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Ecuaciones diferenciales no lineales bajo perturbaciones del dominio

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Ecuaciones diferenciales no lineales bajo perturbaciones del dominio.

(Resumen)

En esta tesis estudiamos el comportamiento de ecuaciones diferenciales no lineales bajo perturbaciones del dominio.

En este sentido, extendemos el teorema de V. Šverák presentando condiciones capacitarias suficientes sobre la convergencia de dominios para asegurar la continuidad de las soluciones del $p(x)-$laplaciano y el $p-$laplaciano fraccionario y luego analizamos la derivada de forma de un funcional de costo que surge en el trabajo con restauración de imágenes.

Key Words: Espacios de exponente variable, Problemas no lineales, Perturbaciones del dominio, Derivada de forma.

Nonlinear differential equations under perturbations of the domain.

(Abstract)

In this thesis we study the behaviour of nonlinear differential equations under perturbations of the domain.

In this sense, we present extensions of V. Šverák's theorem giving capacitary conditions on the convergence of the domains in order to obtain the continuity of the solutions for the $p(x)-$laplacian and the fractional $p-$laplacian and then we analize the shape derivative of a cost functional that appears naturally in image restoration.

Key Words: Exponent variable spaces, Nonlinear problems, Perturbations of the domain, Shape derivative.

Chapter 1

Introducción.

Un problema importante en el área de las ecuaciones diferenciales es la estabilidad de las soluciones con respecto a perturbaciones del dominio. Este problema tiene aplicaciones fundamentales en el cómputo numérico de las soluciones y es fundamental el los problemas de diseño óptimo de formas. Ver [5, 28, 42] y las referencias que incluyen.

El famoso ejemplo de Cioranescu y Murat [10] muestra que este problema presenta severas dificultades cuando se trata en toda generalidad. De hecho, en [10] los autores toman $D = [0,1] \times [0,1] \subset \mathbb{R}^2$, definen los dominios $\Omega_n = D \setminus \cup_{i,j=1}^{n-1} B_{r_n}(x_{i,j}^n)$, donde los centros de las bolas son $x_{i,j}^n = (i/n, j/n)$, $1 \leq i, j \leq n-1$ y los radios $r_n = n^{-2}$, y muestran que estos Ω_n convergen a un conjunto vacío en la topología complementaria de Hausdorff, pero si $u_n \in H_0^1(\Omega_n)$ es la solución de

$$\begin{cases} -\Delta u_n = f & \text{en } \Omega_n, \\ u_n = 0 & \text{en } \partial\Omega_n, \end{cases}$$

entonces $u_n \rightharpoonup u^*$ débil en $H_0^1(D)$ a la solución de

$$\begin{cases} -\Delta u^* + \frac{2}{\pi} u^* = f & \text{en } D, \\ u^* = 0 & \text{en } \partial D. \end{cases}$$

Este ejemplo puede generalizarse a espacios de otras dimensiones, a diferentes conjuntos acotados D y también a diferentes tipos de *holes*. Ver [10] y [48].

Hay algunos casos simples en los que la continuidad puede asegurarse. Por ejemplo, si Ω es convexo y $\{\Omega_n\}_{n\in\mathbb{N}}$ es una sucesión creciente de polígonos convexos tal que $\Omega = \cup_{n\in\mathbb{N}}\Omega_n$, entonces las soluciones de los dominios aproximantes Ω_n convergen a la correspodiente al dominio Ω. Este hecho fue analizado por primera vez a fines de los años 50 y principios de los 60, ver [6, 29, 30, 31].

Algunos años más tarde, en [47], el autor generalizó este resultado y dio condiciones suficientes en términos de la capacidad de las diferencias simétricas de Ω y Ω_n para obtener un resultado positivo. Ver el libro de Henrot, [28] para una prueba.

Más aún, en [47], el autor reemplazó esta condición capacitaria por una geométrica que es más simple de aplicar, pero precisa una restricción de la dimensión N que debe ser igual a 2. Esta restricción proviene del hecho de que en dimensión 2, las curvas tienen capacidad positiva. El problema en dimensiones más altas que 2 permanece abierto. Esta restricción geométrica es que el número de componentes conexas del complemento de Ω_n debe permanecer acotado. Esto debe compararse con el ejemplo de Cioranescu y Murat [10].

En las últimas décadas, ha habido un interés creciente en el estudio de algunos modelos que surgen de los llamados *fluidos no newtonianos*. Se trata de fluidos cuya viscosidad varía con respecto a la velocidad del fluido. El problema modelo para estos fluidos es el llamado operador p-Laplaciano, que se define como $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Este operador es una PDE cuasilineal que resulta singular para $1 < p < 2$, degenerado para $2 < p < \infty$ y cuando $p = 2$ coincide con el operador Laplaciano clásico Δ.

Por lo tanto, las extensiones de los problemas de continuidad con respecto al dominio en el caso del $p-$Laplaciano aparecen naturalmente en el área de las PDE.

El ejemplo dado por Cioranescu-Murat luego se extendió al caso del $p-$Laplaciano en [14] y en [7] los autores probaron la continuidad en el caso no lineal bajo hipótesis similares sobre la convergencia de los dominios a las presentadas en [47]. Más aún, los autores en [7] pudieron obtener la misma condición geométrica que en [47] bajo la restricción $p \geq N$. Una vez más, esta restricción proviene del hecho de que las curvas tienen $p-$capacidad positiva si y sólo si $p \geq N$. Observemos que en el caso del operador Laplaciano, es decir cuando $p = 2$, esta restricción nos deja con $N = 2$.

Finalmente, estudiamos el problema de obtener una imagen que es modelada por una función $u\colon \Omega \to \mathbb{R}$, donde $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, dada una imagen distorsionada $I\colon \Omega \to \mathbb{R}$.

Asumimos que el error introducido, $e = u - I$, es pequeño y el objetivo es recuperar u a partir de I sin hipótesis adicionales sobre el error e. En nuestro trabajo nos enfocaremos en dos métodos basados en EDP. El primero fue introducido por Chambolle y Lions en [8] en 1997. En su paper, la imagen es hallada minimizando el funcional

$$\min \frac{1}{2\beta}\left(\int_{\{|\nabla v|\leq\beta\}} |\nabla v|^2\,dx + \int_{\{|\nabla v|>\beta\}} |\nabla v|\,dx\right) + \frac{\beta}{2}\int_\Omega (v - I)^2\,dx,$$

donde $\beta > 0$ es un parámettro que debe ser ajustado por el operador del método para cada imagen. La idea detrás de este método es que la imagen real debe ser suave en las regiones que no contengan fronteras (interpretadas como regiones en las que las derivadas no son grandes) y, en las que contienen fronteras, la solución debe admitir discontinuidades. El método puede reescribirse como

$$\min \frac{1}{2\beta}\int_\Omega |\nabla v|^{p(|\nabla v|)}\,dx + \frac{\beta}{2}\int_\Omega (v - I)^2\,dx,$$

donde el exponente p se define como

$$p(t) = \begin{cases} 2 & \text{si } t \leq \beta \\ 1 & \text{si } t > \beta. \end{cases}$$

Este método es extremadamente difícil de estudiar rigurosamente ya que el espacio en el que está definido el funcional no es un buen espacio funcional. Ésa es la razón por la cual, en 2006, Chen, Levine y Rao introdujeron en [9] una modificación por la cual el exponente p se computa a partir de I pero es fijo. En este segundo modelo,

$$p(x) = 1 + \frac{1}{1 + k|\nabla G_\sigma * I|^2},$$

donde $G_\sigma(x) = \frac{1}{\sigma}\exp(-|x|^2/4\sigma^2)$ es el filtro Gaussiano, con parámetros $k, \sigma > 0$. Entonces, $p \sim 1$ donde I es discontinuo y $p \sim 2$ donde I es suave.

Luego, el problema a minimizar es

$$\min \frac{1}{2\beta}\int_\Omega |\nabla v|^{p(x)}\,dx + \frac{\beta}{2}\int_\Omega (v - I)^2\,dx.$$

Considerando un exponente fijo regular, los autores pueden utilizar espacios de Sobolev y Lebesgue con exponente variables, ampliamente estudiados desde los sesenta. Ver [18].

1.1 Nuestros resultados

1.1.1 Problemas con crecimiento no standard

En estos problemas el operador es no homogéneo para todo p. Un posible acercamiento fue dado por Orlicz:

$$\Delta_\phi u = -\operatorname{div}\Big(\phi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big)$$

, donde $\phi = \Phi'$ para alguna función Φ. Observemos que si $\Phi = \frac{t^p}{p}$, entonces $\Delta_\phi = \Delta_p$.

Sin embargo, este no es el acercamiento que nosotros buscamos. En nuestro trabajo consideramos $\Phi(x, t)$. Esta función es homogénea para cada x fijo pero la homonegeidad depende del punto.

Uno de los representantes más destacados de estas ecuaciones es el llamado $p(x)-$laplaciano, que se define como $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. Este operador se volvió muy popular debido a muchas nuevas e interesantes aplicaciones, por ejemplo en el modelado matemático de fluidos electrorheológicos (ver [44]) y en procesamiento de imágenes (ver [9]). En estos casos, el exponente $p(x)$ se asume medible y acotado lejos de 1 e infinito.

Por lo tanto, nuestro primer objetivo en esta tesis es presentar una extensión de los resultados de continuidad con respecto al dominio para el caso del exponente variable.

En aplicaciones prácticas, cuando no se tiene control de la sucesión de dominios aproximantes, esta hipótesis no puede chequearse, por lo que se necesita otra condición. Šverák en [47] dio tal condición. De hecho, dado un dominio acotado $D \subset \mathbb{R}^2$ y una sucesión de dominios $\Omega_n \subset D$ tales que $\Omega_n \to \Omega$ en el sentido de la topología complementaria de Hausdorff la condición que garantiza la convergencia de las soluciones en Ω_n a la solución en Ω es que el número de componentes conexas de $D \setminus \Omega_n$ sea acotado. Es interesante comparar con el ejemplo de Cioranescu-Murat.

Recordemos que la razón por la cual el resultado de Šverák vale en dimensión 2 es que la capacidad de las curvas en dimensión 2 es positiva, mientras que en dimensiones más altas las curvas tienen capacidad nula.

El resultado de Šverák luego fue generalizado a ecuaciones elípiticas no lineales del tipo $p-$Laplaciano. De hecho, en [7], los autores prueban la continuidad de las soluciones de

$$\begin{cases} -\Delta_p u_n = f & \text{en } \Omega_n \subset \mathbb{R}^N, \\ u_n = 0 & \text{en } \partial\Omega_n, \end{cases}$$

cuando los dominios Ω_n convergen a Ω en el sentido de la topología complementaria de Hausdorff asumiendo que el número de componentes conexas de sus complementos permanece acotado. La idea de la prueba es similar a la original de Šverák y entonces se termina teniendo la restricción $p > N - 1$ necesaria para que las curvas tengan $p-$capacidad positiva.

Nuestro objetivo en esta tesis es presentar una extensión del resultado de Šverák (y también del de [7]) al caso de exponente variable.

1.1.2 Problemas con difusión fractionaria

En los últimos años, ha habido un creciente interés en problemas no locales debido a que estos operadores demostraron poseer nuevas e interesantes aplicaciones, como algunos modelos para física [19, 21, 25, 34, 40, 52], finanzas [4, 35, 45], dinámica de fluidos [12], ecología [32, 38, 43] y procesamiento de imágenes [27].

La dificultad en este tipo de problemas es que para poder conocer el valor del operador aplicado a la función en cierto punto, es necesario conocer el valor de la función en todo el dominio y no sólo en un entorno del punto.

En particular, el llamado operador $(s, p)-$laplaciano ha sido extensamente estudiado y hasta el momento es casi imposible dar una lista exhaustiva de referencias. Ver por ejemplo [16, 15] y las referencias que contiene.

El operador $(s, p)-$laplaciano se define como

$$(-\Delta_p)^s u(x) := 2 \text{ p.v.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} \, dy,$$

salvo una constante de normalización. El término p.v. debe leerse como *valor principal*.

Dados Ω and f, el problema

$$\begin{cases} (-\Delta_p)^s u = f & \text{en } \Omega \\ u = 0 & \text{en } \mathbb{R}^N \setminus \Omega, \end{cases}$$

tiene una única solución y notamos a esta función u_Ω^f.

