2 = \cos$. The following identity is derived easily

$$|\cos_p(x)|^p + |\sin_p(x)|^p = 1.$$

Observe that if $p \neq 2$, the derivative of $\cos_p$ is not $-\sin_p$.

Now, we enunciate the characterization of eigenvalues and eigenfunctions of (2.2.1). The following result is duo to del Pino, Drábek and Manásevich [DDM99].

Theorem 2.10. *The eigenvalues λ_k and eigenfunctions u_k of equation* (2.2.1) *on the interval $\Omega := (0, \ell)$ are given by*

$$\lambda_k = \frac{\pi_p^p k^p}{\ell^p}, \tag{2.2.6}$$
$$u_k(x) = \sin_p(\pi_p kx/\ell).$$

Remark 2.11. In [DM99], Drábek and Manásevich proved that that they coincide with the variational eigenvalues given by equation (2.1.6). However, let us observe that the notation is different in both papers.

2.2.2 The weighted p−Laplacian in $\mathbb{R}$

As we have seen, using minimax formulas it is possible to construct a sequence of variational eigenvalues of (2.1.2) which approach infinity. In the linear case $p = 2$ those are the only eigenvalues of (2.1.2). In this section we will see that when $p \neq 2$ and $N = 1$ the variational eigenvalues exhaust the full spectrum.

Problem (2.1.2) is well understood in the one dimension case $N = 1$,

$$\begin{cases} -\Delta_p u := -(|u'|^{p-2}u')' = \lambda\rho(x)|u|^{p-2}u & \text{in } \Omega := (0, \ell) \\ u(0) = u(\ell) = 0. \end{cases} \tag{2.2.7}$$

We denote to the spectrum of (2.2.7) by

$$\Sigma := \{\lambda \in \mathbb{R} \colon \text{there exists } u \in W_0^{1,p}(\Omega), \text{ nontrivial solution to (2.2.7)}\}.$$

By means of the critical point theory of Ljusternik–Schnirelmann, in (2.1.7) we have defined the set Σ_{var} of the variational eigenvalues. Moreover, they are given by (2.1.6).

In [ACM02], Anane, Chakrone, and Moussa studied problem (2.2.7) and, among other things, it is proved that any eigenfunction associated to λ_k has exactly k nodal domains (this result had been proved in [Wa98] for the radial p-laplacian). As a consequence of this fact, it is obtained the simplicity of every variational eigenvalue.

Theorem 2.12. *The eigenvalues $\lambda_k \in \Sigma_{var}$ of (2.2.7) satisfy that*

1. *Every eigenfunction corresponding to the k-th eigenvalue λ_k, has exactly $k-1$ zeros in Ω.*
2. *For every k, λ_k is simple and verifies the strict monotonicity property with respect to the weight ρ and the domain Ω.*
3. *The eigenvalues of Σ_{var} are ordered as $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to +\infty$ as $k \to +\infty$.*

From Theorem 2.12 it follows that eigenvalues of problem (2.2.7) are given by the variational ones, i.e., $\Sigma_{var} = \Sigma$.

Theorem 2.13. *Every eigenvalue of (2.2.7) is given by (2.1.6).*

Proof. See [FBP03], Theorem 1.1. □

2.3 The spectral counting function

We denote by $N(\lambda)$ to the number of eigenvalues less than or equal to λ of (2.1.2), i.e.,

$$N(\lambda) = \#\{j \in \mathbb{N} \colon \lambda_j \le \lambda\}.$$

When $\Omega := (0,\ell)$ is a interval, by the characterization (2.2.1) of the eigenvalues of the $p-$Laplacian, it is easy to see that

$$\begin{aligned} N(\lambda,(0,\ell)) &= \#\{j \in \mathbb{N} \colon \lambda_j \le \lambda : \lambda \text{ eigvalalue of (2.2.1)}\} \\ &= \frac{\ell}{\pi_p}\lambda^{1/p} + O(1). \end{aligned}$$

The case where Ω is a disjoint union of intervals, was treated, for instance, in [FBP03]. There, the authors proved the following.

Proposition 2.14. *Let $\Omega = \bigcup_{j\in\mathbb{N}} I_j$, where $\{I_j\}_{j\in\mathbb{N}}$ is a pairwise disjoint family of intervals. Then,*

$$N(\lambda,\Omega) = \sum_{j=1}^{\infty} N(\lambda, I_j). \tag{2.3.1}$$

The following Theorem was proven in [FBP03] and is a suitable generalization of the Dirichlet–Neumann bracketing method of Courant.

Theorem 2.15 ([FBP03], Theorem 2.1)**.** *Let $U_1, U_2 \in \mathbb{R}^n$ be disjoint open sets such that $(\overline{U_1 \cup U_2})^\circ = U$ and $|U \setminus U_1 \cup U_2| = 0$, then*

$$N_D(\lambda, U_1 \cup U_2) \leq N_D(\lambda, U) \leq N_N(\lambda, U) \leq N_N(\lambda, U_1 \cup U_2).$$

Here, $N_D(\lambda, U)$ (resp., $N_N(\lambda, U)$) is the spectral counting function of the Laplace operator in U with Dirichlet boundary conditions on ∂U (resp., with Neumann boundary conditions).

Remark 2.16. In Chapter §3 is also considered the case in which $\Omega \subset \mathbb{R}$ is a open set which is a disjoint union of bounded intervals, $\Omega = \bigcup_{j\in\mathbb{N}} I_j$. Let us suppose that the lengths of the intervals are decreasing and goes to zero,

$$|I_1| \geq |I_2| \geq \cdots \geq |I_j| \geq \cdots \searrow 0.$$

We can assume that there exists some nonincreasing function $g : (0, \infty) \to (0, \infty)$ such that $|I_j| = g(j)$. Here, Ω has *fractal* boundary $\partial\Omega$ with Minkowski dimension $d \in (0, 1)$. In that case, in Section 3.3 it is obtained that

$$\begin{aligned} N(\lambda) &= \#\{j \in \mathbb{N} \colon \lambda_j \leq \lambda : \lambda \text{ eigvalalue of } (2.2.1)\} \\ &= \frac{|\Omega|}{\pi_p}\lambda^{1/p} + \frac{\zeta(d)}{\pi_p^d} f(\lambda^{1/p}) + o(f(\lambda^{1/p})) \end{aligned}$$

with $f(\lambda^{1/p}) = g^{-1}(\lambda^{-1/p})$, for $0 < d < 1$, and ζ is the Riemann Zeta function.

For the weighted $p-$Laplacian (2.2.7) in an bounded interval $\Omega \subset \mathbb{R}$, in [FBP03] Fernández Bonder and Pinasco proved that

$$N(\lambda, \Omega) = \frac{\lambda^{1/p}}{\pi_p} \int_\Omega \rho^{1/p} + o(\lambda^{1/p}). \tag{2.3.2}$$

Remark 2.17. From (2.3.2) it is easy to prove the asymptotic formula for the eigenvalues of (2.2.7). Since $k \sim N(\lambda_k)$, it follows immediately that

$$\lambda_k \sim \Big(\frac{\pi_p}{\int_\Omega \rho^{1/p}}\Big)^p k^p.$$

2.4 Monotone operators

We start this section with the definition and some properties of the so-called monotone operators.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be a bounded domain. We consider the operator $\mathcal{A} \colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ given by

$$\mathcal{A}u := -div(a(x, \nabla u)), \tag{2.4.1}$$

where $a \colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies, for every $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$, the following conditions:

(H0) *measurability:* $a(\cdot,\cdot)$ is a Carathéodory function, i.e. $a(x,\cdot)$ is continuous a.e. $x \in \Omega$, and $a(\cdot,\xi)$ is measurable for every $\xi \in \mathbb{R}^N$.

(H1) *monotonicity:* $0 \leq (a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2)$.

(H2) *coercivity:* $\alpha|\xi|^p \leq a(x,\xi)\xi$.

(H3) *continuity:* $a(x,\xi) \leq \beta|\xi|^{p-1}$.

(H4) *$p-$homogeneity:* $a(x,t\xi) = t^{p-1}a(x,\xi)$ for every $t > 0$.

(H5) *oddness:* $a(x,-\xi) = -a(x,\xi)$.

Let us introduce $\Psi(x,\xi_1,\xi_2) = a(x,\xi_1)\xi_1 + a(x,\xi_2)\xi_2$ for all $\xi_1,\xi_2 \in \mathbb{R}^N$, and all $x \in \Omega$; and let $\delta = min\{p/2,(p-1)\}$.

(H6) *equi-continuity:*

$$|a(x,\xi_1) - a(x,\xi_2)| \leq c\Psi(x,\xi_1,\xi_2)^{(p-1-\delta)/p}(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2)^{\delta/p}$$

(H7) *cyclical monotonicity:* $\sum_{i=1}^{k} a(x,\xi_i)(\xi_{i+1} - \xi_i) \leq 0$, for all $k \geq 1$, and $\xi_1,\ldots,\xi_{k+1}$, with $\xi_1 = \xi_{k+1}$.

(H8) *strict monotonicity:* let $\gamma = \max(2,p)$, then

$$\alpha|\xi_1 - \xi_2|^\gamma \Psi(x,\xi_1,\xi_2)^{1-(\gamma/p)} \leq (a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2).$$

As we will see in Chapter §6, the hypothesis (H1)–(H3) are necessary to ensure the *"convergence"* of (2.4.1). On the other hand, the hypothesis (H4)–(H7) are all important in the context of a well-posed eigenvalue problem. We assume (H8) for technical reasons.

We add that the conditions (H0)–(H8) are not completely independent of each other. It can be seen easily that (H8) implies (H1)–(H2) and that (H4) implies (H3) in addition to the continuity of the coefficient, for details see [BCR06].

Remark 2.18. The prototype for such functions is $a(x,\xi) = A(x)|\xi|^{p-2}\xi$, where $A(\cdot)$ is a measurable function with values in the set of $N \times N$ symmetric matrices which satisfies

$$\alpha'|\xi|^2 \leq A(x)\xi\cdot\xi, \quad |A(x)\xi| \leq \beta'|\xi| \quad \forall\xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega.$$

for some positive constants α' and β'.

In particular, under these conditions, we have the following Proposition due to Baffico, Conca and Rajesh [BCR06]

Proposition 2.19. *Given $a(x,\xi)$ satisfying* (H0)–(H8) *there exists a unique Carathéodory function Φ which is even, $p-$homogeneous strictly convex and differentiable in the variable ξ satisfying*

$$\alpha|\xi|^p \leq \Phi(x,\xi) \leq \beta|\xi|^p \tag{2.4.2}$$

for all $\xi \in \mathbb{R}^N$ a.e. $x \in \Omega$ such that

$$\nabla_\xi \Phi(x,\xi) = pa(x,\xi)$$

and normalized such that $\Phi(x,0) = 0$.

Proof. See Lemma 3.3 in [BCR06]. □

2.5 Eigenvalues of monotone operators

This section is devoted to the study of the following (nonlinear) eigenvalue problem in $\Omega \subset \mathbb{R}^N$, $N \geq 1$

$$\begin{cases} -div(a(x,\nabla u)) = \lambda\rho|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.5.1}$$

where $a(x,\xi)$ verifies (H0)–(H8) and

$$0 < \rho^- \leq \rho(x) \leq \rho^+ < \infty \qquad \text{a.e. in } \Omega. \tag{2.5.2}$$

The purpose of the section is to extend to (2.5.1) the results that are well-known for the $p-$Laplacian case, i.e. the existence of a sequence of variational eigenvalues, the simplicity and isolation of the first eigenvalue, etc.

The methods in the proofs here very much resembles the ones used for the $p-$Laplacian and we refer the reader to the articles [ACM02, AT96, An87, KL06, Li90].

We denote by

$$\Sigma := \{\lambda \in \mathbb{R} \colon \text{there exists } u \in W_0^{1,p}, \text{ nontrivial solution to (2.5.1)}\},$$

the spectrum of (2.5.1). It is immediate to check that $\Sigma \subset (0,+\infty)$ and that it is closed.

By means of the critical point theory of Ljusternik–Schnirelmann it is straight forward to see that we can obtain a discrete sequence of *variational* eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ tending to $+\infty$ (see [CVD90]). We denote by Σ_{var} the sequence of variational eigenvalues.

The kth–variational eigenvalue is given by

$$\lambda_k = \inf_{C\in\Gamma_k} \sup_{v\in C} \frac{\int_\Omega \Phi(x,\nabla v)}{\int_\Omega \rho|v|^p}$$

where $\Phi(x,\xi)$ is the potential function given in Proposition 2.19,

$$\Gamma_k = \{C \subset W_0^{1,p}(\Omega) : C \text{ compact}, C = -C, \ \gamma(C) \geq k\}$$

and $\gamma(C)$ is the Kranoselskii genus.

Below, we define the capacity of a set with the intention of define a Maximum Principle for quasilinear operators.

Definition 2.20. Given a compact set K contained in an open subset U of $\mathbb{R}^N$ and $p \geq 1$, the $W^{1,p}$-*capacity* of the pair (K, U) is defined as

$$\mathrm{Cap}_p(K, U) := \inf\left\{ \int_U |\nabla\varphi|^p : \varphi \in C_0^\infty(U), \varphi \geq 1 \text{ on } K\right\}.$$

If U' is an open subset of U, the corresponding $W^{1,p}$-capacity is defined as

$$\mathrm{Cap}_p(U', U) := \sup\{\mathrm{Cap}_p(K, U) : K \subset U', K \text{ compact}\},$$

and the definition is extended to a general set $E \subset U$ as follows:

$$\mathrm{Cap}_p(E, U) := \inf\{\mathrm{Cap}_p(U', U) : U' \text{ open}, E \subset U' \subset U\}.$$

A set $E \subset \mathbb{R}^N$ is said to be of $W^{1,p}$-*capacity zero*, and we write $\mathrm{Cap}_p(E) = 0$, if $\mathrm{Cap}_p(E \cap U, U) = 0$ for any open set $U \subset \mathbb{R}^N$.

For an extended discussion we also refer to the book of Evans-Gariepy [EG92].

The following maximum principle for quasilinear operators is a generalization of Theorem 2.1, and it was proved in [KLP07] by Kawohl, Lucia and Prashanth. It will be most useful in the sequel.

Definition 2.21. A function $u : \Omega \to \mathbb{R}$ is $W^{1,p}$*–quasi continuous* if for each $\varepsilon > 0$ there is an open set $U \subset \Omega$ such that $\mathrm{Cap}_p(U, \Omega) < \varepsilon$ and $f|_{\Omega\setminus U}$ is continuous.

Theorem 2.22. *Assume that* $u \in W^{1,p}_{loc}(\Omega)$ *satisfies*

$$\int_\Omega a(x, \nabla u)\nabla\phi + \rho|u|^{p-2}u\phi \geq 0, \quad \forall\phi \in C_0^\infty(\Omega),\ \phi \geq 0.$$

Consider its zero set

$$\mathfrak{Z} := \{x \in \Omega\colon \tilde{u}(x) = 0\},$$

where $\tilde{u}$ *is the* $W^{1,p}$*–quasi continuous representative of* u.

Then, either $Cap_p(\mathfrak{Z}) = 0$ *or* $u = 0$.

Proof. See Proposition 3.2 in [KLP07]. □

The positivity of the first eigenfunction together with the simplicity of the first eigenvalue was proved in [KLP07].

Theorem 2.23. *Let* u_1 *be an eigenfunction corresponding to* λ_1*, then* u_1 *does not changes sign on* Ω*. Also, the first eigenvalue is simple, that is, any other eigenfunction* u *associated to* λ_1 *is a multiple of* u_1.

Proof. See Section 6.2 in [KLP07]. □

Next, we show that the first eigenvalue λ_1 is isolated in Σ. The key step in the proof of the isolation is the next result:

Proposition 2.24. *Let $\lambda \in \Sigma$ and let w be an eigenfunction corresponding to $\lambda \neq \lambda_1$. Then, w changes sign on Ω, that is $w^+ \not\equiv 0$ and $w^- \not\equiv 0$. Moreover, there exists a positive constant C independent of w and λ such that*

$$|\Omega^+| \geq C\lambda^{-\gamma}, \quad |\Omega^-| \geq C\lambda^{-\gamma},$$

where $\Omega^\pm$ denotes de positivity and the negativity set of w respectively, γ is a positive parameter, and C depends on N, p, ρ^+ and the coercivity constant α in (H2)*. Here, $\gamma = (N-p)/p$ if $p < N$, $\gamma = 1$ if $p = N$, and $\gamma = (p-N)/N$ if $p > N$.*

Proof. Let w be an eigenfunction corresponding to $\lambda \neq \lambda_1$ and let u be an eigenfunction corresponding to λ_1.

Assume that w does not changes sign on Ω. We can assume that $w \geq 0$ and $u \geq 0$ in Ω. For each $k \in \mathbb{N}$, let us truncate u as follows:

$$u_k(x) := \min\{u(x), k\}$$

and for each $\varepsilon > 0$ we consider the function $u_k^p/(w+\varepsilon)^{p-1} \in W_0^{1,p}(\Omega)$. We get

$$\int_\Omega a(x,\nabla u)\nabla u - a(x,\nabla w)\nabla\Big(\frac{u_k^p}{(w+\varepsilon)^{p-1}}\Big) = \int_\Omega \lambda_1\rho u^p - \lambda\rho w^{p-1}\frac{u_k^p}{(w+\varepsilon)^{p-1}}. \tag{2.5.3}$$

We claim that the integral in the left hand side in (2.5.3) is non-negative. Indeed, let Φ be the potential function given by Proposition 2.19. Then, as Φ is $p-$homogeneous in the second variable we have (see [KLP07], p.19, 5.15)

$$\begin{aligned} &a(x,\nabla u)\nabla u - a(x,\nabla w)\nabla\Big(\frac{u_k^p}{(w+\varepsilon)^{p-1}}\Big) = \\ &p\Big\{\Phi(x,\nabla u) + (p-1)\Phi(x,\frac{u_k}{w+\varepsilon}\nabla w) - a(x,\frac{u_k}{w+\varepsilon}\nabla w)\nabla u_k\Big\}. \end{aligned} \tag{2.5.4}$$

By using the property that $\xi \mapsto \Phi(x,\xi)$ is convex, we easily deduce that (2.5.4) is nonnegative. Therefore, coming back to (2.5.3) we get

$$\int_\Omega \lambda_1\rho u^p - \lambda\rho w^{p-1}\frac{u_k^p}{(w+\varepsilon)^{p-1}} \geq 0. \tag{2.5.5}$$

Since by the strong maximum principle for quasilinear operators (Theorem 2.22) the set $\{\tilde{w} = 0\}$, where $\tilde{w}$ is the $p-$quasi continuous representative of w, is of measure zero then (2.5.5) is equivalent to

$$\int_{\{w>0\}} \lambda_1\rho u^p - \lambda\rho w^{p-1}\frac{u_k^p}{(w+\varepsilon)^{p-1}} \geq 0. \tag{2.5.6}$$

Now, letting $\varepsilon \to 0$ and $k \to \infty$ in (2.5.6), we get

$$(\lambda_1 - \lambda) \int_\Omega \rho |u|^p \geq 0$$

which is a contradiction. Therefore w changes sign on Ω.

The second part of the proof follows almost exactly as in the $p-$Laplacian case. Let us suppose first that $p < N$. In fact, as w changes sign, we can use w^+ as a test function in the equation satisfied by w to obtain

$$\begin{aligned}
\int_\Omega a(x, \nabla w)\nabla w^+ &= \lambda \int_\Omega \rho |w|^{p-2} w w^+ \\
&= \lambda \int_{\Omega^+} \rho |w|^p \\
&\leq \lambda \rho^+ \int_{\Omega^+} |w|^p \\
&\leq \lambda \rho^+ \|w^+\|^p_{L^{p^*}(\Omega)} |\Omega^+|^{p/(N-p)} \\
&\leq \lambda \rho^+ K_p |\Omega^+|^{p/(N-p)} \int_\Omega |\nabla w^+|^p,
\end{aligned}$$

where K_p is the optimal constant in the Sobolev-Poincaré inequality.

Now, by (H2), it follows that

$$\int_\Omega a(x, \nabla w)\nabla w^+ \geq \alpha \int_\Omega |\nabla w^+|^p.$$

Combining these two inequalities, we obtain

$$|\Omega^+| \geq \Big(\frac{\alpha}{K_p \lambda \rho^+}\Big)^{(N-p)/p}.$$

The estimate for $|\Omega^-|$ follows in the same way.

The remaining cases are similar: $p = N$ follows by using the Sobolev's inclusion $W_0^{1,N}(\Omega) \subset L^N(\Omega)$, and the case $p > N$ follows from Morrey's inequality (see [Ev10]). □

Now we are ready to prove the isolation of λ_1.

Theorem 2.25. *The first eigenvalue λ_1 is isolated. That is, there exists $\delta > 0$ such that $(\lambda_1, \lambda_1 + \delta) \cap \Sigma = \emptyset$.*

Proof. Assume by contradiction that there exists a sequence $\lambda_j \in \Sigma$ such that $\lambda_j \to \lambda_1$ as $j \to \infty$. Let u_j be the associated eigenfunctions normalized such that

$$\int_\Omega \rho |u_j|^p = 1.$$

By (H2) it follows that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$ so, passing to a subsequence if necessary, there exists $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}
u_j &\rightharpoonup u \qquad \text{weakly in } W_0^{1,p}(\Omega) \\
u_j &\to u \qquad \text{strongly in } L^p(\Omega).
\end{aligned}$$

Now, as the functional

$$v \mapsto \int_\Omega \Phi(x, \nabla v)$$

is weakly sequentially lower semicontinuous (see [BCR06]), it follows that u is an eigenfunction associated to λ_1.

Now, by Theorem 2.23, we can assume that $u \geq 0$ and by Proposition 2.24 we have $|\{u = 0\}| > 0$. But this is a contradiction to the strong maximum principle in [KLP07], Theorem 2.22. □

As a consequence of Theorem 2.25 it makes sense to define the second eigenvalue Λ_2 as the infimum of the eigenvalues greater than λ_1. Next, we show that this second eigenvalue Λ_2 coincides with the second variational eigenvalue λ_2. This result is known to hold for the weighted $p-$Laplacian (see Theorem 2.8) and we extended here for the general case (2.5.1).

Theorem 2.26. *Let λ_2 be the second variational eigenvalue, and let Λ_2 be defined as*

$$\Lambda_2 = \inf\{\lambda > \lambda_1 : \lambda \in \Sigma\}.$$

Then

$$\lambda_2 = \Lambda_2.$$

Proof. The proof of this Theorem follows closely the one in [FBR02] where the analogous result for the Steklov problem for the $p-$Laplacian is analyzed.

Let us call

$$\mu = \inf\left\{\int_\Omega \Phi(x, \nabla u) \colon \|\rho u\|^p_{L^p(\Omega)} = 1 \text{ and } |\Omega^\pm| \geq c_{\lambda_2}\right\},$$

where $c_{\lambda_2} := C\lambda_2^{-\gamma}$ and C, γ are given by Proposition 2.24.

If we take u_2 an eigenfunction of (2.5.1) associated with Λ_2 such that $\|\rho u\|^p_{L^p(\Omega)} = 1$, by Proposition 2.24, we have that u_2 is admissible in the variational characterization of μ. It follows that $\mu \leq \Lambda_2$. The proof will follows if we show that $\mu \geq \lambda_2$. The inverse of μ can be written as

$$\frac{1}{\mu} = \sup\left\{\int_\Omega \rho|u|^p \colon \int_\Omega \Phi(x, \nabla u) = 1 \text{ and } |\Omega^\pm| \geq c_{\lambda_2}\right\}.$$

The supremum is attained by a function $w \in W_0^{1,p}(\Omega)$ such that $\int_\Omega \Phi(x, \nabla w) = 1$ and $|\Omega^\pm| \geq c_{\lambda_2}$. As w^+ and w^- are not identically zero, if we consider the set

$$C = span\{w^+, w^-\} \cap \{u \in W_0^{1,p}(\Omega) \colon \|u\|_{W_0^{1,p}(\Omega)} = 1\},$$

then $\gamma(C) = 2$. Hence, we obtain

$$\frac{1}{\lambda_2} \geq \inf_{u \in C} \int_\Omega \rho|u|^p \tag{2.5.7}$$

but, as w^+ and w^- have disjoint support, it follows that the infimum (2.5.7) can be computed by minimizing the two variable function

$$G(a, b) := |a|^p \int_\Omega \rho|w^+|^p + |b|^p \int_\Omega \rho|w^-|^p$$

with the restriction

$$H(a,b) := |a|^p \int_\Omega \Phi(x, \nabla w^+) + |b|^p \int_\Omega \Phi(x, \nabla w^-) = 1.$$

Now, an easy computation shows that

$$\frac{1}{\lambda_2} \geq \min\left\{ \frac{\int_\Omega \rho |w^+|^p}{\int_\Omega \Phi(x, \nabla w^+)}, \frac{\int_\Omega \rho |w^-|^p}{\int_\Omega \Phi(x, \nabla w^-)} \right\}.$$

We can assume that the minimum in the above inequality is realized with w^+. Then, for $t > -1$ the function $w + tw^+$ is admissible in the variational characterization of μ, hence if we denote

$$Q(t) := \frac{\int_\Omega \rho |w + tw^+|^p}{\int_\Omega \Phi(x, \nabla w + t\nabla w^+)},$$

we get

$$0 = Q'(0) = p \int_\Omega \rho |w|^{p-2} w w^+ - \frac{p}{\mu} \int_\Omega a(x, \nabla w) \nabla w^+,$$

therefore

$$\frac{\int_\Omega \rho |w^+|^p}{\int_\Omega \Phi(x, \nabla w^+)} = \frac{1}{\mu}$$

and the result follows. □

2.5.1 Monotone operators in one dimension

When we consider a function $a : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying properties (H0)–(H8), it can be explicitly expressed as $a(x, \xi) = a(x)|\xi|^{p-2}\xi$, where a is bounded uniformly away from zero and infinity.

In this case problem (2.5.1) is reduced to (for simplicity we take $\Omega = (0, 1)$)

$$\begin{cases} -(a(x)|u'|^{p-2}u')' = \lambda \rho |u|^{p-2} u & \text{in } \Omega := (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{2.5.8}$$

where ρ satisfies (2.5.2) and a is such that for $\alpha < \beta$ positive constants,

$$0 < \alpha \leq a(x) \leq \beta < +\infty \qquad \text{a.e. in } \Omega. \tag{2.5.9}$$

We denote by

$$\Sigma := \{\lambda \in \mathbb{R} \colon \text{there exists } u \in W_0^{1,p}, \text{ nontrivial solution to (2.5.8)}\},$$

the spectrum of (2.5.1). It is immediate to check that $\Sigma \subset (0, +\infty)$ and that it is closed.

Observe that here, if λ_k is the $k-$th variational eigenvalue,

$$\lambda_k = \inf_{C \in \Gamma_k} \sup_{v \in C} \frac{\int_\Omega a(x)|u'|^p}{\int_\Omega \rho(x)|v|^p}. \tag{2.5.10}$$

As we have seen in Section 2.1, the question of whether $\Sigma_{\text{var}} = \Sigma$ or not is only known to hold in the liner setting and also for the $p-$Laplacian in one space dimension. It is an open problem in any other situation.

We have the following result about the simplicity of the eigenvalues of (2.5.8):

Theorem 2.27. *Every eigenfunction corresponding to the $k-$th eigenvalue λ_k of (2.5.8) has exactly $k-1$ zeroes. Moreover, for every k, λ_k is simple, consequently the eigenvalues are ordered as $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \nearrow +\infty$.*

Moreover, the spectrum of (2.5.8) coincides with the variational spectrum. In fact, we have:

Theorem 2.28. *$\Sigma = \Sigma_{var}$, i.e., every eigenvalue of problem (2.5.8) is given by (2.5.10).*

In order to prove Theorems 2.27 and 2.28, in the following remark we observe that equations of the kind (2.5.8) involving a diffusion function $a(x)$ and a weight function ρ can be converted in one equation involving only a weight function.

Remark 2.29. Through the following change of variables, problem (2.5.8) can be converted in one of the form (2.2.7): we define

$$P(x) = \int_0^x \frac{1}{a(s)^{1/(p-1)}} ds$$

and be the change of variables $(x, u) \to (y, v)$ where

$$y = P(x), \qquad v(y) = u(x).$$

By simple computations we get

$$\begin{cases} -(|\dot{v}|^{p-2}\dot{v})^{\cdot} = \lambda Q(y)|v|^{p-2}v, & y \in [0, L] \\ v(0) = v(L) = 0 \end{cases}$$

where $\cdot = d/dy$, with

$$L = \int_0^1 \frac{1}{a(s)^{1/(p-1)}} ds = \overline{a^{\frac{-1}{p-1}}},$$

and

$$Q(x) = a(x)^{1/(p-1)} \rho(x).$$

Now, we rescale to the unit interval defining

$$w(z) = v(Lz), \qquad z \in I$$

and get

$$\begin{cases} -(|\dot{w}|^{p-2}\dot{w})^{\cdot} = L^p \lambda Q(Lz)|w|^{p-2}w & \text{in } I \\ w(0) = w(1) = 0. \end{cases}$$

So if we denote $\mu = L^p\lambda$ and $g(z) = Q(Lz)$, we get that w verifies

$$\begin{cases} -(|\dot{w}|^{p-2}\dot{w})^{\cdot} = \mu g(z)|w|^{p-2}w & \text{in } I \\ w(0) = w(1) = 0. \end{cases}$$

Having in mind Remark 2.29, the proof of Theorem 2.27 it follows from Theorem 2.12 and the proof of Theorem 2.28 it is completely analogous to that of Theorem 2.13.

3

An application: Refined asymptotic for eigenvalues on domains of infinite measure

3.1 Introduction

Can one hear the shape of a drum? asked M. Kac [Ka66] in 1966. What he meant was the following inverse problem: consider the eigenvalue problem of the $p-$Laplacian with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1.1}$$

where Ω is a open bounded set in $\mathbb{R}^N$, $N \geq 1$. As we have seen in Chapter §2, the spectrum of (3.1.1) is a discrete sequence $\{\lambda_k\}_k \in \mathbb{R}^+$ tending to $+\infty$. Which geometrical information concerning Ω could be recovered from the sole knowledge of the spectrum? All information about the spectrum can be obtained from the *eigenvalue counting function* $N(\lambda)$ defined as

$$N(\lambda) = \#\{j \in \mathbb{N} \colon \lambda_j \leq \lambda\},$$

that is, it counts the number of eigenvalues of (3.1.1) up to λ, counted according to multiplicity (see Section 2.3 for properties).

Generalizing Weyl's classic asymptotic formula, Métivier [Me76] proves that

$$N(\lambda) = (1 + o(1))\varphi(\lambda), \qquad \text{as } \lambda \to +\infty$$

where the *Weyl term* $\varphi(\lambda)$ is given by

$$\varphi(\lambda) = \frac{1}{(2\pi)^N}\omega_N|\Omega|_N\lambda^{N/2}$$

with ω_N is the volume of the unit ball in $\mathbb{R}^N$ and $|A|_N$ denotes the $N-$dimensional Lebesgue measure of $A \subset \mathbb{R}^N$. According to this formula, one can hear the *area* of a drum. A conjecture about the second term in the asymptotic expansion of $N(\lambda)$ it was made by H. Weyl [We12] as follows

$$N(\lambda) = \varphi(\lambda) - C_N|\partial\Omega|_{N-1}\lambda^{(N-1)/2} + o(\lambda^{(N-1)/2}), \qquad \text{as } \lambda \to +\infty \tag{3.1.2}$$

for the case of $\partial\Omega$ *sufficiently regular*.

What happens if the boundary is *non-smooth*? M. V. Berry [Be79, Be80] conjectured that if $\partial\Omega$ is *fractal*, then

$$N(\lambda) = \varphi(\lambda) - C_{H,N}\mathcal{H}(H;\partial\Omega)\lambda^{H/2} + o(\lambda^{H/2}), \quad \text{as } \lambda \to +\infty \tag{3.1.3}$$

where $C_{N,H}$ is a positive constant depending only on N and H, H denotes the Hausdorff dimension of the boundary $\partial\Omega$ and $\mathcal{H}(H;\partial\Omega)$ the $H-$dimensional Hausdorff measure of $\partial\Omega$. Observe that if $\partial\Omega$ is smooth, say C^1, then $H = N - 1$ and we recover (3.1.2) from (3.1.3). Unfortunately, Berry's conjecture has turned out to be false. Brossard and Carmona [BC86] disproved it and suggested that the Minkowski dimension was more appropriate than the Hausdorff dimension to measure the *roughness* of the boundary $\partial\Omega$. A reformulation of Berry's conjecture on $N(\lambda)$ was made by Lapidus [La91]:

$$N(\lambda) = \varphi(\lambda) - C_{N,d}\mathcal{M}(d;\partial\Omega)\lambda^{d/2} + o(\lambda^{d/2}), \quad \text{as } \lambda \to +\infty \tag{3.1.4}$$

where Ω has fractal boundary $\partial\Omega$ with Minkowski dimension $d \in (N-1, N)$ and $C_{N,d}$ is a positive constant depending only on N and d.

In [La93], Conjecture (3.1.4) it was proved for $N = 1$: if $\Omega \subset \mathbb{R}$ has fractal boundary $\partial\Omega$ which is Minkowski measurable and has Minkowski dimension $d \in (0, 1)$, then

$$N(\lambda) = \varphi(\lambda) - C_{1,d}\mathcal{M}(d;\partial\Omega)\lambda^{d/2} + o(\lambda^{d/2}), \quad \text{as } \lambda \to +\infty \tag{3.1.5}$$

where the constant $C_{1,d}$ is given by

$$C_{1,d} = \frac{1}{2^{1-d}\pi^d}(1-d)(-\zeta(d)), \quad \varphi(\lambda) = \frac{1}{\pi}|\Omega|_1\lambda^{1/2}$$

and ζ denotes the Riemann zeta function.

He and Lapidus in [HL97] extend these theorems by using *gauge functions* more general than power functions (see Section 3.2.1) as follows. Let us consider an open set $\Omega \subset \mathbb{R}$ which is a disjoint union of bounded intervals, $\Omega = \bigcup_{j\in\mathbb{N}} I_j$. Let us suppose that the lengths of the intervals are decreasing and goes to zero,

$$|I_1| \geq |I_2| \geq \cdots \geq |I_j| \geq \cdots \searrow 0.$$

We can assume that there exists some nonincreasing function $g : (0,\infty) \to (0,\infty)$ such that

$$|I_j| = g(j).$$

Now, we may consider the following problems:

- A Lattice Point Problem: to estimate, for $x \nearrow \infty$, the number of lattice points below the curve $xg(t)$,

$$N(x) = \#\{(j,k) \in \mathbb{N}\times\mathbb{N} : k \leq xg(j)\} = \sum_{j=1}^{\infty}[xg(j)]. \tag{3.1.6}$$

- An Eigenvalue Counting Problem: to estimate, for $\lambda \nearrow \infty$, the number of eigenvalues less than or equal to λ of $-u'' = \lambda u$ in Ω with zero Dirichlet boundary conditions on $\partial\Omega$,

$$N(\lambda) = \#\{j \in \mathbb{N} \colon \lambda_j \le \lambda\},$$

The first one is called a *plane multiplicative problem*, following Krätzel [Kr88], and generalizes the *Dirichlet's divisor problem*, that is, to count the asymptotic number of divisors of the integers less than or equal to x, which is equivalent to count the number of lattice points below the hyperbola $y = x/t$ in the first quadrant.

The second one is a one dimensional variant of the old problem, *Can one hear the dimension of a drum?* The idea behind this name is the following: the square root of the eigenvalues of the Laplace operator in $\Omega \subset \mathbb{R}^2$ coincide with the musical notes of a membrane with the shape of Ω, and we can ask about the geometric properties of Ω which can be inferred from the sequence of eigenvalues Here, we are interested in the dimension of the boundary of a *fractal string* Ω, as Lapidus called this kind of sets [La91].

Indeed, both problems are the same: the eigenvalues of $-u'' = \lambda u$ in I_j are $\{\frac{\pi^2 k^2}{g(j)^2}\}_{k\ge1}$, and we have

$$\begin{aligned} N(\lambda) &= \sum_{j=1}^{\infty} \#\Big\{k \in \mathbb{N} \colon \frac{\pi^2 k^2}{g(j)^2} \le \lambda\Big\} \\ &= \sum_{j=1}^{\infty} \#\Big\{k \in \mathbb{N} \colon k \le \frac{g(j)\lambda^{1/2}}{\pi}\Big\} \\ &= \sum_{j=1}^{\infty} \Big[\frac{g(j)\lambda^{1/2}}{\pi}\Big] \end{aligned} \tag{3.1.7}$$

So, calling $x = \frac{\lambda^{1/2}}{\pi}$, this expression coincides with equation (3.1.6), and we see that there exists a connection between the Dirichlet problem and the asymptotic behavior of eigenvalues. Let us mention that the eigenvalue counting problem for the Laplacian when Ω is the unit square in $\mathbb{R}^2$ coincide with the Gauss Circle Problem, i.e., to estimate the number of lattice points inside an expanding circle (see [He76]).

Under these considerations, He and Lapidus in [HL97] prove that for $0 < d < 1$,

$$N(\lambda) = \frac{|\Omega|_1}{\pi}\lambda^{1/2} - \frac{\zeta(d)}{\pi^d} f(\sqrt{\lambda}) + o(f(\sqrt{\lambda})), \qquad \text{as } \lambda \to +\infty$$

where $g(x) := h^{-1}(1/x)$ and $f(x) := 1/h(1/x)$. Particularly, when $h(x) = x^d$ it is recovered (3.1.5) and the Minkowski dimension.

Our aim in this chapter is extend the results of He and Lapidus for the one-dimensional $p-$Laplacian operator. When the measure of Ω is finite and $0 < d < 1$, in Section 3.3 we obtain that

$$N(\lambda) = \#\{j \in \mathbb{N} \colon \lambda_j \le \lambda\} = \frac{|\Omega|}{\pi_p}\lambda^{1/p} + \frac{\zeta(d)}{\pi_p^d} f(\lambda^{1/p}) + o(f(\lambda^{1/p}))$$

with $f(\lambda^{1/p}) = g^{-1}(\lambda^{-1/p})$ and ζ is the Riemann Zeta function. The term $f(\lambda^{1/p})$ is connected with a generalized notion of fractal dimension, and we have $f(\lambda^{1/p}) = \lambda^{d/p}$ when the Minkowski dimension of $\partial\Omega$ is d. The precise definitions and properties of g and related functions is given in Section 3.2, together with the definitions of the generalized Minkowski content and dimension.

The proofs in those works depends on difficult estimates of the remainder terms of certain convergent series. We present in Section 3.3 a simplified proof based on the equivalence of the two problems stated above and some arguments from number theory. When the lengths of the intervals satisfy $|I_j| \sim j^{-1/d}$, as in [La93], this ideas were used in [Pi06].

However, as a by-product of the number theoretic methods, we are able to extend those results to fractal strings Ω with infinite measure, and this is the main aim of our work. Let us observe that the sum in equation (3.1.7) is well defined whenever $g(t) \searrow 0$ for $t \nearrow \infty$, even when $\sum_{j=1}^{\infty} g(j)$ diverges.

So, in Section 3.4 we characterize the growth of the number of eigenvalues $N(\lambda)$ in terms of the decay of the lengths of the intervals when the measure of Ω is not finite. We obtain the following non-standard asymptotic formula

$$N(\lambda) = \#\{j \in \mathbb{N} \colon \lambda_j \leq \lambda\} = \frac{\zeta(d)}{\pi_p^d} f(\lambda^{1/p}) + o(f(\lambda^{1/p})),$$

where now $d > 1$.

In the finite measure case, the term depending on f can be thought as a boundary contribution. The measure of Ω gives the main term of the asymptotic of the number of lattice points, and the second term can be understood as the number of points which are close to the boundary and enter when we dilate slightly the domain. Now, when the measure of Ω is infinite, the main term is still a boundary term, which shows the asymptotic growth of the measure of the domain; in this case, when we dilate slightly the domain, a huge number of lattice points enter although it has exactly the same form that the second term in the other case.

The discreteness of the spectrum of an elliptic operator is not well understood yet when the domain has infinite measure. We refer the interested reader to [VL01, CH67, He74, He75, Si83] where a special class of sets in $\mathbb{R}^N$ is considered (*horn-shaped domains*, a $N-1$ dimensional set scaled in the other dimension). In [CH67, He74, He75], an upper bound for the growth of $N(\lambda)$ was derived by using a trace estimate in the class of Hilbert-Schmidt operators, obtained with the aid of some inequalities for the Green function of an elliptic operator. In [Si83] the asymptotic behavior of eigenvalues was refined by using the Trotter product formula in order to obtain another trace estimate by generalizing the Golden-Thompson inequality, and in [VL01] were obtained more terms in the asymptotic expansion by exploiting certain self-similarity of the horns. In Section 3.5 we apply our previous results to this kind of problems in $\mathbb{R}^2$. The main novelty here is the precise order of growth of $N(\lambda, \Omega)$ for horns which are not decaying as powers, although is less precise for this kind of horns since the precise constant in the main term is known, see the paper of van den Berg and Lianantonakis [VL01].

3.2 Generalized Minkowski content and Minkowski dimension

3.2.1 Minkowski dimension and content

We denote by $|A|$ the Lebesgue measure of the set $A \subset \mathbb{R}^n$. Let A_ε denote the tubular neighborhood of radius ε of a set $A \subset \mathbb{R}^n$, i.e.

$$A_\varepsilon = \{x \in \mathbb{R}^n \colon \operatorname{dist}(x, A) \leq \varepsilon\}. \tag{3.2.1}$$

We recall the classical definition of Minkowski dimension and content (see [Fa90, HL97, La91, Tr82]).

Given $d > 0$, the $d-$dimensional upper Minkowski content of $\partial\Omega$ is defined as

$$M^*(d; \partial\Omega) := \limsup_{\varepsilon\to 0^+} \varepsilon^{-(n-d)} |(\partial\Omega)_\varepsilon \cap \Omega|. \tag{3.2.2}$$

Similarly, the $d-$dimensional lower Minkowski content, $M_*(d, \partial\Omega)$, is defined changing the upper by the lower limit in (3.2.2).

The Minkowski dimension of $\partial\Omega$ is then defined by

$$\dim(\partial\Omega) := \inf\{d \geq 0 \colon M^*(d; \partial\Omega) < \infty\} = \sup\{d \geq 0 \colon M^*(d; \partial\Omega) = \infty\}. \tag{3.2.3}$$

We will further say that $\partial\Omega$ is $d-$Minkowski measurable if

$$0 < M_*(d; \partial\Omega) = M^*(d; \partial\Omega) < \infty \qquad \text{for some } d > 0,$$

and we call this value $M(d; \partial\Omega)$ the $d-$dimensional Minkowski content of $\partial\Omega$. Following [La91], we say that $\partial\Omega$ is *fractal* if $d \in (n-1, n]$, and *non-fractal* otherwise.

3.2.2 Dimension functions

In this paper we will be interested in a suitable generalization of the previous concepts. To this end, given $0 < d < 1$ we define G_d to be the class of functions $h \colon (0, \infty) \to (0, \infty)$ continuous such that

(H1) h is strictly increasing and

$$\lim_{x\to 0^+} h(x) = 0, \quad \lim_{x\to\infty} h(x) = \infty.$$

(H2) For any $t > 0$,

$$\lim_{x\to 0^+} \frac{h(tx)}{h(x)} = t^d,$$

uniformly in t on compact subsets of $(0, \infty)$.

(H3) h is sublinear at 0, i.e.

$$\lim_{x\to 0^+} \frac{h(x)}{x} = \infty.$$

One can check that the functions

$$h(x) = \frac{x^d}{(\log(\frac{1}{x}+1))^a} \text{ and } h(x) = \frac{x^d}{(\log(\log(\frac{1}{x}+1)))^a} \tag{3.2.4}$$

are in G_d for all $d \in (0,1)$ and $a \geq 0$.

Remark 3.1. Let $i : (0,\infty) \to (0,\infty)$ be the function $i(x) = x^{-1}$. From now on, given $h \in G_d$, we will always let

$$g(x) := (h^{-1} \circ i)(x) = h^{-1}(1/x), \quad f(x) := (i \circ h \circ i)(x) = \frac{1}{h(1/x)}. \tag{3.2.5}$$

With this notations let us now define the generalized Minkowski content and dimension that was introduced by He and Lapidus in [HL97].

Definition 3.2. Let $\Omega \subset \mathbb{R}^n$ be an open set with finite Lebesgue measure. Let $h \in G_d$ be a dimension function. The upper $h-$Minkowski content of $\partial\Omega$ is defined by

$$M^*(h;\partial\Omega) := \limsup_{\varepsilon\to 0^+} \varepsilon^{-n} h(\varepsilon)|(\partial\Omega)_\varepsilon \cap \Omega|. \tag{3.2.6}$$

We define the lower $h-$Minkowski content $M_*(h;\partial\Omega)$ by taking the lower limit in (3.2.6). We further say that $\partial\Omega$ is $h-$Minkowski measurable if

$$0 < M_*(h;\partial\Omega) = M^*(h;\partial\Omega) < \infty$$

and denote this value as $M(h;\partial\Omega)$ the $h-$Minkowski content of $\partial\Omega$.

Let Ω be an open set in $\mathbb{R}$. Then, $\Omega = \bigcup_{j=1}^\infty I_j$, where I_j is an interval of length l_j. We can assume that

$$l_1 \geq l_2 \geq \cdots \geq l_j \geq \cdots > 0.$$

In [HL97], the authors obtained the following relation between the lengths l_j and the Minkowski measurability of $\partial\Omega$:

Theorem 3.3. *Let $\Omega = \bigcup_{j=1}^\infty I_j$. Then, $\partial\Omega$ is $h-$Minkowski measurable if and only if $l_j \sim Lg(j)$. Moreover, in this case, the $h-$Minkowski content of $\partial\Omega$ is given by*

$$M(h;\partial\Omega) = \frac{2^{1-d}}{1-d}L^d.$$

Note that d being positive and less than one implies the integrability at infinity of the function g, which in turn implies that the Lebesgue measure of the set Ω is finite. Therefore, the $h-$Minkowski content and dimension are well-defined concepts.

The following proposition, that can be found in [La93], is a usefull estimate in our arguments in order to compute the constants appearing from the Euler-McLaurin formula.

Proposition 3.4. *Suppose $h \in G_d$ for some $d \in (0,1)$. Then,*

$$\lim_{x\to\infty} \frac{\int_x^\infty g(u)\,du}{xg(x)} = \frac{d}{1-d}.$$

3.2.3 Nonintegrable Dimension Functions

We now consider the analogous of the dimension functions defined in the previous subsection to the case $d > 1$.

To this end we define the class G_d to be the class of functions $h\colon (0,\infty) \to (0,\infty)$ continuous such that (H1) and (H2) are satisfied and, instead of (H3) we require superlinearity at 0, i.e.

$$\lim_{x\to 0} \frac{h(x)}{x} = 0. \tag{H3'}$$

Remark 3.5. As in the previous subsection, we let $i\colon (0,\infty) \to (0,\infty)$ given by $i(x) = x^{-1}$ and

$$g(x) := (h^{-1} \circ i)(x) = h^{-1}(1/x), \quad f(x) := (i \circ h \circ i)(x) = \frac{1}{h(1/x)}.$$

Now we prove an analogous of Proposition 3.4 to this case.

Proposition 3.6. *Suppose $h \in G_d$ for some $d > 1$. Then,*

$$\lim_{x\to\infty} \frac{\int_1^x g(u)\,du}{xg(x)} = \frac{d}{d-1}.$$

Proof. First, we need to show that hypotheses (H1), (H2) and (H3') imply

$$\lim_{x\to\infty} \frac{g(sx)}{g(x)} = s^{-1/d} \tag{3.2.7}$$

uniformly on $[s_0, \infty)$ for any $s_0 > 0$ and

$$\lim_{x\to\infty} xg(x) = \infty. \tag{3.2.8}$$

Equation (3.2.8) is immediate from (H3'). Now, to prove (3.2.7) we first observe that it is equivalent to

$$\lim_{x\to 0+} \frac{h^{-1}(sx)}{h^{-1}(x)} = s^{1/d}, \tag{3.2.9}$$

on compact sets of $(0,\infty)$. In order to prove (3.2.9), let us note that (H2) implies

$$h(sx) = s^d h(x) + o(1),$$

uniformly on x and in $s \in [0, s_0]$. Then, by the monotonicity of h,

$$h^{-1}(s^d h(x) - \varepsilon) \leq sx \leq h^{-1}(s^d h(x) + \varepsilon).$$

Finally, if we call $y = h(x)$ and $t = s^d$,

$$h^{-1}(ty - \varepsilon) \leq t^{1/d} h^{-1}(y) \leq h^{-1}(ty + \varepsilon),$$

which trivially implies (3.2.9) and hence (3.2.7).

With these observations, now the proof of the Proposition follows easily. In fact, by (3.2.8), it is enough to prove

$$\lim_{x\to\infty} \frac{\int_{x_0}^{x} g(u)\,du}{xg(x)} = \frac{d}{d-1}$$

for x_0 large enough. Now, by (3.2.7),

$$\frac{\int_{x_0}^{x} g(u)\,du}{xg(x)} = \int_{x_0/x}^{1} \frac{g(xs)}{g(x)}\,ds = \int_{x_0/x}^{1} s^{-1/d} + o(1)\,ds = \frac{d}{d-1} + o(1).$$

This fact completes the proof. □

Remark 3.7. Let $\Omega = \bigcup_{j=1}^{\infty} I_j$ where I_j are disjoint open intervals of length $l_j \asymp g(j)$ where g is associated to a function $h \in G_d$ with $d > 1$.

In this case, since g is not integrable at infinity, one can check that $|(\partial\Omega)_\varepsilon \cap \Omega| = \infty$ for every $\varepsilon > 0$. So, we cannot define the corresponding h−Minkowski content or dimension in this case.

Nevertheless, in the computation of the asymptotic behavior of the eigenvalues, we obtain an order of growth for $N(\lambda)$ which depends on $f = (i \circ h \circ i)$.

So, in some sense, h can be considered as certain *spectral dimension* for $\partial\Omega$. That is why we refer to h as a *nonintegrable dimension function* even though there is no concept of dimension associated to it. See Remark 3.16 at the end of Section §5.

3.3 The finite measure case: $0 < d < 1$

To prove the results in this Section will be very useful the followings

We begin this section recalling the well known summation formula of Euler-MacLaurin, which will be very useful in this Chapter, see [Kr88] for a proof:

Theorem 3.8. *Let $f(t)$ be a non negative, continuous and monotonically decreasing function tending to zero when $t \to +\infty$. Then, there exist $C \in \mathbb{R}$, depending only on f, such that*

$$\sum_{j=a}^{b} f(j) = \int_a^b f(t)\,dt + C + O(f(b)), \tag{3.3.1}$$

when $b \to +\infty$. In particular

$$\lim_{b\to+\infty} \Big(\sum_{j=a}^{b} f(j) - \int_a^b f(t)\,dt\Big) = C. \tag{3.3.2}$$

An estimate of the number of eigenvalues of the p−Laplacian equation (1.5.4) relies on Lemma 3.9 below. This Lemma has been proved in [HL97] but we provide here a different proof that will allow us, in the next section, to deal with the infinite measure case.

Lemma 3.9. *Let $\{l_j\}_{j\in\mathbb{N}}$ be an arbitrary nonincreasing positive sequence such that for some $h \in G_d$ we have that $l_j = g(j)$. Then*

$$\sum_{j=1}^{\infty}[l_j x] = \sum_{j=1}^{\infty} l_j x + \zeta(d) f(x) + o(f(x)), \quad as \quad j \to \infty.$$

Proof. First, we need to control the difference between $\sum [l_j x]$ and $\sum l_j x$.

To this end, we firs observe that $[l_j x] = 0$ if $l_j x < 1$. Therefore, the first sum is finite.

Let $J \in \mathbb{R}$ such that $xg(J) = 1$. Therefore,

$$J = g^{-1}\Big(\frac{1}{x}\Big) = \frac{1}{h(1/x)} = f(x).$$

As $[g(j)x] = 0$ if $j > J$, we get

$$\sum_{j=1}^{\infty}[l_j x] = \sum_{j=1}^{J}[g(j)x] = \sum_{j=1}^{J} g(j)x + O(J).$$

Observe that this equation immediately gives

$$\sum_{j=1}^{\infty}[l_j x] = \sum_{j=1}^{\infty} l_j x + O(f(x)).$$

The rest of the proof will consists in refining the error term.

To improve the remainder estimate, we use Dirichlet's argument for the number of lattice points below the hyperbola: we count the points below the graph of the function $xg(t)$ and below its inverse $g^{-1}(t/x)$, up to the intersection point of these graphs and deleting the size of the square which we counted twice.

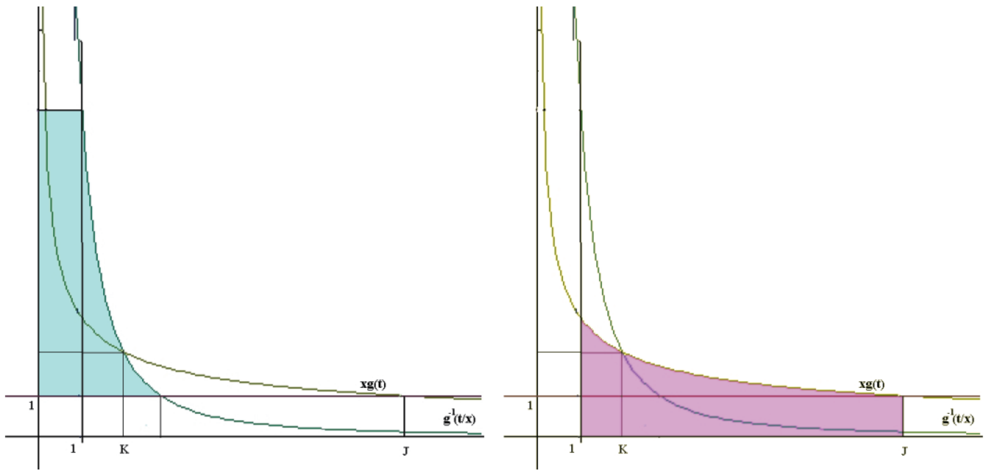

Figure 3.1: Symmetry argument in the proof.

So, let $K \in \mathbb{R}$ be such that

$$xg(K) = g^{-1}\Big(\frac{K}{x}\Big) = K.$$

Then

$$K = g^{-1}\Big(\frac{K}{x}\Big) = \frac{1}{h(\frac{K}{x})} = f\Big(\frac{x}{K}\Big).$$

By symmetry we have:

$$\begin{aligned}\sum_{j=1}^{J}[g(j)x] &= \sum_{j=1}^{K}[g(j)x] + \sum_{j=K}^{J}[g(j)x]\\ &= \sum_{j=1}^{K}[g(j)x] + \sum_{j=1}^{K}\Big[g^{-1}\Big(\frac{j}{x}\Big)\Big] - [K]^2\\ &= \sum_{j=1}^{K} g(j)x + \sum_{j=1}^{K} g^{-1}\Big(\frac{j}{x}\Big) - K^2 + O(K).\end{aligned}$$

Applying the Euler-McLaurin summation formula (3.3.1)

$$\begin{aligned}\sum_{j=1}^{J}[g(j)x] = \int_1^K g(t)xdt + A(x) + O(g(K)x)\\ + \int_1^K g^{-1}\Big(\frac{t}{x}\Big)dt + B(x) + O\Big(g^{-1}\Big(\frac{K}{x}\Big)\Big) - K^2 + O(K).\end{aligned}$$

Clearly

$$O(K) = O(xg(K)) = O\Big(g^{-1}\Big(\frac{K}{x}\Big)\Big).$$

By symmetry (see Figure 1)

$$\int_1^J g = \int_1^K g + \int_K^J g = \int_1^K g + \int_1^K g^{-1} - K^2 + J,$$

then replacing $\int_1^K g(t)x\,dt + \int_1^K g^{-1}\big(\frac{t}{x}\big)\,dt$ in the previous equation we have

$$\sum_{j=1}^{J}[g(j)x] = \int_1^J xg(t)\,dt - J + A(x) + B(x) + O\Big(f\Big(\frac{x}{K}\Big)\Big). \tag{3.3.3}$$

Being the integral convergent, we may write the equation (3.3.3) as

$$\sum_{j=1}^{J}[g(j)x] = \int_1^\infty xg(t)\,dt - \int_J^\infty xg(t)\,dt - J + A(x) + B(x) + O\Big(f\Big(\frac{x}{K}\Big)\Big)$$

and again, by using the Euler-MacLaurin summation formula (3.3.1), we obtain

$$\sum_{j=1}^{J}[g(j)x] = \sum_{j=1}^{\infty} xg(j) - \int_J^\infty xg(t)\,dt - J + B(x) + O\Big(f\Big(\frac{x}{K}\Big)\Big). \tag{3.3.4}$$

Using that as $x \to \infty$, $K \to \infty$ and by (H2), we obtain $f(x/K) = K \sim f(x)^{1/(1+d)}$. Then

$$\sum_{j=1}^{J}[g(j)x] = x\left(\sum_{j=1}^{\infty} g(j)\right) - x\int_J^\infty g(t)\,dt - J + B(x) + O(f^{1/(1+d)}(x)). \tag{3.3.5}$$

To compute the integral we use the Proposition 3.4 to obtain

$$\int_{f(x)}^{\infty} g(u)\,du = f(x)g(f(x))\Big(\frac{d}{1-d} + o(1)\Big).$$

Hence, using that $J = f(x)$ and that $g(f(x)) = \frac{1}{x}$ we arrive at

$$x\int_{J}^{\infty} g(t)\,dt = f(x)\Big(\frac{d}{1-d} + o(1)\Big), \quad \text{as } x \to \infty. \tag{3.3.6}$$

Replacing in equation (3.3.5) we obtain

$$\sum_{j=1}^{J}[g(j)x] = x\left(\sum_{j=1}^{\infty} g(j)\right) - \frac{1}{1-d}f(x) + B(x) + o(f(x)). \tag{3.3.7}$$

Our last task is to determinate the value of $B(x)$. For $b > 1$ fixed, we have,

$$\sum_{j=1}^{b} g^{-1}\Big(\frac{j}{x}\Big) - \int_1^b g^{-1}\Big(\frac{t}{x}\Big)\,dt = B(x) + O\Big(g^{-1}\Big(\frac{b}{x}\Big)\Big)$$

Taking x big enough and remembering that $g^{-1}(t/x) = 1/h(t/x)$, $f(x) = 1/h(1/x)$,

$$\sum_{j=1}^{b} \frac{h(\frac{1}{x})}{h(\frac{j}{x})h(\frac{1}{x})} - \int_1^b \frac{h(\frac{1}{x})}{h(\frac{t}{x})h(\frac{1}{x})}\,dt = B(x) + O\Big(g^{-1}\Big(\frac{b}{x}\Big)\Big).$$

By (H2), for x large we have

$$\frac{h\big(\frac{1}{x}\big)}{h\big(\frac{t}{x}\big)} = t^{-d} + o(1).$$

When $b \to \infty$, as g^{-1} is decreasing, $O(g^{-1}(\frac{b}{x})) \to 0$. Hence,

$$B(x) = f(x)(1 + o(1)) \lim_{b\to\infty}\Big(\sum_{j=1}^{b} j^{-d} - \int_1^b t^{-d}dt\Big), \tag{3.3.8}$$

or equivalently, $B(x) \sim Cf(x)$ as $x \to +\infty$. In order to find the constant C, we use the next expression for the Riemann zeta function, see [La93]:

$$\lim_{b\to\infty}\Big(\sum_{j=1}^{b} j^{-d} - \int_1^b t^{-d}dt\Big) = \zeta(d) - \frac{1}{d-1}.$$

Hence, replacing in (3.3.7) the expression $B(x) = f(x)(\zeta(d) - \frac{1}{d-1} + o(1))$ we have

$$\begin{aligned}\sum_{j=1}^{\infty}[g(j)x] &= \sum_{j=1}^{J}[g(j)x] \\ &= x\left(\sum_{j=1}^{\infty} g(j)\right) - \frac{1}{1-d}f(x) + B(x) + o(f(x)) \\ &= x\left(\sum_{j=1}^{\infty} g(j)\right) + \zeta(d)f(x) + o(f(x))\end{aligned}$$

and the proof is complete. □

Now, we can prove our first theorem:

Theorem 3.10. *Let $\Omega = \bigcup_{j\in\mathbb{N}} I_j \subset \mathbb{R}$ where I_j are disjoint open intervals. Assume that there exist $d \in (0,1)$ and $h \in G_d$ such that $|I_j| = g(j)$. Then,*

$$N(\lambda,\Omega) = \frac{|\Omega|}{\pi_p}\lambda^{1/p} + \frac{\zeta(d)}{\pi_p^d} f(\lambda^{1/p}) + o(f(\lambda^{1/p})) \quad as \quad \lambda \to \infty.$$

Proof. As $\Omega = \bigcup_{j\in\mathbb{N}} I_j$ with $|I_j| = g(j)$, from Proposition 2.14,

$$N(\lambda,\Omega) = \sum_{j=1}^{\infty} \Big[\frac{g(j)}{\pi_p}\lambda^{1/p}\Big].$$

Now the proof follows by a direct application of Lemma 3.9 with $x = \lambda^{1/p}/\pi_p$. In fact,

$$N(\lambda,\Omega) = \sum_{j=1}^{\infty} \Big[\frac{g(j)}{\pi_p}\lambda^{1/p}\Big] = \frac{|\Omega|}{\pi_p}\lambda^{1/p} + \zeta(d) f\Big(\frac{\lambda^{1/p}}{\pi_p}\Big) + o(f(\lambda^{1/p})),$$

as we wanted to prove. □

Remark 3.11. Observe that the assumptions of Theorem 3.10 implies the length of the intervals I_j must be strictly decreasing. This is not desirable for many applications (for instance, complements of Cantor-type sets).

However, a simple inspection of the arguments show that it suffices to assume that $|I_j| \sim g(j)$. Therefore, for example, complements of Cantor-type sets are included in our result. See [Pi06] for the details and also the next section.

3.4 The infinite measure case: $d > 1$

We begin with a couple of lemmas in the spirit of Lemma 3.9.

Lemma 3.12. *Given $\{l_j\}_{j\in\mathbb{N}}$ a sequence of positive numbers and $h \in G_d$ for some $d > 1$. Then, if $l_j \asymp g(j)$, we have*

$$\sum_{j=1}^{\infty} [l_j x] \asymp f(x) \quad as\ x \to +\infty.$$

Proof. Since $l_j \asymp g(j)$, there exist two positive constants c_1, c_2 such that $c_1 g(j) \le l_j \le c_2 g(j)$. Then

$$c_1 x g(j) - 1 \le [c_1 x g(j)] \le [l_j x] \le [c_2 x g(j)] \le c_2 x g(j).$$

So, if we denote $J_i = f(c_i x)$, $i = 1, 2$ we have that $l_j x < 1$ for $j > J_2$. Then

$$\sum_{j=1}^{J_1} c_1 x g(j) - J_1 \le \sum_{j=1}^{J_2} [l_j x] = \sum_{j=1}^{\infty} [l_j x] \le \sum_{j=1}^{J_2} c_2 x g(j). \tag{3.4.1}$$

From the summation formula (3.8), we get

$$\sum_{j=1}^{J_i} c_i x g(j) = c_i x \int_1^{J_i} g(t)\,dt + Cx + O(xg(J_i)). \tag{3.4.2}$$

Applying Proposition 3.6, since $J_i \to \infty$ as, $x \to \infty$

$$\frac{\int_1^{J_i} g(t)\,dt}{J_i g(J_i)} = \frac{d}{d-1} + o(1).$$

Also, as $J_i = f(c_i x)$, we have that $c_i x g(J_i) = 1$. Moreover, by (H3'), $x = o(f(x))$. Collecting all these facts, we arrive at

$$\sum_{j=1}^{J_i} c_i x g(j) = \frac{d}{d-1} J_i + o(J_i).$$

Replacing in (3.4.1) we get

$$\frac{1}{d-1} J_1 + o(J_1) \le \sum_{j=1}^{\infty} [l_j x] \le \frac{d}{d-1} J_2 + o(J_2).$$

Finally, it is easy to see (from (H2)) that $J_i = f(c_i x) \asymp f(x)$ so (1) follows. □

Lemma 3.13. *Given $\{l_j\}_{j\in\mathbb{N}}$ a sequence of positive numbers and $h \in G_d$ for some $d > 1$. Then, if $l_j \sim g(j)$, we have*

$$\sum_{j=1}^{\infty} [l_j x] = \zeta(d) f(x) + o(f(x)) \quad \textit{as } x \to +\infty.$$

Proof. Since $l_j \sim g(j)$, for a fixed $\epsilon > 0$ there exists j_0 such that, for $j > j_0$,

$$1 - \epsilon < \frac{l_j}{g(j)} < 1 + \epsilon. \tag{3.4.3}$$

From Theorem 2.10 and Proposition 2.14

$$\sum_{j=1}^{\infty} [l_j x] = \sum_{j=1}^{j_0} [g(j)x] + \sum_{j=1}^{j_0} \Big([l_j x] - [g(j)x] \Big) + \sum_{j=j_0+1}^{\infty} [l_j x]. \tag{3.4.4}$$

Now, from (3.4.3) and (3.4.4) we get

$$\sum_{j=1}^{\infty} [(1-\varepsilon) g(j) x] \le \sum_{j=1}^{\infty} [l_j x] - \sum_{j=1}^{j_0} \Big([l_j x] - [g(j)x] \Big) \le \sum_{j=1}^{\infty} [(1+\varepsilon) g(j) x].$$

Now, if $K_\pm$ is such that

$$(1 \pm \varepsilon) g(K_\pm) x = g^{-1}(K_\pm / x(1 \pm \varepsilon)) = K_\pm,$$

arguing as in Lemma 3.9, we arrive at

$$\sum_{j=1}^{\infty} [(1 \pm \varepsilon) g(j) x] = \sum_{j=1}^{K_\pm} (1 \pm \varepsilon) g(j) x + \sum_{j=1}^{K_\pm} g^{-1}(j / x(1 \pm \varepsilon)) - K_\pm^2 + O(K_\pm).$$

Applying the Euler-McLaurin summation formula (3.8), we get

$$\begin{aligned}\sum_{j=1}^{\infty}[(1\pm\varepsilon)g(j)x] = &\int_1^{K_\pm}(1\pm\varepsilon)g(t)x\,dt + \int_1^{K_\pm} g^{-1}(t/x(1\pm\varepsilon))\,dt\\ &+ A(x) + B(x) - K_\pm^2 + O(K_\pm),\end{aligned}$$

where $A(x) = C(1\pm\varepsilon)x$ and $B(x)$ are the constants from the Euler-McLaurin formula (3.8) for $(1\pm\varepsilon)g(t)x$ and $g^{-1}(t/x(1\pm\varepsilon))$ respectively.

Again, as in Lemma 3.9

$$\int_1^{K_\pm} g + \int_1^{K_\pm} g^{-1} = \int_1^{J_\pm} g + K_\pm(K_\pm - 1) - J_\pm,$$

where $J_\pm$ is given by $(1\pm\varepsilon)xg(J_\pm) = 1$.

Therefore, we arrive at

$$\sum_{j=1}^{\infty}[(1\pm\varepsilon)g(j)x] = \int_1^{J_\pm}(1\pm\varepsilon)xg(t)\,dt + A(x) + B(x) - J_\pm + O(K_\pm).$$

Applying now Proposition 3.6 and the definition of $J_\pm$ we obtain

$$\begin{aligned}\sum_{j=1}^{\infty}[(1\pm\varepsilon)g(j)x] =&(1\pm\varepsilon)xJ_\pm g(J_\pm)\Big(\frac{d}{d-1} + o(1)\Big)\\ &+ A(x) + B(x) - J_\pm + O(K_\pm)\\ =&J_\pm\Big(\frac{1}{d-1} + o(1)\Big) + A(x) + B(x) + O(K_\pm)\\ =&\frac{1}{d-1}f((1\pm\varepsilon)x) + B(x) + o(f(x)),\end{aligned}$$

where we have used that $A(x) = C(1\pm\varepsilon)x$, $x = o(f(x))$ and $K_\pm = f(x(1\pm\varepsilon)/K_\pm) = o(f(x))$.

It remains to estimate $B(x)$, but this follows exactly as in the proof of the finite measure case, Proposition 3.9. So

$$B(x) = f((1\pm\varepsilon)x)(1+o(1))\lim_{b\to\infty}\Big(\sum_{j=1}^{b} j^{-d} - \int_1^b t^{-d}\,dt\Big).$$

In this case, both terms are convergent, and we easily get

$$B(x) = \Big(\zeta(d) - \frac{1}{d-1}\Big)f((1\pm\varepsilon)x) + o(f(x)).$$

Hence, we finally get

$$\sum_{j=1}^{\infty}[(1\pm\varepsilon)g(j)x] = \zeta(d)f((1\pm\varepsilon)x) + o(f(x)).$$

As $\varepsilon > 0$ is arbitrary, the proof follows. □

Now, we can prove our second theorem:

Theorem 3.14. *Let $\Omega = \bigcup_{j\in\mathbb{N}} I_j$, and $h \in G_d$ for some $d > 1$. Then*

1. *if $|I_j|_1 \asymp g(j)$, we have*
$$N(\lambda, \Omega) = O(f(\lambda^{1/p})) \quad \text{as } \lambda \to +\infty.$$

2. *if $|I_j|_1 \sim g(j)$, we have*
$$N(\lambda, \Omega) = \frac{\zeta(d)}{\pi_p^d} f(\lambda^{1/p}) + o(f(\lambda^{1/p})) \quad \text{as } \lambda \to +\infty.$$

Proof. The proofs follow from Lemmas 3.12 and 3.13 replacing x by $\lambda^{1/p}/\pi_p^{1/p}$. □

We close this section with the following estimate for the eigenvalues.

Corollary 3.15. *Let $h \in G_d$ for some $d > 1$ and let $\Omega = \bigcup_{j\in\mathbb{N}} I_j$ be such that $|I_j| \sim g(j)$. Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the sequence of eigenvalues of problem (1.5.4) in Ω. Then,*

$$\lambda_k \sim \Big[g\Big(\frac{\pi_p^d k}{\zeta(d)}\Big)\Big]^{-p}.$$

Proof. Since
$$k = N(\lambda_k, \Omega) \sim \frac{\zeta(d)}{\pi_p^d} f(\lambda_k^{1/p}) = \frac{\zeta(d)}{\pi_p^d} g^{-1}(\lambda_k^{-1/p}),$$
we get
$$\Big[g\Big(\frac{\pi_p^d k}{\zeta(d)}\Big)\Big]^{-p} \sim \lambda_k$$
and the proof is finished. □

Remark 3.16. Let us note that, for $h(t) = t^d$, we have that $g(t) = t^{-1/d}$, so

$$\lambda_k \sim \Big(\frac{\pi_p^d k}{\zeta(d)}\Big)^{p/d} = \frac{\pi_p^p k^{p/d}}{\zeta(d)^{p/d}}.$$

For $p = 2$, the eigenvalues of the Laplace operator with Dirichlet boundary condition in any bounded open set $U \subset \mathbb{R}^N$ satisfy
$$\lambda_k \sim ck^{2/N}.$$
Hence, seems natural to consider h as a *spectral dimension* for $\partial\Omega$ despite the fact that $\Omega = \bigcup_{j\in\mathbb{N}} I_j \subset \mathbb{R}$ and $d > 1$.

3.5 Two–dimensional horns

For simplicity, we only consider here two dimensional domains. First we derive a simple proof of the upper bound for the eigenvalue counting function of the Laplace operator on horns. Then, we give a lower bound with the same order of growth although with a different constant in the leading term.

Let $h \in G_d$, with $d > 1$, and $g(x) = h^{-1}(1/x)$. Let $\Omega \subset \mathbb{R}^2$ be defined as

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x \geq 1;\ |y| \leq g(x)\}.$$

Clearly, the measure of Ω is infinite.

Let us consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5.1}$$

Since $g(x) \searrow 0$ as $x \nearrow \infty$, the domain is quasibounded, namely,

$$\lim_{|x|\to\infty} d(x, \mathbb{R}^2 \setminus \Omega) = 0,$$

and the spectrum is discrete, consisting of a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \cdots \nearrow \infty$, repeated according their multiplicity.

We want to estimate the order of growth of

$$N(\lambda, \Omega) = \#\{n \in \mathbb{N} \colon \lambda_n \leq \lambda\}.$$

To this end, let us introduce a family of rectangles $\{Q^j\}_{j\in\mathbb{N}}$, and an open set V such that $\Omega \subset V$:

$$Q^j = [j, j+1] \times [-g(j), g(j)], \qquad V = \Big(\bigcup_{j=1}^{\infty} Q^j\Big)^{\circ}.$$

Also, the set V is quasibounded and the spectrum of the Laplace operator in V is a sequence $\mu_1 \leq \mu_2 \leq \cdots \nearrow \infty$, repeated according their multiplicity. Moreover, the monotonicity of eigenvalues respect to the domain gives

$$\mu_n \leq \lambda_n, \qquad n \geq 1.$$

We have the following inclusions of Sobolev spaces:

$$H_0^1(\Omega) \subset H_0^1(V) \subset \bigoplus_{j=1}^{\infty} H_*^1(Q^j),$$

where

$$H_*^1(Q^j) = \{u \in H^1(Q^j) : u(x, \pm g(j)) = 0\}.$$

We can compute by separation of variables the eigenfunctions and eigenvalues of the Laplace operator in each Q^j with mixed boundary conditions. We get

$$\lambda_{h,k}^{Q^j} = h^2\pi^2 + \frac{k^2\pi^2}{4g(j)^2}, \quad u_{h,k}^{Q_j}(x, y) = \cos(h\pi y)\sin(k\pi y/2g(j)), \quad h \geq 0, k \geq 1.$$

Hence, we define the eigenvalue counting function

$$N_{mixed}(\lambda, Q^j) = \#\Big\{(h,k) : h^2\pi^2 + \frac{k^2\pi^2}{4g(j)^2} \le \lambda, \quad h \ge 0, k \ge 1\Big\}.$$

Let us note that we can assign to each eigenvalue a lattice point (h,k) with $h > 0$ and the square $(h-1, h] \times (k-1, k]$, and the number of eigenvalues with $h = 0$ is $[2g(j)\lambda^{1/2}/\pi]$. By using the area of the ellipse which contains those squares, we get

$$N_{mixed}(\lambda, Q^j) \le \frac{g(j)}{2\pi^2}\lambda + \frac{2g(j)}{\pi}\lambda^{1/2} = g(j)\Big(\frac{\lambda}{2\pi^2} + \frac{2\lambda^{1/2}}{\pi}\Big). \tag{3.5.2}$$

Now, the Dirichlet-Neumann bracketing (2.15) together with Proposition 2.14 implies

$$N(\lambda, \Omega) \le \sum_{j=1}^{\infty} N_{mixed}(\lambda, Q^j),$$

but we cannot replace the previous bound yet. Let us note that $N_{mixed}(\lambda, Q^j) = 0$ if

$$\lambda_{0,1}^{Q^j} = \frac{\pi^2}{4g(j)^2} > \lambda,$$

i.e., for $j > g^{-1}(\pi/2\lambda^{1/2}) = f(2\lambda^{1/2}/\pi)$. Hence, by using the estimate (3.5.2), the Euler-McLaurin formula (3.8) and Proposition 3.6, we obtain

$$\begin{aligned}
N(\lambda, \Omega) &\le \sum_{j=1}^{f(2\lambda^{1/2}/\pi)} N_{mixed}(\lambda, Q^j) \\
&\le \sum_{j=1}^{f(2\lambda^{1/2}/\pi)} g(j)\Big(\frac{\lambda}{2\pi^2} + \frac{2\lambda^{1/2}}{\pi}\Big) \\
&=\Big(\frac{\lambda}{2\pi^2} + \frac{2\lambda^{1/2}}{\pi}\Big)\left(\int_1^{f(2\lambda^{1/2}/\pi)} g(t)dt + A + O\Big(g\Big(f\Big(\frac{2\lambda^{1/2}}{\pi}\Big)\Big)\Big)\right) \\
&=\Big(\frac{\lambda}{2\pi^2} + \frac{2\lambda^{1/2}}{\pi}\Big)f\Big(\frac{2\lambda^{1/2}}{\pi}\Big)g\Big(f\Big(\frac{2\lambda^{1/2}}{\pi}\Big)\Big)\Big(\frac{d}{d-1} + o(1)\Big) + O(\lambda^{1/2}) \\
&=\Big(\frac{\lambda}{2\pi^2} + \frac{2\lambda^{1/2}}{\pi}\Big)f\Big(\frac{2\lambda^{1/2}}{\pi}\Big)\frac{\pi}{2\lambda^{1/2}}\Big(\frac{d}{d-1} + o(1)\Big) + O(\lambda^{1/2}) \\
&=\frac{d}{4\pi(d-1)}\lambda^{1/2} f\Big(\frac{2\lambda^{1/2}}{\pi}\Big) + o\left(\lambda^{1/2} f\Big(\frac{2\lambda^{1/2}}{\pi}\Big)\right).
\end{aligned}$$

Therefore, we have proved the following Theorem:

Theorem 3.17. *Let $h \in G_d$, with $d > 1$, and $\Omega \subset \mathbb{R}^2$ be defined as*

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x \ge 1;\ |y| \le g(x)\}.$$

Then, the eigenvalue counting function of the eigenvalue problem (3.5.1) satisfies

$$N(\lambda, \Omega) \le \frac{d}{d-1}\lambda^{1/2} f\Big(\frac{2\lambda^{1/2}}{\pi}\Big) + o\left(\lambda^{1/2} f\Big(\frac{2\lambda^{1/2}}{\pi}\Big)\right).$$

Remark 3.18. When $h(t) = t^d$ with $d > 1$, then $g(t) = t^{-1/d}$ and $f(t) = t^d$. So, we have

$$N(\lambda, \Omega) \leq \frac{d}{d-1}\Big(\frac{2}{\pi}\Big)^d \lambda^{\frac{d+1}{2}} + o(\lambda^{\frac{d+1}{2}}).$$

Following [VL01], the order of growth cannot be improved, since this is the right one for horn-shaped domains.

In much the same way we prove the following lower bound:

Theorem 3.19. *Let $h \in G_d$, with $d > 1$, and $\Omega \subset \mathbb{R}^2$ be defined as*

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x \geq 1;\ |y| \leq g(x)\}.$$

Then, the eigenvalue counting function of the eigenvalue problem (3.5.1) *satisfies*

$$N(\lambda, \Omega) \geq \frac{1}{d-1}\frac{\lambda^{1/2}}{\pi} f\Big(\frac{\lambda^{1/2}}{2\pi}\Big) + o\left(\lambda^{1/2} f\Big(\frac{\lambda^{1/2}}{2\pi}\Big)\right).$$

Proof. As before, let us introduce a family of rectangles $\{Q_j\}_{j\in\mathbb{N}}$ and $U \subset \Omega$, where

$$Q_j = [j, j+1] \times [-g(j+1), g(j+1)], \qquad U = \Big(\bigcup_{j=1}^{\infty} Q_j\Big)^{\circ}.$$

Then,

$$\bigoplus_{j=1}^{\infty} H_0^1(Q_j) \subset H_0^1(U),$$

and the Dirichlet-Neumann bracketing (2.15) together with Proposition 2.14 implies

$$\sum_{j=1}^{\infty} N_D(\lambda, Q_j) \leq N(\lambda, \Omega).$$

The eigenfunctions and eigenvalues of the Laplace operator in Q_j with Dirichlet boundary conditions are

$$\lambda_{h,k}^{Q_j} = h^2\pi^2 + \frac{k^2\pi^2}{4g(j+1)^2}, \quad u_{h,k}^{Q_j}(x, y) = \sin(k\pi x/2g(j))\sin(h\pi y), \quad h, k \geq 1.$$

Therefore, the counting function $N_D(\lambda, Q_j)$ is

$$N_D(\lambda, Q_j) = \#\Big\{(h, k) : h^2\pi^2 + \frac{k^2\pi^2}{4g(j+1)^2} \leq \lambda, \quad h, k \geq 1\Big\}.$$

Let us assign to each eigenvalue the lattice point (h, k) with $h, k \geq 1$, and the square $Q_{h,k} = [h, h+1) \times [k, k+1)$. Hence,

$$N_D(\lambda, Q_j) = \Big|\Big(\bigcup_{\lambda_{h,k}^{Q_j} \leq \lambda} Q_{h,k}\Big)\Big|.$$

Clearly,

$$N_D(\lambda, Q_j) \geq \frac{g(j)\lambda}{2\pi^2} - \frac{\lambda^{1/2}}{\pi} - \frac{2g(j)\lambda^{1/2}}{\pi} - 1,$$

since in the first quadrant, the ellipse of semi-axes $\lambda^{1/2}/\pi$ and $2g(j)\lambda^{1/2}/\pi$ is covered by the squares $Q_{h,k}$ and the rectangles $[0,1)\times[0,\lambda^{1/2})$, $[0,[2g(j)\lambda^{1/2}]+1)\times[0,1)$.