Nuestro segundo objetivo en esta tesis es extender los resultados de contiuidad con respecto del domino para estos operadores no locales. La pregunta que nos hacemos es la siguiente. Asumiendo que la sucesión de dominios $\{\Omega_k\}_{k\in\mathbb{N}}$ es tal que $\Omega_k \to \Omega$ en una noción de convergencia de conjuntos apropiada. Entonces es cierto que $u^f_{\Omega_k} \to u^f_\Omega$ en algún sentido? O, más generalmente, dar condiciones necesarias y/o suficientes para que ello ocurra.

Recientemente [23] extendió el contraejemplo de Cioranescu-Murat al $(s,p)-$laplaciano así que, como en el caso clásico, no se puede esperar una respuesta positiva en toda generalidad.

Luego, nuestro objetivo en este trabajo es hallar condiciones capacitarias sobre las diferencias simétricas $\Omega_k \triangle \Omega$ para poder obtener la convergencia de las soluciones $u^f_{\Omega_k} \to u^f_\Omega$.

1.1.3 Un modelo en restauración de imágenes

Finalmente, presentamos una aplicacón a restauración de imágenes relacionada al $p(x)$-laplaciano.

Damos un método que aproxima al creado por Chambolle y Lions preservando las buenas propiedades funcionales dada por el método de Chen, Levine y Rao. Comenzamos por dividir la región Ω en dos subregiones D_1 y D_2 tales que $i = 1, 2$,

$$D_i \subset \Omega \text{ es abierto, } \overset{\circ}{\overline{D_i}} = D_i,\ D_1 \cap D_2 = \emptyset, \text{ y } \overline{\Omega} = \overline{D_1} \cup \overline{D_2}. \tag{1.1}$$

Gracias a esta partición, nos aseguramos que D_1 contiene las regiones con fronteras de la imagen y D_2 su complemento. Una manera de crear esta partición es la siguiente:

$$D_1 = \{x \in \Omega\colon |\nabla G_\sigma * I| > \beta\}, \quad D_2 = \{x \in \Omega\colon |\nabla G_\sigma * I| < \beta\}.$$

Definimos ahora un exponente $p\colon \Omega \to \mathbb{R}$ dado por

$$p(x) = \begin{cases} 1+\epsilon & \text{si } x \in D_1 \\ 2 & \text{si } x \in D_2. \end{cases}$$

Luego computamos u minimizando el funcional

$$J(v) = \frac{1}{2\beta}\int_\Omega |\nabla v|^{p(x)}\,dx + \frac{\beta}{2}\int_\Omega (v-I)^2\,dx.$$

Para mejorar la imagen hallada, se aplica luego un *steepest descent type method* iterativo.

Un problema relacionado fue estudiado en [1]. En ese artículo se demuestra qye los minimizantes son hölder-continuos a través de la interfase.

Por lo tanto, para computar el gradiente de $J(u)$ con respecto a D_i, consideramos la derivada de forma (o derivada de Hadamard), que describimos a continuación. Dado $V\colon \mathbb{R}^N \to \mathbb{R}^N$ un campo de deformación Lipschitz, el flujo asociado $\{\Phi_t\}_{t\in\mathbb{R}}$ se define como

$$\begin{cases} \frac{d}{dt}\Phi_t(x) = V(\Phi_t(x)), & t \in \mathbb{R},\ x \in \mathbb{R}^N \\ \Phi_0(x) = x & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

Observemos que $\Phi_t \colon \mathbb{R}^N \to \mathbb{R}^N$ es un grupo de difeomorfismos. Esto es, $\Phi_t \circ \Phi_s = \Phi_{t+s}$ y $\Phi_t^{-1} = \Phi_{-t}$.

Asumiremos que $\mathrm{supp}(V) \subset \Omega$, luego $\Phi_t(\Omega) = \Omega$ para todo $t \in \mathbb{R}$.

Entonces, las regiones D_i son deformadas por Φ_t y obtenemos una familia de particiones $D_i^t = \Phi_t(D_i), \quad i = 1, 2$ que verifica (6.10) y definimos

$$p_t(x) = \begin{cases} 1 + \epsilon & \text{si } x \in D_1^t \\ 2 & \text{si } x \in D_2^t. \end{cases}$$

Observemos que $p_t = p \circ \Phi_{-t}$.

Luego, para cada $t \in \mathbb{R}$ definimos el funcional

$$J_t(v) = \frac{1}{2\beta} \int_\Omega |\nabla v|^{p_t(x)}\, dx + \frac{\beta}{2} \int_\Omega (v - I)^2\, dx,$$

Sea u_t el minimizante de J_t. Podemos considerar la función $j \colon \mathbb{R} \to \mathbb{R}$ dada por $j(t) = J_t(u_t)$.

La derivada de forma consiste entonces en computar $j'(0)$.

En esta tesis hallamos una buena expresión para tal derivada de forma tal que es posible computar el campo de deformaciones V que la hace tan negativo como sea posible y entonces elegir el campo de deformaciones óptimo para luego iterar

$$D_i^{\Delta t} \simeq (id + \Delta t V)(D_i).$$

Chapter 2

Introduction.

2.1 History of the problem.

One important problem in partial differential equations is the stability of solutions with respect to perturbations on the domain. This problem has fundamental applications in numerical computations of the solutions and is also fundamental in optimal shape design problems. See [5, 28, 42] and references therein.

The famous example of Cioranescu and Murat [10] shows that this problem presents severe difficulties when treated in full generality. In fact, in [10] the authors take $D = [0,1]\times[0,1] \subset \mathbb{R}^2$ and define the domains $\Omega_n = D \setminus \cup_{i,j=1}^{n-1} B_{r_n}(x_{i,j}^n)$ where the centers of the balls $x_{i,j}^n = (i/n, j/n)$, $1 \le i, j \le n-1$ and the radius $r_n = n^{-2}$. Then these domains Ω_n converge to the empty set in the Hausdorff complementary topology, but if $u_n \in H_0^1(\Omega_n)$ is the solution to

$$\begin{cases} -\Delta u_n = f & \text{in } \Omega_n, \\ u_n = 0 & \text{on } \partial\Omega_n, \end{cases}$$

then $u_n \rightharpoonup u^*$ weakly in $H_0^1(D)$ to the solution of

$$\begin{cases} -\Delta u^* + \frac{2}{\pi} u^* = f & \text{in } D, \\ u^* = 0 & \text{on } \partial D. \end{cases}$$

This example can be generalized to other space dimensions, to different bounded sets D and also to different types of *holes*. See the original work [10] and also [48].

There are some simple cases where the continuity can be granted. For instance, if Ω is convex and $\{\Omega_n\}_{n\in\mathbb{N}}$ is an increasing sequence of convex polygons such that $\Omega = \cup_{n\in\mathbb{N}}\Omega_n$, then the solutions of the approximating domains Ω_n converge to the one of Ω. This fact can be traced back to the late 50's and the beginning of the 60's, see [6, 29, 30, 31].

Some years later, in [47], the author generalized this result and gave sufficient conditions in terms of the capacity of the symmetric differences of Ω and Ω_n in order to get a positive result. See the book of Henrot, [28] for a proof.

Moreover, in [47], the author was able to replace this capacitary condition for a geometric one that is simpler to apply, but he ended up with the restriction on the dimension n that has to be equal to 2. This restriction comes from the fact that in dimension 2, curves have positive capacity. The problem in dimensions higher that 2 is still open. This geometric constraint is that the number of connected components of the complement of Ω_n has to remain bounded. This has to be compared with the example of Cioranescu and Murat [10].

In the past decades, there has been an increasing interest in the study of some models coming from the study of the so-called non-newtonian fluids. These are fluids which viscosity varies with respect to the velocity of the flow. The model problem for these fluids is the so-called p-Laplace operator, that is defined as $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$. This operator is a quasilinear PDE that is singular for $1 < p < 2$, degenerate for $2 < p < \infty$ and when $p = 2$ agrees with the classical Laplace operator Δ.

Therefore, the extensions of the continuity problems with respect to the domain to the $p-$Laplacian case appeared naturally in the PDE community.

The example given by Cioranescu-Murat was later extended to the $p-$Laplacian case in [14] and in [7] the authors proved the continuity for the non linear case under similar assumptions on the convergence of the domains to that of [47]. Moreover, the authors in [7] were able to obtain the same geometric condition of [47] under the restriction that $p \geq N$. Again, this restriction comes from the fact that curves have positive $p-$capacity if and only if $p \geq N$. Observe that in the case of the Laplacian operator, i.e. $p = 2$, this restriction leaves us with $N = 2$.

Finally, we study the problem of obtaining an image which is modeled by a function $u\colon \Omega \to \mathbb{R}$, where $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$, given a distorted image $I\colon \Omega \to \mathbb{R}$.

We assume that the introduced error, $e = u - I$, is small and the objective is to recover u from I without making any further assumptions on the error e. We will focus on two EDPs based methods. The first one was introduced by Chambolle and Lions in [8] in 1997. In their paper, the image is found by minimizing the functional

$$\min \frac{1}{2\beta}\left(\int_{\{|\nabla v|\leq\beta\}} |\nabla v|^2\,dx + \int_{\{|\nabla v|>\beta\}} |\nabla v|\,dx\right) + \frac{\beta}{2}\int_\Omega (v-I)^2\,dx,$$

where $\beta > 0$ is a parameter that needs to be adjusted by the operator of the method for each image. The idea behind this method is that the real image must be smooth in regions where there are no boundaries (which are interpreted as regions where the derivatives are not big) and, in the ones which contains boundaries, the solution must admit discontinuities. This method can be re-written as follows

$$\min \frac{1}{2\beta}\int_\Omega |\nabla v|^{p(|\nabla v|)}\,dx + \frac{\beta}{2}\int_\Omega (v-I)^2\,dx,$$

where the exponent p is defined as

$$p(t) = \begin{cases} 2 & \text{if } t \leq \beta \\ 1 & \text{if } t > \beta. \end{cases}$$

This method is extremely difficult to study rigorously since the space where the functional is defined is not a good functional space. That is why, in 2006, Chen, Levine and Rao introduced in [9] a modification by which the exponent p is computed from I but it is fixed. In this second model,

$$p(x) = 1 + \frac{1}{1 + k|\nabla G_\sigma * I|^2},$$

where $G_\sigma(x) = \frac{1}{\sigma}\exp(-|x|^2/4\sigma^2)$ is the Gaussian filter, with $k, \sigma > 0$ parameters. Therefore, $p \sim 1$ where I is discontinuous and $p \sim 2$ where I is smooth.

Then, the problem to be minimized is

$$\min \frac{1}{2\beta}\int_\Omega |\nabla v|^{p(x)}\,dx + \frac{\beta}{2}\int_\Omega (v - I)^2\,dx.$$

By considering a fixed regular exponent, the authors can use the Sobolev and Lebesgue spaces with variable exponent, thoroughly studied since the sixties. See [18].

2.2 Our results

2.2.1 Problems with non standard growth

In these problems the operator is not homogeneous for any p. A possible approach was given by Orlicz:

$$\Delta_\phi u = -\operatorname{div}\left(\phi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right)$$

, where $\phi = \Phi'$ for some function Φ. Observe that if $\Phi = \frac{t^p}{p}$, then $\Delta_\phi = \Delta_p$.

However this is not the approach we were looking for. We will consider $\Phi(x, t)$. This function is homonegeous for each fixed x but the homogenity depends of the point.

One of the most representative of such equations is the so-called $p(x)-$laplacian, that is defined as $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. This operator became very popular due to many new interesting applications, for instance in the mathematical modeling of electrorheological fluids (see [44]) and also in image processing (see [9]). Here, the exponent $p(x)$ is assumed to be measurable and bounded away from 1 and infinity.

So, our first purpose in this thesis is to present an extension of the results of continuity with respect to the domain for the variable exponent setting.

In practical applications, when one does not have control on the sequence of approximating domains, this hypothesis is uncheckable, so a different condition is needed. Šverák in [47] gave such a condition. In fact, given a bounded domain $D \subset \mathbb{R}^2$ and a sequence of domains $\Omega_n \subset D$ such that $\Omega_n \to \Omega$ in the sense of the Hausdorff complementary topology the condition that guarantees the convergence of the solutions in Ω_n to the one in Ω is that the number of connected components of $D \setminus \Omega_n$ be bounded. c.f. with the example of Cioranescu-Murat.