We consider only $j\le f(\lambda^{1/2}/2\pi)$ (if not, $\frac{g(j)\lambda}{2\pi^2}-\frac{\lambda^{1/2}}{\pi}<0$, and $N_D(\lambda,Q_j)$ is nonnegative) and we get

$$N(\lambda,\Omega)\ge\sum_{j=1}^{\infty}N_D(\lambda,Q_j)\ge\sum_{j=1}^{f(\lambda^{1/2}/2\pi)}\frac{g(j)\lambda}{2\pi^2}-\frac{\lambda^{1/2}}{\pi}-\frac{2g(j)\lambda^{1/2}}{\pi}-1.$$

Finally, as in the previous proof,

$$\begin{aligned}
N(\lambda,\Omega)\ge&\sum_{j=1}^{f(\lambda^{1/2}/2\pi)}\frac{g(j)\lambda}{2\pi^2}-f\Big(\frac{\lambda^{1/2}}{2\pi}\Big)\frac{\lambda^{1/2}}{\pi}+O\Big(\sum_{j=1}^{f(\lambda^{1/2}/2\pi)}2g(j)\lambda^{1/2}\Big)\\
=&\frac{\lambda}{2\pi^2}f\Big(\frac{\lambda^{1/2}}{2\pi}\Big)g\Big(f\Big(\frac{\lambda^{1/2}}{2\pi}\Big)\Big)\Big(\frac{d}{d-1}+o(1)\Big)-\frac{\lambda^{1/2}}{\pi}f\Big(\frac{\lambda^{1/2}}{2\pi}\Big)\\
&+o\Big(\lambda^{1/2}f\Big(\frac{\lambda^{1/2}}{2\pi}\Big)\Big)\\
=&\frac{\lambda}{2\pi^2}f\Big(\frac{\lambda^{1/2}}{2\pi}\Big)\frac{2\pi}{\lambda^{1/2}}\Big(\frac{d}{d-1}+o(1)\Big)-\frac{\lambda^{1/2}}{\pi}f\Big(\frac{\lambda^{1/2}}{2\pi}\Big)+o\Big(\lambda^{1/2}f\Big(\frac{\lambda^{1/2}}{2\pi}\Big)\Big)\\
=&\frac{\lambda^{1/2}}{\pi(d-1)}f\Big(\frac{\lambda^{1/2}}{2\pi}\Big)+o\Big(\lambda^{1/2}f\Big(\frac{\lambda^{1/2}}{2\pi}\Big)\Big)
\end{aligned}$$

and the proof is finished. □

Remark 3.20. From Theorems 3.17 and 3.19 we obtain that

$$c\lambda^{1/2}f\Big(\frac{\lambda^{1/2}}{2\pi}\Big)\le N(\lambda,\Omega)\le C\lambda^{1/2}f\Big(\frac{2\lambda^{1/2}}{\pi}\Big),$$

for horn-shaped domains

$$\Omega=\{(x,y)\in\mathbb{R}^2: x\ge1;\ |y|\le g(x)\},$$

with $f(x)=g^{-1}(1/x)$, and g monotonically decreasing continuous function.

Observe that, as h satisfies (H2), we have

$$N(\lambda,\Omega)\asymp\lambda^{1/2}f(\lambda^{1/2}).$$

This result improves the upper bounds obtained in [CH67, He74, He75], which only gives an upper bound for $N(\lambda,\Omega)$ whenever $g(x)=x^{-1/d}$.

It would be desirable to obtain a better knowledge of the asymptotic behavior, namely, $N(\lambda,\Omega)\sim c\lambda^{1/2}f(\lambda^{1/2})$ (for certain constant c) as in [Si83], and even a second term as in [VL01]. However, without imposing more restrictions on the functions h or g, we believe that this cannot be possible, since the main term can oscillate, as the following one–dimensional example suggest. This example is borrowed from [Pi06].

Example 3.21. Let $\Omega=\bigcup_{k\in\mathbb{N}}\Omega_k$, where Ω_k consist of m^k intervals of lengths n^{1-k}, for $m>n$. Then, the spectral counting function of problem (1.5.4) satisfies

$$N(\lambda,\Omega)=\frac{\lambda^{d/p}}{m}s(\log(\lambda))-O(\lambda^{1/p}),$$

where $s(\log(\lambda))$ is a bounded periodic function, and $d=\frac{\log(m)}{\log(n)}$.

Proof. Since

$$N(\lambda,\Omega) = \sum_{j=1}^{\infty} m^j \left[\frac{\lambda^{1/p}}{\pi_p n^{j-1}}\right] = \sum_{-\infty}^{\infty} m^j \left[\frac{\lambda^{1/p}}{n^{j-1}\pi_p}\right] - O(\lambda^{1/p}).$$

By changing variables,

$$k = \frac{\log(\lambda^{1/p}) - \log(\pi_p)}{\log(n)},$$

we get $n^k = \lambda^{1/p}/\pi_p$ and $m^k = (\lambda^{1/p}/\pi_p)^d$, for $d = \frac{\log(m)}{\log(n)}$, and we obtain

$$N(\lambda,\Omega) = \frac{\lambda^{d/p}}{m} \sum_{j=-\infty}^{\infty} m^{j-k}[n^{y-k}] - O(\lambda^{1/p}) = \frac{\lambda^{d/p}}{m} s(\log(\lambda)) - O(\lambda^{1/p})$$

and, as $j-(k+1) = (j+1)-k$, $s(\log(\lambda))$ is a periodic function with period equal to one. □

This example can be extended to $\mathbb{R}^2$, by defining $\Omega = \bigcup_{k\in\mathbb{N}} \Omega_k$, where Ω_k consists of m^k disjoints squares of sides n^{1-k}. When Ω has finite measure, similar examples were considered in [FV93, LV96, **?**], where oscillating second term were obtained for the spectral counting function of the Laplace operator in Ω with Dirichlet boundary conditions in the boundary of each square. It is not difficult to extend those arguments to the infinite measure case (that is, $m^2 > n$), to obtain in this way a quasibounded set with an oscillating main term. However, the set obtained in this way is not a horn.

4

Homogenization: preliminaries

In order to answer the questions formulated in the Introduction, and others related to a more general class of problems, we follow the approach which uses the theory of $H-$ and $G-$convergence.

The notion of $H-$convergence was introduced by Murat and Tartar [Ta78] to study a wide class of homogenization problem for possibly non-symmetric elliptic equations. $G-$convergence was introduced by Spagnolo [Sp68], [Sp76], De Giorgi and Spagnolo [SP73], and it is restricted to symmetric operators.

In the first part of this Chapter we introduce the $H-$ and $G-$convergence for second order linear uniformly elliptic operators.

Then, we emphasize in the important case of periodic homogenization, namely, when we deal with families of matrices of the form $A^{\varepsilon}(x) = A(x/\varepsilon)$, where $A(x)$ are $Q-$periodic functions, Q being the unit cube in $\mathbb{R}^N$ and ε a real parameter tending to zero. Here, it is possible to find an explicit form of the limit operator.

Finally, we define the notion of $G-$convergence of nonlinear monotone operators in the general setting, that is, a more general family of operator $a_{\varepsilon}(x, \xi)$ satisfying certain properties and whose prototypical example is $a_{\varepsilon}(x, \xi) = A^{\varepsilon}(x)|\xi|^{p-2}\xi$, related with the $p-$Laplacian operator. Here, we also deal with the periodic case and some remarks about the homogenization of nonlinear periodic monotone operators.

4.1 $H-$convergence of linear equations

In this section, we deal with linear elliptic operators of the form $\mathcal{A}_{\varepsilon}u = -div(A^{\varepsilon}(x)\nabla u)$ where $A^{\varepsilon}(x) = (a^{\varepsilon}_{ij}(x))$ is an elliptic symmetric matrix. Let $\mathcal{M}_N$ be the linear space of square real matrices of order N. Given α, β two positive constants, we define a subspace of $\mathcal{M}_N$ made of coercive matrices with coercive inverses

$$\mathcal{M}_{\alpha,\beta} = \{M \in \mathcal{M}_N : M\xi \cdot \xi \geq \alpha|\xi|^2, \quad M^{-1}\xi \cdot \xi \geq \beta|\xi|^2 \quad \forall \xi \in \mathbb{R}^N\}. \tag{4.1.1}$$

A coercive matrix with coercive inverse is also bounded. Indeed, if $M \in \mathcal{M}_{\alpha,\beta}$, introducing $\eta = M^{-1}\xi$, we deduce from (4.1.1)

$$\beta|M\eta|^2 \leq M\eta \cdot \eta.$$

Applying Cauchy-Schwarz inequality, we obtain

$$|M\eta| \leq \beta^{-1}|\eta| \qquad \forall \eta \in \mathbb{R}^N. \tag{4.1.2}$$

Similarly, we have

$$|M^{-1}\eta| \leq \alpha^{-1}|\eta| \qquad \forall \eta \in \mathbb{R}^N.$$

From (4.1.1) and (4.1.2) we deduce that a necessary condition for $M \in \mathcal{M}_{\alpha,\beta}$ is that

$$\alpha|\xi|^2 \leq M\xi \cdot \xi \leq \beta^{-1}|\xi|^2 \qquad \forall \xi \in \mathbb{R}^N.$$

Therefore, $\mathcal{M}_{\alpha,\beta}$ is nonempty if and only if it is satisfied the condition $\alpha\beta \leq 1$.

$H-$convergence is a notion of convergence for the coefficients of an elliptic partial differential equation, which is defined through some convergence properties of the solution of this equation.

Definition 4.1. It is said that a sequence of matrices $A^\varepsilon(x) \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ *$H-$converges* to an homogenized limit matrix $A^*(x) \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ (called *$H-$limit*) if, for any $f \in W^{-1,2}(\Omega)$, the sequence u^ε of solutions of

$$\begin{cases} -div(A^\varepsilon(x)\nabla u^\varepsilon) = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1.3}$$

satisfies

$$\begin{cases} u^\varepsilon \rightharpoonup u \text{ weakly in } H_0^1(\Omega) & \\ A^\varepsilon\nabla u^\varepsilon \rightharpoonup A^*\nabla u & \text{weakly in } L^2(\Omega)^N \end{cases} \tag{4.1.4}$$

where u is the solution of the homogenized equation

$$\begin{cases} -div(A^*\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1.5}$$

This definition is justified by the following compactness theorem.

Theorem 4.2. *For any sequence $A^\varepsilon(x)$ of matrices in $L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ there exist a subsequence, still denoted by A^ε, and an homogenized matrix $A^*(x) \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ such that A^ε H-converges to A^*.*

Proof. See Theorem 7.4 in [De]. □

Remark 4.3. Let us observe the following remarks:

1. By the definition of $H-$convergence, the homogenized matrix A^* does not depend on the source term f.

2. Theorem 4.2 implies that the set $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ is closed under $H-$convergence, i.e., the coercivity constants α, β are the same for the sequence A^ε and its $H-$limit A^*.

3. By definition, if a sequence A^ε $H-$converges to a limit A^*, then any subsequence also $H-$converges to A^*.

4. Since $L^2(\Omega)$ is dense in $H^{-1}(\Omega)$ the source term in the Definition 4.1 can belong to $L^2(\Omega)$ instead $H^{-1}(\Omega)$.

The next theorem is about the localization of the $H-$convergence:

Theorem 4.4. *Let $A^\varepsilon(x)$ and $B^\varepsilon(x)$ be two sequences of matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$, which $H-$converges to A^* and B^*, respectively. Let ω be an open subset compactly embedded in Ω, i.e., $\bar{\omega} \subset \Omega$. If $A^\varepsilon(x) = B^\varepsilon(x)$ in ω, then $A^*(x) = B^*(x)$ in ω.*

Proof. See Proposition 1 in [MT97]. □

Remark 4.5. Even though in Definition 4.1 we define the $H-$converge with Dirichlet boundary conditions. It can be proved that the $H-$limit is independent of the boundary conditions (see, for instance [Al02] Proposition 1.2.19).

4.1.1 The periodic case

To define the concept of homogenization in the periodic framework we need some definitions.

Definition 4.6. Let $Y = (0, \ell_1) \times \cdots (0, \ell_n)$ be an interval in $\mathbb{R}^N$, where $\ell_1, \cdots, \ell_N$ are given positive numbers. We will refer to Y as the *reference cell*.

Definition 4.7. A function f defined a.e. on $\mathbb{R}^N$ is called *Y-periodic* if and only if

$$f(x + k\ell_i e_i) = f(x) \quad \text{a.e. on } \mathbb{R}^N, \quad \forall k \in \mathbb{Z}, \quad \forall i \in \{1, \cdots, N\},$$

where $\{e_1, \cdots, e_N\}$ is the canonical basis of $\mathbb{R}^N$.

In the one-dimensional case $N = 1$, we simply say that f is *ℓ_1-periodic*.

In the study of periodic oscillating functions is essential the definition of the average of a periodic function.

Definition 4.8. Let Ω be a bounded open set of $\mathbb{R}^N$ and f a function in $L^1(\Omega)$. The *average* of f over Ω is the real number $\bar{f}$ given by

$$\bar{f} = \frac{1}{|\Omega|} \int_\Omega f(y) dy.$$

The next result is related to the convergence in the weak sense of periodic functions in L^p.

Theorem 4.9 (Weak limit of rapidly oscillating periodic functions)**.** *Let $1 \le p \le +\infty$ and f be a Y−periodic function in $L^p(Y)$. Set*

$$f_\varepsilon(x) = f(\tfrac{x}{\varepsilon}) \quad \text{a.e. on } \mathbb{R}^N.$$

Then, if $p < +\infty$, as $\varepsilon \to 0$

$$f_\varepsilon \rightharpoonup \bar{f} = \frac{1}{|Y|}\int_Y f(y)dy \quad \text{weakly in } L^p(\omega)$$

for any bounded open subset ω of $\mathbb{R}^N$.

If $p = +\infty$, one has

$$f_\varepsilon \rightharpoonup \bar{f} = \frac{1}{|Y|}\int_Y f(y)dy \quad \text{weakly* in } L^\infty(\omega).$$

Proof. See, for instance, Theorem 2.6 in [CD99]. See also Chapter §4 where the rate of that convergence it is obtained. □

Remark 4.10. Let us point out some features of the weak convergence. Let us consider the following example. Let $Y = (0, 2\pi)$ and $f(x) = \sin x$. Let ε be a sequence of positive real numbers tending to zero. By Theorem 4.9 we have that $f_\varepsilon(x) = \sin(x/\varepsilon) \rightharpoonup 0$ weakly* in $L^\infty(Y)$ (hence weakly in $L^2(Y)$). Particularly,

$$\int_0^{2\pi} f_\varepsilon(x)\,dx \to \frac{1}{2\pi}\int_0^{2\pi} \sin y\,dy = 0,$$

i.e., the average of f_ε converges to 0. Furthermore,

$$\|f_\varepsilon - 0\|^2_{L^2(Y)} = \int_0^{2\pi} \sin^2(\tfrac{x}{\varepsilon})\,dx \to \Big(\frac{1}{\pi}\int_0^{\pi} \sin^2 y\,dy\Big)2\pi = \pi \ne 0,$$

which shows that we do not have convergence of f_ε of f in the strong topology of $L^2(Y)$.

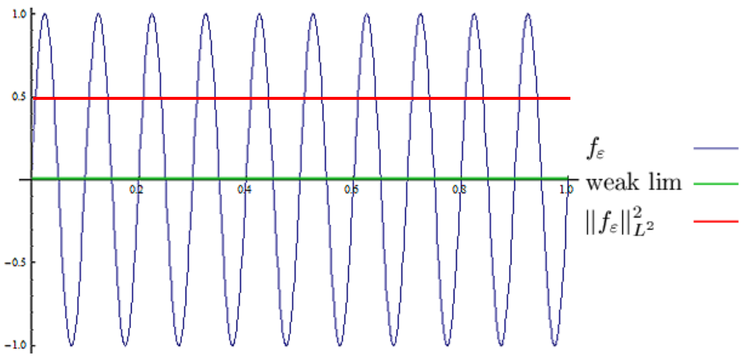

Figure 4.1: $f_\varepsilon(x) = \sin(2\pi x/\varepsilon)$ with $Y = (0, 1)$ and $\varepsilon = 0.1$.

This simple example shows a mathematical difficulty one meets by handling weak convergent sequences. More precisely, if two sequences and their products converge in the weak topology, the limit of the product is not equal, in general to the product of the limits. Indeed, this example proves that $f_\varepsilon^2 = f_\varepsilon f_\varepsilon$ does not converge weakly in $L^2(Y)$ to 0.

Let us consider functions $A : \mathbb{R}^N \to R^{N\times N}$, with $A(x) = (a_{ij}(x))$ such that $A \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ and the functions a_{ij} are Y-periodic $\forall i, j = 1, \dots, N$.

We consider equation (4.1.3) in the periodic framework, i.e., A^ε is a $Y-$periodic matrix defined by

$$A^\varepsilon(x) = A(\tfrac{x}{\varepsilon}) \quad \text{a.e. on } \mathbb{R}^N \tag{4.1.6}$$

where $A(\frac{x}{\varepsilon}) = (a^\varepsilon_{ij}(x))_{1\le i,j\le N}$ a.e. on $\mathbb{R}^N$ with $a^\varepsilon_{ij}(x) = a_{ij}(\frac{x}{\varepsilon})$ a.e. on $\mathbb{R}^N$, for all $i, j = 1, \dots, N$. Note that the functions a^ε_{ij} are $\varepsilon Y-$periodic on $\mathbb{R}^N$.

When we deal with a family of $Y-$periodic matrices of the form (4.1.6), it is possible to find an explicit expression of the limit matrix A^* in term of certain auxiliary functions. In the following Theorem a characterization of the *homogenized coefficients* is given.

Theorem 4.11. *The sequence $A^\varepsilon = A(\frac{x}{\varepsilon})$ $H-$converges to a constant homogenized matrix $A^* \in \mathcal{M}_{\alpha,\beta}$ defined by its entries*

$$A^*_{i,j} = \int_Y A(y)(e_i + \nabla\omega_i)\cdot(e_j + \nabla\omega_j)dy,$$

where $(e_i)_{1\le i\le N}$ is the canonical basis of $\mathbb{R}^N$, and $(w_i)_{1\le i\le N}$ is the family of unique solution in $H^1_\#(Y)/\mathbb{R}$ of the cell problems

$$\begin{cases} -divA(y)(e_i + \nabla\omega_i(y)) = 0 & \text{in } Y \\ y \to \omega_i(y) & Y\text{-periodic} \end{cases} \tag{4.1.7}$$

with

$$H^1_\#(Y) = \{f \in H^1_{loc}(\mathbb{R}^N) \text{ such that } f \text{ is } Y\text{-periodic}\}$$

Proof. See Theorem 1.3.18 in [Al02]. □

4.2 $G-$convergence of linear equations

In the case of symmetric operators, i.e., when the matrix A^ε is symmetric, a notion of operator convergence was introduced by Spagnolo [Sp76] under the name of $G-$convergence. It is a little simpler than $H-$convergence due to the symmetry hypothesis. From a historical point of view, let us mention that the G stands for *Green*, since the original proof of the compactness theorem for $G-$convergence used Green functions.

Let $\mathcal{M}^s_N$ be the linear space of symmetric real matrices of order N. For any positive constants α, β, we define a subspace made of coercive matrices with coercive inverses,

$$\mathcal{M}^s_{\alpha,\beta} = \{M \in \mathcal{M}^s_N : M\xi\cdot\xi \ge \alpha|\xi|^2, \quad M^{-1}\xi\cdot\xi \ge \beta|\xi|^2 \quad \forall\xi \in \mathbb{R}^N\}. \tag{4.2.1}$$

Given $\Omega \subset \mathbb{R}^N$ a bounded open set, we introduce the space $L^\infty(\Omega; \mathcal{M}^s_{\alpha,\beta})$ of admissible symmetric coefficient matrices.

Definition 4.12. A sequence of matrices $A^\varepsilon(x) \in L^\infty(\Omega, \mathcal{M}^s_{\alpha,\beta})$ is said that *$G-$converges* to an homogenized limit matrix, $A^*(x) \in L^\infty(\Omega, \mathcal{M}^s_{\alpha,\beta})$ (called *$G-$limit*) if, for any $f \in H^{-1}(\Omega)$, the sequence u^ε of solutions of

$$\begin{cases} -div(A^\varepsilon(x)\nabla u^\varepsilon) = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2.2}$$

satisfies that $u^\varepsilon \rightharpoonup u$ weakly in $H^1_0(\Omega)$, where u is the unique solution of the homogenized equation

$$\begin{cases} -div(A^*\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2.3}$$

This definition is justified by the following compactness theorem.

Theorem 4.13. *For any sequence A^ε of matrices in $L^\infty(\Omega, \mathcal{M}^s_{\alpha,\beta})$ there exist a subsequence, still denoted by A^ε, and an homogenized matrix $A^*(x) \in L^\infty(\Omega, \mathcal{M}_{\alpha,\beta})$ such that A^ε G-converges to A^*.*

Proof. See [Al02], Lemma 1.3.9. □

The main difference between $H-$ and $G-$convergence is that the latter does not require the convergence of the flux $A^\varepsilon\nabla u^\varepsilon$. Then $G-$convergence is a weaker notion than $H-$convergence in the sense that if a sequence of symmetric matrices A^ε $H-$converges to a symmetric homogenized matrix A^*, then it automatically $G-$converges to the same limit. This is an obvious consequence of the following lemma.

Lemma 4.14. *Let A^ε be a sequence of (not necessarily symmetric) matrices in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$. If A^ε $H-$converges to a limit A^* in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$, then the adjoint, or transposed, sequence $(A^\varepsilon)^t$ $H-$converges to the adjoint limit $(A^*)^t$ in $L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$.*

Proof. See [Al02], Lemma 1.3.10. □

By Lemma 4.14, if A^ε is symmetric and $H-$converges to A^*, then automatically A^* is symmetric, and thus A^ε also $G-$converges to A^*. The following Proposition is the converse of that Lemma. This fact give the equivalence between $H-$ and $G-$convergence for symmetric matrices.

Proposition 4.15. *A sequence of A^ε of symmetric matrices in $L^\infty(\Omega; \mathcal{M}^s_{\alpha,\beta})$ $G-$converges to a limit $A^* \in L^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$ if and only if it $H-$converges to A^*.*

Proof. See [Al02], Proposition 1.3.11. □

Remark 4.16. For symmetric matrices A^ε, the convergence of the flux $A^\varepsilon\nabla u^\varepsilon$ is a consequence of the convergence of the solutions u^ε. If the matrices A^ε are not symmetric, this is no longer true. In particular, for nonsymmetric operator, the notion of $G-$convergence is inconsistent, since it does not guarantee the uniqueness of the $G-$limit.

4.2.1 The periodic case

Here we deal with functions $A : \mathbb{R}^N \to R^{N\times N}$, with $A(x) = (a_{ij}(x))$ such that $A \in L^\infty(\Omega; \mathcal{M}^s_{\alpha,\beta})$ with a_{ij} Y-periodic functions $\forall i, j = 1, \dots, N$.

First, we deal with the one-dimensional problem and then we will see difficulties that arises in the generalization to the case $N > 1$.

The one-dimensional case

This problem was studied by Spagnolo (1967). Let $\Omega = (0,1)$ be an interval in $\mathbb{R}$. In the one-dimensional case equation (4.2.2) is reduced to

$$\begin{cases} -(a_\varepsilon(x)(u^\varepsilon)')' = f & \text{en } \Omega \\ u^\varepsilon(0) = u^\varepsilon(1) = 0 \end{cases} \tag{4.2.4}$$

where $' := \frac{d}{dx}$ and $a_\varepsilon(x) := A(\frac{x}{\varepsilon})$. We assume that a is a positive $1-$periodic function in $L^\infty(\Omega)$ such that for some constants α, β

$$0 < \alpha \leq a(x) \leq \beta < +\infty, \qquad \text{for a.e. } x \in \mathbb{R}. \tag{4.2.5}$$

The weak form of (4.2.4) is

$$\begin{cases} \int_0^1 a_\varepsilon (u^\varepsilon)'\varphi' = \int_0^1 f\varphi & \text{for every } \varphi \in W^{1,2}_0(\Omega) \\ u^\varepsilon \in W^{1,2}_0(\Omega). \end{cases} \tag{4.2.6}$$

Let us observe that by Holder's inequality

$$\alpha \|u^\varepsilon\|^2_{L^2(\Omega)} \leq \int_0^1 a_\varepsilon |u^\varepsilon|^2 = \int_0^1 f u^\varepsilon \leq \|f\|_{W^{-1,2}(\Omega)} \|u^\varepsilon\|_{W^{1,2}_0(\Omega)}. \tag{4.2.7}$$

By the Poincaré inequality for functions with zero boundary values we have that

$$\|u^\varepsilon\|_{L^2(\Omega)} \leq \|(u^\varepsilon)'\|_{L^2(\Omega)}.$$

That implies that

$$\|u^\varepsilon\|_{W^{1,2}_0(\Omega)} \leq \frac{1}{\alpha}\|f\|_{L^2(\Omega)}. \tag{4.2.8}$$

Since $W^{1,2}_0$ is a reflexive space, there exists a subsequence still denoted by ε such that

$$u^\varepsilon \rightharpoonup u \qquad \text{weakly in } W^{1,2}_0(\Omega) \tag{4.2.9}$$

and since $W^{1,2}_0(\Omega)$ is compactly embedded in $L^2(\Omega)$, we have by the Rellich Embedding Theorem (see for instance [Ev10]) that

$$u^\varepsilon \to u \qquad \text{strongly in } L^2(\Omega).$$

Being that is an $1-$periodic function we have that the sequence a_ε converges weakly in $L^\infty(\Omega)$ to its average (and hence weakly in $L^2(\Omega)$), i.e.,

$$a_\varepsilon \overset{*}{\rightharpoonup} \bar{a} = \int_0^1 a \qquad \text{weakly* in } L^\infty(\Omega). \tag{4.2.10}$$

From (4.2.6),(4.2.9) and (4.2.10) it would be reasonable that in the limit we have that u must be a solution of:

$$\begin{cases} \int_0^1 \bar{a}u'\varphi' = \int_0^1 f\varphi & \text{for every } \varphi \in W_0^{1,2}(\Omega) \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{4.2.11}$$

However this **is not true** in general, since $a_\varepsilon(u^\varepsilon)'$ is the product of two weakly converging sequences. This is the main difficulty in the limit process. To obtain the correct answer we proceed as follows.

Let $\xi_\varepsilon = a_\varepsilon(u^\varepsilon)'$. By (4.2.8) we have that the sequence $\{\xi_\varepsilon\}$ is bounded in $L^2(\Omega)$ and (4.2.6) implies that $-\xi_\varepsilon' = f$ in Ω. Moreover, from the estimate on u^ε and (4.2.5) one has

$$\|\xi_\varepsilon\|_{L^2(\Omega)} \leq \frac{\beta}{\alpha}\|f\|_{L^2(\Omega}.$$

Hence up to a subsequence we get

$$\xi_\varepsilon \rightharpoonup \xi \quad \text{weakly in } L^2(\Omega).$$

Then, we can pass to the limit in (4.2.6) to get

$$\int_0^1 \xi\varphi' = \int_0^1 f\varphi \quad \text{for every } \varphi \in W_0^{1,2}(\Omega)$$

i.e.

$$-\frac{d\xi}{dx} = f \quad \text{in } \Omega.$$

Clearly, we obtain

$$\|\xi_\varepsilon\|_{W^{1,2}(\Omega)} \leq \|f\|_{L^2(\Omega)}(1+\frac{\beta}{\alpha}).$$

Hence, ξ_ε is bounded in $W^{1,2}(\Omega)$ and by Rellich's Theorem there exists a subsequence still denoted by ε, such that

$$\xi_\varepsilon \to \xi \quad \text{strongly in } L^2(\Omega).$$

Since $\{\frac{1}{a_\varepsilon}\}$ converges to $\overline{\frac{1}{a}}$ weakly* in $L^\infty(\Omega)$ (and hence weakly in $L^2(\Omega)$), we can pass to the limit in the *weak-strong* product

$$(u^\varepsilon)' = \frac{1}{a_\varepsilon}\xi_\varepsilon \rightharpoonup \overline{\frac{1}{a}}\xi \quad \text{weakly in } L^2(\Omega). \tag{4.2.12}$$

Thus, by (4.2.9) and (4.2.12), we obtain that

$$\xi = (\overline{a^{-1}})^{-1}u'. \tag{4.2.13}$$

Now we can pass to the limit in (4.2.6) obtaining

$$\begin{cases} \int_0^1 a^*u'\varphi' = \int_0^1 f\varphi & \text{for every } \varphi \in W_0^{1,2}(\Omega) \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

where $a^* = (\overline{a^{-1}})^{-1}$. Being $1/\beta \leq \overline{a^{-1}} \leq 1/\alpha$ we conclude that the homogenized equation has a unique solution and thus that the whole sequence $\{u^\varepsilon\}$ converges. Finally u is solution of the equation

$$\begin{cases} -a^* u'' = f & \text{en } \Omega \\ u(0) = u(1) = 0. \end{cases}$$

Observe that in the one dimensional case since a^* is a constant, one can compute explicitly the limit solution u:

$$u(x) = -\frac{1}{a^*} \int_0^x dy \int_0^y f(t)\, dt + \frac{1}{a^*} \Big(\int_0^1 dy \int_0^y f(t)\, dt \Big).$$

Remark 4.17. Note that the value of a^* obtained is the particular case $p = 2$ of the homogenized coefficient of the $p-$Laplacian equation

$$a^* = \Big(\overline{a^{\frac{1}{1-p}}} \Big)^{-(p-1)}$$

given in Section 4.3.1.

In the N-dimensional case with $N > 1$, it is more difficult to obtain an expression of the homogenized matrix and it is no longer obtained by means of algebraic formulas from A.

The $N-$dimensional case

Let Ω be a bounded open subset of $\mathbb{R}^N$. For a fixed $\varepsilon > 0$, let us consider the Dirichlet boundary value problem on Ω

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f & \text{en } \Omega \\ u^\varepsilon = 0 & \text{en } \partial\Omega \end{cases} \tag{4.2.14}$$

where $f \in W^{-1,2}(\Omega)$ is a function on Ω.

The variational weak formulation of (4.2.14) becomes then: find $u^\varepsilon \in W_0^{1,2}(\Omega)$ such that

$$\begin{cases} \int_\Omega A^\varepsilon \nabla u^\varepsilon \cdot \nabla v = \int_\Omega f v & \text{for all } v \in W_0^{1,2}(\Omega) \\ u^\varepsilon \in W_0^{1,2}(\Omega). \end{cases} \tag{4.2.15}$$

Remark 4.18. The existence and uniqueness of (4.2.15) it follows from the Lax-Milgram lemma: if we define the bilinear form $a^\varepsilon : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ by

$$a^\varepsilon(u, v) = \int_\Omega A^\varepsilon u \cdot v \qquad \text{for all } u, v \in W_0^{1,2}(\Omega).$$

we observe that from the boundedness assumption and Holder's inequality it follows

$$|a^\varepsilon(u, v)| \leq c \|u\|_{W_0^{1,2}(\Omega)} \|v\|_{W_0^{1,2}(\Omega)} \qquad \text{for all } u, v \in W_0^{1,2}(\Omega).$$

Moreover, from the ellipticity condition we get

$$a^\varepsilon(u, v) \leq \alpha \|u\|^2_{W_0^{1,2}(\Omega)}. \tag{4.2.16}$$

Hence, a^ε defines a bilinear continuous and coercive form on $W_0^{1,2}(\Omega)$ and the existence and uniqueness is guaranteed.

From the estimate of the Lax-Milgram lemma we get

$$\|u^\varepsilon\|_{W_0^{1,2}(\Omega)} \leq \frac{1}{\alpha}\|f\|_{W^{-1,2}(\Omega)}.$$

Consequently, it follows that there exists a subsequence still denoted by ε and an element $u \in W_0^{1,2}(\Omega)$ such that

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } W_0^{1,2}(\Omega).$$

Like in the one-dimensional case, to investigate the limit u we define

$$\xi_\varepsilon = A^\varepsilon \nabla u^\varepsilon$$

which satisfies

$$\int_\Omega \xi_\varepsilon \nabla v = \int_\Omega f v \quad \text{for all } v \in W_0^{1,2}(\Omega). \tag{4.2.17}$$

Since $A \in L^\infty(\Omega; \mathcal{M}^s_{\alpha,\beta})$ and (4.2.16) it follows that

$$\|\xi_\varepsilon\|_{L^2\Omega} \leq \frac{\beta}{\alpha}\|f\|_{W^{-1,2}(\Omega)}.$$

Then, there exist a subsequence, still denoted by $\{\xi_\varepsilon\}$, and an element $\xi \in L^2(\Omega)$ such that

$$\xi_\varepsilon \rightharpoonup \xi \quad \text{weakly in } (L^2(\Omega))^N.$$

Hence, we can pass to the limit in (4.2.17), to get

$$\int_\Omega \xi \nabla v = \int_\Omega f v \quad \text{for all } v \in W_0^{1,2}(\Omega),$$

i.e.

$$-div\xi = f \quad \text{in } \Omega.$$

To obtain the limit equation which u is solution it is necessary to describe ξ in terms of u. In the one-dimensional case one can easily give the relation between u and ξ as we have seen in (4.2.13). In the $N-$dimensional case the situation is completely different since the coefficients of A^* are no longer obtained as algebraic formulas from A. Indeed, they are defined by means of some functions which are solutions in the reference cell Y of certain boundary value problems.

The classical convergence result states that:

Theorem 4.19. *Let $f \in W^{-1,2}(\Omega)$ and u^ε be the solution of (4.2.14) with A^ε satisfying $(H1)-(H4)$. Then*

1. *$u^\varepsilon \rightharpoonup u$ weakly in $W_0^{1,2}(\Omega)$.*

2. *$A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^* \nabla u$ weakly in $(L^2(\Omega))^N$.*

where u^0 is the unique solution in $W_0^{1,2}(\Omega)$ of the homogenized problem

$$\begin{cases} -div(A^*\nabla u) = f & en\ \Omega \\ u^\varepsilon = 0 & en\ \partial\Omega. \end{cases}$$

The matrix $A^* = (a^*_{ij})$ is constant, elliptic and given by

$$a^*_{ij} = \int_Y (a_{ij}(y) + \sum_{k=1}^N a_{ik}(y)\frac{\partial \omega_k(y)}{\partial y_k} dy)$$

where ω_k is the unique solution to the local problem

$$\begin{cases} \int_Y A(y)(e_k + \nabla\omega_k(y)) \cdot \nabla v(y)dy = 0 & \textit{for every } v \in W^{1,2}_{per}(Y) \\ \omega_k \in W^{1,2}_{per}(Y). \end{cases}$$

This well-known result can be proved by different methods. One of them is the variational method of oscillating test functions due to Tartar [Ta77], [Ta78]. Another way to prove it is by using the two-scale method of Nguetseng [Ng89] and Allaire [Al02]. Also, can be used the formal method of asymptotic expansions, known as the multiple scale method.

Tartar's method is based on the construction of a suitable set of oscillating test functions which allows us to pass to the limit in problem (4.2.14) and this is related to the notion of compensated compactness.

In another way, the two-scale method take into account the two scales of the problem and introduces a new notion of convergence, the *two-scale convergence*, tested on functions of the form $\psi(x, x/\varepsilon)$.

The *multiple scale method* suggests looking for a formal asymptotic expansion of the form

$$u^\varepsilon(x) = u(x, \tfrac{x}{\varepsilon}) + \varepsilon u_1(x, \tfrac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \tfrac{x}{\varepsilon}) + \cdots$$

with $u_j(x, y)$ for $j \in \mathbb{N}$ such that $u_j(x, y)$ is defined for $x \in \Omega$ and $y \in Y$, and $u_j(\cdot, y)$ is $Y-$periodic. The two variables x and $\frac{x}{\varepsilon}$ take into account the two scales of the homogenization phenomenon; the x variable is the macroscopic variable, whereas the $\frac{x}{\varepsilon}$ variable takes into account the microscopic geometry.

4.3 $G-$convergence of monotone operators

In this section we deal with the $G-$convergence of sequences of nonlinear monotone operators. Given a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$ we consider the family of operators $\mathcal{A}_\varepsilon \colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined in Section 2.4 by

$$\mathcal{A}_\varepsilon u := -div(a_\varepsilon(x, \xi)), \tag{4.3.1}$$

where $a_\varepsilon \colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies, for every $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$ conditions (H0)–(H8).

Definition 4.20. Let Ω be a bounded open domain in $\mathbb{R}^N$, $N \geq 1$. We say that $a_\varepsilon(x, \nabla u)$ *$G-$converges* to $a(x, \nabla u)$ if for every $f \in W^{-1,p'}(\Omega)$ and for every f_ε strongly convergent to f in $W^{-1,p'}(\Omega)$, the solutions u^ε of the problem

$$\begin{cases} -div(a_\varepsilon(x, \nabla u^\varepsilon)) = f_\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{4.3.2}$$

satisfy the following conditions

$$\begin{aligned} u^\varepsilon &\rightharpoonup u && \text{weakly in } W_0^{1,p}(\Omega), \\ a_\varepsilon(x, \nabla u^\varepsilon) &\rightharpoonup a(x, \nabla u) && \text{weakly in } (L^p(\Omega))^N, \end{aligned}$$

where u is the solution to the equation

$$\begin{cases} -div(a(x, \nabla u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For instance, in the linear periodic case, in Theorem 4.19 we have seen that the family $A(x/\varepsilon)\nabla u$ $G-$converges to a limit $A^*\nabla u$ where A^* is a constant matrix.

Remark 4.21. For each positive value of ε there exists a unique solution $u^\varepsilon \in W_0^{1,p}(\Omega)$ of (4.3.2). For a proof we refer, for instance, to [KS00], Chapter III, Corollary 1.8 or to [Li69], Chapter 2, Theorem 2.1.

Remark 4.22. It can be proved that this definition of $G-$convergence is independent of the boundary condition. A proof of this fact can be found, for instance, in [CVD90], Theorem 3.8.

It is shown in [BCR06] that properties (H0)–(H8) are stable under $G-$convergence, i.e.

Theorem 4.23. *If $a_\varepsilon(x, \nabla u)$ $G-$converges to $a(x, \nabla u)$ and $a_\varepsilon(x, \xi)$ satisfies* (H0)–(H8)*, then $a(x, \xi)$ also satisfies* (H0)–(H8).

Proof. See [BCR06], Theorem 2.3. □

In the general case, one has the following compactness result due to [CVD90].

Proposition 4.24. *Assume that $a_\varepsilon(x, \xi)$ satisfies* (H1)–(H3) *then, up to a subsequence, $a_\varepsilon(x, \xi)$ $G-$converges to a maximal monotone operator $a(x, \xi)$ which also satisfies* (H1)–(H3).

Proof. See [CVD90], Theorem 4.1. □

Remark 4.25. In the one-dimensional case, as we have seen in Chapter §2, Section 2.5.1, equation (4.3.2) becomes

$$\begin{cases} -(a_\varepsilon(x)|(u^\varepsilon)'|^{p-2}(u^\varepsilon)')' = f_\varepsilon & \text{in } I := (0, 1) \\ u^\varepsilon(0) = u^\varepsilon(1) = 0 \end{cases} \tag{4.3.3}$$

with a_ε satisfying (2.5.9).

4.3.1 The periodic case

Now, we deal with the homogenization of a sequence of nonlinear monotone operators $\mathcal{A}_\varepsilon$ defined in (4.3.1) in the periodic case, i.e., $a_\varepsilon(x,\xi) = a(x/\varepsilon,\xi)$, and $a(\cdot,\xi)$ is $Q-$periodic for every $\xi \in \mathbb{R}^N$.

One has that $\mathcal{A}_\varepsilon$ $G-$converges to the homogenized operator $\mathcal{A}_h = -div(a_h(\nabla))$. But now, due to the periodicity, $a_h : \mathbb{R}^N \to \mathbb{R}^N$ can be characterized in term of certain auxiliary functions. The following result is a generalization of Theorem 4.19 stated for linear periodic operators.

Theorem 4.26. *Let $f \in W^{-1,2}(\Omega)$ such that f_ε strongly convergent to f in $W^{-1,p'}(\Omega)$ and u^ε be the solution of*

$$\begin{cases} -div(a(\frac{x}{\varepsilon},\nabla u^\varepsilon)) = f_\varepsilon & en\ \Omega \\ u^\varepsilon \in W_0^{1,p}(\Omega). \end{cases} \tag{4.3.4}$$

with $a(\cdot,\cdot)$ satisfying (H1)–(H8). Then

1. *$u^\varepsilon \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$.*
2. *$a_\varepsilon(x,\varepsilon\nabla u^\varepsilon) \rightharpoonup a^*(\nabla u)$ weakly in $(L^p(\Omega))^N$.*

where u is the unique solution in $W_0^{1,p}(\Omega)$ of the homogenized problem

$$\begin{cases} -div(a^*(\nabla u)) = f & en\ \Omega \\ u = 0 & en\ \partial\Omega. \end{cases}$$

where $a^* : \mathbb{R}^N \to \mathbb{R}^N$ can be characterized by

$$a^*(\xi) = \lim_{s\to\infty} \frac{1}{s^N} \int_{Q_s(z_s)} a(x,\nabla\chi_s^\xi + \xi)dx \tag{4.3.5}$$

where $\xi \in \mathbb{R}^N$, $Q_s(z_s)$ is the cube of side length s centered at z_s for any family $\{z_s\}_{s>0}$ in $\mathbb{R}^N$, and χ_s^ξ is the solution of the following auxiliary problem

$$\begin{cases} -div(a(x,\nabla\chi_s^\xi + \xi)) = 0 & in\ Q_s(z_s) \\ \chi_s^\xi \in W_0^{1,p}(Q_s(z)), \end{cases} \tag{4.3.6}$$

Proof. See [BCD92], Section 2. □

An example of $G-$convergence

We finish this section by explicitly computing the $G-$limit operator in one space dimension in the periodic case.

In the periodic linear case, (see Section 4.2.1) it is known that the family $a(\frac{x}{\varepsilon})$ $G-$converges to a_2^* with

$$a_2^* = \Big(\int_I a(x)^{-1}dx\Big)^{-1}.$$

In the non-linear case $p \neq 2$, the $G-$limit of (4.3.3) in the periodic case is given in the following Proposition.