Let us recall that the reason why Šverák's result holds in dimension 2 is because the capacity of curves in dimension 2 is positive, while in higher dimension curves have zero capacity.

Šverák's result was later generalized to nonlinear elliptic equations of $p-$Laplace type. In fact, in [7], the authors prove the continuity of the solutions of

$$\begin{cases} -\Delta_p u_n = f & \text{in } \Omega_n \subset \mathbb{R}^N, \\ u_n = 0 & \text{on } \partial\Omega_n, \end{cases}$$

when the domains Ω_n converges to Ω in the Hausdorff complementary topology under the assumption that the number of connected components of its complements remains bounded. The idea of the proof is similar to the original one of Šverák and so they end up with the restriction $p > N - 1$ that is needed for the curves to have positive $p-$capacity.

Our purpose in this thesis is to present an extension of the result of Šverák (and also the results of [7]) to the variable exponent setting.

2.2.2 Problems with fractional diffusion

In recent years, there has been an increasing amount of attention in nonlocal problems due to some interesting new applications that these operators have shown to possess, such as some models for physics [19, 21, 25, 34, 40, 52], finances [4, 35, 45], fluid dynamics [12], ecology [32, 38, 43] and image processing [27].

The dificulty in this kind of problems is that in order to know the value of the operator applied to a function in a certain point, we need to know the value of the function in all of the domain instead of only in a neighbourhood of the point.

In particular, the so-called $(s, p)-$laplacian operator have been extensively studied and up to date is almost impossible to give an exhaustive list of references. See for instance [16, 15] and references therein.

The $(s, p)-$laplace operator is defined as

$$(-\Delta_p)^s u(x) := 2 \text{ p.v.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} \, dy,$$

up to some normalization constant. The term p.v. stands for *principal value*.

Given Ω and f, the problem

$$\begin{cases} (-\Delta_p)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

has a unique solution and we denote this function by u^f_Ω.

Our second purpose in these thesis is to extend the continuity results respect to the domain for these non local operators. The question that we adress is the following. Assume that we have a sequence of domains $\{\Omega_k\}_{k\in\mathbb{N}}$ such that $\Omega_k \to \Omega$ in a suitable defined notion of convergence of sets. Is it then true that $u^f_{\Omega_k} \to u^f_\Omega$ in some sense? Or more generally, give necessary and/or sufficient conditions for the above statement to hold true.

Recently [23] extended the counterexample of Cioranescu-Murat to the $(s, p)-$laplacian so, as in the classical setting, one cannot expect a positive answer in full generality.

Therefore, our purpose in this work is to find some capacitary conditions on the symmetric diference $\Omega_k \triangle \Omega$ in order to have convergence of the solutions $u^f_{\Omega_k} \to u^f_{\Omega}$.

2.2.3 A model in image restoration

Finally, we present an application to image restoration related to the $p(x)$-laplacian.

We give a method that approximates the one created by Chambolle and Lions preserving the good functional properties given by the one presented by Chen, Levine and Rao. We start by dividing the region Ω into two sub regions D_1 and D_2 such that for $i = 1, 2$,

$$D_i \subset \Omega \text{ is open, } \overline{\mathring{D_i}} = D_i,\ D_1 \cap D_2 = \emptyset, \text{ and } \overline{\Omega} = \overline{D_1} \cup \overline{D_2}. \tag{2.1}$$

By this partition, we make sure that D_1 contains the regions with boundaries of the image and D_2 its complement. One way of creating this partition is the following:

$$D_1 = \{x \in \Omega \colon |\nabla G_\sigma * I| > \beta\}, \quad D_2 = \{x \in \Omega \colon |\nabla G_\sigma * I| < \beta\}.$$

We define an exponent $p\colon \Omega \to \mathbb{R}$ given by

$$p(x) = \begin{cases} 1 + \epsilon & \text{if } x \in D_1 \\ 2 & \text{if } x \in D_2. \end{cases}$$

Then we compute u by minimizing the functional

$$J(v) = \frac{1}{2\beta} \int_\Omega |\nabla v|^{p(x)}\, dx + \frac{\beta}{2} \int_\Omega (v - I)^2\, dx.$$

In order to improve the image found, we then apply an iterative *steepest descent type method*.

A related problem was studied in [1]. In that article it is shown that minimizers are hölder-continuous across the interfase.

So, to compute the gradient of $J(u)$ with respect to D_i, we consider the shape derivative (or Hadamard derivative), which we describe now. Given $V\colon \mathbb{R}^N \to \mathbb{R}^N$ a Lipschitz deformation field, the associated flow $\{\Phi_t\}_{t\in\mathbb{R}}$ is defined by

$$\begin{cases} \frac{d}{dt}\Phi_t(x) = V(\Phi_t(x)), & t \in \mathbb{R},\ x \in \mathbb{R}^N \\ \Phi_0(x) = x & x \in \mathbb{R}^N. \end{cases} \tag{2.2}$$

Let us observe that $\Phi_t\colon \mathbb{R}^N \to \mathbb{R}^N$ is a group of diffeomorfisms. That is, $\Phi_t \circ \Phi_s = \Phi_{t+s}$ and $\Phi_t^{-1} = \Phi_{-t}$.

We will assume that $\text{supp}(V) \subset \Omega$, so that $\Phi_t(\Omega) = \Omega$ for every $t \in \mathbb{R}$.

Then, the regions D_i are deformed by Φ_t and we obtain a family of partitions $D_i^t = \Phi_t(D_i), \quad i = 1, 2$ that verify (6.10) and we define

$$p_t(x) = \begin{cases} 1+\epsilon & \text{if } x \in D_1^t \\ 2 & \text{if } x \in D_2^t. \end{cases}$$

Observe that $p_t = p \circ \Phi_{-t}$.

Then, for each $t \in \mathbb{R}$ we define the following functional

$$J_t(v) = \frac{1}{2\beta}\int_\Omega |\nabla v|^{p_t(x)}\, dx + \frac{\beta}{2}\int_\Omega (v - I)^2\, dx,$$

Let u_t be the minimizer of J_t. We can consider the function $j\colon \mathbb{R} \to \mathbb{R}$ given by $j(t) = J_t(u_t)$.

The shape derivative consists then in computing $j'(0)$.

In this thesis we find a good expression for such derivative, so that is possible to compute the deformations field V which makes it as negative as possible and so choose the optimal deformation field to then iterate

$$D_i^{\Delta t} \simeq (id + \Delta t V)(D_i).$$

2.3 Thesis Description.

After this introduction, the thesis is organized as follows.

In Chapter 2, we revise the definitions and results on variable exponent spaces, fractional order Sobolev spaces and fractional capacities that are needed in the rest of the thesis.

In Part 1, we study the stability of solutions of nonlinear problems with respect to perturbations of the domain. This part of the thesis is divided in Chapter 3, in which we present the fractional case, and Chapter 4, in which we present the variable exponent case.

In Chapter 3 we present our main result on fractional order Sobolev spaces.

Chapter 4 is organized as follows.

In section 4.1. we study the Dirichlet problem for the $p(x)-$laplacian, the main result being the continuity of the solution with respect to the source. Although some of the results are well known, we decided to present the proofs of all of the results since we were unable to find a reference for these.

In section 4.2. we analyze the dependence of the solution of the Dirichlet problem for the $p(x)-$laplacian with respect to variations on the domain. Our two main theorems here are Theorem 4.17 where a capacity condition on the sequence of approximating domains is given in order for the continuity of solutions to hold, and Theorem 4.18 where it is shown that the continuity only depends on the approximating domains and not on the source term.

In section 4.3. after giving some capacity estimates that we needed, we collect all of our results and prove the main result of this part of the thesis, namely the extension of Šverák's result to the variable exponent setting, i.e. Theorem 4.28.

In Part 2, we study the Hadamard derivative in image restoration problems.

Chapter 3

Preliminaries.

Many of the results that we present in the following Preliminaries are wellknown but we include them for this thesis to be selfcontained.

3.1 Variable Exponent Spaces.

The standard reference for this is the book [18]. Some results are slight variations of the ones found in [18] and in these cases we present full proofs of those facts (c.f. Theorem 3.54).

3.1.1 Definitions and well-known results

Given $\Omega \subset \mathbb{R}^N$ an open set, we consider the class of exponents $\mathcal{P}(\Omega)$ given by

$$\mathcal{P}(\Omega) := \{p\colon \Omega \to [1,\infty)\colon p \text{ is measurable and bounded}\}.$$

Given a variable exponent $p \in \mathcal{P}(\Omega)$, the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) := \Big\{u \in L^1_{\mathrm{loc}}(\Omega)\colon \rho_{p(x)}(u) < \infty\Big\},$$

where the modular $\rho_{p(x)}$ is given by

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)}\,dx.$$

This space is endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \|u\|_{p(x),\Omega} = \|u\|_{p(x)} := \inf\Big\{\lambda > 0\colon \rho_{p(x)}(\tfrac{u}{\lambda}) < 1\Big\}.$$

The infimum and the supremum of the exponent p play an important role in the estimates as the next elementary proposition shows. For further references, the following notation will be imposed

$$1 \le p_- := \operatorname*{ess\,inf}_\Omega p \le \operatorname*{ess\,sup}_\Omega p =: p_+ < \infty.$$

The proof of the following proposition can be found in [22, Theorem 1.3, p.p. 427]. We include it for convenience of the reader.

Proposition 3.1. *Let $u \in L^{p(x)}(\Omega)$, then*

$$\min\{\|u\|_{p(x)}^{p_-}, \|u\|_{p(x)}^{p_+}\} \leq \rho_{p(x)}(u) \leq \max\{\|u\|_{p(x)}^{p_-}, \|u\|_{p(x)}^{p_+}\}.$$

Remark 3.2. Proposition 3.1, is equivalent to

$$\min\{\rho_{p(x)}(u)^{\frac{1}{p_-}}, \rho_{p(x)}(u)^{\frac{1}{p_+}}\} \leq \|u\|_{p(x)} \leq \max\{\rho_{p(x)}(u)^{\frac{1}{p_-}}, \rho_{p(x)}(u)^{\frac{1}{p_+}}\}.$$

We first present the statement and proof of a lemma that will be used in the proof of Proposition 3.1.

Lemma 3.3. *Sea $u \in L^{p(x)}(\Omega)$. Entonces,*

1. *Si $\|u\|_{p(x)} > 1$, entonces $\|u\|_{p(x)}^{p_-} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p_+}$.*
2. *Si $\|u\|_{p(x)} < 1$, entonces $\|u\|_{p(x)}^{p_+} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p_-}$.*

Proof of Lemma 3.3. We will prove 1. The proof of 2 is analogous.

Let us call $a = \|u\|_{p(x)}$. Since $a > 1$, we know that $\frac{1}{a^{p_+}} \leq \frac{1}{a^{p(x)}} \leq \frac{1}{a^{p_-}}$ for all $x \in \Omega$. Then,

$$\frac{1}{a^{p_+}}\rho_{p(x)}(u) = \frac{1}{a^{p_+}}\int_\Omega |u(x)|^{p(x)}dx \leq \int_\Omega \left|\frac{u(x)}{a}\right|^{p(x)} dx \leq \frac{1}{a^{p_-}}\int_\Omega |u(x)|^{p(x)}dx = \frac{1}{a^{p_-}}\rho_{p(x)}(u).$$

Since $\|u\|_{p(x)} = a$, then $\displaystyle\int_\Omega \left|\frac{u(x)}{a}\right|^{p(x)} dx = \rho_{p(x)}\Big(\frac{u}{a}\Big) = 1.$

We therefore obtain that $\frac{1}{a^{p_+}}\rho_{p(x)}(u) \leq 1$ and $1 \leq \frac{1}{a^{p_-}}\rho_{p(x)}(u)$, as we wanted to prove. □

Proof of Proposition 3.1. It is enough to note that, by Lemma 3.3, we have that

$$\min\{\|u\|_{p(x)}^{p_+}, \|u\|_{p(x)}^{p_-}\} \leq \rho_{p(x)}(u) \leq \max\{\|u\|_{p(x)}^{p_+}, \|u\|_{p(x)}^{p_-}\},$$

from where we deduce the lemma. □

Proposition 3.4. *Given $u \in L^{p(x)}(\Omega)$, then*

$$\|u\|_{L^{p(x)}(\Omega)} \leq 1 \Leftrightarrow \rho_{p(x)}(u) \leq 1.$$

Proof. Let us assume that $\rho_{p(x)}(u) \leq 1$. Then $1 \in \{\lambda > 0\colon \rho_{p(x)}(\frac{u}{\lambda}) \leq 1\}$ and therefore $\|u\|_{L^{p(x)}(\Omega)} = \inf\{\lambda > 0\colon \rho_{p(x)}(\frac{u}{\lambda}) \leq 1\} \leq 1$.

Let us now assume that $\|u\|_{L^{p(x)}(\Omega)} = \inf\{\lambda > 0\colon \rho_{p(x)}(\frac{u}{\lambda}) \leq 1\} \leq 1$. Then, $\rho_{p(x)}(\frac{u}{\lambda}) \leq 1$ for every $\lambda > 1$. So we can conclude that $\rho_{p(x)}(u) = \lim_{\lambda\to 1^+}\rho_{p(x)}\Big(\frac{u}{\lambda}\Big) \leq 1$, which completes the proof. □

We will use the following form of Hölder's inequality for variable exponents. The proof, which is an easy consequence of Young's inequality, can be found in [18, Lemma 3.2.20].

Proposition 3.5 (Hölder's inequality)**.** *Assume $p_- > 1$. Let $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, then*

$$\int_\Omega |uv|\,dx \le 2\|u\|_{p(x)}\|v\|_{p'(x)},$$

where $p'(x)$ is, as usual, the conjugate exponent, i.e. $p'(x) := p(x)/(p(x)-1)$.

Proof. Let us assume first that $\|u\|_{p(x)} = \|v\|_{p'(x)} = 1$. Then, by Young's inequality, we have

$$\int_\Omega |u(x)v(x)|\,dx \le \int_\Omega \frac{|u(x)|^{p(x)}}{p(x)}\,dx + \int_\Omega \frac{|v(x)|^{p'(x)}}{p'(x)}\,dx \le \int_\Omega |u(x)|^{p(x)}\,dx + \int_\Omega |v(x)|^{p'(x)}\,dx = 2$$

Now, for the general case, let us consider $\tilde{u} = \frac{u}{\|u\|_{p(x)}}$ and $\tilde{v} = \frac{v}{\|v\|_{p'(x)}}$. Then, the proof follows from applying the previous case.

□

Proposition 3.6. *Let $p, q \in \mathcal{P}(\Omega)$ such that $1 < p \le q$ a.e. Ω. Assume that Ω is a set of finite measure. Then,*

$$L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$$

with continuous inclusions.

Proof. The proof is an immediate consecuence of Hölder's inequality. In fact, if $u \in L^{q(x)}(\Omega)$, by Proposition 3.5 and Remark 3.2,

$$\int_\Omega |u(x)|^{p(x)}dx \le 2\||u|^{p(x)}\|_{\frac{q(x)}{p(x)}}\|1\|_{\left(\frac{q(x)}{p(x)}\right)'} \le c\Big(\int_\Omega |u(x)|^{q(x)}dx\Big)^{\alpha_{p,q}}.$$

Since $u \in L^{q(x)}(\Omega)$, we have that $u \in L^{p(x)}(\Omega)$. □

The proof of the next lemma can be found in [18, Proposición 4.6.3, página 125].

Lemma 3.7. *Let K the standard mollifier and $p \in \mathcal{P}^{log}(\mathbb{R}^N)$. Then, for every $u \in L^{p(x)}(\mathbb{R}^N)$*

$$\|u * K_\varepsilon\|_{p(x)} \le c\|u\|_{p(x)}\|K\|_1.$$

Proposition 3.8. *Let K be the standard mollifier, $p \in \mathcal{P}^{log}(\mathbb{R}^N)$ y $u \in L^{p(x)}(\mathbb{R}^N)$. Then, $u * K_\varepsilon \to u$ en $L^{p(x)}(\mathbb{R}^N)$.*

Proof. Let $\delta > 0$. By the density of simple functions in $L^{p(x)}(\mathbb{R}^N)$, there is a simple function v such that $\|u - v\|_{p(x)} \le \delta$. Then,

$$\|u * K_\varepsilon - u\|_{p(x)} \le \|v * K_\varepsilon - v\|_{p(x)} + \|(u - v) * K_\varepsilon - (u - v)\|_{p(x)}.$$

Since g is simple, we know that $g \in L^1(\mathbb{R}^N) \cap L^{p_+}(\mathbb{R}^N)$ and, in consecuence, $v * K_\varepsilon \to v$ in $L^1(\mathbb{R}^N) \cap L^{p_+}(\mathbb{R}^N)$.

By Proposition 3.6, we have that $v * K_\varepsilon \to v$ in $L^{p(x)}(\mathbb{R}^N)$.

By Lemma 3.7,

$$\|(u-v) * K_\varepsilon - (u-v)\|_{p(x)} \leq c\|u-v\|_{p(x)} \leq c\delta.$$

Se we can conclude that

$$\limsup \|u * K_\varepsilon - u\|_{p(x)} \leq c\delta.$$

Therefore $\|u * K_\varepsilon - u\|_{p(x)} \to 0$ when $\varepsilon \to 0$, as we wanted to proove. □

The variable exponent Sobolev space $W^{1,p(x)}$ is defined by

$$W^{1,p(x)}(\mathbb{R}^N) := \Big\{u \in W^{1,1}_{\text{loc}}(\mathbb{R}^N) \colon u \in L^{p(x)}(\mathbb{R}^N) \text{ and } \partial_i u \in L^{p(x)}(\mathbb{R}^N)\ i = 1, \dots, N\Big\},$$

where $\partial_i u$ stands for the $i-$th partial weak derivative of u.

This space posses a natural modular given by

$$\rho_{1,p(x)}(u) := \int_{\mathbb{R}^N} |u|^{p(x)} + |\nabla u|^{p(x)}\, dx,$$

so $u \in W^{1,p(x)}(\mathbb{R}^N)$ if and only if $\rho_{1,p(x)}(u) < \infty$.

The corresponding Luxemburg norm associated to this modular is

$$\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = \|u\|_{1,p(x)} := \inf\Big\{\lambda > 0 \colon \rho_{1,p(x)}(\tfrac{u}{\lambda}) < 1\Big\}.$$

Observe that this norm turns out to be equivalent to $\|u\| := \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$.

In most applications is very helpful to have test functions to be dense in $W^{1,p(x)}(\mathbb{R}^N)$. It is well known, see [18], that this property fails in general, even for continuous exponents $p(x)$. In order to have this desired property one need to impose some regularity conditions on the exponent $p(x)$. This condition was introduced in [51].

Definition 3.9. We say that $p \colon \Omega \to \mathbb{R}$ is (locally) log $-$*Hölder continuous* in Ω if

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \log(e + |x-y|^{-1})|p(x) - p(y)| < \infty. \tag{3.1}$$

Set $\mathcal{P}^{\log}(\Omega) = \{p \in \mathcal{P}(\Omega) \colon p \text{ satisfies } (3.1)\}$.

Theorem 3.10. *Let $p \in \mathcal{P}^{log}(\Omega)$. Then $C^\infty(\Omega) \cap W^{1,p(x)}(\Omega)$ is dense in $W^{1,p(x)}(\Omega)$.*

Proof. Let $u \in W^{1,p(x)}(\Omega)$. We fix $\varepsilon > 0$, define $\Omega_0 = \emptyset$ and, for $m = 1, 2, \dots$ and $x_0 \in \Omega$ fixed, we denote

$$\Omega_m := \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > \frac{1}{m}\} \cap B(x_0, m)$$

and $U_m := \Omega_{m+1} \setminus \overline{\Omega}_{m-1}$.

Consider $\{\xi_m\}_{m\in\mathbb{N}}$ in $C_c^\infty(U_m)$ such that $\sum_{m=1}^\infty \xi_m(x) = 1$ for every $x \in \Omega$.

Let K be the standard mollifier. Then, for each m there is δ_m such that

$$\mathrm{supp}((\xi_m u) * K_{\delta_m}) \subset U_m.$$

By Proposition 3.8, choosing a smaller δ_m if necessary, we obtain that

$$\|\xi_m u - (\xi_m u) * K_{\delta_m}\|_{1,p(x)} \leq \frac{\varepsilon}{2^m}.$$

Let us denote

$$u_\varepsilon := \sum_{m=1}^{\infty} (\xi_m u) * K_{\delta_m}. \tag{3.2}$$

Since $x \in \Omega$ has a neighbourhood where the summ 3.2 only has finite terms that are not zero and, therefore, $u_\varepsilon \in C^\infty(\Omega)$. More over,

$$\begin{aligned}
\|u - u_\varepsilon\|_{1,p(x)} &= \|u - \sum_{m=1}^{\infty} (\xi_m u) * K_{\delta_m}\|_{1,p(x)} \\
&= \|u \sum_{m=1}^{\infty} \xi_m - \sum_{m=1}^{\infty} (\xi_m u) * K_{\delta_m}\|_{1,p(x)} \\
&= \|\sum_{m=1}^{\infty} u\xi_m - (\xi_m u) * K_{\delta_m}\|_{1,p(x)} \\
&\leq \sum_{m=1}^{\infty} \|u\xi_m - (\xi_m u) * K_{\delta_m}\|_{1,p(x)} \\
&\leq \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} \leq \varepsilon.
\end{aligned}$$

So the proof is completed. □

One important subspace of $W^{1,p(x)}(\mathbb{R}^N)$ is the functions with zero boundary values. This is the content of the next definition.

Definition 3.11. We define $W_0^{1,p(x)}(\Omega)$ as the closure in $W^{1,p(x)}(\Omega)$ of functions with compact support.

Now we present the result that shows that, if the exponent is log −Hölder continuous, we have the density of smooth functions.

Theorem 3.12 (Theorem 9.1.6 in [18])**.** *Assume that $p \in \mathcal{P}^{\log}(\Omega)$, then $W_0^{1,p(x)}(\Omega)$ coincides withf the closure of $C_c^\infty(\Omega)$ with respect to the norm $W^{1,p(x)}(\Omega)$.*

Proof. It is clear that any element of the closure of $C_c^\infty(\Omega)$ with respect of the norm of $W^{1,p(x)}(\Omega)$ is an element of $W_0^{1,p(x)}(\Omega) \subset W^{1,p(x)}(\Omega)$.

Let $u \in W_0^{1,p(x)}(\Omega)$. By the Theorem 3.10, we can consider $\{u_i\}_{i\in\mathbb{N}} \subset C^\infty(\Omega) \cap W^{1,p(x)}(\Omega)$ such that u_i converges to u in $W^{1,p(x)}(\Omega)$.

Let $\psi \in C_c^\infty(\Omega)$ such that $0 \le \psi \le 1$ y $\psi = 1$ in the support of u. Since $u_i - \psi u_i = 0$ in the support u, we have

$$\begin{aligned}\|u_i - \psi u_i\|_{W^{1,p(x)}(\Omega)} &= \|u_i - \psi u_i\|_{W^{1,p(x)}(\Omega\setminus \mathrm{supp}(u))}\\ &\le c_\psi \|u_i\|_{W^{1,p(x)}(\Omega\setminus \mathrm{supp}(u))}\\ &= c_\psi \|u_i - u\|_{W^{1,p(x)}(\Omega\setminus \mathrm{supp}(u))}\\ &\le c_\psi \|u_i - u\|_{W^{1,p(x)}(\Omega)}.\end{aligned}$$

Then,

$$\|u - \psi u_i\|_{W^{1,p(x)}(\Omega)} \le \|u - u_i\|_{W^{1,p(x)}(\Omega)} + \|u_i - \psi u_i\|_{W^{1,p(x)}(\Omega)} \le (1 + c_\psi)\|u - u_i\|_{W^{1,p(x)}(\Omega)}.$$

And so, since $\|u_i - u\|_{W^{1,p(x)}(\Omega)}$ converges to 0, we can conclude that ψu_i converges to u in $W^{1,p(x)}(\Omega)$.

Since each $u \in W_0^{1,p(x)}(\Omega)$ can be approximated by functions Sobolev of compact support, we can find a sequence in $C_c^\infty(\Omega)$ that converges to u, as we wanted to prove. □

Now we present the Rellich Kondrachov theorem for the variable exponent case.