Proposition 4.27. *Let $a \in L^\infty(\mathbb{R})$ be $1-$periodic function such that for $\alpha < \beta$ two constants it holds that $0 < \alpha \leq a(x) \leq \beta < \infty$. Then $a(x/\varepsilon)$ $G-$converges to $a_p^* \in \mathbb{R}$ given by*

$$a_p^* = \Big(\int_I a(x)^{-\frac{1}{p-1}} \Big)^{-(p-1)}.$$

Proof. Let $f_\varepsilon \in W^{-1,p'}(I)$ be such that $f_\varepsilon \to f$ in $W^{-1,p'}(I)$.

Let $g_\varepsilon(x) := \langle f_\varepsilon, \chi_{(0,x)} \rangle$, then $g_\varepsilon \in L^p(I)$, $g_\varepsilon' = f_\varepsilon$ and $g_\varepsilon \to g := \langle f, \chi_{(0,x)} \rangle$ in $L^p(I)$.

Let u^ε be the weak solution to

$$\begin{cases} -(a(\frac{x}{\varepsilon})|(u^\varepsilon)'|^{p-2}(u^\varepsilon)')' = f_\varepsilon & \text{in } I \\ u^\varepsilon(0) = u^\varepsilon(1) = 0. \end{cases}$$

Then, there exists a constant c_ε such that $a(x/\varepsilon)|(u^\varepsilon)'|^{p-2}(u^\varepsilon)' = c_\varepsilon - g_\varepsilon$.

Let $\varphi_p(x) = |x|^{p-2}x$. Then φ_p is invertible and so

$$(u^\varepsilon)' = \varphi_p^{-1}(c_\varepsilon - g_\varepsilon) a(\tfrac{x}{\varepsilon})^{\frac{1}{1-p}}. \tag{4.3.7}$$

Since $(u^\varepsilon)_{\varepsilon > 0}$ is bounded in $W_0^{1,p}(I)$, we can assume that is weakly convergent to some $u \in W_0^{1,p}(I)$ and, since $a(\frac{x}{\varepsilon})^{\frac{1}{1-p}} \rightharpoonup \overline{a^{\frac{1}{1-p}}} := \int_I a^{\frac{1}{1-p}}$ weakly * in $L^\infty(I)$ and $g_\varepsilon \to g$ in $L^p(I)$, we can assume that there exists c such that $c_\varepsilon \to c$.

Now we can pass to the limit in (4.3.7) and obtain

$$u' = \varphi_p^{-1}(c - g)\overline{a^{\frac{1}{1-p}}}.$$

The proof is now complete. □

5

Oscillating integrals

In this section we prove some results concerning to the estimate of integrals of periodic functions with rapidly oscillating coefficients. They allows us to replace an integral involving a rapidly oscillating function with one that involves its average in the unit cube. As we will see in Chapters §6 and §7, it is essential to estimate the rate of convergence of eigenvalues in homogenization problems.

Let u be a smooth function and let g be a Q-periodic function, being Q the unit cube in $\mathbb{R}^N$. Let us consider the following integral

$$\int_Q g(\tfrac{x}{\varepsilon})u(x)dx, \tag{5.0.1}$$

where ε is a real positive parameter. Note that if ε is small then $g(x/\varepsilon)$ is a rapidly oscillating function. Our propose is to obtain an expansion of the integral (5.0.1) in terms of the ε. We recall the well-known result of Bensoussan, Lions and Papanicolaou [BLP78], which characterize the asymptotic behavior of (5.0.1) as

$$\lim_{\varepsilon\to 0}\int_Q g(\tfrac{x}{\varepsilon})u(x) = \frac{1}{|Q|}\int_Q g(x)dx \int_Q u(x)dx.$$

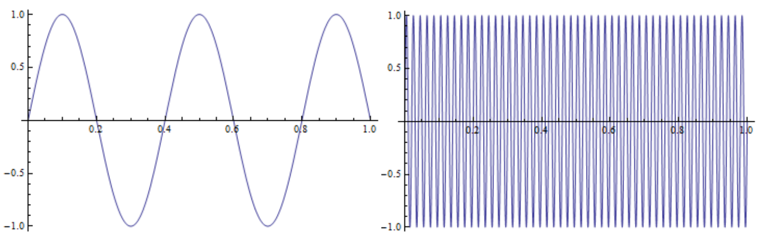

Figure 5.1: $g(\frac{x}{\varepsilon}) = \sin(2\pi x/\varepsilon)$ for $\varepsilon = 0.4$ and $\varepsilon = 0.02$.

We would wish to obtain more information about the second term in the expansion.

There are some works related to the estimate of oscillating integrals. The following result gives an asymptotic expansion of oscillating one-dimensional integrals in terms of the parameter ε, here $\varepsilon = m^{-1}$ with m a positive integer.

Theorem 5.1. *If $g \in L^1([0,1])$, $u \in C^1([0,1])$ then*

$$\int_0^1 g\left(\frac{x}{\varepsilon}\right)u(x)dx = \int_0^1 g(x)dx \int_0^1 u(x)dx + \varepsilon \int_0^1 \widetilde{p}\left(\frac{x}{\varepsilon}\right)u'(x)dx \tag{5.0.2}$$

where $\widetilde{p}(x) = \int_0^x [\bar{g} - g(t)]\, dt$ and $\bar{g} = \int_0^1 g(t)dt$.

Proof. See [BLL87], Corollary 3.3. □

In higher dimensions some similar results are known. When the parameter ε is a negative power of a positive integer, oscillating integrals can be estimated in the unit cube of $\mathbb{R}^N$.

Theorem 5.2. *Let $g \in L^p(Q)$, $p > 1$ be a Q-periodic function and $u \in C^\infty(Q)$ then*

$$\int_Q g\left(\frac{x}{\varepsilon}\right)u(x) = \int_Q g \int_Q u + \varepsilon a(g) \int_Q D^\gamma u \tag{5.0.3}$$

where $a(g)$ is a function depending on g and $\gamma \in \mathbb{R}^n$ is such that $|\gamma| = 1$.

Proof. See [IL88], Proposition 5. □

When the parameter ε is not of the form m^{-1} with m a positive integer, up to our knowledge, there are no equalities of the kind (5.0.2) or (5.0.3). However, when ε is a real positive parameter the following result due to Oleĭnik, Shamaev and Yosifian.

Theorem 5.3. *Let $g \in L^\infty(\mathbb{R}^N)$ be a $Q-$periodic function such that $0 < g^- \leq g \leq g^+ < \infty$ and let $\bar{g} = ⨍_Q g$. Then,*

$$\left| \int_\Omega (g(\tfrac{x}{\varepsilon}) - \bar{g})uv \right| \leq c\varepsilon \|u\|_{W^{1,2}(\Omega)} \|v\|_{W^{1,2}(\Omega)}$$

holds for every $u, v \in W^{1,2}(\Omega)$ where c is a constant independent of ε, u and v.

Proof. See [OSY92], Lemma 1.6. □

Our aim in this Chapter is to give some generalizations of Oleĭnik-Shamaev-Yosifian's result for $p \neq 2$.

We give two independent proofs for functions in $W_0^{1,p}(\Omega)$, the first one for the case $N \geq 1$ and the second one for $N = 1$. The need for this second proof comes from estimate explicitly the constants in our result. We are unable to do that in higher dimensions.

Finally, requesting more regularly to the domain, we give a similar result to Theorem 5.3 for functions belonging to $W^{1,p}(\Omega)$ with $p \neq 2$ and $N \geq 1$.

In the following, we assume that $1 \leq p < +\infty$, the function $g \in L^\infty(\mathbb{R}^N)$ be a $Q-$periodic function uniformly bounded away from zero and infinity, being Q the unit cube in $\mathbb{R}^N$, i.e., for certain constants $g^\pm$,

$$0 < g^- \leq g \leq g^+ < \infty.$$

Also, we will denote by $\bar{g}$ the average of g over Q.

The first result reads:

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain such that Hardy's inequality holds, for instance Lipschitz boundary. Then,*

$$\Big|\int_\Omega (g(\tfrac{x}{\varepsilon}) - \bar{g})|u|^p\Big| \le \|g-\bar{g}\|_{L^\infty(\mathbb{R}^N)}\varepsilon\|\nabla u\|^p_{L^p(\Omega)}\Big[\frac{p}{\mu_1^{p-1}}c_1 + C_{H,p}(\Omega)N^{p/2}\varepsilon^{p-1}\Big]$$
$$\le C\varepsilon\|\nabla u\|^p_{L^p(\Omega)},$$

for every $u \in W^{1,p}_0(\Omega)$. The constant $C_{H,p}(\Omega)$ is the best constant in Hardy's inequality (5.1.1), c_1 is the optimal constant in Poincaré's inequality in $L^1(Q)$ and μ_1 is the first eigenvalue of the $p-$Laplacian in Ω.

By the methods employed in the proof, the constant obtained in Theorem 5.4 can not be estimated. In one space dimension we can employ a more direct approach in order to obtain the explicit constant in the case $\Omega = I$, where I is the unit interval in $\mathbb{R}$.

Theorem 5.5. *Let $I := (0,1)$. Then, for every $u \in W^{1,p}_0(I)$ we have that*

$$\Big|\int_I (g(\tfrac{x}{\varepsilon}) - \bar{g})|u|^p\Big| \le \|g-\bar{g}\|_{L^\infty(\mathbb{R})}\varepsilon\|u'\|^p_{L^p(I)}\Big[\frac{p}{\pi_p^{p-1}} + \frac{\varepsilon^{p-1}}{p}\Big].$$

The case in which the space function is $W^{1,p}(\Omega)$ with $\Omega \subset \mathbb{R}^N$ and $N \ge 1$, the arguments of the proof of Theorem 5.4 do not work. The fact that we enlarge the set of test functions is reflected in the need for more regularity of the domain Ω. In Theorem 5.4 test functions are in $W^{1,p}_0(\Omega)$ and the proof works in a domain with very little regularity, let us say Lipschitz boundary or less (see Remark 5.10). Instead, when we want to prove a similar result for test functions belonging to $W^{1,p}(\Omega)$ it is necessary a little bit more of regularity in the domain, for instance a domain with C^1 boundary.

We have the following result:

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary. Then for every $u \in W^{1,p}(\Omega)$ there exists a constant C independent of ε such that*

$$\Big|\int_\Omega (g(\tfrac{x}{\varepsilon}) - \bar{g})u\Big| \le C\varepsilon\|u\|_{W^{1,p}(\Omega)}.$$

Remark 5.7. In fact, the regularity of the domains in Theorem 5.4 and Theorem 5.6 is given by the regularity needed in Lemmas 5.9 and 5.19, respectively.

As in Theorem 5.4, we are not able to estimate the constant in Theorem 5.6 in N space dimension. However, in one space dimension, by similar techniques as in Theorem 5.5, we can get an estimate for this constant.

Theorem 5.8. *Let $I := (0,1)$. Then, for every $u \in W^{1,p}(I)$ we have that*

$$\Big|\int_I (g(\tfrac{x}{\varepsilon}) - \bar{g})u\Big| \le \|g-\bar{g}\|_{L^\infty(\mathbb{R})}\varepsilon\|u\|_{W^{1,p}(I)}\beta\Big(4 + \tfrac{p}{(p-1)\pi_p}\Big).$$

5.1 Proof of Theorem 5.4

We first prove a couple of lemmas in order to prove Theorem 5.4.

Lemma 5.9. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary and, for $\delta > 0$, let G_δ be a tubular neighborhood of $\partial\Omega$, i.e. $G_\delta = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$.*

Then

$$\|v\|^p_{L^p(G_\delta)} \le C_{H,p}(\Omega)\delta^p \|\nabla v\|^p_{L^p(\Omega)},$$

for every $v \in W^{1,p}_0(\Omega)$, where $C_{H,p}(\Omega)$ is the best constant in the Hardy inequality (see [Ma85])

$$\int_\Omega \frac{|v|^p}{d^p} \le C_{H,p}(\Omega) \int_\Omega |\nabla u|^p \tag{5.1.1}$$

and $d(x) = dist(x, \partial\Omega)$.

Proof. The proof follows by noticing that if $x \in G_\delta$, then $d(x) \le \delta$, so, by (5.1.1) we get

$$\int_{G_\delta} |v|^p = \int_{G_\delta} \frac{|v|^p}{d^p} d^p \le \delta^p \int_\Omega \frac{|v|^p}{d^p} \le C_{H,p}(\Omega)\delta^p \int_\Omega |\nabla u|^p.$$

The proof is now complete. □

Remark 5.10. Observe that the only requirement on the regularity of $\partial\Omega$ is the validity of Hardy's inequality (5.1.1). Therefore, much less than Lipschitz will do. We refer the reader to the book of Maz'ja [Ma85].

Now we need an easy Lemma that computes the Poincaré constant on the cube of side ε in terms of the Poincaré constant of the unit cube. Although this result is well known and its proof follows directly by a change of variables, we choose to include it for the sake of completeness.

Lemma 5.11. *Let Q be the unit cube in $\mathbb{R}^N$ and let c_q be the Poincaré constant in the unit cube in L^q, $q \ge 1$, i.e.*

$$\|u - \bar{u}_Q\|_{L^q(Q)} \le c_q \|\nabla u\|_{L^q(Q)}, \qquad \text{for every } u \in W^{1,q}(Q),$$

where $\bar{u}_Q$ is the average of u over Q. Then, for every $u \in W^{1,q}(Q_\varepsilon)$ we have

$$\|u - \bar{u}_{Q_\varepsilon}\|_{L^q(Q_\varepsilon)} \le c_q \varepsilon \|\nabla u\|_{L^q(Q_\varepsilon)},$$

where $Q_\varepsilon = \varepsilon Q$.

Proof. Let $u \in W^{1,q}(Q_\varepsilon)$. We can assume that $(u)_{Q_\varepsilon} = 0$. Now, if we denote $u^\varepsilon(y) = u(\varepsilon y)$, we have that $u^\varepsilon \in W^{1,q}(Q)$ and by the change of variables formula, we get

$$\int_{Q_\varepsilon} |u|^q = \int_Q |u^\varepsilon|^q \varepsilon^N \le c_q^q \varepsilon^N \int_Q |\nabla u^\varepsilon|^q = c_q^q \varepsilon^q \int_{Q_\varepsilon} |\nabla u|^q.$$

The proof is now complete. □

The next Lemma is the final ingredient in the estimate of Theorem 5.4.

Lemma 5.12. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and denote by Q the unit cube in $\mathbb{R}^N$. Let $g \in L^\infty(\mathbb{R}^N)$ be a Q-periodic function such that $\bar{g} = 0$. Then the inequality*

$$\left|\int_{\Omega_1} g(\tfrac{x}{\varepsilon})v\right| \leq \|g\|_{L^\infty(\mathbb{R}^N)} c_1 \varepsilon \|\nabla v\|_{L^1(\Omega)}$$

holds for every $v \in W_0^{1,1}(\Omega)$, where c_1 is the Poincaré constant given in Lemma 5.11 and $\Omega_1 \subset \Omega$ is given by

$$\Omega_1 = \bigcup Q_{z,\varepsilon}, \qquad Q_{z,\varepsilon} := \varepsilon(z+Q) \subset \Omega, \qquad z \in \mathbb{Z}^N.$$

Proof. Denote by I^ε the set of all $z \in \mathbb{Z}^N$ such that $Q_{z,\varepsilon} := \varepsilon(z+Q) \subset \Omega$. Let us consider the function $\bar{v}_\varepsilon$ given by the formula

$$\bar{v}_\varepsilon(x) = \frac{1}{\varepsilon^N}\int_{Q_{z,\varepsilon}} v(y)dy$$

for $x \in Q_{z,\varepsilon}$. Then we have

$$\int_{\Omega_1} g_\varepsilon v = \int_{\Omega_1} g_\varepsilon(v - \bar{v}_\varepsilon) + \int_{\Omega_1} g_\varepsilon \bar{v}_\varepsilon. \tag{5.1.2}$$

Now, by Lema 5.11 we get

$$\begin{aligned}\|v - \bar{v}_\varepsilon\|_{L^1(\Omega_1)} &= \sum_{z\in I^\varepsilon}\int_{Q_{z,\varepsilon}} |v - \bar{v}_\varepsilon| dx \\ &\leq c_1\varepsilon \sum_{z\in I^{z,\varepsilon}}\int_{Q_{z,\varepsilon}} |\nabla v(x)| dx \\ &\leq c_1 \varepsilon \|\nabla u\|_{L^1(\Omega)}.\end{aligned} \tag{5.1.3}$$

Finally, since $\bar{g} = 0$ and since g is $Q-$periodic, we get

$$\int_{\Omega_1} g_\varepsilon \bar{v}_\varepsilon = \sum_{z\in I^\varepsilon} \bar{v}_\varepsilon\, |_{Q_{z,\varepsilon}} \int_{Q_{z,\varepsilon}} g_\varepsilon = 0. \tag{5.1.4}$$

Now, combining (5.1.3) and (5.1.4) we can bound (5.1.2) by

$$\left|\int_{\Omega_1} g_\varepsilon v\right| \leq \|g\|_{L^\infty(\mathbb{R}^N)} c_1 \varepsilon \|\nabla v\|_{L^1(\Omega)}.$$

This finishes the proof. □

Now, we are ready to prove Theorem 5.4:

Proof of Theorem 5.4. Let $\varepsilon > 0$ be fixed, and let Ω_1 be the set defined in Lemma 5.12.

Denote by $G := \Omega \setminus \Omega_1$ and observe that $G \subset G_{\sqrt{N}\varepsilon}$. In fact, with the notations of Lemma 5.12, if $x \in G$ then there exists a cube $Q = Q_{z,\varepsilon}$ such that $x \in Q$ and $Q \cap \partial\Omega \neq \emptyset$. Therefore, $dist(x, \partial\Omega) \leq diam(Q) = \sqrt{N}\varepsilon$.

Now, denote by $h = g - \bar{g}$ and so, by Lemma 5.9,

$$\Big| \int_G h_\varepsilon |u|^p \Big| \leq C_{H,p}(\Omega) \|h\|_{L^\infty(\mathbb{R}^N)} (\sqrt{N}\varepsilon)^p \|\nabla u\|^p_{L^p(\Omega)}. \tag{5.1.5}$$

Now, to bound the integral in Ω_1 we use Lemma 5.12 to obtain

$$\Big| \int_{\Omega_1} h_\varepsilon |u|^p \Big| \leq \|h\|_{L^\infty(\mathbb{R}^N)} c_1 \varepsilon \|\nabla(|u|^p)\|_{L^1(\Omega)}. \tag{5.1.6}$$

An easy computation shows that

$$\|\nabla(|u|^p)\|_{L^1(\Omega)} \leq p \|u\|^{p-1}_{L^p(\Omega)} \|\nabla u\|_{L^p(\Omega)} \leq \frac{p}{\mu_1^{p-1}} \|\nabla u\|^p_{L^p(\Omega)}. \tag{5.1.7}$$

Finally, combining (5.1.5), (5.1.6) and (5.1.7) we obtain the desired result. □

5.2 Proof of Theorem 5.5

The following lemma is well known, but we included it for the sake of completeness.

Lemma 5.13. *Let $J = (0, \ell)$ be an interval in $\mathbb{R}$ and let $v \in W^{1,q}(J)$, $1 \leq q < \infty$, be such that $v(0) = 0$. Then*

$$\|v\|^q_{L^q(J)} \leq \frac{\ell^q}{q} \|v'\|^q_{L^q(J)}.$$

Proof. We have

$$v(x) = \int_0^x v' \leq \|v'\|_{L^q(J)} x^{1/q'}.$$

Integrating, we obtain

$$\|v\|^q_{L^q(J)} \leq \|v'\|^q_{L^q(J)} \int_0^\ell x^{q/q'} = \|v'\|^q_{L^q(J)} \frac{\ell^q}{q},$$

as we wanted to show. □

Now, we can show two immediate consequences of Lemma 5.13

Corollary 5.14. *Let $v \in W^{1,q}_0(I)$. Then*

$$\int_0^\delta |v|^q \leq \frac{\delta^q}{q} \|v'\|^q_{L^q(I)}.$$

Proof. Immediate from Lemma 5.13. □

Corollary 5.15. *Let $J = (0, \ell)$ and let $v \in W^{1,q}(J)$, $1 \leq q < \infty$. Assume that there exists $x_0 \in I$ such that $u(x_0) = 0$. Then*

$$\|v\|^q_{L^q(J)} \leq \frac{\ell^q}{q} \|v'\|^q_{L^q(J)}.$$

Proof. Let $J_1 = (0, x_0)$ and $J_2 = (x_0, \ell)$. Then, by Lemma 5.13 we have that

$$\|v\|^q_{L^q(J_1)} \leq \frac{x_0^p}{p}\|v'\|^q_{L^q(J_1)} \leq \frac{\ell^p}{p}\|v'\|^q_{L^q(J_1)} \tag{5.2.1}$$

and

$$\|v\|^q_{L^q(J_2)} \leq \frac{(\ell - x_0)^p}{p}\|v'\|^q_{L^q(J_2)} \leq \frac{\ell^p}{p}\|v'\|^q_{L^q(J_2)}. \tag{5.2.2}$$

Now, adding (5.2.1) and (5.2.2) we obtain the desired result. □

From Corollary 5.15 we can obtain the following

Corollary 5.16. *Let $J = (0, \ell)$ and let $v \in W^{1,q}(J)$. Then*

$$\|v - \bar{v}\|^q_{L^q(J)} \leq \frac{\ell^q}{q}\|v'\|^q_{L^q(J)},$$

where $\bar{v}$ stands for the average of v over J

Proof. Just notice that for $w = v - \bar{v}$ there exists $x_0 \in J$ such that $w(x_0) = 0$. Then, we use Corollary 5.15. □

Remark 5.17. Corollary 5.16 is the well known Poincaré inequality. As far as we known, the optimal constant in the Poincaré inequality is unknown even in this one dimensional setting. See [CW06] for a discussion on this. So, the purpose of Corollary 5.16 is to provide with a rough estimate on this constant and any improvements on the computation of the optimal constant will automatically give an improvement in the constant entering in our result.

Finally, we need a Lemma that controls the oscillating behavior of the weight function.

Lemma 5.18. *Let $v \in W^{1,1}(I)$ and let $g \in L^\infty(\mathbb{R})$ be a 1−periodic function such that $\bar{g} = \int_I g = 0$. Let $\varepsilon > 0$ and denote by $m = [1/\varepsilon]$ the integer part of $1/\varepsilon$. Then*

$$\Big| \int_0^{m\varepsilon} g(\tfrac{x}{\varepsilon})v \Big| \leq \|g\|_{L^\infty(\mathbb{R})}\varepsilon\|v'\|_{L^1(I)}.$$

Proof. Let $I_j^\varepsilon = ((j-1)\varepsilon, j\varepsilon]$ and

$$J^\varepsilon = (0, m\varepsilon] = \bigcup_{j=1}^m I_j^\varepsilon.$$

Let v_ε be defined as

$$v_\varepsilon(x) = \frac{1}{\varepsilon}\int_{I_j^\varepsilon} v(y)\,dy \qquad \text{for } x \in I_j^\varepsilon.$$

Now, as $\bar{g} = 0$ and v_ε is constant on each I_j^ε we have that

$$\int_{I_j^\varepsilon} g(\tfrac{x}{\varepsilon})v = \int_{I_j^\varepsilon} g(\tfrac{x}{\varepsilon})(v - v_\varepsilon),$$

so, by Corollary 5.16,

$$\Big| \int_{I_j^\varepsilon} g(\tfrac{x}{\varepsilon}) v \Big| \le \|g\|_{L^\infty(\mathbb{R})} \varepsilon \int_{I_j^\varepsilon} |v'|.$$

Therefore

$$\Big| \int_0^{m\varepsilon} g(\tfrac{x}{\varepsilon}) v \Big| \le \sum_{j=1}^m \Big| \int_{I_j^\varepsilon} g(\tfrac{x}{\varepsilon}) v \Big| \le \|g\|_{L^\infty(\mathbb{R})} \varepsilon \sum_{j=1}^m \int_{I_j^\varepsilon} |v'| \le \|g\|_{L^\infty(\mathbb{R})} \varepsilon \int_I |v'|.$$

The proof is now complete. □

With these preliminaries, we arrive at the key estimate in our main result.

Proof of Theorem 5.5. For every $\varepsilon > 0$ we denote, as in the previous Lemma, $I_j^\varepsilon = ((j-1)\varepsilon, j\varepsilon]$ and

$$J^\varepsilon = \bigcup_{j=1}^m I_j^\varepsilon = (0, \varepsilon m]$$

where $m = [1/\varepsilon]$ is the integer part of $1/\varepsilon$.

It is immediate to see that $G_\varepsilon := I \setminus J^\varepsilon = (\varepsilon m, 1) \subset (1-\varepsilon, 1)$.

Then

$$\int_I g(\tfrac{x}{\varepsilon}) |u|^p = \int_{G_\varepsilon} g(\tfrac{x}{\varepsilon}) |u|^p + \int_{J_\varepsilon} g(\tfrac{x}{\varepsilon}) |u|^p.$$

Now, by Corollary 5.14,

$$\Big| \int_{G_\varepsilon} g(\tfrac{x}{\varepsilon}) |u|^p \Big| \le \|g\|_{L^\infty(\mathbb{R})} \int_{1-\varepsilon}^1 |u|^p \le \|g\|_{L^\infty(\mathbb{R})} \frac{\varepsilon^p}{p} \|u'\|_{L^p(I)}^p.$$

If we notice that $|u|^p \in W_0^{1,1}(I)$, by Lemma 5.18 we get

$$\Big| \int_{J_\varepsilon} g(\tfrac{x}{\varepsilon}) |u|^p \Big| \le \|g\|_{L^\infty(\mathbb{R})} \varepsilon \int_I |(|u|^p)'|.$$

Finally, the proof is complete once we observe that

$$\begin{aligned} \int_I |(|u|^p)'| = p \int_I |u|^{p-1} |u'| &\le p \Big(\int_I |u|^p \Big)^{\frac{p-1}{p}} \Big(\int_I |u'|^p \Big)^{\frac{1}{p}} \\ &= p \|u\|_{L^p(I)}^{p-1} \|u'\|_{L^p(I)} \le p \pi_p^{1-p} \|u'\|_{L^p(I)}^p. \end{aligned}$$

The proof is finished. □

5.3 Proof of Theorem 5.6

The next lemma is a generalization for $p \ge 2$ of Oleĭnik-Shamaev-Yosifian's Lemma [OSY92] and it is essential to prove Theorem 5.6.

Lemma 5.19. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary and, for $\delta > 0$, let G_δ be a tubular neighborhood of $\partial\Omega$, i.e. $G_\delta = \{x \in \Omega\colon dist(x, \partial\Omega) < \delta\}$. Then there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ and every $v \in W^{1,p}(\Omega)$ we have*

$$\|v\|_{L^p(G_\delta)} \leq c\delta^{\frac{1}{p}}\|v\|_{W^{1,p}(\Omega)},$$

where c is a constant independent of δ and v.

Proof. Let $G_\delta = \{x \in \Omega\colon dist(x, \partial\Omega) < \delta\}$, it follows that $S_\delta = \partial G_\delta$ are uniformly smooth surfaces.

By the Sobolev trace Theorem we have

$$\|v\|^p_{L^p(S_\delta)} = \int_{S_\delta} |v|^p dS \leq c_3\|v\|^p_{W^{1,p}(\Omega_\delta)} \leq c_3\|v\|^p_{W^{1,p}(\Omega)} \quad \delta \in [0, \delta_0],$$

where c_3 is a constant independent of δ. Integrating this inequality with respect to δ we get

$$\|v\|^p_{L^p(G_\delta)} = \int_0^\delta \Big(\int_{S_\tau} |v|^p dS\Big) d\tau \leq c_3\delta\|v\|^p_{W^{1,p}(\Omega)}$$

and the Lemma is proved. □

Now, we are able to prove the following key Theorem:

Theorem 5.20. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and denote by Q the unit cube in $\mathbb{R}^n$. Let g be a Q-periodic function such that $\bar{g} = 0$ over Q and $0 < \alpha \leq g \leq \beta < +\infty$ for α, β constants. Then the inequality*

$$\left|\int_\Omega g\left(\frac{x}{\varepsilon}\right) uv\right| \leq c\varepsilon\|u\|_{W^{1,p}(\Omega)}\|v\|_{W^{1,p'}(\Omega)}$$

holds for every $u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$, where c is a constant independent of ε, u, v and p, p′ are conjugate exponents.

Proof. Denote by I^ε the set of all $z \in \mathbb{Z}^n$ such that $Q_{z,\varepsilon} := \varepsilon(z + Q) \subset \Omega$. Set $\Omega_1 = \bigcup_{z \in I^\varepsilon} Q_{z,\varepsilon}$ and $G = \Omega \setminus \bar{\Omega}_1$. As in Lemma 5.12 let us consider the functions $\bar{v}$ and $\bar{u}$ given by the formulas

$$\bar{v}(x) = \frac{1}{\varepsilon^n}\int_{Q_{z,\varepsilon}} v(x)dx, \quad \bar{u}(x) = \frac{1}{\varepsilon^n}\int_{Q_{z,\varepsilon}} u(x)dx$$

for $x \in Q_{z,\varepsilon}$. Then we have

$$\begin{aligned}\int_\Omega g_\varepsilon uv &= \int_G g_\varepsilon uv + \int_{\Omega_1} g_\varepsilon uv \\ &= \int_G g_\varepsilon uv + \int_{\Omega_1} g_\varepsilon(u - \bar{u})v + \int_{\Omega_1} g_\varepsilon\bar{u}(v - \bar{v}) + \int_{\Omega_1} g_\varepsilon\bar{v}\bar{u}.\end{aligned} \tag{5.3.1}$$

The set G is a δ-neighborhood of $\partial\Omega$ with $\delta = c\varepsilon$ for $c = \text{diam}Q_1 = \sqrt{n}$, and therefore according to Lemma 5.9 we have

$$\begin{aligned}\|u\|_{L^p(G)} &\leq c\varepsilon^{\frac{1}{p}}\|u\|_{W^{1,p}(\Omega)}, \\ \|v\|_{L^{p'}(G)} &\leq c\varepsilon^{\frac{1}{p'}}\|v\|_{W^{1,p'}(\Omega)}.\end{aligned} \tag{5.3.2}$$

Then we get

$$\int_G g_\varepsilon uv \leq c\|u\|_{L^p(G)}\|v\|_{L^{p'}(G)} \leq c\varepsilon\|u\|_{W^{1,p}(\Omega)}\|v\|_{W^{1,p'}(\Omega)}. \tag{5.3.3}$$

Now, by Lema 5.11 we get

$$\begin{aligned}\|u-\bar{u}\|_{L^p(\Omega_1)} &= \left(\sum_{z\in I^\varepsilon}\int_{Q_{z,\varepsilon}}|u-\bar{u}|^p dx\right)^{\frac{1}{p}} \leq c_p\varepsilon\left(\sum_{z\in I^{z,\varepsilon}}\int_{Q_{z,\varepsilon}}|\nabla u(x)|^p dx\right)^{\frac{1}{p}} \\ &= c_p\varepsilon\|\nabla u\|_{L^p(\Omega_1)}.\end{aligned} \tag{5.3.4}$$

Analogously

$$\|v-\bar{v}\|_{L^{p'}(\Omega_1)} \leq c_{p'}\varepsilon\|\nabla v\|_{L^{p'}(\Omega_1)}. \tag{5.3.5}$$

By the definition of $\bar{u}(x)$ we get

$$\begin{aligned}\|\bar{u}\|^p_{L^p(\Omega_1)} &= \sum_{z\in I^\varepsilon}\int_{Q_{z,\varepsilon}}|\bar{u}|^p = \sum_{z\in I^\varepsilon}\varepsilon^n\Big(\varepsilon^{-n}\int_{Q_{z,\varepsilon}}u\Big)^p \\ &\leq \varepsilon^{n-np}\sum_{z\in I^\varepsilon}|Q_{z,\varepsilon}|^{p/p'}\int_{Q_{z,\varepsilon}}|u|^p = \varepsilon^{n-np+np/p'}\sum_{z\in I^\varepsilon}\int_{Q_{z,\varepsilon}}|u|^p \\ &= \int_{\Omega_1}|u|^p = \|u\|^p_{L^p(\Omega_1)}.\end{aligned} \tag{5.3.6}$$

Finally, since $\int_{Q_1} g = 0$ and since g is $Q-$periodic, we get

$$\int_{\Omega_1} g_\varepsilon\bar{u}\bar{v} = \sum_{z\in I^\varepsilon}\bar{u}\bar{v}\int_{Q_{z,\varepsilon}} g_\varepsilon = 0. \tag{5.3.7}$$

Now, combining (5.3.3), (5.3.4), (5.3.5), (5.3.6) and (5.3.7) we can bound (5.3.1) by

$$\int_\Omega g_\varepsilon uv \leq C\varepsilon\|u\|_{W^{1,p}(\Omega)}\|v\|_{W^{1,p'}(\Omega)}.$$

This finishes the proof. □

We are ready to proof Theorem 5.6:

Proof of Theorem 5.6: The result follows applying Theorem 5.20 to $\tilde{g}_\varepsilon = g_\varepsilon - \bar{g}$ and taking $v \equiv 1$. □

5.4 Proof of Theorem 5.8

First, we prove two auxiliary lemmas.

Lemma 5.21. *Let $v \in W^{1,p}(I)$, where $I = (0,1)$. Then for each $x_0 \in I$ we have that*

$$|v(x_0)| \leq \|v\|_{W^{1,p}(I)}.$$

Proof. For each $0 < x_0 < x < 1$ we have that

$$|v(x_0)| \le |v(x)| + \left| \int_{x_0}^{x} v' \right|.$$

Integrating between 0 and 1 and applying Hölder's inequality we get

$$|v(x_0)| \le \int_0^1 |v(x)| + \int_0^1 \int_{x_0}^{x} |v'| \le \int_0^1 |v(x)| + \int_0^1 |v'| \tag{5.4.1}$$

$$\le \left(\int_0^1 |v(x)|^p \right)^{\frac{1}{p}} + \left(\int_0^1 |v'|^p \right)^{\frac{1}{p}} = \|v\|_{W^{1,p}(I)}. \tag{5.4.2}$$

□

Lemma 5.22. *Let $v \in W^{1,p}(I)$, $I = (0, 1)$ and for $\delta > 0$ small let $G_\delta = (0, \delta)$. We have that*

$$\|v\|_{L^p(G_\delta)} \le 2\delta^{\frac{1}{p}} \|v\|_{W^{1,p}(I)}.$$

Proof. For each $0 < x < x_0 < 1$ we have that

$$|v(x)| \le |v(x_0)| + \int_x^{x_0} |v'|.$$

It follows that

$$|v(x)|^p \le 2^{p-1} \left(|v(x_0)|^p + \left(\int_0^1 |v'| \right)^p \right).$$

Now, by Holder's inequality we get

$$|v(x)|^p \le 2^{p-1} \left(|v(x_0)|^p + \int_0^1 |v'|^p \right).$$

Integrating between 0 and δ and applying Lemma 5.21 we obtain

$$\begin{aligned} \int_0^\delta |v(x)|^p &\le 2^{p-1} \delta \left(|v(x_0)|^p + \int_0^1 |v'|^p \right) \\ &\le 2^p \delta \|v\|^p_{W^{1,p}(I)}. \end{aligned}$$

It follows that

$$\|v\|_{L^p((0,\delta))} \le 2\delta^{\frac{1}{p}} \|v\|_{W^{1,p}(I)}.$$

Now, the proof is complete. □

Now, we are able to prove the following Theorem which is essential to prove Theorem 5.8.

Theorem 5.23. *Let $I = (0, 1)$ and g be a $1-$periodic function such that $\bar{g} = 0$ over $(0, 1)$ and $0 < \alpha \le g \le \beta < +\infty$ for α, β constants. Then the inequality*

$$\left| \int_\Omega g(\tfrac{x}{\varepsilon}) uv \right| \le c\varepsilon \|u\|_{W^{1,p}(I)} \|v\|_{W^{1,p'}(I)}$$

holds for every $u \in W^{1,p}(I)$ and $v \in W^{1,p'}(I)$, where

$$c := \beta \left(4 + \tfrac{p}{(p-1)\pi_p} \right).$$

Proof. Now the proof is similar to that of Theorem 5.20. Let $I_j^\varepsilon = ((j-1)\varepsilon, j\varepsilon]$ and

$$J^\varepsilon = (0, m\varepsilon] = \bigcup_{j=1}^{m} I_j^\varepsilon, \qquad G^\varepsilon = I \setminus J^\varepsilon = (\ell - \varepsilon, \ell).$$

For $x \in I_j^\varepsilon$ let $u_\varepsilon, v_\varepsilon$ be defined as

$$u_\varepsilon(x) = \frac{1}{\varepsilon}\int_{I_j^\varepsilon} u(y)\,dy, \qquad v_\varepsilon(x) = \frac{1}{\varepsilon}\int_{I_j^\varepsilon} v(y)\,dy.$$

We have that

$$\begin{aligned}\int_I g_\varepsilon uv &= \int_{G^\varepsilon} g_\varepsilon uv + \int_{J^\varepsilon} g_\varepsilon uv \\ &= \int_{G^\varepsilon} g_\varepsilon uv + \int_{J^\varepsilon} g_\varepsilon (u-u_\varepsilon)v + \int_{J^\varepsilon} g_\varepsilon u_\varepsilon (v - v_\varepsilon) + \int_{J^\varepsilon} g_\varepsilon v_\varepsilon u_\varepsilon.\end{aligned} \tag{5.4.3}$$

By using Lemma 5.22 we have

$$\begin{aligned}\|u\|_{L^p(G^\varepsilon)} &\le 2\varepsilon^{\frac{1}{p}}\|u\|_{W^{1,p}(I)}, \\ \|v\|_{L^{p'}(G^\varepsilon)} &\le 2\varepsilon^{\frac{1}{p'}}\|v\|_{W^{1,p'}(I)}.\end{aligned} \tag{5.4.4}$$

Then we get

$$\int_{G^\varepsilon} g_\varepsilon uv \le \beta\|u\|_{L^p(G^\varepsilon)}\|v\|_{L^{p'}(G^\varepsilon)} \tag{5.4.5}$$

$$\le 4\beta\varepsilon\|u\|_{W^{1,p}(I)}\|v\|_{W^{1,p'}(I)}. \tag{5.4.6}$$

Now, by Lema 5.11 we get

$$\begin{aligned}\|u - u_\varepsilon\|_{L^p(J^\varepsilon)} &= \left(\sum_{j=1}^{m}\int_{I_j^\varepsilon} |u - u_\varepsilon|^p dx\right)^{\frac{1}{p}} \le c_p\varepsilon\left(\sum_{j=1}^{m}\int_{I_j^\varepsilon}|u'(x)|^p dx\right)^{\frac{1}{p}} \\ &= c_p\varepsilon\|u'\|_{L^p(J^\varepsilon)}.\end{aligned} \tag{5.4.7}$$

Analogously

$$\|v - v_\varepsilon\|_{L^{p'}(\Omega_1)} \le c_{p'}\varepsilon\|v'\|_{L^{p'}(J^\varepsilon)}. \tag{5.4.8}$$

By the definition of u_ε we get

$$\begin{aligned}\|u_\varepsilon\|^p_{L^p(J^\varepsilon)} &= \sum_{j=1}^{m}\int_{I_j^\varepsilon}|u_\varepsilon|^p = \sum_{j=1}^{m}\varepsilon\Big(\varepsilon^{-1}\int_{I_j^\varepsilon} u\Big)^p \\ &\le \varepsilon^{1-p}\sum_{j=1}^{m}|I_j^\varepsilon|^{p/p'}\int_{I_j^\varepsilon}|u|^p = \varepsilon^{1-p+p/p'}\sum_{j=1}^{m}\int_{I_j^\varepsilon}|u|^p \\ &= \int_{J^\varepsilon}|u|^p = \|u\|^p_{L^p(J^\varepsilon)}.\end{aligned} \tag{5.4.9}$$

Finally, since g is 1−periodic with $\bar{g} = 0$, we get

$$\int_{J^\varepsilon} u_\varepsilon v_\varepsilon = \sum_{j=1}^{m} u_\varepsilon v_\varepsilon \int_{I_j^\varepsilon} g_\varepsilon = 0. \tag{5.4.10}$$

Now, combining (5.4.5), (5.4.7), (5.4.8), (5.4.9) and (5.4.10) we can bound (5.4.3) by

$$\int_\Omega g_\varepsilon uv \leq \beta(4 + c_p + c_{p'})\varepsilon \|u\|_{W^{1,p}(I)} \|v\|_{W^{1,p'}(I)},$$

where $c_p = 1/\pi_p$, $c_{p'} = 1/\pi_{p'}$.

By using the relation (2.2.2) it follows that $\pi_{p'} = (p-1)\pi_p$ and this finishes the proof. □

We are ready to proof Theorem 5.8:

Proof of Theorem 5.8: The result follows applying Theorem 5.23 to $\tilde{g}_\varepsilon = g_\varepsilon - \bar{g}$ and taking $v \equiv 1$. □

6

Eigenvalue homogenization for quasilinear elliptic operators

In this Chapter we study the asymptotic behavior (as $\varepsilon \to 0$) of the eigenvalues of the following problem

$$\begin{cases} -div(a_\varepsilon(x, \nabla u^\varepsilon)) = \lambda^\varepsilon \rho_\varepsilon |u^\varepsilon|^{p-2} u^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, ε is a positive real number, and λ^ε is the eigenvalue parameter.