Theorem 3.13 (Rellich Kondrachov Theorem.)**.** *Given $p, q \in \mathcal{P}(\Omega)$ continuous exponents such that $p^+ < N$ and $q(x) < \frac{Np(x)}{N-p(x)}$ for each $x \in \bar{\Omega}$.*

Then, the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.

The proof of the following theorem can be found in [18, Theorem 8.2.4].

Theorem 3.14 (Poincaré's inequality.)**.** *Let $p \in \mathcal{P}(\Omega)$ a continuous exponent. Then there exists a constant $c > 0$ such that*

$$\|u\|_{p(x)} \le c\|\nabla u\|_{p(x)}, \quad u \in W_0^{1,p(x)}(\Omega).$$

Proof. Let us suppose that for every n exists $u_n \in W_0^{1,p(x)}(\Omega)$ such that $\|u_n\|_{p(x)} > n\|\nabla u_n\|_{p(x)}$.

Let us consider $v_n = \frac{u_n}{\|u_n\|_{p(x)}}$. Then,

$$\|\nabla v_n\|_{p(x)} = \frac{\|\nabla u_n\|_{p(x)}}{\|u_n\|_{p(x)}} < \frac{1}{n}. \tag{3.3}$$

Then, $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$ and so there exists a subsequence, which we denote $\{v_n\}_{n\in\mathbb{N}}$ and $v \in L^{p(x)}(\Omega)$ such that $v_n \rightharpoonup v$ in $W_0^{1,p(x)}(\Omega)$ and also, by Theorem 3.13,

$$v_n \to v \text{ en } L^{p(x)}(\Omega). \tag{3.4}$$

Let $\varphi \in C_c^\infty(\Omega)$, by Proposition 3.5, (3.3) and (3.4), we have

$$\int_\Omega v(x)\varphi_{x_i}(x)\,dx = \lim_{n\to\infty}\int_\Omega v_n(x)\varphi_{x_i}(x)\,dx$$
$$= -\lim_{n\to\infty}\int_\Omega (v_n)_{x_i}(x)\varphi(x)\,dx = 0$$

So, $\nabla v = 0$ and, then, v is constant. Since $v \in W_0^{1,p(x)}(\Omega)$, we conclude $v = 0$, which is an absurd because $\|v\|_{p(x)} = 1$ (is it enough to observe that $\|v_n\|_{p(x)} = 1$ by definition), which completes the proof. □

Remark 3.15. Thanks to Poincaré inequality, as usual, in $W_0^{1,p(x)}(\Omega)$ with $p \in \mathcal{P}^{\log}(\Omega)$ the following norm will be used,

$$\|u\|_{W_0^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)}.$$

This norm, is equivalent to the usual norm in $W^{1,p(x)}(\Omega)$ for functions $u \in W_0^{1,p(x)}(\Omega)$.

Definition 3.16. We denote by $W^{-1,p'(x)}(\Omega)$ the topological dual space of $W_0^{1,p(x)}(\Omega)$.

The duality product between $u \in W^{-1,p'(x)}(\Omega)$ and $v \in W_0^{1,p(x)}(\Omega)$ will be denoted, as usual, by $\langle u, v\rangle$.

The norm in this space will be denoted by

$$\|u\|_{W^{-1,p'(x)}(\Omega)} = \|u\|_{-1,p'(x)} := \sup\{\langle u, v\rangle \colon v \in W_0^{1,p(x)}(\Omega),\ \|\nabla v\|_{p(x)} \le 1\}.$$

We now present a result which we will find most useful later.

Proposition 3.17. *Given Ω of finite measure, the space $L^\infty(\Omega)$ is dense in $W^{-1,p'(x)}(\Omega)$.*

Proof. By Hölder's inequality we have that $W_0^{1,p_+}(\Omega) \subset W_0^{1,p(x)}(\Omega) \subset W_0^{1,p_-}(\Omega)$ with continuous embeddings. Since $C_c^\infty(\Omega) \subset W_0^{1,p_+}(\Omega)$ and $p \in \mathcal{P}^{\log}(\Omega)$ we have the embeddings are dense. Therefore,

$$W^{-1,(p_-)'}(\Omega) \subset W^{-1,p'(x)}(\Omega) \subset W^{-1,(p_+)'}(\Omega),$$

with dense embeddings. Finally, since $L^\infty(\Omega)$ is dense in $W^{-1,(p_-)'}(\Omega)$, we have that $L^\infty(\Omega)$ is dense in $W^{-1,p'(x)}(\Omega)$. □

Analogous to the constant exponent case, we have the following characterization of $W^{-1,p'(x)}(\Omega)$.

Proposition 3.18. *Let $f \in W^{-1,p'(x)}(\Omega)$. Then, there exists $\{f_i\}_{i=0}^N \subset L^{p'(x)}(\Omega)$ such that*

$$\langle f, u\rangle = \int_\Omega f_0 u\,dx - \sum_{i=1}^N \int_\Omega f_i \partial_i u\,dx.$$

We will then say that $f = f_0 + \sum_{i=1}^N \partial_i f_i$. Moreover, the norm

$$\|f\|_* = \inf\left\{\sum_{i=0}^N \|f_i\|_{p'(x)} \colon f = f_0 + \sum_{i=1}^N \partial_i f_i,\ f_i \in L^{p'(x)}(\Omega), i = 0,\dots,N\right\},$$

verifies

$$\|f\|_{-1,p'(x)} \leq C\|f\|_*.$$

Proof. The characterization of $W^{-1,p'(x)}(\Omega)$ follows exactly as in the constant exponent case. It remains to see the inequality $\|f\|_{-1,p'(x)} \leq C\|f\|_*$.

Observe that $\|\cdot\|_*$ clearly defines a norm in $W^{-1,p'(x)}(\Omega)$.

Let us now take $f_0, f_1, \dots, f_n \in L^{p'(x)}(\Omega)$ such that $f = f_0 + \sum_{i=1}^N \partial_i f_i$ and consider $v \in W_0^{1,p(x)}(\Omega)$ such that $\|\nabla v\|_{p(x)} = 1$.

By Hölder's inequality (Proposition 3.5) and Poincaré's inequality (Theorem 3.14), we have

$$\begin{aligned}\langle f, v\rangle &= \int_\Omega \left(f_0 v + \sum_{i=1}^N f_i \partial_i v \right) dx \\ &\leq 2\|f_0\|_{p'(x)}\|v\|_{p(x)} + 2\sum_{i=1}^N \|f_i\|_{p'(x)}\|\partial_i v\|_{p(x)} \\ &\leq C\left(\|f_0\|_{p'(x)} + \sum_{i=1}^N \|f_i\|_{p'(x)} \right).\end{aligned}$$

Therefore,

$$\|f\|_{-1,p'(x)} = \sup_{\|\nabla v\|_{p(x)}=1} \langle f, v\rangle \leq C\left(\|f_0\|_{p'(x)} + \sum_{i=1}^N \|f_i\|_{p'(x)} \right),$$

so

$$\|f\|_{-1,p'(x)} \leq C\|f\|_*$$

, which completes the proof. □

Remark 3.19. Let now $D \subset \mathbb{R}^N$ be a bounded, open set and let $\Omega \subset D$ be open. Then, we have that $W_0^{1,p(x)}(\Omega) \subset W_0^{1,p(x)}(D)$, the inclusion being canonical, extending by zero. This inclusion induces $W^{-1,p'(x)}(D) \subset W^{-1,p'(x)}(\Omega)$ by restriction. Therefore, when dealing with sets Ω that are subsets of D, if one is considering $f \in W^{-1,p'(x)}(D)$ and $u \in W_0^{1,p(x)}(\Omega)$ there is no ambiguity in the notation $\langle f, u\rangle$.

3.1.2 $p(x)$-capacity and pointwise properties of Sobolev functions

We need the concept of capacity modified to deal with pointwise properties of functions in $W_0^{1,p(x)}(\Omega)$. This is the concept of $p(x)-$capacity. See [18, Chapter 10].

Throughout this section we assume that Ω is a bounded open set of $\mathbb{R}^N$.

Definition 3.20. Given $E \subset \mathbb{R}^N$, we consider the set

$$S_{p(x)}(E) = \left\{ u \in W^{1,p(x)}(\mathbb{R}^N) \colon u \geq 0 \text{ and } u \geq 1 \text{ a.e. in an open set containing } E \right\}.$$

If $S_{p(x)}(E) \neq \emptyset$, we define $p(x)-$*Sobolev capacity* of E as follows

$$\mathrm{cap}_{p(x)}(E) = \inf_{u\in S_{p(x)}(E)} \int_{\mathbb{R}^N} |u|^{p(x)} + |\nabla u|^{p(x)} dx = \inf_{u\in S_{p(x)}(E)} \rho_{1,p(x)}(u).$$

If $S_{p(x)}(E) = \emptyset$, we set $\mathrm{cap}_{p(x)}(E) = \infty$.

Definition 3.21. Let $p \in \mathcal{P}^{\log}(\Omega)$ and $K \subset \Omega$ compact, we define the $p(x)-$*relative capacity* as

$$\mathrm{cap}^*_{p(x)}(K, \Omega) = \inf_{u\in R_{p(x)}(K,\Omega)} \rho_{p(x)}(|\nabla u|)$$

where $R_{p(x)}(K, \Omega) = \{u \in W_0^{1,p(x)}(\Omega)\colon u > 1$ a.e. in K and $u \geq 0$ a.e. in $\Omega\}$.

If $U \subset \Omega$ is an open set, we define $\mathrm{cap}_{p(x)}(U, \Omega) = \sup\limits_{\substack{K\subset U\\ K \text{ compact}}} \mathrm{cap}^*_{p(x)}(K, \Omega)$.

Finally, if $E \subset \Omega$ is arbitrary, we define the $p(x)-$ relative capacity of E with respect to Ω as

$$\mathrm{cap}_{p(x)}(E, \Omega) = \inf_{\substack{E\subset U\subset\Omega\\ U \text{ open}}} \mathrm{cap}_{p(x)}(U, \Omega).$$

The main advantage of the relative capacity is the fact that it is possible to obtain a *capacitary potential*, i.e. a function whose modular gives the capacity of a set.

To this end, let $D \subset \mathbb{R}^N$ be a bounded open set and let $A \subset D$. Consider the class

$$\Gamma_A = \overline{\left\{v \in W_0^{1,p(x)}(D)\colon v \geq 1 \text{ a.e. in an open set containing } A\right\}},$$

the closure being taken in $W_0^{1,p(x)}(D)$.

Remark 3.22. Observe that since $\Gamma_A \subset W_0^{1,p(x)}(D)$ is closed and convex (the closure of a convex set is convex), it follows that is weakly closed. This fact will be used in the next proposition.

Now we show that the relative capacity of a set is realized by a function in Γ_A.

Proposition 3.23. *Let $p \in \mathcal{P}^{\log}(D)$ be such that $p_- > 1$. If $\Gamma_A \neq \emptyset$, then there exists a unique $u_A \in \Gamma_A$ such that*

$$\mathrm{cap}_{p(x)}(A, D) = \int_D |\nabla u_A|^{p(x)}\, dx.$$

Proof. Consider $\{v_n\}_{n\in\mathbb{N}} \subset W_0^{1,p(x)}(D)$ such that $v_n \geq 1$ a.e. in an open set containing A and

$$\int_D |\nabla v_n|^{p(x)}\, dx \to \mathrm{cap}_{p(x)}(A, D).$$

By Theorem 3.14 and Proposition 3.1, we have

$$\|\nabla v_n\|_{p(x)} \leq \max\{\rho_{p(x)}(\nabla v_n)^{\frac{1}{p_+}}, \rho_{p(x)}(\nabla v_n)^{\frac{1}{p_-}}\}.$$

Then, $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(D)$, which is a reflexive space. By Alaoglu's Theorem, there is a subsequence $v_{n_j} \rightharpoonup v_\infty$ in $W_0^{1,p(x)}(D)$. By Remark 3.22, $v_\infty \in \Gamma_A$.

Observe that

$$\int_D |\nabla v_\infty|^{p(x)}\,dx \le \liminf \int_D \left|\nabla v_{n_j}\right|^{p(x)}\,dx = \mathrm{cap}_{p(x)}(A, D).$$

Since the reverse inequality is obvious, the first part of the Proposition is proved.