The weight functions $\rho_\varepsilon(x)$ are assumed to be positive and uniformly bounded away from zero and infinity and the family of operators $a_\varepsilon(x, \xi)$ satisfies hypothesis (H0)–(H8) of Section 2.4, Chapter §2, but the prototypical example is

$$-div(a_\varepsilon(x, \nabla u^\varepsilon)) = -div(A^\varepsilon(x)|\nabla u^\varepsilon|^{p-2} \nabla u^\varepsilon), \tag{6.0.2}$$

with $1 < p < +\infty$, and $A^\varepsilon(x)$ is a family of uniformly elliptic matrices (both in $x \in \Omega$ and in $\varepsilon > 0$).

The study of this type of problems have a long history due to its relevance in different fields of applications. The problem of finding the asymptotic behavior of the eigenvalues of (6.0.1) is an important part of what is called *Homogenization Theory*. Homogenization Theory is applied in composite materials in which the physical parameters such as conductivity and elasticity are oscillating. Homogenization Theory try to get a good approximation of the macroscopic behavior of the heterogeneous material by letting the parameter $\varepsilon \to 0$. The main references for the homogenization theory of periodic structures are the books by Bensoussan-Lions-Papanicolaou [BLP78], Sanchez–Palencia [SP70], Oleĭnik-Shamaev-Yosifian [OSY92] among others.

In the linear setting (i.e., $a_\varepsilon(x, \xi)$ as in (6.0.2) with $p = 2$) this problem is well understood. It is known that, up to a subsequence, there exists a limit operator $a_h(x, \xi) = A^h(x)\xi$ and a limit function $\bar\rho$ such that the spectrum of (6.0.1) converges to that of the limit problem (see Section 4.2.1)

$$\begin{cases} -div(a_h(x, \nabla u)) = \lambda \bar\rho |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.0.3}$$

In the important case of periodic homogenization, i.e. when $\rho_\varepsilon(x) = \rho(x/\varepsilon)$ and $A^\varepsilon(x) = A(x/\varepsilon)$

where $\rho(x)$ and $A(x)$ are $Q-$periodic functions, Q being the unit cube in $\mathbb{R}^N$, the limit problem can be fully characterized and so the entire sequence $\varepsilon \to 0$ is convergent. See Section 4.2.1.

In the general nonlinear setting, recently Baffico, Conca and Donato [BCR06], relying on the $G-$convergence results of Chiadó Piat, Dal Maso and Defranceschi [CVD90] for monotone operators, study the convergence problem of the principal eigenvalue of (6.0.1). The concept of $G-$convergence of linear elliptic second order operators was introduced by Spagnolo in [Sp68]. See Section 4.3 for the precise definitions.

Up to our knowledge, no further investigation was made in the quasilinear non-uniformly elliptic case. One of the reasons why in [BCR06] only the principal eigenvalue was studied is that, as long as we know, no results are available for higher order eigenvalues of (6.0.1).

The principal eigenvalue of (6.0.1) was studied by Kawohl, Lucia and Prashanth in [KLP07] where, among other things, they prove its existence together with the simplicity and positivity of the associated eigenfunction.

In Section 2.5, we have continued with this investigation. We have extended some results for higher eigenvalues that are well known in the $p-$Laplacian case, to (6.0.1). Namely, the isolation of the principal eigenvalue, the existence of a sequence of (variational) eigenvalues growing to $+\infty$ and a variational characterization of the second eigenvalue.

Using the results of Section 2.5, in Section 6.2 we give a new simpler proof of the convergence of the principal eigenvalues of (6.0.1) to the principal eigenvalue of the limit problem (6.0.3). Moreover we can prove the convergence of the second eigenvalues of (6.0.1) to the second eigenvalue of (6.0.3). These two results rely on a more general one that says that the limit of any sequence of eigenvalues of (6.0.1) is an eigenvalue of (6.0.3). Although this result was already proved in [BCR06], we provide here a simplified proof of this fact.

Convergence of eigenvalues in the multidimensional linear case was studied in 1976 by Boccardo and Marcellini [BM76] for general bounded matrices. Kesavan [Ke79b] studied the problem in a periodic setting.

Now, we turn our attention to the order of convergence of the eigenvalues. Clearly, the question of order of convergence cannot be treated with the previous generality. To this end, we restrict ourselves to the problems

$$\begin{cases} -div(a(x,\nabla u^\varepsilon)\nabla u^\varepsilon) = \lambda^\varepsilon \rho_\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.0.4}$$

where the family of weight functions ρ_ε are given in terms of a single bounded $Q-$periodic function ρ in the form $\rho_\varepsilon(x) := \rho(x/\varepsilon)$, Q being the unit cube of $\mathbb{R}^N$.

The limit problem is then given by

$$\begin{cases} -div(a(x,\nabla u)\nabla u) = \lambda\bar{\rho}|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.0.5}$$

where $\bar{\rho}$ is the average of ρ in Q.

The first result in this problem, for the linear case, can be found in Chapter III, section 2 of [OSY92]. By estimating the eigenvalues of the inverse operator, which is compact, and using tools from functional analysis in Hilbert spaces, they deduce that

$$|\lambda_k^\varepsilon - \lambda_k| \leq \frac{C\lambda_k^\varepsilon(\lambda_k)^2}{1-\lambda_k\beta_k^\varepsilon}\,\varepsilon^{\frac{1}{2}} < C_k k^{\frac{6}{N}}\varepsilon^{\frac{1}{2}}.$$

Here, C is a positive constant, C_k is a constant depending on k and β_ε^k satisfies

$$0 \leq \beta_\varepsilon^k < \lambda_k^{-1},$$

and

$$\lim_{\varepsilon\to 0}\beta_\varepsilon^k = 0$$

for each $k \geq 1$.

Again, in linear case with ε dependence in the operator $a(x,\xi)$, Santonsa and Vogelius [SaVo93] by using eigenvalues expansion proved that

$$|\lambda_k^\varepsilon - \lambda_k| \leq C\varepsilon$$

where C depends on k.

More recently, Kenig, Lin and Shen [KLS11] studied the linear problem in any dimension (allowing an ε dependance in the diffusion matrix of the elliptic operator) and proved that for Lipschitz domains Ω one has

$$|\lambda_k^\varepsilon - \lambda_k| \leq C\varepsilon|\log(\varepsilon)|^{\frac{1}{2}+\sigma}$$

for any $\sigma > 0$, C depending on k and σ.

Moreover, the authors show that if the domain Ω is more regular ($C^{1,1}$ is enough) they can get rid of the logarithmic term in the above estimate. However, no explicit dependance of C on k is obtained in that work.

In Section 6.3, we analyze the order of convergence of eigenvalues of (6.0.4) (either with Dirichlet or Neumann boundary conditions) to the ones of the correspondent limit problem, and prove that

$$|\lambda_k^\varepsilon - \lambda_k| \leq C k^{\frac{2p}{N}}\varepsilon$$

with C independent of k and ε. In this result, by λ_k^ε and λ_k we refer to the variational eigenvalues of problems (6.0.4) and (6.0.5) respectively with the correspondent boundary data.

Some remarks are in order:

1. Classical estimates on the eigenvalues of second order, N-dimensional problems, show that λ_k and λ_k^ε behaves like $ck^{\frac{2}{N}}$, with c depending only on the coefficients of the operator and N. Hence, the order of growth of the right-hand side in the estimate of [OSY92] is

   $$\frac{\lambda_k^\varepsilon(\lambda_k)^2\varepsilon^{\frac{1}{2}}}{1-\lambda_k\beta_k^\varepsilon} \sim \frac{k^{\frac{6}{N}}\varepsilon^{\frac{1}{2}}}{1-\lambda_k\beta_k^\varepsilon} \geq k^{\frac{6}{N}}\varepsilon^{\frac{1}{2}}.$$

   Moreover, the constant involved in their bound are unknown.

2. In our result very low regularity on the domain Ω is assumed in this work. We only required the validity of the Hardy inequality (see [Ma85])
$$\int_\Omega \frac{|u|^p}{d^p} \le C \int_\Omega |\nabla u|^p,$$
where $d(x) = dist(x, \partial\Omega)$ and $u \in W_0^{1,p}(\Omega)$. For instance, Lipschitz regularity will do. So we get an improvement of the result in [KLS11]. However, we recall that the result in [KLS11] allows for a dependence in ε on the operator. Nevertheless, our result includes nonlinear eigenvalue problems, such as the $p-$Laplacian eigenvalues.

6.1 About the convergence of the spectrum

In this section we analyze the convergence of the spectrum Σ_ε of (6.0.1) to the spectrum Σ_h of the homogenized limit problem (6.0.3)

In the linear case, it is known (see [Al02]) that the $G-$convergence of the operators implies the convergence of their spectra in the sense that the kth–variational eigenvalue λ_k^ε converges to the kth–variational eigenvalue of the limit problem.

We want to study the convergence of the spectrum in the non-linear case. We begin with a general result for bounded sequences of eigenvalues. This result was already proved in [BCR06] but we present here a simpler proof.

Here, and in all this Chapter we will assume that ρ is a $Q-$periodic function defined over a bounded domain $\Omega \subset \mathbb{R}^N$, being Q the unit cube in $\mathbb{R}^N$, such that for some constants $\rho^- < \rho^+$,
$$0 < \rho^- \le \rho(x) \le \rho^+ < +\infty \qquad \text{a.e. } \Omega.$$
We will assume that the family of operators $a(\cdot, \xi)$ satisfies properties (H0)–(H8) defined in Section 2.4 and the associated potential $\Phi_\varepsilon(x, \xi)$ satisfies (2.4.2).

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^N$ be bounded. Let $\lambda^\varepsilon \in \Sigma_\varepsilon$ be a sequence of eigenvalues of the problems (6.0.1) with $\{u^\varepsilon\}_{\varepsilon>0}$ associated normalized eigenfunctions.*

Assume that the sequence of eigenvalues is convergent
$$\lim_{\varepsilon\to 0^+} \lambda^\varepsilon = \lambda.$$
Then, $\lambda \in \Sigma_h$ and there exists a sequence $\varepsilon_j \to 0^+$ such that
$$u^{\varepsilon_j} \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega)$$
with u a normalized eigenfunction associated to λ.

Remark 6.2. In most applications, we take the sequence λ^ε to be the sequence of the kth–variational eigenvalue of (6.0.1). In this case, it is not difficult to check that the sequence $\{\lambda_k^\varepsilon\}_{\varepsilon>0}$ is bounded and so, up to a subsequence, convergent.

In fact, by using the variational characterization of λ_k^ε, (2.4.2) and our assumptions on ρ we have that

$$\frac{\alpha}{\rho^+}\frac{\int_\Omega |\nabla v|^p}{\int_\Omega |v|^p} \leq \frac{\int_\Omega \Phi_\varepsilon(x,\nabla v)}{\int_\Omega \rho_\varepsilon |v|^p} \leq \frac{\beta}{\rho^-}\frac{\int_\Omega |\nabla v|^p}{\int_\Omega |v|^p},$$

therefore

$$\frac{\alpha}{\rho^+}\mu_k \leq \lambda_k^\varepsilon \leq \frac{\beta}{\rho^-}\mu_k$$

where μ_k is the kth variational eigenvalue of the $p-$Laplacian.

Proof. As λ_ε is bounded and u^ε is normalized, by (H2) it follows that the sequence $\{u^\varepsilon\}_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$.

Therefore, up to some sequence $\varepsilon_j \to 0$, we have that

$$\begin{aligned} u^{\varepsilon_j} &\rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega)\\ u^{\varepsilon_j} &\to u \quad \text{strongly in } L^p(\Omega). \end{aligned} \tag{6.1.1}$$

with u also normalized.

We define the sequence of functions $f_\varepsilon := \lambda^\varepsilon \rho_\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon$. By using the fact that $\rho_\varepsilon \rightharpoonup \bar\rho$ $*$-weakly in $L^\infty(\Omega)$ together with (6.1.1) it follows that

$$f_{\varepsilon_j} \rightharpoonup f := \lambda\bar\rho|u|^{p-2}u \quad \text{weakly in } L^p(\Omega)$$

and therefore

$$f_{\varepsilon_j} \to f \quad \text{strongly in } W^{-1,p'}(\Omega).$$

By Proposition 4.24 we deduce that u^{ε_j} converges weakly in $W_0^{1,p}(\Omega)$ to the unique solution v of the homogenized problem

$$\begin{cases} -div(a_h(x,\nabla v)) = \lambda\bar\rho|u|^{p-2}u & \text{in } \Omega\\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

By uniqueness of the limit, $v = u$ is a normalized eigenfunction of the homogenized problem. □

Remark 6.3. In the case where the sequence λ^ε is the sequence of the kth–variational eigenvalues of (6.0.1) it would be desirable to prove that it converges to the kth–variational eigenvalue of the homogenized problem (6.0.3) (see Remark 6.2).

Unfortunately, we are able to prove this fact only for the first and second variational eigenvalues in the general setting.

In the one dimensional case, one can be more precise and this fact holds true. See Chapter §7.

In section 6.3, we address this problem (even with Neumann boundary conditions) in the more specific situation of $a_\varepsilon(x,\xi) = a(x,\xi)$ and $\rho_\varepsilon(x) = \rho(x/\varepsilon)$ and prove that this fact also holds true and, moreover, we provide with an estimate for the error term $|\lambda_k^\varepsilon - \lambda_k|$.

6.2 Convergence of the first and second eigenvalue

The first eigenvalue of (6.0.1) is the infimum of the Rayleigh quotient (see Section 2.5)

$$\lambda_1^\varepsilon = \inf_{v \in W_0^{1,p}(\Omega)} \frac{\int_\Omega \Phi_\varepsilon(x, \nabla v)}{\int_\Omega \rho_\varepsilon |v|^p}.$$

In the following result we prove the convergence of λ_1^ε when ε tends to zero.

Theorem 6.4. *Let be λ_1^ε the first eigenvalue of* (6.0.1) *and λ_1 the first eigenvalue of the limit problem* (6.0.3)*, then*

$$\lim_{\varepsilon \to 0} \lambda_1^\varepsilon = \lambda_1.$$

Moreover, if u_1^ε and u_1 are the (normalized) nonnegative eigenfunctions of (6.0.1) *and* (6.0.3) *associated to λ_1^ε and λ_1 respectively, then*

$$u_1^\varepsilon \rightharpoonup u_1 \quad \textit{weakly in } W_0^{1,p}(\Omega).$$

Remark 6.5. In [BCR06] using the theory of convergence of monotone operators the authors obtain the conclusions of Theorem 6.4. We propose here a simple proof of this result which exploits the fact that the first eigenfunction has constant sign.

Proof. Let u_1^ε be the nonnegative normalized eigenfunction associated to λ_1^ε, the uniqueness of u_1^ε follows from Theorem 2.23.

By Theorem 6.1, up to some sequence, u_1^ε converges weakly in $W_0^{1,p}(\Omega)$ to u, an eigenfunction of the homogenized eigenvalue problem associated to $\lambda = \lim_{\varepsilon \to 0} \lambda_1^\varepsilon$.

But then, u is a nonnegative normalized eigenfunction of the homogenized problem (6.0.3) and so $u = u_1$. Therefore $\lambda = \lambda_1$ and the uniqueness imply that the whole sequences λ_1^ε and u_1^ε are convergent. □

Now we turn our attention to the second eigenvalue. For this purpose we use the fact that eigenfunctions associated to the second variational eigenvalue of problems (6.0.1) and (6.0.3) have, at least, two nodal domains (cf. Proposition 2.24).

Theorem 6.6. *Let λ_2^ε be the second eigenvalue of* (6.0.1) *and λ_2 be the second eigenvalue of the homogenized problem* (6.0.3)*. Then*

$$\lim_{\varepsilon \to 0} \lambda_2^\varepsilon = \lambda_2.$$

Proof. Let u_2 be a normalized eigenfunction associated to λ_2 and let $\Omega^\pm$ be the positivity and the negativity sets of u_2 respectively.

We denote by $u_\pm^\varepsilon$ the first eigenfunction of (6.0.1) in $\Omega^\pm$ respectively. Extending $u_\pm^\varepsilon$ to Ω by 0, those functions have disjoint supports and therefore they are linearly independent in $W_0^{1,p}(\Omega)$.

Let S be the unit sphere in $W_0^{1,p}(\Omega)$ and we define the set C_2^ε as

$$C_2^\varepsilon := \operatorname{span}\{u_+^\varepsilon, u_-^\varepsilon\} \cap S.$$

Clearly C_2^ε is compact, symmetric and $\gamma(C_2^\varepsilon) = 2$. Hence,

$$\lambda_2^\varepsilon = \inf_{C\in\Gamma_2} \sup_{v\in C} \frac{\int_\Omega \Phi_\varepsilon(x,\nabla v)}{\int_\Omega \rho_\varepsilon |v|^p} \leq \sup_{v\in C_2^\varepsilon} \frac{\int_\Omega \Phi_\varepsilon(x,\nabla v)}{\int_\Omega \rho_\varepsilon |v|^p}.$$

As C_2^ε is compact, the supremum is achieved for some $v^\varepsilon \in C_2^\varepsilon$ which can be written as

$$v^\varepsilon = a_\varepsilon u_+^\varepsilon + b_\varepsilon u_-^\varepsilon$$

with $a_\varepsilon, b_\varepsilon \in \mathbb{R}$ such that $|a_\varepsilon|^p + |b_\varepsilon|^p = 1$. Since the functions u_+^ε and u_-^ε have disjoint supports, we obtain, using the $p-$homogeneity of Φ_ε (see Proposition 2.19),

$$\lambda_2^\varepsilon \leq \frac{\int_\Omega \Phi_\varepsilon(x,\nabla v^\varepsilon)}{\int_\Omega \rho_\varepsilon |v^\varepsilon|^p} = \frac{|a_\varepsilon|^p \int_{\Omega^+} \Phi_\varepsilon(x,\nabla u_+^\varepsilon) + |b_\varepsilon|^p \int_{\Omega^-} \Phi_\varepsilon(x,\nabla u_-^\varepsilon)}{\int_\Omega \rho_\varepsilon |v^\varepsilon|^p}.$$

Using the definition of $u_\pm^\varepsilon$, the above inequality can be rewritten as

$$\lambda_2^\varepsilon \leq \frac{|a_\varepsilon|^p \lambda_{1,+}^\varepsilon \int_{\Omega^+} \rho_\varepsilon |u_+^\varepsilon|^p + |b_\varepsilon|^p \lambda_{1,-}^\varepsilon \int_{\Omega^-} \rho_\varepsilon |u_-^\varepsilon|^p}{\int_\Omega \rho_\varepsilon |v^\varepsilon|^p} \leq \max\{\lambda_{1,+}^\varepsilon, \lambda_{1,-}^\varepsilon\} \tag{6.2.1}$$

where $\lambda_{1,\pm}^\varepsilon$ is the first eigenvalue of (6.0.1) in the nodal domain $\Omega^\pm$ respectively.

Now, using Theorem 6.4, we have that $\lambda_{1,\pm}^\varepsilon \to \lambda_{1,\pm}$ respectively, where $\lambda_{1,\pm}$ are the first eigenvalues of (6.0.3) in the domains $\Omega^\pm$ respectively. Moreover, we observe that these eigenvalues $\lambda_{1,\pm}$ are both equal to the second eigenvalue λ_2 in Ω, therefore from (6.2.1), we get

$$\lambda_2^\varepsilon \leq \lambda_2 + \delta$$

for δ arbitrarily small and ε tending to zero. So,

$$\limsup_{\varepsilon\to 0} \lambda_2^\varepsilon \leq \lambda_2. \tag{6.2.2}$$

On the other hand, suppose that $\lim_{\varepsilon\to 0} \lambda_2^\varepsilon = \lambda$ where $\lambda \in \Sigma_h$. We claim that $\lambda > \lambda_1$.

In fact, we have that $u_2^\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ where u is a normalized eigenfunction associated to λ. As the measure of the positivity and negativity sets of u_2^ε are bounded below uniformly in $\varepsilon > 0$ (see Proposition 2.24), we have that either u changes sign or $|\{u = 0\}| > 0$. In any case, this implies our claim.

Then, as $\lambda > \lambda_1$ it must be $\lambda \geq \lambda_2$. Then

$$\lambda_2 \leq \lambda = \lim_{\varepsilon\to 0} \lambda_2^\varepsilon. \tag{6.2.3}$$

Combining (6.2.2) and (6.2.3) we obtain the desired result. □

6.3 Rate of convergence

In this section we consider the eigenvalue problem in which the operator is independent on ε and the dependance on ε only appears in the oscillating weights. We consider the equation

$$-div(a(x, \nabla u^\varepsilon)) + V_\varepsilon |u|^{p-2}u = \lambda^\varepsilon \rho_\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon \quad \text{in } \Omega \tag{6.3.1}$$

either with Dirichlet or Neumann boundary conditions. We assume that the weights ρ and V satisfies

$$\begin{aligned} &0 < \rho^- \le \rho(x) \le \rho^+ < \infty && \text{a.e. in } \Omega \\ &0 < V^- \le V(x) \le V^+ < \infty && \text{a.e. in } \Omega. \end{aligned} \tag{6.3.2}$$

When $\varepsilon \to 0$ we obtain the following limit problem

$$-div(a(x, \nabla u^\varepsilon)) + \bar{V}|u|^{p-2}u = \lambda^\varepsilon \bar{\rho} |u^\varepsilon|^{p-2}u^\varepsilon \quad \text{in } \Omega \tag{6.3.3}$$

with Dirichlet or Neumann boundary conditions, respectively. We will prove that in this case the kth–variational eigenvalue of problem (6.3.1) converges to the kth–variational eigenvalue of the limit problem (6.3.3).

Our goal is to estimate the rate of convergence between the eigenvalues. That is, we want to find explicit bounds for the error $|\lambda_k^\varepsilon - \lambda_k|$.

Using the results of Section 5 concerning to oscillating integrals, we prove our main result of this section.

Theorem 6.7. *Let λ_k^ε be the kth–variational eigenvalue associated to equation* (6.3.1) *and let be λ_k be the kth–variational eigenvalue associated to the limit problem* (6.3.3)*. Then there exists a constant $C > 0$ independent of the parameters ε and k such that*

$$|\lambda_k - \lambda_k^\varepsilon| \le C k^{\frac{2p}{N}} \varepsilon.$$

Proof of Theorem 6.7. Let us observe that variational eigenvalues of (6.3.1) and (6.3.3), according to Section 2.5, are characterized as

$$\lambda_k^\varepsilon = \inf_{C \in \Gamma_k} \sup_{v \in C} \frac{\int_\Omega \Phi(x, \nabla v) + V_\varepsilon |v|^p}{\int_\Omega \rho_\varepsilon |v|^p}, \qquad \lambda_k = \inf_{C \in \Gamma_k} \sup_{v \in C} \frac{\int_\Omega \Phi(x, \nabla v) + \bar{V} |v|^p}{\int_\Omega \bar{\rho} |v|^p} \tag{6.3.4}$$

where, in the case of Dirichlet boundary conditions

$$\Gamma_k = \{C \subset W_0^{1,p}(\Omega) : C \text{ compact}, C = -C, \ \gamma(C) \ge k\}$$

however, when Neumann boundary conditions are considered

$$\Gamma_k = \{C \subset W^{1,p}(\Omega) : C \text{ compact}, C = -C, \ \gamma(C) \ge k\}.$$

The proofs in both cases are very similar. We prove the Neumann case and then we note the main differences in the proof of the Dirichlet case.

Let $\delta > 0$ and let $G_\delta^k \subset W^{1,p}(\Omega)$ be a compact, symmetric set of genus k such that

$$\lambda_k = \sup_{u \in G_\delta^k} \frac{\int_\Omega \Phi(x, \nabla u) + \bar{V}|u|^p}{\bar{\rho} \int_\Omega |u|^p} + O(\delta). \tag{6.3.5}$$

We use now the set G_δ^k, which is admissible in the variational characterization of the kth–eigenvalue (6.3.1) in order to found a bound for it as follows,

$$\lambda_k^\varepsilon \leq \sup_{u \in G_\delta^k} \frac{\int_\Omega \Phi(x, \nabla u) + V_\varepsilon |u|^p}{\bar{\rho} \int_\Omega |u|^p} \frac{\bar{\rho} \int_\Omega |u|^p}{\int_\Omega \rho_\varepsilon |u|^p}. \tag{6.3.6}$$

To bound λ_k^ε we look for bounds of the two quotients in (6.3.6). For every function $u \in G_\delta^k \subset W^{1,p}(\Omega)$ we can apply Theorem 5.6 and we obtain that

$$\frac{\int_\Omega \Phi(x, \nabla u) + V_\varepsilon |u|^p}{\bar{\rho} \int_\Omega |u|^p} \leq \frac{\int_\Omega \Phi(x, \nabla u) + \bar{V}|u|^p}{\bar{\rho} \int_\Omega |u|^p} + C\varepsilon \frac{\| |u|^p \|_{W^{1,1}(\Omega)}}{\bar{\rho} \int_\Omega |u|^p}. \tag{6.3.7}$$

By using Young's inequality

$$\begin{aligned}
\| |u|^p \|_{W^{1,1}(\Omega)} &= \| |u|^p \|_{L^1(\Omega)} + p \| |u|^{p-1} \nabla(u) \|_{L^1(\Omega)} \\
&= \|u\|_{L^p(\Omega)}^p + p \| |u|^{p-1} \nabla u \|_{L^1(\Omega)} \\
&\leq p \|u\|_{L^p(\Omega)}^p + \|\nabla u\|_{L^p(\Omega)}^p \\
&\leq p \|u\|_{W^{1,p}(\Omega)}^p.
\end{aligned}$$

Now, by (2.5.2) and (2.4.2), we have for each $u \in G_\delta^k$

$$\begin{aligned}
\frac{\| |u|^p \|_{W^{1,1}(\Omega)}}{\bar{\rho} \int_\Omega |u|^p} &\leq p \frac{\|u\|_{W^{1,p}(\Omega)}^p}{\bar{\rho} \int_\Omega |u|^p} \\
&\leq \frac{p}{\bar{V}} \frac{\bar{V} \int_\Omega |u|^p + \frac{\bar{V}}{\alpha} \int_\Omega \Phi(x, \nabla u)}{\bar{\rho} \int_\Omega |u|^p} \\
&\leq c_1 \frac{\bar{V} \int_\Omega |u|^p + \int_\Omega \Phi(x, \nabla u)}{\bar{\rho} \int_\Omega |u|^p}
\end{aligned} \tag{6.3.8}$$

where $c_1 = \frac{p}{\bar{V}} \max\{\frac{\bar{V}}{\alpha}, 1\}$.

Then, by (6.3.8) and (6.3.5)

$$\begin{aligned}
\frac{\| |u|^p \|_{W^{1,1}(\Omega)}}{\bar{\rho} \int_\Omega |u|^p} &\leq c_1 \sup_{v \in G_\delta^k} \frac{\bar{V} \int_\Omega |v|^p + \int_\Omega \Phi(x, \nabla v)}{\bar{\rho} \int_\Omega |v|^p} \\
&= c_1 (\lambda_k + O(\delta)).
\end{aligned} \tag{6.3.9}$$

Moreover, by (6.3.5) we get

$$\frac{\int_\Omega \Phi(x, \nabla u) + \bar{V}|u|^p}{\bar{\rho} \int_\Omega |u|^p} \leq \sup_{v \in G_\delta^k} \frac{\int_\Omega \Phi(x, \nabla v) + \bar{V}|v|^p}{\bar{\rho} \int_\Omega |v|^p} = \lambda_k + O(\delta). \tag{6.3.10}$$

Again, since $u \in G^k_\delta \subset W^{1,p}(\Omega)$, by applying Theorem 5.6 we obtain that

$$\frac{\bar\rho \int_\Omega |u|^p}{\int_\Omega \rho_\varepsilon |u|^p} \le 1 + C\varepsilon \frac{\||u|^p\|_{W^{1,1}(\Omega)}}{\int_\Omega \rho_\varepsilon |u|^p}, \tag{6.3.11}$$

and by (6.3.9),

$$\frac{\||u|^p\|_{W^{1,1}(\Omega)}}{\int_\Omega \rho_\varepsilon |u|^p} \le \frac{\bar\rho}{\rho^-} \frac{\||u|^p\|_{W^{1,1}(\Omega)}}{\int_\Omega \bar\rho |u|^p} \le \frac{\bar\rho}{\rho^-} c_1(\lambda_k + O(\delta)). \tag{6.3.12}$$

Then combining (6.3.6), (6.3.9), (6.3.10) and (6.3.12) we find that

$$\lambda_k^\varepsilon \le (\lambda_k + O(\delta) + C\varepsilon(\lambda_k + O(\delta)))\,(1 + C\varepsilon(\lambda_k + O(\delta)))\,.$$

Letting $\delta \to 0$ we get

$$\lambda_k^\varepsilon - \lambda_k \le C\varepsilon(\lambda_k^2 + \lambda_k) + C\varepsilon^2\lambda_k^2. \tag{6.3.13}$$

In a similar way, interchanging the roles of λ_k and λ_k^ε, we obtain

$$\lambda_k - \lambda_k^\varepsilon \le C\varepsilon((\lambda_k^\varepsilon)^2 + \lambda_k^\varepsilon) + C\varepsilon^2(\lambda_k^\varepsilon)^2. \tag{6.3.14}$$

So, from (6.3.13) and (6.3.14), we arrive at

$$|\lambda_k^\varepsilon - \lambda_k| \le C\varepsilon \max\{\lambda_k^2 + \lambda_k, (\lambda_k^\varepsilon)^2 + \lambda_k^\varepsilon\}.$$

In order to complete the proof of the Theorem, we need an estimate on λ_k and λ_k^ε. But this follows by comparison with the kth–variational eigenvalue of the p−Laplacian, μ_k and the bound for μ_k proved in [GAP88].

In fact, from (2.4.2) we have

$$\frac{\min\{\alpha, \bar V\}}{\bar\rho} \frac{\int_\Omega |\nabla u|^p + |u|^p}{\int_\Omega |u|^p} \le \frac{\int_\Omega \Phi(x, \nabla u) + \bar V |u|^p}{\int_\Omega \bar\rho |u|^p} \le \frac{\max\{\beta, \bar V\}}{\bar\rho} \frac{\int_\Omega |\nabla u|^p + |u|^p}{\int_\Omega |u|^p},$$

$$\frac{\min\{\alpha, V^-\}}{\rho^+} \frac{\int_\Omega |\nabla u|^p + |u|^p}{\int_\Omega |u|^p} \le \frac{\int_\Omega \Phi(x, \nabla u) + V_\varepsilon |u|^p}{\int_\Omega \rho_\varepsilon |u|^p} \le \frac{\max\{\beta, V^+\}}{\rho^-} \frac{\int_\Omega |\nabla u|^p + |u|^p}{\int_\Omega |u|^p},$$

from where it follows that

$$\frac{\min\{\alpha, \bar V\}}{\bar\rho}\mu_k \le \lambda_k \le \frac{\max\{\beta, \bar V\}}{\bar\rho}\mu_k, \qquad \frac{\min\{\alpha, V^-\}}{\rho^+}\mu_k \le \lambda_k^\varepsilon \le \frac{\max\{\beta, V^+\}}{\rho^-}\mu_k,$$

where μ_k is the k−th eigenvalue of

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \mu |u|^{p-2}u & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.3.15}$$

Observe that $u \in W^{1,p}(\Omega)$ is solution of (6.3.15) if and only if u is solution of

$$\begin{cases} -\Delta_p u = \tilde\mu |u|^{p-2}u & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\tilde{\mu} = \mu - 1$, which satisfies that [GAP88]

$$\tilde{\mu}_k \leq Ck^{p/N}, \tag{6.3.16}$$

and so the proof is complete in the Neumann boundary condition case.

The main difference in the Dirichlet case is the fact that in the variational characterization (6.3.4) of the eigenvalues, functions are taken in $W_0^{1,p}(\Omega)$ instead of $W^{1,p}(\Omega)$. This leads to use Theorem (5.6) instead Theorem 5.4 to estimate the oscillating integrals.

Being functions belonging to $W_0^{1,p}(\Omega)$ we can apply Theorem 5.4 and obtain an analogous equation to (6.3.7)

$$\frac{\int_\Omega \Phi(x,\nabla u) + V_\varepsilon |u|^p}{\bar{\rho}\int_\Omega |u|^p} \leq \frac{\int_\Omega \Phi(x,\nabla u) + \bar{V}|u|^p}{\bar{\rho}\int_\Omega |u|^p} + C\varepsilon \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\bar{\rho}\int_\Omega |u|^p}. \tag{6.3.17}$$

Moreover, instead (6.3.11) we have

$$\frac{\bar{\rho}\int_\Omega |u|^p}{\int_\Omega \rho_\varepsilon |u|^p} \leq 1 + C\varepsilon \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\int_\Omega \rho_\varepsilon |u|^p}, \tag{6.3.18}$$

Now, the only difference appears in the estimate of the quotients $\|\nabla u\|^p_{L^p(\Omega)}/\bar{\rho}\int_\Omega |u|^p$ and $\|\nabla u\|^p_{L^p(\Omega)}/\int_\Omega \rho_\varepsilon |u|^p$.

By (2.5.2), (2.4.2) we get

$$\begin{aligned} \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\int_\Omega \rho_\varepsilon |u|^p} &\leq \frac{\bar{\rho}}{\rho^-} \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\int_\Omega \bar{\rho}|u|^p} \\ &\leq \frac{\bar{\rho}}{\rho^-}\frac{1}{\alpha} \frac{\int_\Omega \Phi(x,\nabla u) + \bar{V}|u|^p}{\int_\Omega \bar{\rho}|u|^p} \\ &\leq \frac{\bar{\rho}}{\rho^-}\frac{1}{\alpha}(\lambda_k + O(\delta)). \end{aligned} \tag{6.3.19}$$

and

$$\frac{\|\nabla u\|^p_{L^p(\Omega)}}{\bar{\rho}\int_\Omega |u|^p} \leq \frac{\rho^+}{\bar{\rho}} \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\int_\Omega \rho_\varepsilon |u|^p}. \tag{6.3.20}$$

Changing equations (6.3.7), (6.3.11), (6.3.8) and (6.3.12) by (6.3.17), (6.3.18), (6.3.19) and (6.3.20) the result in the Dirichlet case follows. □

Remark 6.8. Observe that the result holds even for $V \equiv 0$. Actually, in this case u is solution of equation (6.3.1) (either with Dirichlet or Neumann boundary conditions) if and only if u is solution of

$$-div(a(x,\nabla u^\varepsilon)) + \rho_\varepsilon |u|^{p-2}u = \tilde{\lambda}^\varepsilon \rho_\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon \quad \text{in } \Omega$$

with the corresponding boundary condition, where $\tilde{\lambda}^\varepsilon = \lambda^\varepsilon + 1$.

In relation to equation (6.3.1) with Dirichlet boundary condition we make the following remarks.

Remark 6.9. As we mentioned in the introduction, in the linear case and in one space dimension Castro and Zuazua [CZ00b] prove that, for $k < C\varepsilon^{-1}$,

$$|\lambda_k^\varepsilon - \lambda_k| \leq Ck^4\varepsilon.$$

If we specialize our result to this case, we get the same bound. The advantage of our method is that very low regularity on ρ is needed (only L^∞). However, the method in [CZ00b], making use of the linearity of the problem, gives precise information about the behavior of the eigenfunctions u_k^ε.

Remark 6.10. In [KLS11], Kenig, Lin and Shen studied the linear case in any space dimension (allowing a periodic oscillation diffusion matrix) and prove the bound

$$|\lambda_k^\varepsilon - \lambda_k| \leq C\varepsilon|\log\varepsilon|^{1+\sigma}.$$

for some $\sigma > 0$ and C depending on σ and k. The authors can get rid off the logarithmic term assuming more regularity on Ω). If we specialize our result to this case, we cannot treat an ε dependance on the operator, but we get an explicit dependance on k on the estimate and assuming very low regularity on Ω (Lipschitz is more than enough) we get a better dependance on ε.

7

Eigenvalue homogenization for quasilinear elliptic operators in one space dimension

7.1 Introduction

In this Chapter we study the asymptotic behavior (as $\varepsilon \to 0$) of the eigenvalues of the following problem

$$\begin{cases} -(a(\frac{x}{\varepsilon})|(u^\varepsilon)'|^{p-2}(u^\varepsilon)')' = \lambda^\varepsilon \rho(\frac{x}{\varepsilon})|u^\varepsilon|^{p-2}u^\varepsilon & \text{in } I := (0,1) \\ u^\varepsilon(0) = u^\varepsilon(1) = 0, \end{cases} \tag{7.1.1}$$

where the diffusion coefficient $a(x)$ and the weight function $\rho(x)$ are $1-$periodic functions, bounded away from zero and infinity and $\varepsilon > 0$ is a real parameter. In this Chapter we will denote by I to the unit interval $(0, 1)$.

This type of problems have been considered extensively in the literature due to its many applications in different fields.

Homogenization of one-dimensional periodic linear problems was studied in the late 60's by Spagnolo [Sp68] and De Giorgi [DGSP73] and generalized to the linear multi-dimensional case in the mid-70's by Sanchez-Palencia [SP70], Bensoussan, Lions and Papanicolaou [BLP78] among others. Likewise, the study of eigenvalue problems with oscillating coefficients started with the works of Boccardo and Marcellini [BM76], and Kesavan [Ke79b, Ke79]. See Chapter §4.

Problem (7.1.1) has a natural limit problem as $\varepsilon \to 0$ given by

$$\begin{cases} -(a_p^*|u'|^{p-2}u')' = \lambda\bar{\rho}|u|^{p-2}u & \text{in } I \\ u(0) = u(1) = 0 \end{cases} \tag{7.1.2}$$

where $\bar{\rho}$ is the average of ρ in the interval I and a_p^* is given by

$$a_p^* := \Big(\int_I \frac{1}{a(s)^{1/(p-1)}}\,ds\Big)^{-(p-1)},$$

see Proposition 4.27.

Now, what we are interested in is on the convergence of the eigenvalues of problem (7.1.1) to the ones of problem (7.1.2); more specifically, on the order of convergence of the eigenvalues, i.e. we find explicit bounds on ε and k for the difference

$$|\lambda_k^\varepsilon - \lambda_k|$$

where λ_k^ε and λ_k are the $k-$th eigenvalue of problem (7.1.1) and (7.1.2) respectively.

In the Chapter §6 we have dealt of the $N-$dimensional case for the quasilinear problem with diffusion coefficients independent of ε, and we obtain the bound

$$|\lambda_k^\varepsilon - \lambda_k| \leq Ck^{\frac{2p}{N}}\varepsilon$$

with C independent on k and ε for any Lipschitz domain. The constant C is unknown.

It is expected that in the one dimensional case one can be more precise with the estimates. In fact, Castro and Zuazua in [CZ00, CZ00b], for the linear problem using the so-called WKB method which relays on asymptotic expansions of the solutions of the problem, and the explicit knowledge of the eigenfunctions and eigenvalues of the constant coefficient limit problem, proved

$$|\lambda_k^\varepsilon - \lambda_k| \leq Ck^4\varepsilon$$

and they also presented a variety of results on correctors for the eigenfunction approximation. Let us mention that their method needs higher regularity on the weight ρ and on the diffusion a, which must belong at least to C^2 and that the bound holds for $k \sim \varepsilon^{-1}$. Also, the value of the constant C entering in the estimate is unknown.

The main result of this Chapter is the following Theorem:

Theorem 7.1. *There exists a constant C depending only on p, a and ρ such that*

$$|\lambda_k^\varepsilon - \lambda_k| \leq Ck^{2p}\varepsilon.$$

Moreover, C can be estimated explicitly in terms of the functions a and ρ, and p.

A useful tool used in the proof of Theorem 7.1 is the variational characterization of the eigenvalues of (7.1.1) and (7.1.2). Also, it will be essential that the *variational* eigenvalues, for the one dimensional problem, exhaust the whole spectrum of (7.1.1) and (7.1.2). These facts are collected in Section 2.5.

In the course of our arguments, a general result on the convergence of eigenvalues is used. Namely, we prove that the eigenvalues of

$$\begin{cases} -(a_\varepsilon(x)|(u^\varepsilon)'|^{p-2}(u^\varepsilon)')' = \lambda^\varepsilon\rho_\varepsilon(x)|u^\varepsilon|^{p-2}u^\varepsilon & \text{in } I \\ u^\varepsilon(0) = u^\varepsilon(1) = 0, \end{cases} \tag{7.1.3}$$

converges to the ones of the limit problem

$$\begin{cases} -(a_h(x)|(u^\varepsilon)'|^{p-2}(u^\varepsilon)')' = \lambda^\varepsilon\rho_h(x)|u^\varepsilon|^{p-2}u^\varepsilon & \text{in } I \\ u^\varepsilon(0) = u^\varepsilon(1) = 0, \end{cases} \tag{7.1.4}$$

where ρ_h is the weak* limit of ρ_ε and a_h is the $G-$limit of a_ε.

In the linear case ($p = 2$), and in $N-$dimensional space, Kesavan in [Ke79b, Ke79] proved that if a_ε $G-$converges to a_h and $\rho_\varepsilon \rightharpoonup \rho_h$ weakly* in L^∞ then the sequence of the $k-$th eigenvalues of (7.1.1) converges to the $k-$th eigenvalue of (7.1.2).

In the general quasilinear setting, for $N-$dimensional space, the first result we are aware of is by Baffico, Conca and Rajesh, [BCR06], where the authors prove that the limit of any convergent sequence of eigenvalues of (7.1.1) is an eigenvalue of (7.1.2) and, moreover, that the sequence of the first eigenvalues of (7.1.1) converges to the first eigenvalue of (7.1.2).

In Chapter §6 we studied the same problem, again in $N-$dimensional space, and prove that the first and second eigenvalues of (7.1.1) converges to those of the limit operator (7.1.2). Moreover, when the diffusion coefficient a_ε is independent of ε, we prove that the sequence of the $k-$th *variational* eigenvalues of (7.1.1) converges to the $k-$th *variational* eigenvalue of (7.1.2).

In one space dimension one can be more precise and we can prove the following

Theorem 7.2. *Assume that a_ε $G-$converges to a_h and that $\rho_\varepsilon \rightharpoonup \rho_h$ weakly* in $L^\infty(I)$.*

For each $k \geq 1$ let λ_k^ε be the k-th eigenvalue of (7.1.1). Then we have that

$$\lim_{\varepsilon\to 0} \lambda_k^\varepsilon = \lambda_k,$$

where λ_k the $k-$th eigenvalue of (7.1.2).

Moreover, up to a subsequence, an eigenfunction u_k^ε associated to λ_k^ε converges weakly in $W_0^{1,p}(I)$ to u_k, an eigenfunction associated to λ_k.

7.2 Convergence of eigenvalues

In order to prove Theorem 7.2 we need some preliminaries.

From Chapter §2, according to Theorem 2.28, we denote by $\Sigma_\varepsilon := \{\lambda_k^\varepsilon\}_{k\in\mathbb{N}}$ the full sequence of eigenvalues of problem (7.1.3) and by $\Sigma_h := \{\lambda_k\}_{k\in\mathbb{N}}$ those of its limit problem (7.1.4). They can be written as

$$\lambda_k^\varepsilon = \inf_{C\in\Gamma_k} \sup_{u\in C} \frac{\int_I a_\varepsilon(x)|u'|^p}{\int_I \rho_\varepsilon(x)|u|^p}, \qquad \lambda_k = \inf_{C\in\Gamma_k} \sup_{u\in C} \frac{\int_I a_h(x)|u'|^p}{\int_I \rho_h(x)|u|^p}.$$

We assume that a and ρ are 1−periodic functions defined on I such that for some constants $\alpha < \beta, \rho_- < \rho_+$,

$$\begin{aligned} &0 < \alpha \leq a(x) \leq \beta < +\infty && \text{a.e. } I \\ &0 < \rho^- \leq \rho(x) \leq \rho^+ < +\infty && \text{a.e. } I. \end{aligned} \tag{7.2.1}$$

We begin by stating a general result for bounded sequences of eigenvalues that can be found in [BCR06] (see also Theorem 7.3, where a simplified proof of this result is given).

Theorem 7.3. *Let $\lambda^\varepsilon \in \Sigma_\varepsilon$ be a sequence of eigenvalues of the problem (7.1.3) with $\{u^\varepsilon\}_{\varepsilon>0}$ associated normalized eigenfunctions. Assume that the sequence of eigenvalues is convergent*

$$\lim_{\varepsilon\to 0+} \lambda^\varepsilon = \lambda.$$

Then, $\lambda \in \Sigma_h$ and there exists a sequence $\varepsilon_j \to 0$ such that

$$u^{\varepsilon_j} \rightharpoonup u \text{ weakly in } W_0^{1,p}(I)$$

with u a normalized eigenfunction associated to λ.

Assume now that we take the family of the $k-$th eigenvalue of (7.1.1) $\{\lambda_k^\varepsilon\}_{\varepsilon>0}$. It is not difficult to see that this family is bounded, in fact as

$$\frac{\alpha}{\rho_+}\frac{\int_0^1 |v'|^p}{\int_0^1 |v|^p} \le \frac{\int_0^1 a_\varepsilon(x)|v'|^p}{\int_0^1 \rho_\varepsilon(x)|v|^p} \le \frac{\beta}{\rho_-}\frac{\int_0^1 |v'|^p}{\int_0^1 |v|^p},$$

we have

$$\frac{\alpha}{\rho_+}\mu_k \le \lambda_k^\varepsilon \le \frac{\beta}{\rho_-}\mu_k \tag{7.2.2}$$

where $\mu_k = \pi_p^p k^p$ is the $k-$th eigenvalue of the one dimensional $p-$Laplacian (see Chapter §2, Theorem 2.10).

Therefore, up to a subsequence, λ_k^ε converges to $\lambda \in \Sigma_h$. The main tool that allows us to prove that $\lambda = \lambda_k$ is Theorem 2.27 that says that any eigenfunction associated to the $k-$th eigenvalue of (7.1.1) has exactly k nodal domains.

Moreover, we need a refinement of this result, namely an estimate on the measure of each nodal domain independent on ε. This is the content of the next Lemma.

Lemma 7.4. *Let λ_k^ε be a eigenvalue of (7.1.3) with corresponding eigenfunction u_k^ε. Let $\mathcal{N} = \mathcal{N}(k,\varepsilon)$ be a nodal domain of u_k^ε. We have that*

$$|\mathcal{N}| > C$$

where $C = C(k)$ is a positive constant independent of ε.

Proof. We can write λ_k^ε as

$$\lambda_k^\varepsilon(I) = \lambda_1^\varepsilon(\mathcal{N}) = \inf_{u\in W_0^{1,p}(\mathcal{N})} \frac{\int_{\mathcal{N}} a_\varepsilon(x)|u'|^p}{\int_{\mathcal{N}} \rho_\varepsilon(x)|u|^p},$$

by our assumptions (2.5.2) we get

$$\lambda_k^\varepsilon(I) \ge \frac{\alpha}{\rho_+}\mu_1(\mathcal{N}) = \frac{\alpha}{\rho_+}\frac{\pi_p^p}{|\mathcal{N}|^p}$$

where $\mu_1(\mathcal{N})$ is the first eigenvalue of the $p-$Laplacian on $\mathcal{N}$. Moreover,

$$\lambda_k^\varepsilon(I) \le \frac{\beta}{\rho_-}\mu_k(I) = \frac{\beta}{\rho_-}\pi_p^p k^p.$$

Combining both inequalities we get

$$|\mathcal{N}|^p \geq \frac{\alpha}{\rho_+}\frac{\pi_p^p}{\lambda_k^\varepsilon(\Omega)} \geq \frac{\alpha}{\beta}\frac{\rho_-}{\rho_+}\frac{1}{k^p}$$

and the result follows. □

Now we are ready to establish the main result of this section:

Proof of Theorem 7.2. Let u_k be a normalized eigenfunction associated to λ_k and according to Theorem 2.27, let I_i, $i = 1, \ldots, k$ be the nodal domains of u_k.

We denote by u_i^ε the first eigenfunction of (7.1.3) in I_i respectively. Extending u_i^ε to I by 0, these function have disjoint supports and therefore they are linearly independent in $W_0^{1,p}(I)$.

Let S be the unit sphere in $W_0^{1,p}(I)$ and we define the set C_k^ε as

$$C_k^\varepsilon := \operatorname{span}\{u_1^\varepsilon, \ldots, u_k^\varepsilon\} \cap S.$$

Clearly C_k^ε is compact, symmetric and $\gamma(C_k^\varepsilon) = k$. Hence,

$$\lambda_k^\varepsilon = \inf_{C\in\Gamma_k}\sup_{v\in C}\frac{\int_I a_\varepsilon(x)|v'|^p}{\int_I \rho_\varepsilon|v|^p} \leq \sup_{v\in C_k^\varepsilon}\frac{\int_I a_\varepsilon(x)|v'|^p}{\int_I \rho_\varepsilon|v|^p}.$$

As C_k^ε is compact, the supremum is achieved for some $v^\varepsilon \in C_k^\varepsilon$ which can be written as

$$v^\varepsilon = \sum_{i=1}^k a_i^\varepsilon u_i^\varepsilon$$

with $a_i^\varepsilon \in \mathbb{R}$ such that $\sum_{i=1}^k |a_i^\varepsilon|^p = 1$. Since the functions u_i^ε have non-overlapping supports, we obtain

$$\lambda_k^\varepsilon \leq \frac{\int_I a_\varepsilon(x)|v^{\varepsilon\prime}|^p}{\int_I \rho_\varepsilon|v^\varepsilon|^p} = \frac{\sum_{i=1}^k |a_i^\varepsilon|^p \int_{I_i} a_\varepsilon(x)|u_i^{\varepsilon\prime}|^p}{\int_I \rho_\varepsilon|v^\varepsilon|^p}.$$

Using the definition of u_i^ε, the above inequality can be rewritten as

$$\lambda_k^\varepsilon \leq \frac{\sum_{i=1}^k |a_i^\varepsilon|^p \lambda_{1,i}^\varepsilon \int_{I_i} \rho_\varepsilon|u_i^\varepsilon|^p}{\int_I \rho_\varepsilon|v^\varepsilon|^p} \leq \max_{1\leq i\leq k}\{\lambda_{1,i}^\varepsilon\} \tag{7.2.3}$$

where $\lambda_{1,i}^\varepsilon$ is the first eigenvalue of (7.1.1) in the nodal domain Ω_i respectively.

Now, using that $\lambda_{1,i}^\varepsilon \to \lambda_{1,i}$ respectively , where $\lambda_{1,i}$ are the first eigenvalues of (7.1.2) in the domains I_i respectively (see Theorem 4.4, [FBPS12]). Moreover, we observe that these eigenvalues $\lambda_{1,i}$ are all equal to the $k-$th eigenvalue λ_k in I, therefore from (7.2.3), we get

$$\lambda_k^\varepsilon \leq \lambda_k + \delta$$

for δ arbitrarily small and ε tending to zero. So

$$\limsup_{\varepsilon\to 0}\lambda_k^\varepsilon \leq \lambda_k. \tag{7.2.4}$$

On the other hand, suppose that $\lim_{\varepsilon\to 0}\lambda_k^\varepsilon = \lambda$. By Lemma 7.4 the k nodal domains of u_k^ε have positive measure independent of ε. Then it must be $\lambda \geq \lambda_k$. It follows that

$$\lambda_k \leq \lambda = \lim_{\varepsilon\to 0}\lambda_k^\varepsilon. \tag{7.2.5}$$

Combining (7.2.4) and (7.2.5) we obtain the desired result. □

7.3 Rates of convergence. The periodic case

In this section, we focus on the limit behavior of eigenvalues of

$$-(a(\tfrac{x}{\varepsilon})|(u^\varepsilon)'|^{p-2}(u^\varepsilon)')' = \lambda^\varepsilon\rho(\tfrac{x}{\varepsilon})|u^\varepsilon|^{p-2}u^\varepsilon \quad \text{in } I \tag{7.3.1}$$

either with Dirichlet or Neumann boundary conditions, where a and ρ are $1-$periodic functions satisfying (7.2.1). In fact, from the results of Section 7.2, it follows that the $k-$th eigenvalue of (7.3.1) converges to the $k-$th eigenvalue of the limit problem

$$-(a_p^*|(u)'|^{p-2}(u)')' = \lambda\bar{\rho}|u|^{p-2}u \quad \text{in } I \tag{7.3.2}$$

with the corresponding boundary condition.

In order to clarify the statement of the main Theorem we introduce the following notation:

Definition 7.5. Let $g:\mathbb{R}\to\mathbb{R}$ be a measurable function such that $0 < g^- \leq g \leq g^+ < \infty$. We define the oscillation of g as

$$\omega_g = \frac{g^+}{g^-}.$$

The main results of this Section are the following Theorems:

Theorem 7.6. *Let λ_k^ε be the $k-$th eigenvalue of problem (7.3.1) with Dirichlet boundary conditions and let λ_k be the $k-$th eigenvalue of its limit problem (7.3.2). Then we have*

$$\begin{aligned}|\lambda_k^\varepsilon - \lambda_k| \leq & (a_p^*)^{p/(p-1)}\frac{\omega_\rho\omega_a^{1/(p-1)}-1}{\rho^-\alpha^{1/(p-1)}}\omega_\rho\omega_a^{1/(p-1)}\pi_p^{2p}\varepsilon k^{2p}\Big[\frac{p}{\pi_p^{p-1}}+\frac{\varepsilon^{p-1}}{p}\Big] \\ &+\frac{\beta}{\rho_-}p\pi_p^p k^p\varepsilon(1+\varepsilon)^{p-1}C(\varepsilon),\end{aligned}$$

where ω_ρ and ω_a are given by Definition 7.5 and

$$C(\varepsilon) = \begin{cases} 0 & \textit{if } \varepsilon = 1/j,\ j\in\mathbb{N} \\ 1 & \textit{else}. \end{cases} \tag{7.3.3}$$

Remark 7.7. As a consequence of Theorem 7.6 we obtain the rate of convergence of the nodal domains of the eigenfunctions of problem (7.3.1) to those of the limit problem, c.f. Theorem 7.15.

We state a simple corollary of Theorem 7.6.

Corollary 7.8. *Let* $C(p,a,\rho) = (a_p^*)^{p/(p-1)} \frac{\omega_\rho \omega_a^{1/(p-1)} - 1}{\rho^- \alpha^{1/(p-1)}} \omega_\rho \omega_a^{1/(p-1)} p\pi_p^{p+1}$ *then for* $\varepsilon < \varepsilon_0$ *we have*

$$|\lambda_k^\varepsilon - \lambda_k| \leq 2C(p,a,\rho)\varepsilon k^{2p}. \tag{7.3.4}$$

Remark 7.9. The constant 2 in (7.3.4) can be replaced by any other constant greater than 1.

Remark 7.10. Corollary 7.8 is exactly Theorem 7.1.

We obtain similar results for the Neumann problem.

Theorem 7.11. *Let* λ_k^ε *be the* $k-$*th eigenvalue of problem* (7.3.1) *with Neumann boundary conditions and let* λ_k *be the* $k-$*th eigenvalue of its limit problem* (7.3.2)*. Then we have*

$$\begin{aligned} |\lambda_k^\varepsilon - \lambda_k| \leq & 4(a_p^*)^{p/(p-1)} c\varepsilon p \max\{\tfrac{1}{\rho^-}\alpha^{\frac{1}{1-p}}, \omega_\rho \omega_a^{\frac{1}{p-1}}\}^3 \pi_p^{2p}(k-1)^{2p} \\ & + \frac{\beta}{\rho_-} p\pi_p^p (k-1)^p \varepsilon (1+\varepsilon)^{p-1} C(\varepsilon), \end{aligned}$$

where ω_ρ *and* ω_a *are given by Definition 7.5,*

$$c \leq \beta\left(4 + \frac{p}{(p-1)\pi_p}\right)$$

and $C(\varepsilon)$ *is given in* (7.3.3)*.*

Corollary 7.12. *Let*

$$C(p,a,\rho) = 4p\beta\left(4 + \frac{p}{(p-1)\pi_p}\right)(a_p^*)^{p/(p-1)} \max\{\tfrac{1}{\rho^-}\alpha^{\frac{1}{1-p}}, \omega_\rho \omega_a^{\frac{1}{p-1}}\}^3 \pi_p^{2p}$$

then for $\varepsilon < \varepsilon_0$ *we have*

$$|\lambda_k^\varepsilon - \lambda_k| \leq 2C(p,a,\rho)\varepsilon(k-1)^{2p}. \tag{7.3.5}$$

7.3.1 Proof of Theorem 7.6. The case $a = 1$

In order to deal with Theorem 7.6 we first analyze the case where the diffusion coefficient is equal to 1 and then show how the general case can be reduced to this one.

Theorem 7.13. *Let* $g \in L^\infty(\mathbb{R})$ *be a* $1-$*periodic function such that*

$$0 < g^- \leq g \leq g^+ < \infty. \tag{7.3.6}$$

Consider the eigenvalue problem

$$\begin{cases} -(|u'|^{p-2}u')' = g(\frac{x}{\varepsilon})\lambda^\varepsilon |u|^{p-2}u & \text{in } I \\ u(0) = u(1) = 0 \end{cases} \tag{7.3.7}$$

and its limit problem

$$\begin{cases} -(|u'|^{p-2}u')' = \bar{g}\lambda |u|^{p-2}u & \text{in } I \\ u(0) = u(1) = 0. \end{cases} \tag{7.3.8}$$

Let $\{\lambda_k^\varepsilon\}_{k\geq 1}$ and $\{\lambda_k\}_{k\geq 1}$ be the eigenvalues of (7.3.7) and (7.3.8) respectively.

Then, we have

$$|\lambda_k^\varepsilon - \lambda_k| \leq \frac{g^+ - \bar{g}}{g^- \bar{g}} \frac{g^+}{g^-} \pi_p^{2p} \varepsilon k^{2p} \Big[\frac{p}{\pi_p^{p-1}} + \frac{\varepsilon^{p-1}}{p} \Big].$$

Proof. The proof follows from Theorem 5.5. In fact, for every $\delta > 0$, let $G_\delta^k \in \Gamma_k$ be such that

$$\lambda_k = \frac{\pi_p^p k^p}{\bar{g}} = \inf_{G\in\Gamma_k} \sup_{v\in G} \frac{\int_I |v'|^p}{\bar{g}\int_I |v|^p} = \sup_{v\in G_\delta^k} \frac{\int_I |v'|^p}{\bar{g}\int_I |v|^p} + O(\delta). \tag{7.3.9}$$

We bound the eigenvalues of (7.3.7) as follows

$$\begin{aligned}
\lambda_k^\varepsilon &= \inf_{G\in\Gamma_k} \sup_{v\in G} \frac{\int_I |v'|^p}{\int_I g(\frac{x}{\varepsilon})|v|^p} \\
&\leq \sup_{v\in G_\delta^k} \frac{\int_I |v'|^p}{\bar{g}\int_I |v|^p} \frac{\bar{g}\int_I |v|^p}{\int_I g(\frac{x}{\varepsilon})|v|^p} \\
&\leq (\lambda_k + O(\delta)) \sup_{v\in G_\delta^k} \frac{\bar{g}\int_I |v|^p}{\int_I g(\frac{x}{\varepsilon})|v|^p}.
\end{aligned} \tag{7.3.10}$$

Now, by Theorem 5.5 we bound the quotient

$$\frac{\bar{g}\int_I |v|^p}{\int_I g(\frac{x}{\varepsilon})|v|^p} \leq 1 + (g^+ - \bar{g})\varepsilon \Big[\frac{p}{\pi_p^{p-1}} + \frac{\varepsilon^{p-1}}{p} \Big] \frac{\int_I |v'|^p}{\int_I g(\frac{x}{\varepsilon})|v|^p}. \tag{7.3.11}$$

By (7.3.6) we have, as $v \in G_\delta^k$,

$$\frac{\int_I |v'|^p}{\int_I g(\frac{x}{\varepsilon})|v|^p} \leq \frac{\bar{g}}{g^-} \frac{\int_I |v'|^p}{\int_I \bar{g}|v|^p} \leq \frac{\bar{g}}{g^-}(\lambda_k + O(\delta)). \tag{7.3.12}$$

By using (7.3.12) we bound (7.3.11) as

$$1 + \frac{(g^+ - \bar{g})\bar{g}}{g^-} \varepsilon \Big[\frac{p}{\pi_p^{p-1}} + \frac{\varepsilon^{p-1}}{p} \Big] (\lambda_k + O(\delta)). \tag{7.3.13}$$

Finally, combining (7.3.10), (7.3.11) and (7.3.13) and letting $\delta \to 0$ we find that

$$\lambda_k^\varepsilon - \lambda_k \leq \frac{g^+ - \bar{g}}{g^-} \bar{g} \varepsilon \lambda_k^2 \Big[\frac{p}{\pi_p^{p-1}} + \frac{\varepsilon^{p-1}}{p} \Big]. \tag{7.3.14}$$

Arguing in much the same way, we get the inequality

$$\lambda_k - \lambda_k^\varepsilon \leq \frac{g^+ - \bar{g}}{\bar{g}} g^+ \varepsilon (\lambda_k^\varepsilon)^2 \Big[\frac{p}{\pi_p^{p-1}} + \frac{\varepsilon^{p-1}}{p} \Big]. \tag{7.3.15}$$

Now, from (7.3.6) and the variational characterization of the eigenvalues, we get the estimates

$$\frac{\bar{g}}{g^+}\lambda_k \leq \lambda_k^\varepsilon \leq \frac{\bar{g}}{g^-}\lambda_k. \tag{7.3.16}$$

Using (7.3.16) in (7.3.15), together with (7.3.14), we obtain

$$|\lambda_k^\varepsilon - \lambda_k| \leq \frac{g^+ - \bar{g}}{g^-}\frac{g^+}{g^-}\bar{g}\varepsilon\lambda_k^2\Big[\frac{p}{\pi_p^{p-1}} + \frac{\varepsilon^{p-1}}{p}\Big] \tag{7.3.17}$$

and from the explicit form of the eigenvalues

$$\lambda_k = \frac{\pi_p^p k^p}{\bar{g}},$$

we arrive at

$$|\lambda_k^\varepsilon - \lambda_k| \leq \frac{g^+ - \bar{g}}{g^-\bar{g}}\frac{g^+}{g^-}\pi_p^{2p}\varepsilon k^{2p}\Big[\frac{p}{\pi_p^{p-1}} + \frac{\varepsilon^{p-1}}{p}\Big],$$

as we wanted to proved. □

Remark 7.14. If we replace the unit interval $I = (0, 1)$ by $I_\ell = (0, \ell)$ by a simple change of variables, the estimates of Theorem 7.13 are modified as

$$|\lambda_k^\varepsilon(I_\ell) - \lambda_k(I_\ell)| = \ell^p|\lambda_k^\varepsilon(I) - \lambda_k(I)|. \tag{7.3.18}$$

7.3.2 Proof of Theorem 7.6. The general case

Now we are ready to prove the main result of the section, namely Theorem 7.6

Proof of Theorem 7.6. The proof of the Theorem follows by converting problem (7.1.1) into (7.3.7) by a change of variables.

In fact, if we define

$$P_\varepsilon(x) = \int_0^x \frac{1}{a_\varepsilon(s)^{1/(p-1)}}ds = \varepsilon\int_0^{x/\varepsilon}\frac{1}{a(s)^{1/(p-1)}}ds = \varepsilon P(\tfrac{x}{\varepsilon})$$

and perform the change of variables

$$(x, u) \to (y, v) \tag{7.3.19}$$

where

$$y = P_\varepsilon(x) = \varepsilon P(\tfrac{x}{\varepsilon}), \qquad v(y) = u(x).$$

By simple computations we get

$$\begin{cases} -(|\dot{v}|^{p-2}\dot{v})^{\cdot} = \lambda^\varepsilon Q_\varepsilon(y)|v|^{p-2}v, & y \in [0, L_\varepsilon] \\ v(0) = v(L_\varepsilon) = 0 \end{cases}$$

where

$$\cdot = d/dy,$$

with

$$L_\varepsilon = \int_0^1 \frac{1}{a_\varepsilon(s)^{1/(p-1)}} ds \to L = \overline{a^{\frac{-1}{p-1}}},$$

and

$$\begin{aligned} Q_\varepsilon(y) =& a_\varepsilon(x)^{1/(p-1)}\rho_\varepsilon(x) \\ =& a(P^{-1}(\tfrac{y}{\varepsilon}))^{1/(p-1)}\rho(P^{-1}(\tfrac{y}{\varepsilon})) \\ =& Q(\tfrac{y}{\varepsilon}). \end{aligned}$$

Observe that Q is an $L-$periodic function.

Moreover, it is easy to see that

$$|L_\varepsilon - L| \le \varepsilon L \tag{7.3.20}$$

and that $L_\varepsilon = L$ if $\varepsilon = 1/j$ for some $j \in \mathbb{N}$.

In order to apply Theorem 7.13 we need to rescale to the unit interval. So we define

$$w(z) = v(L_\varepsilon z), \qquad z \in I$$

and get

$$\begin{cases} -(|\dot{w}|^{p-2}\dot{w})^{\cdot} = L_\varepsilon^p \lambda^\varepsilon Q_\varepsilon(L_\varepsilon z)|w|^{p-2}w & \text{in } I \\ w(0) = w(1) = 0. \end{cases}$$

So if we denote $\delta = \varepsilon L/L_\varepsilon$, $\mu^\delta = L_\varepsilon^p \lambda^\varepsilon$ and $g(z) = Q(Lz)$, we get that g is a $1-$periodic function and that w verifies

$$\begin{cases} -(|\dot{w}|^{p-2}\dot{w})^{\cdot} = \mu^\delta g(\tfrac{z}{\delta})|w|^{p-2}w & \text{in } I \\ w(0) = w(1) = 0. \end{cases}$$

Now we can apply Theorem 7.13 to the eigenvalues μ^δ to get

$$|\mu_k^\delta - \mu_k| \le \frac{g^+ - \bar{g}}{g^- \bar{g}} \frac{g^+}{g^-} \pi_p^{2p} \delta k^{2p} \Big[\frac{p}{\pi_p^{p-1}} + \frac{\varepsilon^{p-1}}{p}\Big]. \tag{7.3.21}$$

In the case where $\varepsilon = 1/j$ with $j \in \mathbb{N}$ we directly obtain

$$|\lambda_k^\varepsilon - \lambda_k| \le \frac{1}{L^p} \frac{g^+ - \bar{g}}{g^- \bar{g}} \frac{g^+}{g^-} \pi_p^{2p} \varepsilon k^{2p} \Big[\frac{p}{\pi_p^{p-1}} + \frac{\varepsilon^{p-1}}{p}\Big].$$

Now we observe that $L^{-p} = (a_p^*)^{p/(p-1)}$ and that we have the bounds

$$\rho^- \alpha^{\frac{1}{p-1}} \le g^- \le g \le g^+ \le \rho^+ \beta^{\frac{1}{p-1}}, \tag{7.3.22}$$

$$\alpha^{\frac{1}{p-1}} \bar{\rho} \le \bar{g} \le \beta^{\frac{1}{p-1}} \bar{\rho}. \tag{7.3.23}$$

Therefore, we get

$$|\lambda_k^\varepsilon - \lambda_k| \le (a_p^*)^{\frac{p}{p-1}} \frac{\frac{\rho^+}{\bar{\rho}}(\frac{\beta}{\alpha})^{1/(p-1)} - 1}{\rho^- \alpha^{1/(p-1)}} \frac{\rho^+}{\rho^-} \Big(\frac{\beta}{\alpha}\Big)^{1/(p-1)} \pi_p^{2p} \varepsilon k^{2p} \Big[\frac{p}{\pi_p^{p-1}} + \frac{\varepsilon^{p-1}}{p}\Big].$$

In the general case, one has to measure the defect between L and L_ε. So,

$$
\begin{aligned}
|\lambda_k^\varepsilon - \lambda_k| &\le \frac{1}{L^p}(|\mu_k^\delta - \mu_k| + \lambda_k^\varepsilon |L_\varepsilon^p - L^p|)\\
&\le (a_p^*)^{\frac{p}{p-1}}(|\mu_k^\delta - \mu_k| + \frac{\beta}{\rho_-}\pi_p^p k^p |L_\varepsilon^p - L^p|).
\end{aligned}
$$

From (7.3.20) it is easy to see that

$$|(\tfrac{L_\varepsilon}{L})^p - 1| \le p(1+\varepsilon)^{p-1}\varepsilon.$$

so

$$|L_\varepsilon^p - L^p| = L^p|(\tfrac{L_\varepsilon}{L})^p - 1| \le (1+\varepsilon)^{p-1}\frac{p}{(a_p^*)^{p/(p-1)}}\varepsilon. \tag{7.3.24}$$

Finally, using (7.3.21), (7.3.22) and (7.3.23) we obtain the desired result. □

7.3.3 Convergence of nodal domains

To finish with this section, as a consequence of Theorem 7.1, we prove a result about the convergence of the nodal sets and of the zeroes of the eigenfunctions.

Theorem 7.15. *Let $(\lambda_k^\varepsilon, u_k^\varepsilon)$ and (λ_k, u_k) be eigenpairs associated to equations (7.1.1) and (7.1.2) respectively. We denote by $\mathcal{N}_k^\varepsilon$ and $\mathcal{N}_k$ to a nodal domains of u_k^ε and u_k respectively. Then*

$$|\mathcal{N}_k^\varepsilon| \to |\mathcal{N}_k| \qquad \text{as } \varepsilon \to 0$$

and we have the estimate

$$\Big||\mathcal{N}_k^\varepsilon|^{-p} - |\mathcal{N}_k|^{-p}\Big| \le c\varepsilon(k^{2p}+1).$$

Proof. By using Theorem 7.1, together with (7.3.18) and the explicit form of the eigenvalues of the limit problem we obtain that

$$\lambda_k^\varepsilon(I) = \lambda_1^\varepsilon(\mathcal{N}_k^\varepsilon) \le \lambda_1(\mathcal{N}_k^\varepsilon) + c|\mathcal{N}_k^\varepsilon|^{p-1}\varepsilon \le \frac{\pi_p^p}{\overline{\rho}|\mathcal{N}_k^\varepsilon|^p} + c\varepsilon. \tag{7.3.25}$$

Also,

$$\lambda_k^\varepsilon(I) \ge \lambda_k(I) - ck^{2p}\varepsilon = \frac{k^p\pi_p^p}{\overline{\rho}} - ck^{2p}\varepsilon. \tag{7.3.26}$$

As $u_k(x) = \sin_p(k\pi_p x)$ (see Chapter §2, Theorem 2.10) has k nodal domain in I we must have $|\mathcal{N}_k| = k^{-1}$. Then by (7.3.25) and (7.3.26) we get

$$\frac{\pi_p^p}{\overline{\rho}|\mathcal{N}_k|^p} - ck^{2p}\varepsilon \le \frac{1}{|\mathcal{N}_k^\varepsilon|^p}\frac{\pi_p^p}{\overline{\rho}} + c\varepsilon$$

it follows that

$$|\mathcal{N}_k|^{-p} - |\mathcal{N}_k^\varepsilon|^{-p} \le c\varepsilon(k^{2p}+1). \tag{7.3.27}$$

Similarly we obtain that

$$\frac{\pi_p^p}{\overline{\rho}|\mathcal{N}_k|^p} = \lambda_1(\mathcal{N}_k) = \lambda_k(I) \geq \lambda_k^\varepsilon(I) - c\varepsilon k^{2p} \geq \lambda_1^\varepsilon(\mathcal{N}_k^\varepsilon) - c\varepsilon k^{2p}$$

and using again Theorem 7.1 we get

$$\lambda_1^\varepsilon(\mathcal{N}_k^\varepsilon) \geq \lambda_1(\mathcal{N}_k^\varepsilon) - c\varepsilon = \frac{\pi_p^p}{\overline{\rho}|\mathcal{N}_k^\varepsilon|^p} - c\varepsilon$$

it follows that

$$|\mathcal{N}_k^\varepsilon|^{-p} - |\mathcal{N}_k|^{-p} \leq c\varepsilon(k^{2p} + 1). \tag{7.3.28}$$

Combining (7.3.27) and (7.3.28) the result follows. □

Finally, as a corollary of Theorem 7.15 we are able to prove the individual convergence of the zeroes of the eigenfunctions of (7.1.1) to those of the limit problem (7.1.2).

Corollary 7.16. *Let $(\lambda_k^\varepsilon, u_k^\varepsilon)$ and (λ_k, u_k) be eigenpairs associated to equations (7.1.1) and (7.1.2) respectively. Denote x_j^ε and x_j, $0 \leq j \leq k$ its respective zeroes. Then for each $1 < j < k$*

$$x_j^\varepsilon \to x_j \quad \textit{when } \varepsilon \to 0$$

and

$$|x_j^\varepsilon - x_j| \leq jc\varepsilon(k^{2p} + 1).$$

In particular $x_0^\varepsilon = x_0 = 0$ and $x_k^\varepsilon = x_k = 1$ by the boundary condition.

Proof. With the notation of Theorem 7.15 we have that $|\mathcal{N}_k^\varepsilon| \to |\mathcal{N}_k|$. For the first pair of nodal domains we get

$$|x_1^\varepsilon - x_1| = |x_1^\varepsilon - x_0^\varepsilon - x_1 + x_0| = \left||\mathcal{N}_{k,1}^\varepsilon| - |\mathcal{N}_{k,1}|\right| \leq c\varepsilon(k^{2p} + 1)$$

for the second couple

$$|(x_2^\varepsilon - x_2) - (x_1^\varepsilon - x_1)| = \left||\mathcal{N}_{k,2}^\varepsilon| - |\mathcal{N}_{k,2}|\right| \leq c\varepsilon(k^{2p} + 1)$$

then

$$|x_2^\varepsilon - x_2| \leq c\varepsilon(k^{2p} + 1) + |x_1^\varepsilon - x_1| \leq 2c\varepsilon(k^{2p} + 1).$$

Inductively, for $j < k$

$$|x_j^\varepsilon - x_j| \leq jc\varepsilon(k^{2p} + 1)$$

and the proof is complete. □

7.3.4 Proof of Theorem 7.11.

In order to deal with Theorem 7.11 we first analyze the case where the diffusion coefficient is equal to 1. The general case follows by using the same change of variables before given by (7.3.19)

Theorem 7.17. *Let $g \in L^\infty(\mathbb{R})$ be a 1−periodic function such that*

$$0 < g^- \le g \le g^+ < \infty. \tag{7.3.29}$$

Consider the eigenvalue problem

$$\begin{cases} -(|u'|^{p-2}u')' = g(\frac{x}{\varepsilon})\lambda^\varepsilon |u|^{p-2}u & \text{in } I \\ u'(0) = u'(1) = 0 \end{cases} \tag{7.3.30}$$

and its limit problem

$$\begin{cases} -(|u'|^{p-2}u')' = \bar{g}\lambda |u|^{p-2}u & \text{in } I \\ u'(0) = u'(1) = 0. \end{cases} \tag{7.3.31}$$

Let $\{\lambda_k^\varepsilon\}_{k\ge1}$ and $\{\lambda_k\}_{k\ge1}$ be the eigenvalues of (7.3.30) and (7.3.31) respectively.

Then, we have

$$|\lambda_k^\varepsilon - \lambda_k| \le 4c\varepsilon p \max\{\tfrac{1}{g^-}, \omega_g\}^3 \pi_p^{2p}(k-1)^{2p},$$

where ω_g is given in Definition 7.5 and

$$c \le \beta\left(4 + \frac{p}{(p-1)\pi_p}\right)$$

is the constant given in Theorem 5.8.

Remark 7.18. Let us observe that $u \in W^{1,p}(I)$ is a solution of (7.3.30) if and only if u is a solution of the following equation

$$\begin{cases} -(|u'|^{p-2}u')' + g_\varepsilon |u|^{p-2}u = g_\varepsilon \tilde{\lambda}^\varepsilon |u|^{p-2}u & \text{in } I \\ u'(0) = u'(1) = 0 \end{cases} \tag{7.3.32}$$

where $g_\varepsilon(x) = g(\frac{x}{\varepsilon})$ and $\tilde{\lambda}^\varepsilon = \lambda^\varepsilon + 1$. For convenience, we will work with equation (7.3.32) instead (7.3.30).

Proof of Theorem 7.17. The proof follows the same lines of Theorem 6.7. The kth variational eigenvalue of (7.3.32) and its limit problem can be written as

$$\tilde{\lambda}_k^\varepsilon = \inf_{G\in\Gamma_k} \sup_{u\in G} \frac{\int_I |u'|^p + g_\varepsilon |u|^p}{\int_I g_\varepsilon |u|^p}, \qquad \tilde{\lambda}_k = \inf_{G\in\Gamma_k} \sup_{u\in G} \frac{\int_I |u'|^p + \bar{g}|u|^p}{\bar{g}\int_I |u|^p} \tag{7.3.33}$$

where $\Gamma_k = \{C \subset W^{1,p}(I) : C \text{ compact}, C = -C, \ \gamma(C) \ge k\}$.

Let $\delta > 0$ and let $G_\delta^k \subset W^{1,p}(I)$ be a compact, symmetric set of genus k such that

$$\tilde{\lambda}_k = \sup_{u\in G_\delta^k} \frac{\int_I |u'|^p + \bar{g}|u|^p}{\bar{g}\int_I |u|^p} + O(\delta). \tag{7.3.34}$$

Being the set G_δ^k admissible in the variational characterization of the kth–eigenvalue of the limit problem of (7.3.32), we have

$$\tilde{\lambda}_k^\varepsilon \le \sup_{u \in G_\delta^k} \frac{\int_I |u'|^p + g_\varepsilon |u|^p}{\bar{g} \int_I |u|^p} \frac{\bar{g} \int_I |u|^p}{\int_I g_\varepsilon |u|^p}. \tag{7.3.35}$$

To bound $\tilde{\lambda}_k^\varepsilon$ we look for bounds of the two quotients in (7.3.35). For every function $u \in G_\delta^k \subset W^{1,p}(I)$ we can apply Theorem 5.8 and we obtain that

$$\frac{\int_I |u'|^p + g_\varepsilon |u|^p}{\bar{g} \int_I |u|^p} \le \frac{\int_I |u'|^p + \bar{g} |u|^p}{\bar{g} \int_I |u|^p} + c\varepsilon \frac{\| |u|^p \|_{W^{1,1}(I)}}{\bar{g} \int_I |u|^p}. \tag{7.3.36}$$

where c is given explicitly in Theorem 5.8. By using Young's inequality,

$$\| |u|^p \|_{W^{1,1}(I)} \le p \|u\|_{W^{1,p}(I)}^p.$$

Now, by (7.3.29) we have for each $u \in G_\delta^k$,

$$\frac{\| |u|^p \|_{W^{1,1}(I)}}{\bar{g} \int_I |u|^p} \le p \frac{\|u\|_{W^{1,p}(I)}^p}{\bar{g} \int_I |u|^p} \le c_1 \frac{\bar{g} \int_I |u|^p + \int_I |u'|^p}{\bar{g} \int_I |u|^p} \tag{7.3.37}$$

where $c_1 = p \max\{1, \frac{1}{\bar{g}}\}$. Then, by (7.3.37) and (7.3.34)

$$\frac{\| |u|^p \|_{W^{1,1}(I)}}{\bar{g} \int_I |u|^p} \le c_1 \sup_{v \in G_\delta^k} \frac{\bar{g} \int_I |v|^p + \int_I |v'|^p}{\bar{g} \int_I |v|^p} = c_1 (\tilde{\lambda}_k + O(\delta)). \tag{7.3.38}$$

Again, since $u \in G_\delta^k \subset W^{1,p}(I)$, by applying Theorem 5.8 we obtain that

$$\frac{\bar{g} \int_I |u|^p}{\int_I g_\varepsilon |u|^p} \le 1 + c\varepsilon \frac{\| |u|^p \|_{W^{1,1}(I)}}{\int_I g_\varepsilon |u|^p}, \tag{7.3.39}$$

and by (7.3.38),

$$\frac{\| |u|^p \|_{W^{1,1}(I)}}{\int_I g_\varepsilon |u|^p} \le \frac{\bar{g}}{g^-} \frac{\| |u|^p \|_{W^{1,1}(I)}}{\int_I \bar{g} |u|^p} \le \frac{\bar{g}}{g^-} c_1 (\tilde{\lambda}_k + O(\delta)). \tag{7.3.40}$$

Then combining (7.3.35), (7.3.38), (7.3.40) and letting $\delta \to 0$ we find that

$$\tilde{\lambda}_k^\varepsilon - \tilde{\lambda}_k \le c c_1 \varepsilon (\tfrac{\bar{g}}{g^-} \tilde{\lambda}_k^2 + \tilde{\lambda}_k) + c^2 c_1^2 \tfrac{\bar{g}}{g^-} \varepsilon^2 \lambda_k^2. \tag{7.3.41}$$

In a similar way, interchanging the roles of $\tilde{\lambda}_k$ and $\tilde{\lambda}_k^\varepsilon$, we obtain

$$\tilde{\lambda}_k - \tilde{\lambda}_k^\varepsilon \le c \tilde{c}_1 \varepsilon (\tfrac{g^+}{g^-} (\tilde{\lambda}_k^\varepsilon)^2 + \tilde{\lambda}_k^\varepsilon) + c^2 \tilde{c}_1^2 \tfrac{g^+}{g^-} \varepsilon^2 (\tilde{\lambda}_k^\varepsilon)^2 \tag{7.3.42}$$

with $\tilde{c}_1 = \frac{p}{g^-} \max\{1, g^+\}$. So, from (7.3.41) and (7.3.42), we arrive at

$$|\tilde{\lambda}_k^\varepsilon - \tilde{\lambda}_k| \le c\varepsilon p \max\{\tfrac{1}{g^-}, \tfrac{g^+}{g^-}\} \max\{1, \tfrac{g^+}{g^-}\} \max\{\tilde{\lambda}_k^2 + \tilde{\lambda}_k, (\tilde{\lambda}_k^\varepsilon)^2 + \tilde{\lambda}_k^\varepsilon\}. \tag{7.3.43}$$

In order to complete the proof of the Theorem, we need an estimate on $\tilde\lambda_k$ and $\tilde\lambda_k^\varepsilon$. In fact, from (7.3.29) it follows that

$$\min\{1, \tfrac{1}{\tilde g}\}\mu_k \le \tilde\lambda_k \le \max\{1, \tfrac{1}{\tilde g}\}\mu_k, \qquad \min\{1, \tfrac{1}{g^+}\}\mu_k \le \tilde\lambda_k^\varepsilon \le \max\{1, \tfrac{1}{g^-}\}\mu_k. \tag{7.3.44}$$

where μ_k is the $k-$th eigenvalue of

$$\begin{cases} -|u'|^{p-2}u' + |u|^{p-2}u = \mu|u|^{p-2}u & \text{in } I \\ u'(0) = u'(1) = 0. \end{cases} \tag{7.3.45}$$

Observe that $u \in W^{1,p}(I)$ is solution of (7.3.45) if and only if u is solution of

$$\begin{cases} -(|u'|^{p-2}u')' = \tilde\mu|u|^{p-2}u & \text{in } I \\ u'(0) = u'(1) = 0 \end{cases}$$

where $\tilde\mu = \mu - 1$, and its explicit form is

$$\tilde\mu_k = \pi_p^p(k-1)^p. \tag{7.3.46}$$

From (7.3.43),(7.3.44) and (7.3.46) and using that $|\tilde\lambda_k^\varepsilon - \tilde\lambda_k| = |\lambda_k^\varepsilon - \lambda_k|$ we arrive to the desired result. □

Now we are ready to prove Theorem 7.11.