The uniqueness is an immediate consequence of the strict convexity of the modular, since $p_- > 1$. We leave the details to the reader. □

We can now give the definition of capacitary potential.

Definition 3.24. We define the capacitary potential of A such as the only u_A that verifies

$$\int_D |\nabla u_A|^{p(x)}\,dx = \inf_{v\in\Gamma_A} \int_D |\nabla v|^{p(x)}\,dx = \mathrm{cap}_{p(x)}(A, D).$$

It is well known that when dealing with pointwise properties of Sobolev functions, the concept of *almost everywhere* needs to be changed to *quasi everywhere*. This is the content of the next definition.

Definition 3.25. An statement is valid $p(x)-$*quasi everywhere* ($p(x)-$q.e.) if it is valid except in a set of null Sobolev $p(x)-$capacity.

Definition 3.26. Let $D \subset \mathbb{R}^N$ be an open bounded set, $\Omega \subset D$ is $p(x)-$*quasi open* if there is a decreasing sequence $\{W_n\}_{n\in\mathbb{N}}$ of open sets such that $\mathrm{cap}_{p(x)}(W_n, D)$ converges to 0 and $\Omega \cup W_n$ is an open set for each n.

Definition 3.27. A function $u\colon \Omega \to \mathbb{R}$ is $p(x)-$*quasi continuous* if for every $\varepsilon > 0$, there is an open set U such that $\mathrm{cap}_{p(x)}(U, \Omega) < \varepsilon$ and $u|_{\Omega\setminus U}$ is continuous.

Remark 3.28. Ovserve that if u and v are $p(x)-$cuasicontinuos, then $u + v$, au with $a \in \mathbb{R}$, $\min\{u, v\}$ and $\max\{u, v\}$ are also $p(x)-$cuasicontinuos.

Lemma 3.29. *Let $p \in \mathcal{P}(\mathbb{R}^N)$. Then, for each Cauchy sequence with respect to the norm $W^{1,p(x)}(\mathbb{R}^N)$ of functions in $C(\mathbb{R}^N) \cap W^{1,p(x)}(\mathbb{R}^N)$, there is a subsequence that converges pointwise $p(x)-$ a.e. in $\mathbb{R}^N$.*

More over, the convergence is uniform outside a set of arbitrary small $p(x)-$capacity.

Proof. Let $\{u_i\}_{i\in\mathbb{N}}$ a Cauchy sequence in $C(\mathbb{R}^N) \cap W^{1,p(x)}(\mathbb{R}^N)$.

Considering a subsequence if necessary, we can assume that for every $i \in \mathbb{N}$ we have

$$\|u_i - u_{i+1}\|_{1,p(x)} \le \frac{1}{4^i}.$$

Let us denote $U_i = \{x \in \mathbb{R}^N\colon\ |u_i(x) - u_{i+1}(x)| > \frac{1}{2^i}\}$ and $V_j = \cup_{i\ge j} U_i$.

If $v = 2^i |u_i - u_{i+1}|$, then $v \in W^{1,p(x)}(\mathbb{R}^N)$ and $\|v\|_{1,p(x)} = 2^i \|u_i - u_{i+1}\|_{1,p(x)} \leq \frac{1}{2^i}$.

Then, by Proposition 3.4, $\rho_{1,p(x)}(v) \leq \frac{1}{2^i}$.

Therefore

$$\mathrm{cap}_{p(x)}(V_j) = \mathrm{cap}_{p(x)}(\cup_{i\geq j} U_i) \leq \sum_{i\geq j} \mathrm{cap}_{p(x)}(U_i) \leq \sum_{i\geq j} \rho_{1,p(x)}(v) \leq \sum_{i\geq j} \frac{1}{2^i} \leq 2^{1-j} \to 0$$

So we can conclude that $\lim_{j\to\infty} \mathrm{cap}_{p(x)}(V_j) = 0$.

If $V = \cap_{j=1}^{\infty} V_j$, we have that $\mathrm{cap}_{p(x)}(V) \leq \mathrm{cap}_{p(x)}(V_j) \to 0$.

Therefore, $\{u_i\}_{i\in\mathbb{N}}$ converges point-wise in $\mathbb{R}^N \setminus V$ and $\mathrm{cap}_{p(x)}(V) = 0$.

Now given $\varepsilon > 0$, let us consider j_0 such that $\mathrm{cap}_{p(x)}(V_{j_0}) < \varepsilon$.

Let $x \in \mathbb{R}^N \setminus V_{j_0}$, then $x \in (V_{j_0})^c = (\cup_{i\geq j_0} U_i)^c = \cap_{i\geq j_0} U_i^c$. Therefore for every $i \geq j_0$,

$$|u_i(x) - u_{i+1}(x)| \leq \frac{1}{2^i}.$$

Then, given $k \geq l \geq j_0$,

$$|u_l(x) - u_k(x)| \leq \sum_{i=l}^{k-1} |u_i(x) - u_{i+1}(x)| \leq \sum_{i=l}^{k-1} \frac{1}{2^i} < 2^{1-l}.$$

And so the convergence in $\mathbb{R}^N \setminus V_{j_0}$ s uniform, which completes the proof. □

Corollary 3.30. *Let $p \in \mathcal{P}^{log}(\mathbb{R}^N)$. Therefore, for every $u \in W^{1,p(x)}(\mathbb{R}^N)$, there is $v \in W^{1,p(x)}(\mathbb{R}^N)$ $p(x)-$cuasicontinuous such that $u = v$ a.e. in $\mathbb{R}^N$.*

Proof. Let $u \in W^{1,p(x)}(\mathbb{R}^N)$. By density, we can consider a sequence $\{u_i\}_{i\in\mathbb{N}}$ in $C_c^{\infty}(\mathbb{R}^N)$ such that $u_i \to u$ in $W^{1,p(x)}(\mathbb{R}^N)$.

By Lemma 3.29, there is a subsequence of $\{u_i\}_{i\in\mathbb{N}}$ that converges uniformly outside a set of arbitrary small capacity. Since uniform convergence implies the continuity of the limit function, we get a continuous function outside a set of arbitrary small capacity. □

The proof of the next theorem can be found in [18, Corollary 11.1.5]. We include it for the convenience of the reader.

Theorem 3.31. *Let $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p_- \leq p_+ < \infty$. Then for each $u \in W^{1,p(x)}(\Omega)$ there exists a $p(x)-$quasicontinuous function $v \in W^{1,p(x)}(\Omega)$ such that $u = v$ almost everywhere in Ω.*

Proof. Let $z \in \Omega$. We define

$$\Omega_i = \{x \in \Omega\colon\ \mathrm{dist}(x, \partial\Omega) > \frac{1}{i}\} \cap B(z, i).$$

Let us consider $\{\psi_i\}_{i\in\mathbb{N}} \subset C_c^{\infty}(\Omega)$ such as $\psi_i = 1$ in Ω_i. Then, $u\psi_i \in W^{1,p(x)}(\mathbb{R}^N)$ and $u\psi_i = u$ in Ω_i.

Since $u\psi_i \in W^{1,p(x)}(\mathbb{R}^N)$, by Corollary 3.30, there exists $v_i \in W^{1,p(x)}(\mathbb{R}^N)$ $p(x)-$quasicontinuos such that $v_i = u\psi_i$ almost everywhere $\mathbb{R}^N$.

Let $j > i$. Since $u\psi_i = u$ in Ω_i, we have that $v_i = v_j$ almost everywhere Ω_i. Then, $v_i = v_j$ $p(x)-$ctp Ω_i.

Let $\varepsilon > 0$. Let us consider $V_i \subset \mathbb{R}^N$ such that $v_i|_{\mathbb{R}^N \setminus V_i}$ is continuos, $v_i = v_{i-1}$ in $\Omega_{i-1} \setminus V_i$ and $\mathrm{cap}_{p(x)}(V_i) \le \frac{\varepsilon}{2^i}$.

Let us denote $\tilde{u}(x) = v_i(x)$ where i is the smallest natural number such that $x \in B(z, i)$ y $\mathrm{dist}(x, \partial\Omega) > \frac{1}{i}$.

Observe that we have $\tilde{u} = u$ almost everywhere Ω, $\tilde{u}|_{\Omega \setminus \cup_{i=1}^{\infty} V_i}$ is continuos and verifies

$$\mathrm{cap}_{p(x)}(\cup_{i=1}^{\infty} V_i) \le \sum_{i=1}^{\infty} \mathrm{cap}_{p(x)}(V_i) \le \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} \le \varepsilon.$$

As this holds for every $\varepsilon > 0$, the proof is completed. □

Remark 3.32. It is easy to see that two $p(x)-$quasi continuous representatives of a given function $u \in W^{1,p(x)}(\Omega)$ can only differ in a set of zero $p(x)-$capacity. Therefore, the unique $p(x)-$quasi continuous representative (defined $p(x)-$q.e.) of $u \in W^{1,p(x)}(\Omega)$ will be denoted by $\tilde{u}$.

Proposition 3.33. *Let $p \in \mathcal{P}(\Omega)$. Then,*

$$\mathrm{cap}^*_{p(x)}(K, \Omega) = \inf_{u \in \tilde{R}_{p(x)}(K,\Omega)} \rho_{p(x),\Omega}(\nabla u)$$

where $\tilde{R}_{p(x)}(K, \Omega) = \{u \in W^{1,p(x)}(\Omega) \cap C_0(\Omega) \colon u \ge 1$ en $K\}$.

Proof. Since $R_{p(x)}(K, \Omega) \subset \tilde{R}_{p(x)}(K, \Omega)$, we have

$$\inf_{u \in \tilde{R}_{p(x)}(K,\Omega)} \rho_{p(x),\Omega}(\nabla u) \le \inf_{u \in R_{p(x)}(K,\Omega)} \rho_{p(x),\Omega}(\nabla u) = \mathrm{cap}^*_{p(x)}(K, \Omega).$$

Given $\varepsilon > 0$, there is $u \in \tilde{R}_{p(x)}(K, \Omega)$ such that

$$\rho_{p(x),\Omega}(\nabla u) \le \inf_{u \in \tilde{R}_{p(x)}(K,\Omega)} \rho_{p(x),\Omega}(\nabla u) + \varepsilon.$$

Let us consider $v = (1 + \varepsilon)u > 1$ in K. Observe that $v \in R_{p(x)}(K, \Omega)$. Therefore

$$\begin{aligned}
\mathrm{cap}^*_{p(x)}(K, \Omega) &= \inf_{u \in R_{p(x)}(K,\Omega)} \rho_{p(x),\Omega}(\nabla u) \\
&\le \rho_{p(x),\Omega}(\nabla v) = \int_\Omega |\nabla v(x)|^{p(x)}\, dx \\
&= \int_\Omega (1 + \varepsilon)^{p(x)} |\nabla u(x)|^{p(x)} dx \\
&\le (1 + \varepsilon)^{p_+} \rho_{p(x),\Omega}(\nabla u).
\end{aligned}$$

And so, given $\varepsilon > 0$, we have

$$\text{cap}^*_{p(x)}(K,\Omega) \leq (1+\varepsilon)^{p_+}\Big(\inf_{u\in\tilde{R}_{p(x)}(K,\Omega)} \rho_{p(x),\Omega}(\nabla u) + \varepsilon\Big)$$

Then we can conclude

$$\text{cap}^*_{p(x)}(K,\Omega) = \inf_{u\in\tilde{R}_{p(x)}(K,\Omega)} \rho_{p(x),\Omega}(\nabla u),$$

which completes the proof. □

Proposition 3.34. *Let $p \in \mathcal{P}(\Omega)$ and $K \subset \Omega$ a compact subset. Then,*

$$\text{cap}^*_{p(x)}(K,\Omega) = \text{cap}_{p(x)}(K,\Omega).$$

Proof. From the definition, follows the fact that $\text{cap}^*_{p(x)}(K,\Omega) \leq \text{cap}_{p(x)}(K,\Omega)$.

Given $\varepsilon > 0$, let us consider $u \in R_{p(x)}(K,\Omega)$ such that

$$\rho_{p(x)}(\nabla u) \leq \text{cap}^*_{p(x)}(K,\Omega) + \varepsilon.$$

Observe that $u > 1$ en $U = u^{-1}(1,\infty)$, which is an open set containing K (since $u \in R_{p(x)}(K,\Omega)$).