Proof of Theorem 7.11. The proof follows in the same way of the proof of Theorem 7.13 for the Dirichlet case. The only difference is that we use bounds obtained in Theorem 7.17 instead of the ones from Theorem 7.13. □

7.4 Some examples and numerical results

We consider equation (7.1.1) with weight r and $a \equiv 1$, i.e.,

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda r(x)|u|^{p-2}u & \text{in } I := (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{7.4.1}$$

In this section we present some numerical experiments in the homogenization of the eigenvalues of (7.1.1) in the case $a_\varepsilon(x) \equiv 1$. Using the Prüfer transformation method introduced by Elbert [El82] for the $p-$Laplacian we design an algorithm in order to estimate the eigenfunctions and eigenvalues of (7.1.1).

We define the following Prüfer transformation:

$$\begin{cases} \left(\frac{\lambda r(x)}{p-1}\right)^{1/p} u(x) & = \rho(x)S_p(\varphi(x)), \\ u'(x) & = \rho(x)C_p(\varphi(x)). \end{cases} \tag{7.4.2}$$

As in [Pi07], we can see show that $\rho(x)$ and $\varphi(x)$ are continuously differentiable functions satisfying

$$\begin{cases} \varphi'(x) &= \left(\frac{\lambda r(x)}{p-1}\right)^{\frac{1}{p}} + \frac{1}{p}\frac{r'(x)}{r(x)}|C_p(\varphi(x))|^{p-2}C_p(\varphi(x))S_p(\varphi(x)) \\ \rho'(x) &= \frac{1}{p}\frac{r'(x)}{r(x)}\rho(x)|S_p(\varphi(x))|^p \end{cases} \tag{7.4.3}$$

and we obtain that

$$u_k(x) = \left(\frac{\lambda_k r(x)}{p-1}\right)^{-1/p} \rho_k(x)S_p(\varphi_k(x)), \quad k \geq 1$$

is a eigenfunction of problem (7.4.1) corresponding to λ_k with zero Dirichlet boundary conditions. We propose the following algorithm to compute the eigenvalues of problem (7.4.1) based in the fact that the eigenfunction associate to λ_k has k nodal domain in I, so the phase function φ must vary between 0 and $k\pi_p$. It consists in a shooting method combined with a bisection algorithm (a Newton-Raphson version can be implemented too).

```
Let a < λ < b and let τ be the tolerance
Solve the ODE 7.4.3 and obtain φ_λ and ρ_λ
Let w(x) = (p − 1)^{1/p} (λr(x))^{−1/p} ρ_λ(x)S_p(φ_λ(x))
Let α = w(1)
while (|α| ≥ τ)
    λ = (a + b)/2
    Solve the ODE 7.4.3 and obtain φ_λ and ρ_λ
    Let w(x) = (p − 1)^{1/p} (λr(x))^{−1/p} ρ_λ(x)S_p(φ_λ(x))
    Let β = w(1)
    If (αβ < 0)
        b = (a + b)/2
    else
        a = (a + b)/2
end while
Then λ is the aproximation of eigenvalue with error ≤ τ
```

For example, let us consider $r(x) = 2 + \sin(2\pi x)$. In this case we obtain that $\overline{r} = \int_I 2 + \sin(2\pi x)dx = 2$, and the eigenvalues of the limit problem are given by $\lambda_k^{1/p} = \frac{k\pi_p}{2^{1/p}}$. When ε tends to zero the value of λ^ε tends to the limit value λ displaying oscillations.

When $p = 2$ the first limit eigenvalue is $\sqrt{\lambda_1} = \pi/\sqrt{2} \sim 2.221441469$. We see the oscillating behavior when plot $\sqrt{\lambda_1^\varepsilon}$ as function of ε in Figure 7.1

A more complex behavior can be found in Figure 7.2, where we considered the weight $r(x) = \frac{1}{2+\sin 2\pi x}$. We observe that the sequence tends to

$$\lambda_1 = \pi^2 / \int_I \frac{1}{2 + \sin 2\pi x}dx = \sqrt{3}\pi \sim 17.09465627.$$

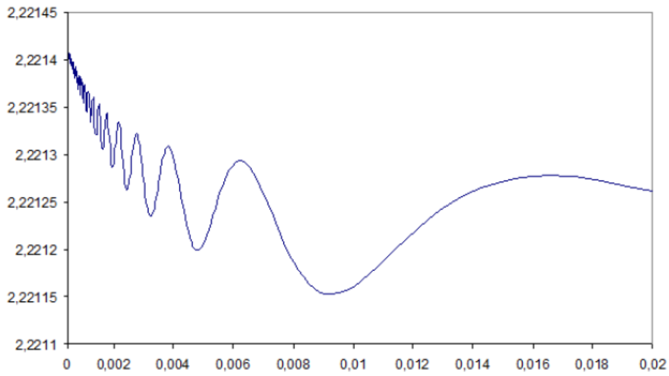

Figure 7.1: The square root of the first eigenvalue as a function of ε.

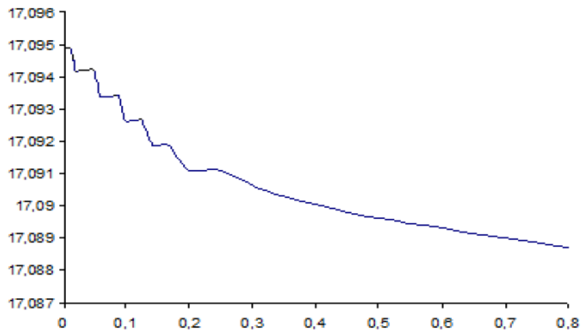

Figure 7.2: The square root of the first eigenvalue as a function of ε.

It is not clear why the convergence of the first eigenvalue display the oscillations and the monotonicity observed (although the monotonicity is reversed for the weight $r(x) = 2 - \sin 2\pi x$). We believe that some Sturmian type comparison theorem with integral inequalities for the weights (instead of point-wise inequalities as usual) is involved. However, we are not able to prove it, and for higher eigenvalues it is not clear what happens.

Turning now to the eigenfunctions, with the weight $r(x) = 2 + \sin(2\pi x)$, the normalized eigenfunction associated to the first eigenvalue of the limit problem is given by $u_1(x) = \pi^{-1}\sin(\pi x)$. Applying the numerical algorithm we obtain that the graph of an eigenfunction associated to the first eigenvalue λ_1^ε intertwine with the graph of $u_1(x)$. When ε decreases, the number of crosses increases, and the amplitude of the difference between them decreases. In Figure 7.3 we can observe this behavior and the difference between u_1 and u_1^ε for different values of ε.

To our knowledge, it is not known any result about the number of the oscillations as ε decreases, nor it is known if those oscillations disappear for ε sufficiently small.

The same behavior seems to hold for the higher eigenfunctions, see in Figure 7.4 the behavior of the fourth eigenfunction u_4^ε when the parameter ε decrease.

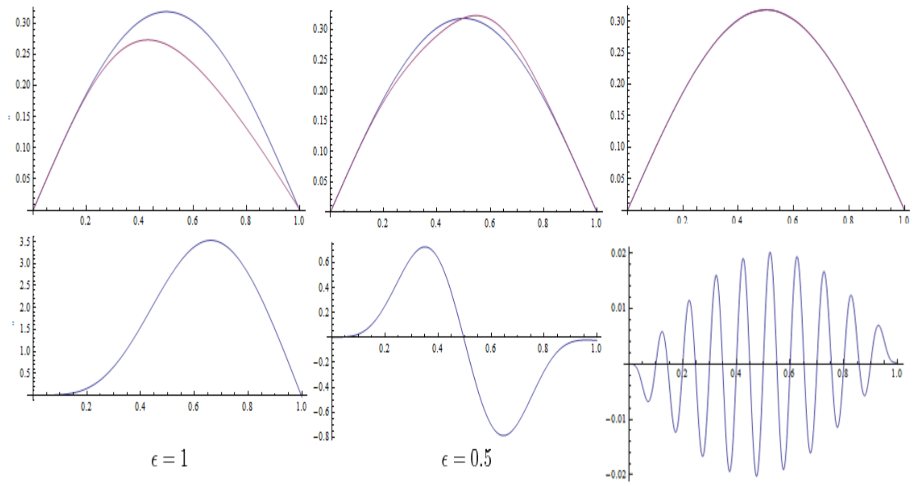

Figure 7.3: The first eigenfunctions and the difference between them for different values of ε.

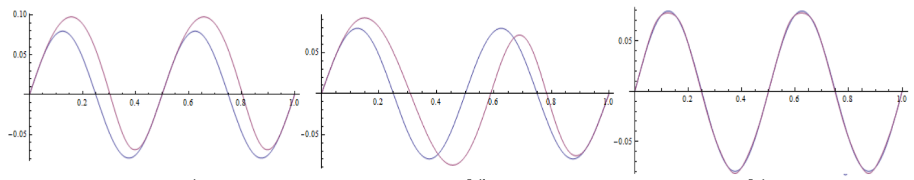

Figure 7.4: The fourth eigenfunctions and the difference between them. for different values of ε.

Here, the convergence of the nodal domains and the fact that the restriction of an eigenfunction to one of its nodal domains $\mathcal{N}$ coincides with the first eigenfunction of the problem in $\mathcal{N}$, together with the continuous dependence of the eigenfunctions on the weight and the length of the domain, suggest that the presence or not of oscillations for the higher eigenfunctions must be the same as for the first one. However, the computations show very complex patterns in the oscillations.

8

Homogenization of the Fučik spectrum

8.1 The Fučik spectrum

Given a bounded domain Ω in $\mathbb{R}^N$, $N \geq 1$ we study the asymptotic behavior as $\varepsilon \to 0$ of the spectrum of the following asymmetric elliptic problem

$$-\Delta_p u_\varepsilon = \alpha_\varepsilon m_\varepsilon (u_\varepsilon^+)^{p-1} - \beta_\varepsilon n_\varepsilon (u_\varepsilon^-)^{p-1} \quad \text{in } \Omega \tag{8.1.1}$$

either with homogeneous Dirichlet or Neumann boundary conditions.

Here, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplacian with $1 < p < \infty$ and $u^\pm := \max\{\pm u, 0\}$. The parameters α_ε and β_ε are reals and depending on $\varepsilon > 0$. We assume that the family of weight functions m_ε and n_ε are positive and uniformly bounded away from zero.

For a moment let us focus problem (8.1.1) for fixed $\varepsilon > 0$ with positive weights $m(x), n(x)$:

$$-\Delta_p u = \alpha m(x)(u^+)^{p-1} - \beta n(x)(u^-)^{p-1} \quad \text{in } \Omega \tag{8.1.2}$$

with Dirichlet or Neumann boundary conditions.

Consider the Fučik spectrum defined as the set

$$\Sigma(m,n) := \{(\alpha,\beta) \in \mathbb{R}^2 : (8.1.2) \text{ has a nontrivial solution}\}.$$

Let us observe that when $r = n = m$ and $\lambda = \alpha = \beta$, equation (8.1.2) becomes

$$-\Delta_p u = \lambda r |u|^{p-2} u \quad \text{in } \Omega \tag{8.1.3}$$

with Dirichlet or Neumann boundary conditions, which is the eigenvalue problem for the $p-$Laplacian. These has been widely studied. See for instance [CFG99, ET06, DGT10] and Chapter §2 of this thesis for more information.

It follows immediately that Σ contain the lines $\lambda_1(m) \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1(n)$. For this reason, we denote by $\Sigma^* = \Sigma^*(m,n)$ the set Σ without these trivial lines. Observe that if $(\alpha,\beta) \in \Sigma^*$ with $\alpha \geq 0$ and $\beta \geq 0$ then $\lambda_1(m) < \alpha$ and $\lambda_1(n) < \beta$.

The study of problem (8.1.2) with Dirichlet boundary conditions have a long history that we briefly describe below. The one-dimensional case with positive constant coefficients (i.e., $m, n \in \mathbb{R}^+$ and $p = 2$) was studied in the 1970s by Fučik [Fu76] and Dancer [Da77] in connection with jumping nonlinearities. Properties and descriptions of the first non-trivial curve on the spectrum of (8.1.2) on $\mathbb{R}^N$ for the general case ($p \neq 2$) without weights can be found in Cuesta, de Figueiredo and Gossez [CFG99], Dancer and Perera [DP01], Drábek and Robinson [DRo02], Perera [Pe04].

The case with positive weights $m(x)$ and $n(x)$ was recently studied, see for instance Rynne and Walter [RW00], Arias and Campos [AC96], Drabek [Dr92], Reichel and Walter [LW99]. For indefinite weights $m(x)$ and $n(x)$ see Alif and Gossez[AG01], Leadi and Marcos [LM07].

The main problem one address is to obtain a description as accurate as possible of the set Σ^*. In the one–dimensional case, $p = 2$, without weights this description is obtained in a precise manner: the spectrum is made of a sequence of hyperbolic like curves in $\mathbb{R}^+ \times \mathbb{R}^+$, see for instance [FH80]. When $m(x)$ and $n(x)$ are non-constants weights, in [AG01] it is proved a characterization of the spectrum in terms of the so-called zeroes-functions.

8.1.1 Dirichlet boundary conditions

Given $\Omega \subset \mathbb{R}^N$ with $N > 1$ let us consider (8.1.2) with Dirichlet boundary conditions, i.e.

$$\begin{cases} -\Delta_p u = \alpha m(x)(u^+)^{p-1} - \beta n(x)(u^-)^{p-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.1.4}$$

Here, only a full description of the first nontrivial curve of Σ is known, which we will denote by $C_1 = C_1(m, n)$. Assuming that $m, n \in L^r(\Omega)$ with

$$r > \tfrac{N}{p} \quad \text{if } p \leq N \qquad \text{and} \qquad r = 1 \quad \text{if } p > N, \tag{8.1.5}$$

in [ACCG02] (see Theorem 33) is proved that C_1 can be characterized by

$$C_1 = \{(\alpha(s), \beta(s)), s \in \mathbb{R}^+\} \tag{8.1.6}$$

where $\alpha(s)$ and $\beta(s)$ are continuous functions defined by

$$\alpha(s) = c(m, sn), \quad \beta(s) = s\alpha(s) \tag{8.1.7}$$

and $c(\cdot, \cdot)$ is given by

$$c(m, n) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma(I)} \frac{A(u)}{B(u)}. \tag{8.1.8}$$

where $I := [-1, +1]$. Here, the functionals A and B are given by

$$A(u) = \int_\Omega |\nabla u|^p dx, \quad B_{m,n} = \int_\Omega m(x)(u^+)^p + n(x)(u^-)^p dx, \tag{8.1.9}$$

with

$$\Gamma = \{\gamma \in C([-1, +1], W_0^{1,p}(\Omega)) : \gamma(-1) \geq 0 \text{ and } \gamma(1) \leq 0\}.$$

The functions $\alpha(s)$ and $\beta(s)$ defined in (8.1.7) satisfy some important properties.

Proposition 8.1. *The functions $\alpha(s)$ and $\beta(s)$ are continuous. Moreover, $\alpha(s)$ is strictly decreasing and $\beta(s)$ is strictly increasing. One also has that $\alpha(s) \to +\infty$ if $s \to 0$ and $\beta(s) \to +\infty$ is $s \to +\infty$.*

Proof. See [ACCG02], Proposition 34. □

If we denote $\alpha_\infty := \lim_{s\to\infty} \alpha(s)$ and $\beta_\infty := \lim_{s\to 0} \beta(s)$, we have the following characterization.

Proposition 8.2. *The asymptotic values α_∞ and β_∞ are equal to $\bar{\alpha}$ and $\bar{\beta}$ respectively, where*

$$\bar{\alpha} := \inf\left\{\int_\Omega |\nabla u^+|p : u \in W_0^{1,p}(\Omega), \int_\Omega m(u^+)^p = 1 \textit{ and } \int_\Omega n(u^-)^p > 0\right\},$$

$$\bar{\beta} := \inf\left\{\int_\Omega |\nabla u^-|p : u \in W_0^{1,p}(\Omega), \int_\Omega n(u^-)^p = 1 \textit{ and } \int_\Omega m(u^+)^p > 0\right\}.$$

Moreover if $p \le N$, then $\bar{\alpha} = \lambda_1(m)$ and $\bar{\beta} = \lambda_1(n)$.

Proof. See [ACCG02], Proposition 35. □

8.1.2 Neumann boundary conditions

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \ge 1$ with Lipschitz boundary and let m, n be two weights satisfying (8.1.5) and bounded uniformly away from zero. We also assume that $m \not\equiv 0$ and $n \not\equiv 0$ in Ω. We consider (8.1.2) with Neumann boundary conditions

$$\begin{cases} -\Delta_p u = \alpha m(x)(u^+)^{p-1} - \beta n(x)(u^-)^{p-1} & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.1.10}$$

where $\partial u/\partial\eta = \nabla u \cdot \eta$ denotes the unit exterior normal.

The Fučik spectrum $\Sigma = \Sigma(m, n)$ clearly contains the lines $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$ and we denote by $\Sigma^* = \Sigma^*(m, n)$ the set $\Sigma(m, n)$ without these two lines.

In this case, when $N > 1$ only a full description of the first nontrivial curve of Σ is known, which we will denote by $C_1 = C_1(m, n)$. Moreover, in [ACCG08] (see Theorem 6.1) a characterization similar to the Dirichlet case is given:

$$C_1 = \{(\alpha(s), \beta(s)), s \in \mathbb{R}^+\} \tag{8.1.11}$$

where $\alpha(s)$ and $\beta(s)$ are continuous functions defined by $\alpha(s) = c(m, sn)$, $\beta(s) = s\alpha(s)$ and $c(\cdot,\cdot)$ is given by

$$c(m, n) = \inf_{\gamma\in\Gamma} \max_{u\in\gamma(J)} \frac{A(u)}{B(u)} \tag{8.1.12}$$

where $J := [0, 1]$, the functionals A and B are given by (8.1.9) and

$$\Gamma = \{\gamma \in C(J, W^{1,p}(\Omega)) : \gamma(0) \ge 0 \text{ and } \gamma(1) \le 0\}.$$

In this case, for a weight function $r(x)$ satisfying (8.1.5) and uniformly bounded away from zero and infinity, clearly 0 is a principal eigenvalue of

$$\begin{cases} -\Delta_p u = \lambda r(x)|u|^{p-2}u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{8.1.13}$$

with the constants as eigenfunctions. Moreover, the condition $r > 0$ guaranties that 0 is the unique nonnegative principal eigenvalue, see [GGP02].

Remark 8.3. In the Neumann case Proposition 8.1 still being valid.

8.2 Homogenization of the spectrum

Up to our knowledge, no investigation was made in the homogenization and rates of convergence of the Fučik Spectrum. We are interested in studying the behavior as $\varepsilon \to 0$ of problem (8.1.1) when $m_\varepsilon(x)$ and $n_\varepsilon(x)$ are general functions depending on ε, and in the special case of rapidly oscillating periodic functions, i.e., $m_\varepsilon(x) = m(x/\varepsilon)$ and $n_\varepsilon(x) = n(x/\varepsilon)$ for two $Q-$periodic functions m, n uniformly bounded away from zero (see assumptions (8.2.1)), Q being the unit cube of $\mathbb{R}^N$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and ε a real positive number. We consider functions $m_\varepsilon, n_\varepsilon$ such that for constants $m_- \le m^+, n^- \le n^+$

$$0 < m_- \le m_\varepsilon(x) \le m_+ \le +\infty \quad \text{and} \quad 0 < n_- \le n_\varepsilon(x) \le n_+ \le +\infty. \tag{8.2.1}$$

Also, we assume that there exist functions $m(x)$ and $n(x)$ satisfying (8.2.1) such that, as $\varepsilon \to 0$,

$$\begin{aligned} m_\varepsilon(x) &\rightharpoonup m(x) && \text{weakly* in } L^\infty(\Omega) \\ n_\varepsilon(x) &\rightharpoonup n(x) && \text{weakly* in } L^\infty(\Omega). \end{aligned} \tag{8.2.2}$$

First we address the problem with Dirichlet boundary conditions.

When $\varepsilon \to 0$ the natural limit problem for (8.1.1) is the following

$$\begin{cases} -\Delta_p u_0 = \alpha_0 m(x)(u_0^+)^{p-1} - \beta_0 n(x)(u_0^-)^{p-1} & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{8.2.3}$$

where m and n are given in (8.2.2).

Our main aim is to study the limit as $\varepsilon \to 0$ of the first nontrivial curve in the spectrum $\Sigma_\varepsilon := \Sigma(m_\varepsilon, n_\varepsilon)$, say $C_1^\varepsilon = \{(\alpha_\varepsilon(s), \beta_\varepsilon(s)), s \in \mathbb{R}^+\}$. We wonder: there exists a limit curve $C_1 = \{(\alpha_0(s), \beta_0(s)), s \in \mathbb{R}^+\}$ such that

$$C_1^\varepsilon \to C_1, \quad \text{as } \varepsilon \to 0\ ?$$

Can this limit curve be characterized like a curve of a limit problem? We will see that the answer is positive.

Therefore, a natural question arises: can the rate of convergence of C_1^ε be estimated? I.e., can we give an estimate of the remainders

$$|\alpha_\varepsilon(s) - \alpha_0(s)| \quad \text{and} \quad |\beta_\varepsilon(s) - \beta_0(s)|?$$

We give positive answers to these questions in the periodic setting . In fact, in Theorems 8.7 and 8.11 we obtain bounds:

$$|\alpha_\varepsilon(s) - \alpha_0(s)| \le C(1+s)\tau(s)\varepsilon, \quad |\beta_\varepsilon(s) - \beta_0(s)| \le Cs(1+s)\tau(s)\varepsilon, \quad s \in \mathbb{R}^+$$

where C is a constant independent of s and ε, and τ is a explicit function depending only of s (see (8.2.6)).

Particularly, for the limit values of the coordinates, we get

$$|\alpha_\varepsilon^\infty - \alpha_0^\infty| \le C\varepsilon, \quad |\beta_\varepsilon^0 - \beta_0^0| \le C\varepsilon$$

where $\alpha_\varepsilon^\infty = \lim_{s\to\infty} \alpha_\varepsilon(s)$, $\alpha_0^\infty = \lim_{s\to\infty} \alpha_0(s)$, $\beta_\varepsilon^0 = \lim_{s\to\infty} \beta_\varepsilon(s)$, $\beta_0^0 = \lim_{s\to\infty} \beta_0(s)$. The constant C is independent of s and ε.

The main result is the following:

Theorem 8.4. *Let $m_\varepsilon, n_\varepsilon$ satisfying* (8.2.1),(8.2.2) *and* (8.1.5). *Then the first non-trivial curve of problem* (8.1.1)

$$C_\varepsilon := C_1(m_\varepsilon, n_\varepsilon) = \{\alpha_\varepsilon(s), \beta_\varepsilon(s), s \in \mathbb{R}^+\}$$

converges to the first non-trivial curve of the limit problem (8.2.3)

$$C := C_1(m_0, n_0) = \{\alpha_0(s), \beta_0(s), s \in \mathbb{R}^+\}$$

as $\varepsilon \to 0$ in the sense that $\alpha_\varepsilon(s) \to \alpha_0(s)$ and $\beta_\varepsilon(s) \to \beta_0(s)$ $\forall s \in \mathbb{R}^+$.

Remark 8.5. Let us consider the weighted $p-$Laplacian problem

$$\begin{cases} -\Delta_p u = \lambda r_\varepsilon(x)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{8.2.4}$$

where r_ε is a function such that $r_\varepsilon(x) \rightharpoonup r(x)$ weakly* in $L^\infty(\Omega)$ as ε tends to zero. It is well-known that the first eigenvalue of (8.2.4) converges to the first eigenvalue of the $p-$Laplacian equation with weight $r(x)$, see for instance [BCR06]. The fact that the trivial lines of Σ are defined by $\lambda_1(m_\varepsilon) \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1(n_\varepsilon)$ it allows us to affirm the convergence of the trivial lines to those of the limit problem.

Remark 8.6. Using the variational characterization of the second variational eigenvalue given in Theorem 2.8, Theorem 8.4 implies the convergence of the second variational eigenvalue of (8.2.4) to those of the limit problem which recover Theorem 6.6 as a particular case.

In the important case of periodic homogenization, i.e., when $m_\varepsilon(x) = m(x/\varepsilon)$ and $n_\varepsilon(x) = n(x/\varepsilon)$ where m and n are $Q-$periodic functions, Q being the unit cube in $\mathbb{R}^N$, problem (8.2.3) becomes

$$\begin{cases} -\Delta_p u_0 = \alpha_0 m_0 (u_0^+)^{p-1} - \beta_0 n_0 (u_0^-)^{p-1} & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{8.2.5}$$

where the real numbers m_0, n_0 are the averages of m and n over Q. In this case besides the convergence of the first nontrivial curve of the spectrum given in Theorem 8.4, we obtain the rate of convergence:

Theorem 8.7. *Under the same considerations of Theorem 8.4, if the weights m_ε and n_ε are given in terms of $Q-$periodic functions m, n in the form $m_\varepsilon(x) = m(\frac{x}{\varepsilon})$ and $n_\varepsilon(x) = n(\frac{x}{\varepsilon})$, for each $s \in \mathbb{R}^+$ we have the following estimates*

$$|\alpha_\varepsilon(s) - \alpha_0(s)| \le c(1+s)\tau(s)\varepsilon, \quad |\beta_\varepsilon(s) - \beta_0(s)| \le cs(1+s)\tau(s)\varepsilon$$

where $c = c(\Omega, p, m, n)$ is a constant independent of ε and s and τ is defined by

$$\tau(s) = \begin{cases} 1 & s \ge 1 \\ s^{-2} & s < 1. \end{cases} \tag{8.2.6}$$

Particularly, with the same arguments uses in the proof of Theorem 6.6 we are able to compute the rate convergence of the trivial lines of Σ_ε:

Theorem 8.8. *The trivial curves converges. Moreover, if the point $p_\varepsilon = (\alpha_\varepsilon, \beta_\varepsilon) \in \mathbb{R}^2$ belongs to a trivial curve of (8.1.1) then*

$$|p_\varepsilon - p_0| \le c\varepsilon$$

where $p_0 = (\alpha_0, \beta_0) \in \mathbb{R}^2$ is the limit point belonging to the trivial curve of (8.2.5) and $c = c(p, \Omega)$ in a constant independent of ε.

Remark 8.9. When $s >> 1$ is a real fixed number, Theorem 8.7 reads

$$|\alpha_\varepsilon(s) - \alpha_0(s)| \sim cs\varepsilon, \quad |\beta_\varepsilon(s) - \beta_0(s)| \sim cs^2\varepsilon$$

and when $s << 1$ is fixed,

$$|\alpha_\varepsilon(s) - \alpha_0(s)| \sim c\varepsilon/s^2, \quad |\beta_\varepsilon(s) - \beta_0(s)| \sim c\varepsilon/s.$$

According to Proposition 8.1 and Proposition 8.2, when $p \le N$ the limits of $\alpha_\varepsilon(s), \alpha_0(s)$ as $s \to \infty$ and $\beta_\varepsilon(s), \beta_0(s)$ as $s \to 0$ can be characterized in terms of the first eigenvalues of weighted $p-$Laplacian problems. Following the same argument for the estimate of the difference of eigenvalues used in the proof of Theorem 8.7, we are able to compute the rate of convergence in the limit cases, namely:

$$\lim_{s\to\infty} |\alpha_\varepsilon(s) - \alpha_0(s)| = |\lambda_1(m_\varepsilon) - \lambda_1(m_0)| \le c\varepsilon$$
$$\lim_{s\to 0} |\beta_\varepsilon(s) - \beta_0(s)| = |\lambda_1(n_\varepsilon) - \lambda_1(n_0)| \le c\varepsilon$$

where c is a constant independent of ε.

Now we focus our attention on the Neumann boundary conditions case.

As we made with the Dirichlet problem (8.1.1), we want to study the behavior of the first nontrivial curve in the spectrum of (8.1.1) with Neumann boundary conditions as $\varepsilon \to 0$. When ε tends to zero the natural limit problem is the following

$$\begin{cases} -\Delta_p u_0 = \alpha_0 m(x)(u_0^+)^{p-1} - \beta_0 n(x)(u_0^-)^{p-1} & \text{in } \Omega \\ \frac{\partial u_0}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.2.7}$$

Analogously to Theorem 8.4, we obtain the following result of convergence:

Theorem 8.10. *Let $m_\varepsilon, n_\varepsilon$ satisfying* (8.2.1) *and* (8.2.2) *such that $m \not\equiv 0$ and $n \not\equiv 0$. Then the first non-trivial curve of problem* (8.1.1)

$$C_1^\varepsilon := C_1(m_\varepsilon, n_\varepsilon) = \{\alpha_\varepsilon(s), \beta_\varepsilon(s), s \in \mathbb{R}^+\}$$

converges to the first non-trivial curve of the limit problem (8.2.7)

$$C_1 := C_1(m_0, n_0) = \{\alpha_0(s), \beta_0(s), s \in \mathbb{R}^+\}$$

as $\varepsilon \to 0$ in the sense that $\alpha_\varepsilon(s) \to \alpha_0(s)$, $\beta_\varepsilon(s) \to \beta_0(s)$ $\forall s \in \mathbb{R}^+$.

When the case of periodic homogenization is considered, like in the Dirichlet case, in addition to the convergence of the first non-trivial curve in the spectrum enunciated in Theorem 8.10, we obtain the order of convergence:

Theorem 8.11. *Under the same considerations of Theorem 8.10, if the weights m_ε and n_ε are given in terms of Q−periodic functions m, n in the form $m_\varepsilon(x) = m(\frac{x}{\varepsilon})$ and $n_\varepsilon(x) = n(\frac{x}{\varepsilon})$, for each $s \in \mathbb{R}^+$ we have the following estimates*

$$|\alpha_\varepsilon(s) - \alpha_0(s)| \le c(1+s)\tau(s)\varepsilon, \quad |\beta_\varepsilon(s) - \beta_0(s)| \le cs(1+s)\tau(s)\varepsilon$$

where $c = c(\Omega, p, m, n)$ is a constant independent of ε and s, and τ is given by (8.2.6).

To prove Theorem 8.11 the arguments of the proof of Theorem 8.7 fail. This is due to the fact that now the functions space is $W^{1,p}(\Omega)$ but Theorem 5.4 holds only for functions in $W_0^{1,p}(\Omega)$. Then, we use Theorem 5.6 which allow us to consider functions in $W^{1,p}(\Omega)$. Observe that the fact of enlarge the set of test functions is reflected in the need for more regularity of the domain Ω.

8.3 Proof of the Dirichlet results

We begin with the proof of the Theorem 8.8 which is analogous to the proof of Theorem 6.7.

Proof of Theorem 8.8: The trivial lines of the spectrum Σ_ε are given by

$$C_{0,1}^\varepsilon = \{(\lambda_1(m_\varepsilon), t), t \in \mathbb{R}\} \quad \text{and} \quad C_{0,2}^\varepsilon = \{(t, \lambda_1(n_\varepsilon), t \in \mathbb{R}\}.$$

The first limit eigenvalue $\lambda_1(m_0)$ can be characterized variationally as

$$\lambda_1(m_0) = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega m_0 |u|^p} = \frac{\int_\Omega |\nabla u_1|^p}{\int_\Omega m_0 |u_1|^p} + o(1) \tag{8.3.1}$$

for some $u_1 \in W_0^{1,p}(\Omega)$. We can bound

$$\lambda_1(m_\varepsilon) = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega m_\varepsilon |u|^p} \le \frac{\int_\Omega |\nabla u_1|^p}{\int_\Omega m_0 |u_1|^p} \frac{\int_\Omega m_0 |u_1|^p}{\int_\Omega m_\varepsilon |u_1|^p}. \tag{8.3.2}$$

By using Theorem 5.4 it follows that

$$\frac{\int_\Omega m_0|u_1|^p}{\int_\Omega m_\varepsilon|u_1|^p} \le 1 + c\varepsilon\frac{\|\nabla u_1\|^p_{L^p(\Omega)}}{\int_\Omega m_\varepsilon|u_1|^p}. \tag{8.3.3}$$

Now, by (8.2.1) and (8.3.1) we have

$$\frac{\|\nabla u_1\|^p_{L^p(\Omega)}}{\int_\Omega m_\varepsilon|u_1|^p} \le c(\lambda_1(m_0) + o(1)). \tag{8.3.4}$$

By replacing (8.3.4) and (8.3.1), in (8.3.2) we get

$$\lambda_1(m_\varepsilon) - \lambda_1(m_0) \le c\varepsilon. \tag{8.3.5}$$

In a similar way, interchanging the roles of $\lambda_1(m_\varepsilon)$ and $\lambda_1(m_0)$ we obtain

$$\lambda_1(m_0) - \lambda_1(m_\varepsilon) \le c\varepsilon. \tag{8.3.6}$$

From equations (8.3.5) and (8.3.6) it follows that

$$|p_\varepsilon - p_0| \le c\varepsilon$$

for $p_\varepsilon \in C^\varepsilon_{0,1}$, $p_0 \in C_{0,1}$ with $c = c(p,\Omega)$ a constant independent of ε. Analogously is obtained a bound for the points of $C^\varepsilon_{0,2}$. This implies the convergence of the trivial lines of the spectrum. □

In the next Lemma we obtain upper bounds for the coordinates of the first curve of $\Sigma^*(m,n)$.

Lemma 8.12. *Let m, n satisfying (8.2.1) and let $(\alpha(s),\beta(s)) \in C_1(m,n)$. Then for each $s \in \mathbb{R}^+$,*

$$\alpha(s) \le \min\{m_-^{-1}, n_-^{-1}\}\mu_2\tau(s), \qquad \beta(s) \le \min\{m_-^{-1}, n_-^{-1}\}\mu_2 s\tau(s)$$

with τ defined by

$$\tau(s) = \begin{cases} 1 & s \ge 1 \\ s^{-1} & s \le 1. \end{cases} \tag{8.3.7}$$

where m_-, n_- are given by (8.2.1) and μ_2 is the second eigenvalue of the $p-$Laplacian equation without weights on Ω with Dirichlet boundary conditions.

Proof. Let $s \in \mathbb{R}^+$. When the parameter $s \ge 1$ we can bound

$$\lambda_1(m) \le \alpha(s) \le \alpha(1) = c(m,n).$$

Let $\lambda_2(m)$ be the second eigenvalue of the problem (8.1.3) with weight $m(x)$. It satisfies that $\alpha(1) \le \min\{\lambda_2(m), \lambda_2(n)\}$. By using the assumptions (8.2.1) over $m(x)$, we can bound $\lambda_2(m)$ by $\mu_2 m_-^{-1}$, where μ_2 is the second eigenvalue of the $p-$Laplacian equation without weights with Dirichlet boundary conditions on Ω. Analogously for $\lambda_2(n)$. We get

$$\alpha(s) \le \alpha(1) \le \min\{m_-^{-1}, n_-^{-1}\}\mu_2, \quad s \ge 1 \tag{8.3.8}$$

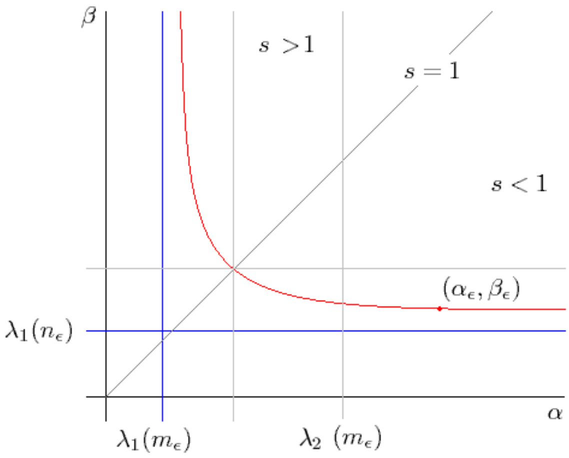

Figure 8.1: The first curve of the spectrum.

When $s \leq 1$ the following bound holds for the second coordinate of C_ε

$$\lambda_1(n) \leq \beta(s) \leq \beta(1). \tag{8.3.9}$$

By multiplying (8.3.9) by s^{-1} and by using that $\beta(s) = s\alpha(s)$ we have

$$s^{-1}\lambda_1(n) \leq \alpha(s) \leq s^{-1}\beta(1).$$

Being $\alpha(1) = \beta(1)$, it follows that

$$\alpha(s) \leq s^{-1}\alpha(1) \leq s^{-1}\min\{m_-^{-1}, n_-^{-1}\}\mu_2, \quad s \leq 1. \tag{8.3.10}$$

By using (8.3.8), (8.3.10) and the relation $\beta(s) = s\alpha(s)$ the conclusions of the lemma follows.

□

The following Proposition gives the monotonicity of $c(\cdot,\cdot)$:

Proposition 8.13. *If $m \leq \tilde{m}$ and $n \leq \tilde{n}$ a.e., then*

$$c(\tilde{m}, \tilde{n}) \leq c(m, n),$$

where $c(\cdot,\cdot)$ is defined by (8.1.8).

Proof. See [ACCG02], Proposition 23. □

In the next Lemma we obtain lower bounds for the coordinates of the first curve of $\Sigma^*(m, n)$.

Lemma 8.14. *Let m, n satisfying* (8.2.1) *and let $(\alpha(s), \beta(s)) \in C(m, n)$. Then for each $s \in \mathbb{R}^+$,*

$$\alpha(s) \geq \tfrac{1}{s}C\omega(s), \qquad \beta(s) \geq C\omega(s)$$

with ω defined by

$$\omega(s) = \begin{cases} 1 & s \geq 1 \\ s & s \leq 1 \end{cases} \tag{8.3.11}$$

where C is a positive constant depending only of the bounds given in (8.2.1).

Proof. Let $s \in \mathbb{R}^+$. When the parameter $s \geq 1$ we can bound bellow

$$\beta(s) \geq \beta(1) = c(m, n), \quad s \geq 1.$$

Using the relation $\beta(s) = s\alpha(s)$ we obtain

$$\alpha(s) \geq s^{-1} c(m, n), \quad s \geq 1.$$

Similarly, when $s \leq 1$ we have

$$\alpha(s) \geq \alpha(1) = c(m, n), \quad s \leq 1,$$

and again, by the relation between $\alpha(s)$ and $\beta(s)$ we get

$$\beta(s) \geq sc(m, n), \quad s \leq 1.$$

Using the bounds (8.2.1) of m,n and Proposition 8.13 we can bound bellow

$$c(m, n) \geq c(m_+, n_+).$$

and the result follows. □

Now we are able to prove Theorem 8.7.

Proof of Theorem 8.7: For each fixed value of $\varepsilon > 0$, by (8.2.1) together with the monotonicity of $c(\cdot, \cdot)$ provided by Proposition 8.13 we can assert that there exist two curves $C_1^+(m_+, n_+)$ and $C_1^-(m_-, n_-)$ such that delimit above and below to the curve $C_1^\varepsilon(m_\varepsilon, n_\varepsilon)$. It follows that for each fixed value of s, $\alpha_\varepsilon(s)$ and $\beta_\varepsilon(s)$ are bounded.

Let $(\alpha_\varepsilon, \beta_\varepsilon)$ be a point belonging to the curve $C_1^\varepsilon(m_\varepsilon, n_\varepsilon)$ and let (α_0, β_0) be the point obtained when $\varepsilon \to 0$. Let us see that it belongs to $C_1(m_0, n_0)$.

Fixed a value of $\varepsilon > 0$ and by using (8.1.8) the inverse of $c(m_\varepsilon, n_\varepsilon)$ can be written as

$$\frac{1}{c(m_\varepsilon, n_\varepsilon)} = \sup_{\gamma \in \Gamma} \inf_{u \in \gamma[-1,+1]} B_{m_\varepsilon, n_\varepsilon}(u) \tag{8.3.12}$$

where

$$\Gamma = \{\gamma \in C(I, H) : \gamma(-1) \geq 0 \text{ and } \gamma(1) \leq 0\}$$

for $I := [-1, +1]$ and

$$H = \{u \in W_0^{1,p}(\Omega) : A(u) = 1\}$$

A and B being the functionals defined in (8.1.9).

By (8.1.7) and (8.3.12) we have the following characterization for the inverse of $\alpha_\varepsilon(s)$

$$\frac{1}{\alpha_\varepsilon(s)} = \frac{1}{c(m_\varepsilon, sn_\varepsilon)} = \sup_{\gamma \in \Gamma} \inf_{u \in \gamma(I)} B_{m_\varepsilon, sn_\varepsilon}(u). \tag{8.3.13}$$

Similarly, we can consider an equation analog to (8.3.13) for the representation of the inverse of $\alpha_0(s)$. Let $\delta > 0$ and $\gamma_1(\delta) \in \Gamma$ such that

$$\frac{1}{\alpha_0(s)} = \inf_{u\in\gamma_1(I)} B_{m_0,sn_0}(u) + O(\delta). \tag{8.3.14}$$

In order to find a bound for a_ε we use $\gamma_1 \in \Gamma_1$, which is admissible in its variational characterization,

$$\frac{1}{\alpha_\varepsilon(s)} \geq \inf_{u\in\gamma_1(I)} B_{m_\varepsilon,sn_\varepsilon}(u). \tag{8.3.15}$$

As $u \in W_0^{1,p}(\Omega)$ it follows that $(u^+)^p$ and $(u^-)^p$ belong to $W_0^{1,1}(\Omega)$. This allows us to estimate the error by replacing the oscillating weights by their averages by using Theorem 5.4. For each fixed function $u \in \gamma_1(I)$ we bound

$$B_{m_\varepsilon,sn_\varepsilon}(u) \geq B_{m_0,sn_0}(u) - c\varepsilon\|\nabla u^+\|^p_{L^p(\Omega)} - c\varepsilon s\|\nabla u^-\|^p_{L^p(\Omega)} \tag{8.3.16}$$

where c the constant given in Theorem 5.4. As $u \in H$ we have

$$\|\nabla u^+\|^p_{L^p(\Omega)} \leq 1, \quad \|\nabla u^-\|^p_{L^p(\Omega)} \leq 1. \tag{8.3.17}$$

So, from (8.3.17) and (8.3.16) we get

$$B_{m_\varepsilon,sn_\varepsilon}(u) \geq B_{m_0,sn_0}(u) - c\varepsilon(1+s). \tag{8.3.18}$$

Taking the infimum over the functions u in $\gamma_1(I)$ together with (8.3.14) and (8.3.15) we obtain

$$\alpha_\varepsilon^{-1}(s) - \alpha_0^{-1}(s) \geq -c\varepsilon(1+s) + O(\delta).$$

Letting $\delta \to 0$ we get

$$\alpha_\varepsilon^{-1}(s) - \alpha_0^{-1}(s) \geq -c\varepsilon(1+s). \tag{8.3.19}$$

In a similar way, interchanging the roles of α_ε and α_0 we obtain the inequality

$$\alpha_\varepsilon^{-1}(s) - \alpha_0^{-1}(s) \leq c\varepsilon(1+s). \tag{8.3.20}$$

From equations (8.3.19) and (8.3.20) it follows that

$$|\alpha_\varepsilon(s) - \alpha_0(s)| \leq c\varepsilon(1+s)\alpha_\varepsilon(s)\alpha_0(s). \tag{8.3.21}$$

By using Lemma 8.12 we can bound the expression (8.3.21) as

$$|\alpha_\varepsilon(s) - \alpha_0(s)| \leq c(\min\{m_-^{-1}, n_-^{-1}\}\mu_2)^2(1+s)\tau(s)^2\varepsilon,$$

where $\tau(s)$ is given by (8.3.7).

From the convergence of α_ε it follows the convergence of β_ε and of the whole curve. □

The proof of Theorem 8.4 is similar to that of Theorem 8.7 but now we need a result analogous to Theorem 5.4 that works without assuming periodicity. This is the content of the next theorem.

Theorem 8.15. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let g_ε be a function such that $0 < g^- \leq g_\varepsilon \leq g^+ < +\infty$ for $g^\pm$ constants and $g_\varepsilon \rightharpoonup g$ weakly* in $L^\infty(\Omega)$. Then for every $u \in W^{1,p}(\Omega)$,*

$$\lim_{\varepsilon\to 0}\int_\Omega (g_\varepsilon - g)|u|^p = 0$$

where $1 < p < +\infty$.

Proof. The weak* convergence of g_ε in $L^\infty(\Omega)$ says that $\int_\Omega g_\varepsilon \varphi \to \int_\Omega g\varphi$ for all $\varphi \in L^1(\Omega)$. Particularly, $u \in W^{1,p}(\Omega)$ implies that $|u|^p \in W^{1,1}(\Omega)$, it follows that $|u|^p \in L^1(\Omega)$ and the result is proved. □

Proof of Theorem 8.4: The argument follows exactly as in the proof of Theorem 8.7 using the Theorem 8.15 instead of the Theorem 5.4. □

8.4 Proof of the Neumann results

Let us start with a simple remark.

Remark 8.16. Let us observe that $u \in W^{1,p}(\Omega)$ is solution of equation (8.1.10) if and only if u is solution of equation

$$-\Delta_p u + m(u^+)^{p-1} + n(u^-)^{p-1} = \tilde{\alpha} m(u^+)^{p-1} - \tilde{\beta} n(u^-)^{p-1} \quad \text{in } \Omega. \tag{8.4.1}$$

with Neumann boundary conditions, where $\tilde{\alpha} = \alpha - 1$ and $\tilde{\beta} = \beta + 1$. The main advantage between consider equations (8.1.10) and (8.4.1) is the fact that in the second one the functional $A(u)$ defined in (8.1.9) becomes in

$$A_{m,n}(u) = \int_\Omega |\nabla u|^p + m(u^+)^p + n(u^-)^p dx, \tag{8.4.2}$$

which involves both ∇u and the function u.

Having in mind the remark (8.16), the proof of Theorem 8.11 is similar to that of Theorem 8.7 for the Dirichlet case.

Proof of Theorem 8.11: The proof is similar to that of Theorem 8.7 for the Dirichlet case. According to Remark 8.16 we consider equation (8.4.1). Let $(\tilde{\alpha}_\varepsilon, \tilde{\beta}_\varepsilon)$ be a point belonging to the curve $C_1^\varepsilon(m_\varepsilon, n_\varepsilon)$ and let $(\tilde{\alpha}_0, \tilde{\beta}_0)$ be the point obtained when $\varepsilon \to 0$. It follows that $(\tilde{\alpha}_0, \tilde{\beta}_0)$ belongs to the spectrum of the limit equation. Let us see that it belongs to $C(m_0, n_0)$. The main difference is that in the characterization (8.1.12) of $c(m_\varepsilon, n_\varepsilon)$, now we are considering

$$\Gamma = \{\gamma \in C(J, W^{1,p}(\Omega)) : \gamma(0) \geq 0 \text{ and } \gamma(1) \leq 0\}.$$

with $J := [0, 1]$. Fixed a value of $\varepsilon > 0$ we write

$$c(m_\varepsilon, n_\varepsilon) = \inf_{\gamma\in\Gamma}\sup_{u\in\gamma}\frac{A_{m_\varepsilon,n_\varepsilon}(u)}{B_{m_\varepsilon,n_\varepsilon}(u)}. \tag{8.4.3}$$

By (8.1.7) and (8.4.3) we have the following characterization of $\tilde{\alpha}_\varepsilon(s)$

$$\tilde{\alpha}_\varepsilon(s) = c(m_\varepsilon, sn_\varepsilon) = \inf_{\gamma\in\Gamma}\sup_{u\in\gamma}\frac{A_{m_\varepsilon,n_\varepsilon}(u)}{B_{m_\varepsilon,sn_\varepsilon}(u)}. \tag{8.4.4}$$

Similarly, we can consider an equation analog to (8.4.4) for the representation of $\tilde{\alpha}_0(s)$. Let $\delta > 0$ and $\gamma_1 = \gamma_1(\delta) \in \Gamma$ such that

$$\tilde{\alpha}_0(s) = \sup_{u\in\gamma_1}\frac{A_{m_0,n_0}(u)}{B_{m_0,sn_0}(u)} + O(\delta). \tag{8.4.5}$$

In order to find a bound for $\tilde{\alpha}_\varepsilon$ we use $\gamma_1 \in \Gamma$, which is admissible in its variational characterization,

$$\tilde{\alpha}_\varepsilon(s) \leq \sup_{u\in\gamma_1}\frac{A_{m_\varepsilon,sn_\varepsilon}(u)}{B_{m_0,sn_0}(u)}\frac{B_{m_0,sn_0}(u)}{B_{m_\varepsilon,sn_\varepsilon}(u)}. \tag{8.4.6}$$

To bound $\tilde{\alpha}_\varepsilon$ we look for bounds of the two quotients in (8.4.6). Since $u \in W^{1,p}(\Omega)$, by Theorem 5.6 we obtain that

$$\frac{A_{m_\varepsilon,n_\varepsilon}(u)}{B_{m_0,sn_0}(u)} \leq \frac{A_{m_0,n_0}(u)}{B_{m_0,sn_0}(u)} + \frac{c\varepsilon\||u^+|^p\|_{W^{1,1}(\Omega)} + c\varepsilon\||u^-|^p\|_{W^{1,1}(\Omega)}}{B_{m_0,sn_0}(u)}.$$

For every function $u \in \gamma_1$ we have that

$$\frac{A_{m_0,n_0}(u)}{B_{m_0,sn_0}(u)} \leq \sup_{u\in\gamma_1}\frac{A_{m_0,n_0}(u)}{B_{m_0,sn_0}(u)} = \tilde{\alpha}_0(s) + O(\delta). \tag{8.4.7}$$

By using Young inequality

$$\begin{aligned}\||v|^p\|_{W^{1,1}(\Omega)} &= \||v|^p\|_{L^1(\Omega)} + p\||v|^{p-1}\nabla v\|_{L^1(\Omega)}\\ &= \|v\|^p_{L^p(\Omega)} + p\||v|^{p-1}\nabla v\|_{L^1(\Omega)}\\ &\leq p\|v\|^p_{L^p(\Omega)} + \|\nabla v\|^p_{L^p(\Omega)}.\end{aligned} \tag{8.4.8}$$

From (8.4.8) it follows that

$$\begin{aligned}\frac{\||u^+|^p\|_{W^{1,1}(\Omega)}}{B_{m_0,sn_0}(u)} &\leq \frac{p\|u^+\|^p_{L^p(\Omega)} + \|\nabla u^+\|^p_{L^p(\Omega)}}{B_{m_0,sn_0}(u)}\\ &\leq c\frac{A_{m_0,n_0}(u)}{B_{m_0,sn_0}(u)}\\ &\leq c\sup_{u\in\gamma_1}\frac{A_{m_0,n_0}(u)}{B_{m_0,sn_0}(u)}\\ &= c(\tilde{\alpha}_0(s) + O(\delta)),\end{aligned} \tag{8.4.9}$$

and similarly

$$\frac{\||u^-|^p\|_{W^{1,1}(\Omega)}}{B_{m_0,sn_0}(u)} \leq c(\tilde{\alpha}_0(s) + O(\delta)). \tag{8.4.10}$$

To bound the second quotient in (8.4.6), let us observe that

$$\begin{aligned}\frac{\int_\Omega m_0|u^+|^p}{B_{m_\varepsilon,sn_\varepsilon}(u)} &\leq \frac{\int_\Omega m_\varepsilon|u^+|^p}{B_{m_\varepsilon,sn_\varepsilon}(u)} + c\varepsilon\frac{\||u^+|^p\|_{W^{1,1}(\Omega)}}{B_{m_\varepsilon,sn_\varepsilon}(u)}\\ &\leq \frac{\int_\Omega m_\varepsilon|u^+|^p}{B_{m_\varepsilon,sn_\varepsilon}(u)} + c\varepsilon\frac{\||u^+|^p\|_{W^{1,1}(\Omega)}}{B_{m_0,sn_0}(u)},\end{aligned} \tag{8.4.11}$$

and similarly

$$\frac{\int_\Omega sn_0|u^-|^p}{B_{m_\varepsilon,sn_\varepsilon}(u)} \le \frac{\int_\Omega sn_\varepsilon|u^+|^p}{B_{m_\varepsilon,sn_\varepsilon}(u)} + sc\varepsilon\frac{\||u^-|^p\|_{W^{1,1}(\Omega)}}{B_{m_0,sn_0}(u)}. \tag{8.4.12}$$

Now, from equations (8.4.11),(8.4.12) together with (8.4.9) and (8.4.10) we get

$$\begin{aligned}\frac{B_{m_0,sn_0}(u)}{B_{m_\varepsilon,sn_\varepsilon}(u)} &= \frac{\int_\Omega m_0|u^+|^p + \int_\Omega sn_0|u^-|^p}{B_{m_\varepsilon,sn_\varepsilon}(u)} \\ &\le 1 + (1+s)c\varepsilon(\tilde{\alpha}_0(s) + O(\delta)).\end{aligned} \tag{8.4.13}$$

Then combining (8.4.6),(8.4.9),(8.4.10) and ,(8.4.13) we find that

$$\tilde{\alpha}_\varepsilon(s) \le ((\tilde{\alpha}_0(s) + O(\delta)) + c\varepsilon(\tilde{\alpha}_0(s) + O(\delta)))\,(1 + (1+s)c\varepsilon(\tilde{\alpha}_0(s) + O(\delta)))\,.$$

Letting $\delta \to 0$ we get

$$\tilde{\alpha}_\varepsilon(s) - \tilde{\alpha}_0(s) \le c\varepsilon(\tilde{\alpha}_0^2(1+s) + \tilde{\alpha}_0). \tag{8.4.14}$$

In a similar way, interchanging the roles of $\tilde{\alpha}_0$ and $\tilde{\alpha}_\varepsilon$, we obtain

$$\tilde{\alpha}_0(s) - \tilde{\alpha}_\varepsilon(s) \le c\varepsilon(\tilde{\alpha}_\varepsilon^2(1+s) + \tilde{\alpha}_\varepsilon). \tag{8.4.15}$$

From (8.4.14) and (8.4.15) we arrive at

$$|\tilde{\alpha}_0(s) - \tilde{\alpha}_\varepsilon(s)| \le c\varepsilon(1+s)\max\{\tilde{\alpha}_0(s)^2, \tilde{\alpha}_\varepsilon(s)^2\}.$$

Now, using Lemma 8.12,

$$|\alpha_\varepsilon(s) - \alpha_0(s)| \le c(1+s)\tau(s)^2\varepsilon,$$

where c is a constant independent of ε and s, and $\tau(s)$ is given by (8.3.7). Here, Lemma 8.12 holds in the Neumann case, but now we have

$$\alpha(s) \le \min\{m_-^{-1}, n_-^{-1}\}\mu_2\tau(s), \qquad \beta(s) \le \min\{m_-^{-1}, n_-^{-1}\}\mu_2 s\tau(s)$$

with μ_2 the second eigenvalue of the $p-$Laplacian equation on Ω with Neumann boundary conditions. From the convergence of α_ε it follows the convergence of β_ε and of the whole curve. □

Proof of Theorem 8.10: The proof is analogous to the proof of Theorem 8.4. □

Bibliography

[AP96] J. A. Aguilar Crespo and I. Peral Alonso. *On an elliptic equation with exponential growth*. Rend. Sem. Mat. Univ. Padova, vol. 96 (1996), pp. 143–175. URL http://www.numdam.org/item?id=RSMUP_1996__96__143_0. 20

[AG01] M. Alif and J.-P. Gossez. *On the Fučík spectrum with indefinite weights*. Differential Integral Equations, vol. 14 (2001), no. 12, pp. 1511–1530. 10, 112

[Al02] G. Allaire. *Shape optimization by the homogenization method*, *Applied Mathematical Sciences*, vol. 146 (Springer-Verlag, New York, 2002). v, vii, 56, 58, 59, 64, 84

[Am72] H. Amann. *Lusternik-Schnirelman theory and non-linear eigenvalue problems*. Math. Ann., vol. 199 (1972), pp. 55–72. v, vii

[An87] A. Anane. *Simplicité et isolation de la première valeur propre du p-laplacien avec poids*. C. R. Acad. Sci. Paris Sér. I Math., vol. 305 (1987), no. 16, pp. 725–728. 27

[ACFK07] A. Anane, O. Chakrone, M. Filali and B. Karim. *Nodal domains for the p-Laplacian*. Adv. Dyn. Syst. Appl., vol. 2 (2007), no. 2, pp. 135–140. 21

[ACM02] A. Anane, O. Chakrone and M. Moussa. *Spectrum of one dimensional p-Laplacian operator with indefinite weight*. Electron. J. Qual. Theory Differ. Equ., (2002), pp. No. 17, pp 11. 24, 27

[AT96] A. Anane and N. Tsouli. *On the second eigenvalue of the p-Laplacian*. In *Nonlinear partial differential equations (Fès, 1994)*, *Pitman Res. Notes Math. Ser.*, vol. 343, pp. 1–9 (Longman, Harlow, 1996). 27

[An83] S. S. Antman. *The influence of elasticity on analysis: modern developments*. Bull. Amer. Math. Soc. (N.S.), vol. 9 (1983), no. 3, pp. 267–291. URL http://dx.doi.org/10.1090/S0273-0979-1983-15185-6. v, vii

[AC96] M. Arias and J. Campos. *Fučik spectrum of a singular Sturm-Liouville problem*. Nonlinear Anal., vol. 27 (1996), no. 6, pp. 679–697. URL http://dx.doi.org/10.1016/0362-546X(95)00071-3. 112

[ACCG02] M. Arias, J. Campos, M. Cuesta and J.-P. Gossez. *Asymmetric elliptic problems with indefinite weights*. Ann. Inst. H. Poincaré Anal. Non Linéaire, vol. 19 (2002), no. 5, pp. 581–616. URL http://dx.doi.org/10.1016/S0294-1449(01)00093-2. 10, 21, 112, 113, 119

[ACCG08] M. Arias, J. Campos, M. Cuesta and J.-P. Gossez. *An asymmetric Neumann problem with weights*. Ann. Inst. H. Poincaré Anal. Non Linéaire, vol. 25 (2008), no. 2, pp. 267–280. URL http://dx.doi.org/10.1016/j.anihpc.2006.07.006. 10, 113

[BCR06] L. Baffico, C. Conca and M. Rajesh. *Homogenization of a class of nonlinear eigenvalue problems*. Proc. Roy. Soc. Edinburgh Sect. A, vol. 136 (2006), no. 1, pp. 7–22. URL http://dx.doi.org/10.1017/S0308210500004418. v, vii, 26, 27, 31, 65, 82, 84, 86, 95, 115

[BLL87] U. Banerjee, L. J. Lardy and A. Lutoborski. *Asymptotic expansions of integrals of certain rapidly oscillating functions*. Math. Comp., vol. 49 (1987), no. 179, pp. 243–249. URL http://dx.doi.org/10.2307/2008261. 69

[BLP78] A. Bensoussan, J.-L. Lions and G. Papanicolaou. *Asymptotic analysis for periodic structures* (AMS Chelsea Publishing, Providence, RI, 2011). Corrected reprint of the 1978 original [MR0503330]. 68, 81, 93

[Be68] M. S. Berger. *Corrections to my paper: "A Sturm-Liouville theorem for non-linear elliptic partial differential equations"*. Ann. Scuola Norm. Sup. Pisa (3), vol. 22 (1968), pp. 351–354. v, vii

[Be79] M. V. Berry. *Distribution of modes in fractal resonators*. In *Structural stability in physics (Proc. Internat. Symposia Appl. Catastrophe Theory and Topological Concepts in Phys., Inst. Inform. Sci., Univ. Tübingen, Tübingen, 1978)*, *Springer Ser. Synergetics*, vol. 4, pp. 51–53 (Springer, Berlin, 1979). 35

[Be80] M. V. Berry. *Some geometric aspects of wave motion: wavefront dislocations, diffraction catastrophes, diffractals*. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pp. 13–28 (Amer. Math. Soc., Providence, R.I., 1980). 35

[BM76] L. Boccardo and P. Marcellini. *Sulla convergenza delle soluzioni di disequazioni variazionali*. Ann. Mat. Pura Appl. (4), vol. 110 (1976), pp. 137–159. 82, 93

[BCD92] A. Braides, V. Chiadò Piat and A. Defranceschi. *Homogenization of almost periodic monotone operators*. Ann. Inst. H. Poincaré Anal. Non Linéaire, vol. 9 (1992), no. 4, pp. 399–432. 66

[BC86] J. Brossard and R. Carmona. *Can one hear the dimension of a fractal?* Comm. Math. Phys., vol. 104 (1986), no. 1, pp. 103–122. URL http://projecteuclid.org/getRecord?id=euclid.cmp/1104114935. 35

[Br65] F. E. Browder. *Infinite dimensional manifolds and non-linear elliptic eigenvalue problems*. Ann. of Math. (2), vol. 82 (1965), pp. 459–477. v, vii

[CZ00] C. Castro and E. Zuazua. *High frequency asymptotic analysis of a string with rapidly oscillating density*. European J. Appl. Math., vol. 11 (2000), no. 6, pp. 595–622. URL http://dx.doi.org/10.1017/S0956792500004307. 16, 94

[CZ00b] C. Castro and E. Zuazua. *Low frequency asymptotic analysis of a string with rapidly oscillating density*. SIAM J. Appl. Math., vol. 60 (2000), no. 4, pp. 1205–1233 (electronic). URL http://dx.doi.org/10.1137/S0036139997330635. 16, 92, 94

[CVD90] V. Chiadò Piat, G. Dal Maso and A. Defranceschi. *G-convergence of monotone operators*. Ann. Inst. H. Poincaré Anal. Non Linéaire, vol. 7 (1990), no. 3, pp. 123–160. 27, 65, 82

[CW06] S.-K. Chua and R. L. Wheeden. *A note on sharp 1-dimensional Poincaré inequalities*. Proc. Amer. Math. Soc., vol. 134 (2006), no. 8, pp. 2309–2316 (electronic). URL http://dx.doi.org/10.1090/S0002-9939-06-08545-5. 74

[CD99] D. Cioranescu and P. Donato. *An introduction to homogenization*, *Oxford Lecture Series in Mathematics and its Applications*, vol. 17 (The Clarendon Press Oxford University Press, New York, 1999). v, vii, 57

[CH67] C. Clark and D. Hewgill. *One can hear whether a drum has finite area*. Proc. Amer. Math. Soc., vol. 18 (1967), pp. 236–237. 37, 52

[CoHi53] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. I* (Interscience Publishers, Inc., New York, N.Y., 1953). v, vii

[Cu01] M. Cuesta. *Eigenvalue problems for the p-Laplacian with indefinite weights*. Electron. J. Differential Equations, (2001), pp. No. 33, 9. 19, 20

[Cu03] M. Cuesta. *Minimax theorems on C^1 manifolds via Ekeland variational principle*. Abstr. Appl. Anal., (2003), no. 13, pp. 757–768. URL http://dx.doi.org/10.1155/S1085337503303100. 20

[CFG99] M. Cuesta, D. de Figueiredo and J.-P. Gossez. *The beginning of the Fučik spectrum for the p-Laplacian*. J. Differential Equations, vol. 159 (1999), no. 1, pp. 212–238. URL http://dx.doi.org/10.1006/jdeq.1999.3645. 111, 112

[Cu98] M. Cuesta, D. G. de Figueiredo and J.-P. Gossez. *Sur le spectre de Fučik du p-Laplacien*. C. R. Acad. Sci. Paris Sér. I Math., vol. 326 (1998), no. 6, pp. 681–684. URL http://dx.doi.org/10.1016/S0764-4442(98)80030-X. 21

[Da77] E. N. Dancer. *On the Dirichlet problem for weakly non-linear elliptic partial differential equations*. Proc. Roy. Soc. Edinburgh Sect. A, vol. 76 (1976/77), no. 4, pp. 283–300. 112

[DP01] N. Dancer and K. Perera. *Some remarks on the Fučík spectrum of the p-Laplacian and critical groups*. J. Math. Anal. Appl., vol. 254 (2001), no. 1, pp. 164–177. URL http://dx.doi.org/10.1006/jmaa.2000.7228. 112

[SP73] E. De Giorgi and S. Spagnolo. *Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine*. Boll. Un. Mat. Ital. (4), vol. 8 (1973), pp. 391–411. 54

[DGSP73] E. De Giorgi and S. Spagnolo. *Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine*. Boll. Un. Mat. Ital. (4), vol. 8 (1973), pp. 391–411. 93

[De] A. Defranceschi. *An introduction to homogenization and g−convergence*. Notes of lectures in the "School on Homogenization" at the ICTP on September 6-8, 1993. URL http://axp.mat.uniroma2.it/~braides/0102/Dottorato/DefranceschiICTP.pdf. 55

[DDM99] M. del Pino, P. Drábek and R. Manásevich. *The Fredholm alternative at the first eigenvalue for the one-dimensional p-Laplacian*. J. Differential Equations, vol. 151 (1999), no. 2, pp. 386–419. URL http://dx.doi.org/10.1006/jdeq.1998.3506. 23

[DSP03] Z. Denkowski, S. Migórski and N. S. Papageorgiou. *An introduction to nonlinear analysis: applications* (Kluwer Academic Publishers, Boston, MA, 2003). 20

[DGT10] A. Derlet, J.-P. Gossez and P. Takáč. *Minimization of eigenvalues for a quasilinear elliptic Neumann problem with indefinite weight*. J. Math. Anal. Appl., vol. 371 (2010), no. 1, pp. 69–79. URL http://dx.doi.org/10.1016/j.jmaa.2010.03.068. 111

[DP05] O. Došlý and P. Řehák. *Half-linear differential equations*, *North-Holland Mathematics Studies*, vol. 202 (Elsevier Science B.V., Amsterdam, 2005). 20

[Dr92] P. Drábek. *Solvability and bifurcations of nonlinear equations*, *Pitman Research Notes in Mathematics Series*, vol. 264 (Longman Scientific & Technical, Harlow, 1992). 10, 112

[DM99] P. Drábek and R. Manásevich. *On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian*. Differential Integral Equations, vol. 12 (1999), no. 6, pp. 773–788. 23

[DRo02] P. Drábek and S. B. Robinson. *On the generalization of the Courant nodal domain theorem*. J. Differential Equations, vol. 181 (2002), no. 1, pp. 58–71. URL http://dx.doi.org/10.1006/jdeq.2001.4070. 112

[ET06] S. El Habib and N. Tsouli. *On the spectrum of the p-Laplacian operator for Neumann eigenvalue problems with weights*. In *Proceedings of the 2005 Oujda International Conference on Nonlinear Analysis*, *Electron. J. Differ. Equ. Conf.*, vol. 14 (Southwest Texas State Univ., San Marcos, TX, 2006) pp. 181–190 (electronic). 111

[El82] Á. Elbert. *Oscillation and nonoscillation theorems for some nonlinear ordinary differential equations*. In *Ordinary and partial differential equations (Dundee, 1982)*, *Lecture Notes in Math.*, vol. 964, pp. 187–212 (Springer, Berlin, 1982). 107

[Ev10] L. C. Evans. *Partial differential equations*, *Graduate Studies in Mathematics*, vol. 19 (American Mathematical Society, Providence, RI, 2010), 2nd ed. 6, 30, 60

[EG92] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics (CRC Press, Boca Raton, FL, 1992). 28

[Fa90] K. Falconer. *Fractal geometry* (John Wiley & Sons Ltd., Chichester, 1990). Mathematical foundations and applications. 38

[FBP03] J. Fernández Bonder and J. P. Pinasco. *Asymptotic behavior of the eigenvalues of the one-dimensional weighted p-Laplace operator*. Ark. Mat., vol. 41 (2003), no. 2, pp. 267–280. URL `http://dx.doi.org/10.1007/BF02390815`. 24, 25

[FBPS10] J. Fernández Bonder, J. P. Pinasco and A. M. Salort. *Refined asymptotics for eigenvalues on domains of infinite measure*. J. Math. Anal. Appl., vol. 371 (2010), no. 1, pp. 41–56. 17

[FBPS12b] J. Fernández Bonder, J. P. Pinasco and A. M. Salort. *Eigenvalue homogenization for quasilinear elliptic operators in one space dimension, preprint*. arXiv:1203.2091v1, (2012). 17

[FBPS12] J. Fernández Bonder, J. P. Pinasco and A. M. Salort. *Eigenvalue homogenization for quasilinear elliptic operators, preprint*. arXiv:1201.1219v2, (2012). 17, 97

[FBR02] J. Fernández Bonder and J. D. Rossi. *A nonlinear eigenvalue problem with indefinite weights related to the Sobolev trace embedding*. Publ. Mat., vol. 46 (2002), no. 1, pp. 221–235. URL `http://dx.doi.org/10.5565/PUBLMAT_46102_12`. 31

[FV93] J. Fleckinger-Pellé and D. G. Vassiliev. *An example of a two-term asymptotics for the "counting function" of a fractal drum*. Trans. Amer. Math. Soc., vol. 337 (1993), no. 1, pp. 99–116. 53

[FH80] H. Fučík and A. Kufner. *Nonlinear differential equations*, *Studies in Applied Mechanics*, vol. 2 (Elsevier Scientific Publishing Co., Amsterdam, 1980). 10, 112

[Fu76] S. Fučík. *Boundary value problems with jumping nonlinearities*. Časopis Pěst. Mat., vol. 101 (1976), no. 1, pp. 69–87. 112

[GAP88] J. García Azorero and I. Peral Alonso. *Comportement asymptotique des valeurs propres du p-laplacien*. C. R. Acad. Sci. Paris Sér. I Math., vol. 307 (1988), no. 2, pp. 75–78. 20, 90, 91

[GGP02] T. Godoy, J.-P. Gossez and S. Paczka. *On the antimaximum principle for the p-Laplacian with indefinite weight*. Nonlinear Anal., vol. 51 (2002), no. 3, pp. 449–467. URL `http://dx.doi.org/10.1016/S0362-546X(01)00839-2`. 114

[HL97] C. Q. He and M. L. Lapidus. *Generalized Minkowski content, spectrum of fractal drums, fractal strings and the Riemann zeta-function*. Mem. Amer. Math. Soc., vol. 127 (1997), no. 608, pp. x+97. 13, 35, 36, 38, 39, 41

[He76] D. A. Hejhal. *The Selberg trace formula and the Riemann zeta function*. Duke Math. J., vol. 43 (1976), no. 3, pp. 441–482. 36

[He74] D. Hewgill. *On the eigenvalues of a second order elliptic operator in an unbounded domain*. Pacific J. Math., vol. 51 (1974), pp. 467–476. 37, 52

[He75] D. E. Hewgill. *On the growth of the eigevalues of an elliptic operator in a quasi-bounded domain*. Arch. Rational Mech. Anal., vol. 56 (1974/75), pp. 367–371. 37, 52

[Hor68] L. Hörmander. *The spectral function of an elliptic operator*. Acta Math., vol. 121 (1968), pp. 193–218. v, vii

[Hor07] L. Hörmander. *The analysis of linear partial differential operators. III*. Classics in Mathematics (Springer, Berlin, 2007). Pseudo-differential operators, Reprint of the 1994 edition. v, vii

[IL88] T. Iwaniec and A. Lutoborski. *Asymptotic expansions of multiple integrals of rapidly oscillating functions*. Math. Comp., vol. 50 (1988), no. 181, pp. 215–228. URL http://dx.doi.org/10.2307/2007925. 69

[Ka66] M. Kac. *Can one hear the shape of a drum?* Amer. Math. Monthly, vol. 73 (1966), no. 4, part II, pp. 1–23. 34

[KL06] B. Kawohl and P. Lindqvist. *Positive eigenfunctions for the p-Laplace operator revisited*. Analysis (Munich), vol. 26 (2006), no. 4, pp. 545–550. URL http://dx.doi.org/10.1524/anly.2006.26.4.545. 27

[KLP07] B. Kawohl, M. Lucia and S. Prashanth. *Simplicity of the principal eigenvalue for indefinite quasilinear problems*. Adv. Differential Equations, vol. 12 (2007), no. 4, pp. 407–434. 28, 29, 31, 82

[KLS11] C. E. Kenig, F. Lin and Z. Shen. *Convergence rates in l^2 for elliptic homogenization problems, to appear in archive for rational mechanics and analysis*. arXiv:1103.0023v1, (2011). 15, 83, 84, 92

[Ke79b] S. Kesavan. *Homogenization of elliptic eigenvalue problems. I*. Appl. Math. Optim., vol. 5 (1979), no. 2, pp. 153–167. URL http://dx.doi.org/10.1007/BF01442551. 82, 93, 95

[Ke79] S. Kesavan. *Homogenization of elliptic eigenvalue problems. II*. Appl. Math. Optim., vol. 5 (1979), no. 3, pp. 197–216. URL http://dx.doi.org/10.1007/BF01442554. 93, 95

[KS00] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*, *Classics in Applied Mathematics*, vol. 31 (Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000). Reprint of the 1980 original. 65

[Kr88] E. Krätzel. *Lattice points*, *Mathematics and its Applications (East European Series)*, vol. 33 (Kluwer Academic Publishers Group, Dordrecht, 1988). 36, 41

[La91] M. L. Lapidus. *Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture*. Trans. Amer. Math. Soc., vol. 325 (1991), no. 2, pp. 465–529. URL http://dx.doi.org/10.2307/2001638. 35, 36, 38

[La93] M. L. Lapidus and C. Pomerance. *The Riemann zeta-function and the one-dimensional Weyl-Berry conjecture for fractal drums*. Proc. London Math. Soc. (3), vol. 66 (1993), no. 1, pp. 41–69. 13, 35, 37, 39, 44

[An06] A. Lê. *Eigenvalue problems for the p-Laplacian*. Nonlinear Anal., vol. 64 (2006), no. 5, pp. 1057–1099. URL http://dx.doi.org/10.1016/j.na.2005.05.056. 20

[LM07] L. Leadi and A. Marcos. *On the first curve in the Fučik spectrum with weights for a mixed p-Laplacian*. Int. J. Math. Math. Sci., (2007), pp. Art. ID 57607, 13. 112

[LW99] R. Lemmert and W. Walter. *Singular nonlinear boundary value problems*. Appl. Anal., vol. 72 (1999), no. 1-2, pp. 191–203. URL http://dx.doi.org/10.1080/00036819908840737. 112

[LV96] M. Levitin and D. Vassiliev. *Spectral asymptotics, renewal theorem, and the Berry conjecture for a class of fractals*. Proc. London Math. Soc. (3), vol. 72 (1996), no. 1, pp. 188–214. 53

[Li90] P. Lindqvist. *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$. Proc. Amer. Math. Soc., vol. 109 (1990), no. 1, pp. 157–164. URL http://dx.doi.org/10.2307/2048375. 27

[Li69] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires* (Dunod, 1969). 65

[Ma85] V. G. Maz'ja. *Sobolev spaces*. Springer Series in Soviet Mathematics (Springer-Verlag, Berlin, 1985). Translated from the Russian by T. O. Shaposhnikova. 71, 84

[Me76] G. Métivier. *Valeurs propres de problèmes aux limites elliptiques irrégulières*. Bull. Soc. Math. France Suppl. Mém., , no. 51–52. 34

[MT97] F. Murat and L. Tartar. *H-convergence*. In *Topics in the mathematical modelling of composite materials*, *Progr. Nonlinear Differential Equations Appl.*, vol. 31, pp. 21–43 (Birkhäuser Boston, Boston, MA, 1997). 56

[Ng89] G. Nguetseng. *A general convergence result for a functional related to the theory of homogenization*. SIAM J. Math. Anal., vol. 20 (1989), no. 3, pp. 608–623. URL http://dx.doi.org/10.1137/0520043. 64

[OSY92] O. A. Oleĭnik, A. S. Shamaev and G. A. Yosifian. *Mathematical problems in elasticity and homogenization*, *Studies in Mathematics and its Applications*, vol. 26 (North-Holland Publishing Co., Amsterdam, 1992). v, vii, 13, 15, 69, 75, 81, 83

[Pe04] K. Perera. *On the Fučík spectrum of the p-Laplacian*. NoDEA Nonlinear Differential Equations Appl., vol. 11 (2004), no. 2, pp. 259–270. URL http://dx.doi.org/10.1007/s00030-003-1061-6. 112

[Pi06] J. P. Pinasco. *Asymptotic of eigenvalues and lattice points*. Acta Math. Sin. (Engl. Ser.), vol. 22 (2006), no. 6, pp. 1645–1650. 37, 45, 52

[Pi07] J. P. Pinasco. *The asymptotic behavior of nonlinear eigenvalues*. Rocky Mountain J. Math., vol. 37 (2007), no. 6, pp. 1981–1988. URL http://dx.doi.org/10.1216/rmjm/1199649834. 108

[SePu84] P. Pucci and J. Serrin. *The strong maximum principle revisited*. J. Differential Equations, vol. 196 (2004), no. 1, pp. 1–66. URL http://dx.doi.org/10.1016/j.jde.2003.05.001. 19

[Ra74] P. H. Rabinowitz. *Variational methods for nonlinear eigenvalue problems*. In *Eigenvalues of non-linear problems (Centro Internaz. Mat. Estivo (C.I.M.E.), III Ciclo, Varenna, 1974)*, pp. 139–195 (Edizioni Cremonese, Rome, 1974). 20

[Ry00] B. P. Rynne. *The Fučík spectrum of general Sturm-Liouville problems*. J. Differential Equations, vol. 161 (2000), no. 1, pp. 87–109. URL http://dx.doi.org/10.1006/jdeq.1999.3661. 10

[RW00] B. P. Rynne. *The Fučík spectrum of general Sturm-Liouville problems*. J. Differential Equations, vol. 161 (2000), no. 1, pp. 87–109. URL http://dx.doi.org/10.1006/jdeq.1999.3661. 112

[Sa12] A. M. Salort. *Convergence rates in a weighted fucik problem, preprint*. arXiv:1205.2075v2, (2012). 17

[SP70] E. Sánchez-Palencia. *Équations aux dérivées partielles dans un type de milieux hétérogènes*. C. R. Acad. Sci. Paris Sér. A-B, vol. 272 (1971), pp. A1410–A1413. 81, 93

[SV93] F. Santosa and M. Vogelius. *First-order corrections to the homogenized eigenvalues of a periodic composite medium*. SIAM J. Appl. Math., vol. 53 (1993), no. 6, pp. 1636–1668. URL http://dx.doi.org/10.1137/0153076. v, vii

[SaVo93] F. Santosa and M. Vogelius. *First-order corrections to the homogenized eigenvalues of a periodic composite medium*. SIAM J. Appl. Math., vol. 53 (1993), no. 6, pp. 1636–1668. URL http://dx.doi.org/10.1137/0153076. 83

[Si83] B. Simon. *Nonclassical eigenvalue asymptotics*. J. Funct. Anal., vol. 53 (1983), no. 1, pp. 84–98. 37, 52

[Sp68] S. Spagnolo. *Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche*. Ann. Scuola Norm. Sup. Pisa (3) 22 (1968), 571-597; errata, ibid. (3), vol. 22 (1968), p. 673. 54, 82, 93

[Sp76] S. Spagnolo. *Convergence in energy for elliptic operators*. In *Numerical solution of partial differential equations, III (Proc. Third Sympos. (SYNSPADE), Univ. Maryland, College Park, Md., 1975)*, pp. 469–498 (Academic Press, New York, 1976). 54, 58

[Sz88] A. Szulkin. *Ljusternik-Schnirelmann theory on C^1-manifolds*. Ann. Inst. H. Poincaré Anal. Non Linéaire, vol. 5 (1988), no. 2, pp. 119–139. URL http://www.numdam.org/item?id=AIHPC_1988__5_2_119_0. 20

[Ta78] L. Tartar. *Quelques remarques sur les systèmes semi-linéaires*. vol. 7 (1978), pp. 198–212. 54, 64

[Ta77] L. Tartar. *Estimation de coefficients homogénéisés*. In *Computing methods in applied sciences and engineering (Proc. Third Internat. Sympos., Versailles, 1977), I*, *Lecture Notes in Math.*, vol. 704, pp. 364–373 (Springer, Berlin, 1979). 64

[Ti46] E. C. Titchmarsh. *Eigenfunction Expansions Associated with Second-Order Differential Equations* (Oxford, at the Clarendon Press, 1946). v, vii

[Tr82] C. Tricot, Jr. *Two definitions of fractional dimension*. Math. Proc. Cambridge Philos. Soc., vol. 91 (1982), no. 1, pp. 57–74. 38

[VL01] M. van den Berg and M. Lianantonakis. *Asymptotics for the spectrum of the Dirichlet Laplacian on horn-shaped regions*. Indiana Univ. Math. J., vol. 50 (2001), no. 1, pp. 299–333. 37, 51, 52

[Wa98] W. Walter. *Sturm-Liouville theory for the radial Δ_p-operator*. Math. Z., vol. 227 (1998), no. 1, pp. 175–185. URL http://dx.doi.org/10.1007/PL00004362. 24

[We12] H. Weyl. *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*. Math. Ann., vol. 71 (1912), no. 4, pp. 441–479. URL http://dx.doi.org/10.1007/BF01456804. 34

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