Let us also observe that $u > 1$ in every compact set $\tilde{K} \subset U \subset \Omega$ and, in consecuence, $u \in R_{p(x)}(\tilde{K},\Omega)$ for every $\tilde{K} \subset U$ compact set. Therefore,

$$\begin{aligned}
\text{cap}_{p(x)}(U,\Omega) &= \sup_{\substack{\tilde{K}\subset U \\ \tilde{K}\text{ compacto}}} \text{cap}^*_{p(x)}(\tilde{K},\Omega) \\
&= \sup_{\substack{\tilde{K}\subset U \\ \tilde{K}\text{ compacto}}} \Big(\inf_{v\in R_{p(x)}(\tilde{K},\Omega)} \rho_{p(x)}(\nabla v)\Big) \\
&\leq \sup_{\substack{\tilde{K}\subset U \\ \tilde{K}\text{ compacto}}} \rho_{p(x)}(\nabla u) \\
&= \rho_{p(x)}(\nabla u).
\end{aligned}$$

So we can conclude that, given $\varepsilon > 0$,

$$\text{cap}_{p(x)}(K,\Omega) \leq \text{cap}_{p(x)}(U,\Omega) \leq \rho_{p(x)}(\nabla u) \leq \text{cap}^*_{p(x)}(K,\Omega) + \varepsilon,$$

which completes the proof. □

Remark 3.35. Given K a compact set, let's see that $\text{cap}_{p(x)}(K,\Omega) = \inf_{\substack{K\subset U \\ U\text{ abierto}}} \text{cap}_{p(x)}(U,\Omega)$.

Let $\varepsilon > 0$, consider $u \in W^{1,p(x)}(\Omega) \cap C_0(\Omega)$ such that $u \geq 1$ in K y $\rho_{p(x)}(\nabla u) \leq \varepsilon + \text{cap}_{p(x)}(K,\Omega)$.

Consider $U = \{(1+\varepsilon)u > 1\}$ an open set and $K_1 = \{(1+\varepsilon)u \geq 1\}$, which is a compact set (it is bounded because u has compact support and it is clearly closed). Then, $K \subset U \subset K_1$ and, in consecuence,

$$\begin{aligned}\inf_{\substack{K\subset U\\ U \text{ abierto}}} \operatorname{cap}_{p(x)}(U,\Omega) &\leq \operatorname{cap}_{p(x)}(U,\Omega)\\ &\leq \operatorname{cap}_{p(x)}(K_1,\Omega)\\ &\leq \rho_{p(x)}(\nabla((1+\varepsilon)u))\\ &\leq (1+\varepsilon)^{p_+}\rho_{p(x)}(\nabla u)\\ &\leq (1+\varepsilon)^{p_+}(\varepsilon + \operatorname{cap}_{p(x)}(K,\Omega))\end{aligned}$$

where we took into account that $(1+\varepsilon)u \in W^{1,p(x)}(\Omega) \cap C_0(\Omega)$ and $(1+\varepsilon)u \geq 1$ in K_1.

Therefore, if $\varepsilon \to 0$, we have that

$$\inf_{\substack{K\subset U\\ U \text{ open}}} \operatorname{cap}_{p(x)}(U,\Omega) \leq \operatorname{cap}_{p(x)}(K,\Omega).$$

The inverse inequality is a direct consecuence of the monotony of the $p(x)-$capacity. In fact, given an open set U such that $K \subset U$, we have $\operatorname{cap}_{p(x)}(K,\Omega) \leq \operatorname{cap}_{p(x)}(U,\Omega)$. Then,

$$\operatorname{cap}_{p(x)}(K,\Omega) \leq \inf_{\substack{K\subset U\\ U \text{ open}}} \operatorname{cap}_{p(x)}(U,\Omega).$$

Let us now present an equivalent definition of relative $p(x)-$capacity that will be very helpful for our next results.

Proposition 3.36. *Let $E \subset \Omega$. Then,*

$$\operatorname{cap}_{p(x)}(E,\Omega) = \inf_{u\in\tilde{R}_{p(x)}(E,\Omega)} \rho_{p(x)}(\nabla u).$$

Proof. Let us denote

$$\widetilde{\operatorname{cap}}_{p(x)}(E,\Omega) = \inf_{u\in\tilde{R}_{p(x)}(E,\Omega)} \rho_{p(x)}(\nabla u).$$

Observe that $\widetilde{\operatorname{cap}}_{p(x)}(\cdot,\Omega)$ is non-decreasing. it is enough to see that, given $E \subset E'$, if $u \geq 1$ in E', then $u \geq 1$ in E. Therefore,

$$\widetilde{\operatorname{cap}}_{p(x)}(E',\Omega) = \inf_{u\in\tilde{R}_{p(x)}(E',\Omega)} \rho_{p(x)}(\nabla u) \geq \inf_{u\in\tilde{R}_{p(x)}(E,\Omega)} \rho_{p(x)}(\nabla u) = \widetilde{\operatorname{cap}}_{p(x)}(E,\Omega).$$

Let us first assume that $E = K$ is a compact set. Then, by Propositions 3.33 and 3.34, we have that

$$\operatorname{cap}_{p(x)}(K,\Omega) = \operatorname{cap}^*_{p(x)}(K,\Omega) = \inf_{u\in\tilde{R}_{p(x)}(K,\Omega)} \rho_{p(x)}(\nabla u) = \widetilde{\operatorname{cap}}_{p(x)}(K,\Omega).$$

Let us now assume that $E = W$ is an open set.

Let $K \subset W$ be a compact set, then

$$\text{cap}_{p(x)}(K,\Omega) = \widetilde{\text{cap}}_{p(x)}(K,\Omega) \leq \widetilde{\text{cap}}_{p(x)}(W,\Omega).$$

Therefore,

$$\text{cap}_{p(x)}(W,\Omega) = \sup_{\substack{K\subset W\\ K \text{ a compact set}}} \text{cap}_{p(x)}(K,\Omega) \leq \widetilde{\text{cap}}_{p(x)}(W,\Omega).$$

It remains to proove the inverse inequality. If $\text{cap}_{p(x)}(W,\Omega) = \infty$, it is clear. Let us assume then that $\text{cap}_{p(x)}(W,\Omega) < \infty$.

Since $\text{cap}_{p(x)}(W,\Omega) = \sup_{\substack{K\subset W\\ K \text{ compacto}}} \text{cap}_{p(x)}(K,\Omega)$, there is a sequence of compact sets $\{K_n\}_{n\in\mathbb{N}}$ in W such that $\text{cap}_{p(x)}(K_n,\Omega) \to \text{cap}_{p(x)}(W,\Omega)$.

Let us asuume, enlarging our K_n if necessary, that $W = \cup_{n=1}^{\infty} K_n$.

We have proven that

$$\text{cap}_{p(x)}(K_n,\Omega) = \inf_{u\in\tilde{R}_{p(x)}(K_n,\Omega)} \rho_{p(x)}(\nabla u).$$

Then, for every n, there is $u_n \in W^{1,p(x)}(\Omega) \cap C_0(\Omega)$ such that $u_n \geq 1$ in K_n and

$$\rho_{p(x)}(\nabla u_n) \leq \text{cap}_{p(x)}(K_n,\Omega) + \frac{1}{n} \leq \text{cap}_{p(x)}(W,\Omega) + \frac{1}{n}.$$

By Remark 3.2, we have that $\{\|\nabla u_n\|_{p(x)}\}_{n\in\mathbb{N}}$ is bounded and, in consecuence, since $u \in W_0^{1,p(x)}(\Omega)$, $\{\|u_n\|_{1,p(x)}\}_{n\in\mathbb{N}}$ is also bounded.

Therefore, since $W_0^{1,p(x)}(\Omega)$ is reflexive, there is a subsequence, which we will still denote $\{\|u_n\|_{1,p(x)}\}_{n\in\mathbb{N}}$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and a.e.

Let us also observe that, since $u_n \geq 1$ in K_n, we have $u_n \geq 1$ in $\cup_{n=1}^{\infty} K_n = W$. Therefore,

$$\widetilde{\text{cap}}_{p(x)}(W,\Omega) = \inf_{u\in\tilde{R}_{p(x)}(W,\Omega)} \rho_{p(x)}(\nabla u) \leq \rho_{p(x)}(\nabla u_n) \leq \text{cap}_{p(x)}(W,\Omega) + \frac{1}{n}.$$

If n goes to ∞, we conclude that $\widetilde{\text{cap}}_{p(x)}(W,\Omega) \leq \text{cap}_{p(x)}(W,\Omega)$, which completes the proof.

Let us consider now an arbitrary set $E \subset \Omega$.

Since $\text{cap}_{p(x)}(E,\Omega) = \inf_{\substack{E\subset W\\ W \text{ open}}} \text{cap}_{p(x)}(W,\Omega)$, we can consider $\{W_n\}_{n\in\mathbb{N}}$ a sequence of open sets containing E such that $\text{cap}_{p(x)}(W_n,\Omega) \to \text{cap}_{p(x)}(E,\Omega)$.

Since $E \subset W_n$, and taking into account that we have already proved the result for open sets, we have that

$$\widetilde{\text{cap}}_{p(x)}(E,\Omega) \leq \widetilde{\text{cap}}_{p(x)}(W_n,\Omega) = \text{cap}_{p(x)}(W_n,\Omega).$$

So we conclude that $\widetilde{\text{cap}}_{p(x)}(E,\Omega) \leq \text{cap}_{p(x)}(E,\Omega)$.

It remains to proove the inverse inequality. If $\widetilde{\text{cap}}_{p(x)}(E,\Omega) = \infty$, it it clear. Let us assume then that $\widetilde{\text{cap}}_{p(x)}(E,\Omega) < \infty$.

By the definition of $\widetilde{\text{cap}}_{p(x)}(E,\Omega)$, we know there is $\{u_n\}_{n\in\mathbb{N}}$ in $W^{1,p(x)}(\Omega)\cap C_0(\Omega)$ such that $u_n\geq 1$ in E and $\rho_{p(x)}(\nabla u_n)\to\widetilde{\text{cap}}_{p(x)}(E,\Omega)$.

Let us observe that

$$\rho_{p(x)}(\nabla u_n)\geq \inf_{u\in S_{p(x)}(E,\Omega)}\rho_{p(x)}(\nabla u)=\widetilde{\text{cap}}_{p(x)}(W_n,\Omega)=\text{cap}_{p(x)}(W_n,\Omega)\geq \text{cap}_{p(x)}(E,\Omega).$$

We can conclude then that $\widetilde{\text{cap}}_{p(x)}(E,\Omega)\geq \text{cap}_{p(x)}(E,\Omega)$, which completes the proof. □

The proof of the next proposition can be found in [18, Section 11.1.11].

Proposition 3.37. *Let $D\subset\mathbb{R}^N$ be open. Let $v_j\to v$ in $W_0^{1,p(x)}(D)$. Then, there is a subsequence $\{v_{j_k}\}_{k\in\mathbb{N}}$ such that $\tilde{v}_{j_k}\to\tilde{v}$ $p(x)$–q.e.*

Proof. Let us consider $\{v_{j_k}\}_{k\in\mathbb{N}}$ subsequence such that $\left\|v_{j_{k+1}}-v_{j_k}\right\|^{p_-}_{W_0^{1,p(x)}(D)}\leq 4^{-kp_+}$.

Let us denote $\Omega_k=\{|\tilde{v}_{j_{k+1}}-\tilde{v}_{j_k}|>2^{-k}\}$ y $W_n=\cup_{k\geq n}\Omega_k$.

Since $\text{cap}_{p(x)}(W_n,D)=\inf_{\substack{W_n\subset W\\ W\text{ abierto}}}\text{cap}_{p(x)}(W,D)$, there exist $\tilde{W}_n$ open sets such that $W_n\subset\tilde{W}_n$ and also

$$\text{cap}_{p(x)}(\tilde{W}_n,D)\leq \text{cap}_{p(x)}(W_n,D)+\frac{1}{2^n}. \tag{3.5}$$

Let us assume that $\tilde{W}_n\subset\tilde{W}_{n+1}$ (if not, it is enough to replace it for $\tilde{W}_n\cap\tilde{W}_{n+1}$).

Observe that, since $2^k|\tilde{v}_{j_{k+1}}-\tilde{v}_{j_k}|>1$ in the open set Ω_k, by Proposición 3.36, we have

$$\begin{aligned}
\text{cap}_{p(x)}(\Omega_k,D) &\leq \int_D|\nabla(2^k|v_{j_{k+1}}(x)-v_{j_k}(x)|)|^{p(x)}dx\\
&\leq 2^{kp_+}\rho_{p(x)}(\nabla(v_{j_{k+1}}-v_{j_k}))\\
&\leq 2^{kp_+}\rho_{1,p(x)}(v_{j_{k+1}}-v_{j_k})\\
&\leq 2^{kp_+}\left\|v_{j_{k+1}}-v_{j_k}\right\|^{p_-}_{W_0^{1,p(x)}(D)}\leq 2^{kp_+}4^{-kp_+}=2^{-kp_+}
\end{aligned}$$

Therefore,

$$\text{cap}_{p(x)}(W_n,D)=\text{cap}_{p(x)}(\cup_{k\geq n}\Omega_k,D)\leq\sum_{k\geq n}\text{cap}_{p(x)}(\Omega_k,D)\leq\sum_{k\geq n}2^{-kp_+}\to 0$$

From (3.5), we can conclude that $\text{cap}_{p(x)}(\tilde{W}_n,D)\to 0$ and so

$$\text{cap}_{p(x)}(\cap_{n=1}^{\infty}\tilde{W}_n,D)=\lim_{n\to\infty}\text{cap}_{p(x)}(\tilde{W}_n,D)=0.$$

Then, $\tilde{v}_{j_k}\to\tilde{v}$ in $D-\cap_{n=1}^{\infty}\tilde{W}_n$, which completes the proof. □

Now we need a characterization of the space $W_0^{1,p(x)}(\Omega)$ as the restriction of quasi continuous functions that vanishes quasi everywhere on $\mathbb{R}^N\setminus\Omega$. This theorem is essentialy contained in [18, Corollary 11.2.5, Proposition 11.2.3]. We include here the proof since a minor modification of the above mentioned result is needed and for the reader's convenience.

Theorem 3.38 (Characterization Theorem)**.** *Let $D \subset \mathbb{R}^N$ be an open set, $\Omega \subset D$ an open subset and $p \in \mathcal{P}^{log}(\Omega)$ with $p_- > 1$. Then,*

$$u \in W_0^{1,p(x)}(\Omega) \Leftrightarrow u \in W_0^{1,p(x)}(D) \text{ and } \tilde{u} = 0 \ p(x) - q.e. \text{ in } D \setminus \Omega.$$

Proof. Let $u \in W_0^{1,p(x)}(\Omega)$, then, it is immediate that $u \in W_0^{1,p(x)}(D)$.

Now, let $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\Omega)$ such that $\varphi_n \to u$ in $W_0^{1,p(x)}(\Omega)$ (and therefore in $W_0^{1,p(x)}(D)$).

Let $\{\varphi_{n_j}\}_{j\in\mathbb{N}} \subset \{\varphi_n\}_{n\in\mathbb{N}}$ be a subsequence such that $\varphi_{n_j} \to \tilde{u}$ $p(x)$–q.e. Then, since $\varphi_{n_j} = 0$ in $D \setminus \Omega$, we have that $\tilde{u} = 0$ $p(x)$–q.e. in $D \setminus \Omega$.

To see the converse, let us assume that $D = \mathbb{R}^N$ (or else, we extend by zero). Since $u = u^+ - u^-$, we can assume that $u \geq 0$. Moreover, since $\min\{u, n\} \in W^{1,p(x)}(\mathbb{R}^N)$ converges to u in $W^{1,p(x)}(\mathbb{R}^N)$, we can assume that u is bounded. Finally, let us consider $\xi \in C_c^\infty(B(0,2))$ such that $0 \leq \xi \leq 1$ and $\xi \equiv 1$ in $B(0,1)$. Setting $\xi_n(x) = \xi(\frac{x}{n})$, we have that $\xi_n u$ converges to u in $W^{1,p(x)}(\mathbb{R}^N)$. Therefore we can assume that $u(x) = 0$ for every $x \in (B(0,R))^c$ with R large enough.

Therefore, we need to prove the converse for bounded, compactly supported and nonnegative functions $u \in W^{1,p(x)}(\mathbb{R}^N)$ such that $\tilde{u} = 0$ $p(x)$–q.e. in Ω^c.

Since $\tilde{u}$ is $p(x)$–quasi continuous, there is a decreasing sequence of open sets $\{W_n\}_{n\in\mathbb{N}}$ such that $\text{cap}_{p(x)}(W_n, D) \to 0$ and $\tilde{u}|_{\mathbb{R}^N\setminus W_n}$ is continuous.

We can assume that W_n contains the set of null capacity of $\mathbb{R}^N \setminus \Omega$ where $\tilde{u} \neq 0$. Therefore, $\tilde{u} = 0$ in $(\Omega \cup W_n)^c = \Omega^c \cap W_n^c$.

Given $\delta > 0$, set $V_n = \{x\colon \tilde{u}(x) < \delta\} \cup W_n$. Since $\tilde{u}$ is continuous in $\mathbb{R}^N \setminus W_n$, V_n is an open set. Therefore, V_n^c is a closed set. It is also bounded since $V_n^c \subset B(0,R)$. Then, V_n^c is compact.

Let u_{W_n} be the capacitary potential of W_n, then $(u-\delta)^+(1-u_{W_n}) = 0$ a.e. in $\Omega \setminus V_n^c$.

Consider now a regularizing sequence $\{\phi_j\}_{j\in\mathbb{N}}$. Therefore, for j sufficiently large we have that

$$\phi_j * [(u-\delta)^+(1-u_{W_n})] \in C^\infty(\Omega).$$

Observe that

$$\rho_{p(x)}(\nabla u_{W_n}) = \text{cap}_{p(x)}(W_n, D) \to 0.$$

By Proposition 3.1, we can conclude that $\|\nabla u_{W_n}\|_{p(x)} \to 0$ and, by Poincaré's inequality,

$$\|u_{W_n}\|_{1,p(x)} \to 0.$$

Therefore, $1 - u_{W_n} \to 1$ in $W^{1,p(x)}(D)$ when $n \to \infty$.

Obviously, $(u-\delta)^+ \to u^+ = u$ in $W^{1,p(x)}(D)$ when $\delta \to 0$ and observe that

$$\begin{aligned}\left\|(u-\delta)^+(1-u_{W_n}) - u\right\|_{1,p(x)} \leq &\left\|1-u_{W_n}\right\|_{1,p(x)} \left\|(u-\delta)^+ - u\right\|_{1,p(x)} \\ &+ \|u\|_{1,p(x)} \left\|u_{W_n}\right\|_{1,p(x)}.\end{aligned}$$

Finally, taking the limit when when $j \to \infty$, $n \to \infty$ and $\delta \to 0$, we have that

$$\phi_j * [(u-\delta)^+(1-u_{W_n})] \to u,$$

which completes the proof. □

We end this subsection with a lemma that will be much helpful in the sequel.

Lemma 3.39. *Let $v \in W^{1,p(x)}(\mathbb{R}^N)$ and $w \in W_0^{1,p(x)}(D)$ such that $|v| \leq w$ a.e. in D. Then, $v \in W_0^{1,p(x)}(D)$.*

Proof. It is enough to see that $v^+ \in W_0^{1,p(x)}(D)$ (for v^- we procede similarly and having shown this result for v^+ and v^-, we can state that is valid for $v = v^+ - v^-$).

Since $w \geq 0$, by density we can consider $\{w_n\}_{n\in\mathbb{N}} \subset C_c^\infty(D)$, $w_n \geq 0$, such that $\{w_n\}_{n\in\mathbb{N}}$ converges to w in $W^{1,p(x)}(D)$.

Therefore, $\inf\{w_n, v^+\}$, which has compact support in D (for each w_n has so) converges to $\inf\{w, v^+\}$ which coincides with v^+ since $|v| \leq w$ a.e. in D.

Then, taking an adequate regularizing sequence, we obtain a sequence of $C_c^\infty(D)$ convergent to v^+, which completes the proof. □

3.2 Fractional Sobolev spaces.

The standard reference for this is [16].

Let $\Omega \subset \mathbb{R}^N$ be an open, connected set. For $0 < s < 1 < p < \infty$, we consider the fractional order Sobolev space $W^{s,p}(\Omega)$ defined as follows

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \colon \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\},$$

endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{s,p;\Omega} = \left(\int_\Omega |u|^p \, dx + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dxdy \right)^{\frac{1}{p}}.$$

The term

$$[u]^p_{W^{s,p}(\Omega)} = [u]^p_{s,p;\Omega} = \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx\, dy$$

is called the *Gagliardo seminorm* of u. We refer the interested reader to [16] for a throughout introduction to these spaces.

When $\Omega = \mathbb{R}^N$, we omit it in the notation, i.e.

$$\|u\|_{s,p;\mathbb{R}^N} = \|u\|_{s,p} \text{ and } [u]_{s,p;\mathbb{R}^N} = [u]_{s,p}.$$

Proposition 3.40. *For $0 < s < 1 < p < \infty$ the fractional Sobolev space $W^{s,p}(\Omega)$ is uniformly convex and reflexive.*

Proof. Let us define $T : W^{s,p}(\Omega) \to L^p(\Omega) \times L^p(\Omega \times \Omega)$ as follows

$$T(u) := \Big(u, \frac{|u(x) - u(y)|}{|x - y|^{\frac{n}{p}+s}}\Big),$$

where $L^p(\Omega) \times L^p(\Omega \times \Omega)$ is equipped with the norm

$$\|(u, v)\| := \Big(\int_\Omega |u|^p + \int_{\Omega\times\Omega} |v(x, y)|^p\Big)^{\frac{1}{p}}.$$

Observe that, since

$$\|T(u)\| = \Big(\int_\Omega |u|^p + \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\Big)^{\frac{1}{p}} = \|u\|_{s,p},$$

T is an isometry, which completes the proof. □

In order to consider Dirichlet boundary conditions, it is customary to define the spaces

$$W_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) \colon u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.?$$

Let us observe that $W_0^{s,p}(\Omega)$ is a closed subset of $W^{s,p}(\mathbb{R}^N)$. Therefore it has the same properties as a functional space. In particular, $(W_0^{s,p}(\Omega), \|\cdot\|_{s,p})$ is a uniformly convex and reflexive Banach space.

An alternative definition for $W_0^{s,p}(\Omega)$ is to consider the closure of $C_c^\infty(\Omega)$ in $W^{s,p}(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_{s,p}$. If Ω is Lipschitz, both definitions are known to coincide (see [16]).

3.2.1 Elementary properties

We will now present some well-known properties of the norm that will be useful for our results. We state the results without proof for future references.

Proposition 3.41 (Poincaré Inequality). *Let $\Omega \subset \mathbb{R}^N$ be an open set of finite measure. Then, there exists a constant $c = c(s, p, n, |\Omega|) > 0$ such that*

$$\|u\|_p \leq c[u]_{s,p} \textit{ for every } u \in W_0^{s,p}(\Omega).$$

Proof. Given $u \in W_0^{s,p}(\Omega)$, we have

$$[u]_{s,p}^p = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy$$

$$\geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N\setminus\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy \geq \int_{\mathbb{R}^N} |u(x)|^p \Big(\int_{R^N\setminus\Omega} \frac{1}{|x - y|^{n+sp}} dy\Big) dx.$$

Then, we need to find an inferior bound for

$$\rho_\Omega(x) := \int_{\mathbb{R}^N\setminus\Omega} \frac{1}{|x-y|^{n+sp}}dy.$$

Let $r > 0$ be such that $|B_r(x)| = |\Omega|$. Then, $r = \left(\frac{|\Omega|}{\omega_n}\right)^{\frac{1}{n}}$.
Therefore,

$$\left|\left(\mathbb{R}^N \setminus \Omega\right) \cap B_r(x)\right| = |B_r(x)| - |\Omega \cap B_r(x)| = |\Omega| - |\Omega \cap B_r(x)| = \left|\Omega \cap \left(\mathbb{R}^N \setminus B_r(x)\right)\right|.$$

Let us rewrite the integral as follows

$$\rho_\Omega(x) = \int_{(R^N\setminus\Omega)\cap B_r(x)} \frac{1}{|x-y|^{n+sp}}dy + \int_{(R^N\setminus\Omega)\cap(\mathbb{R}^N\setminus B_r(x))} \frac{1}{|x-y|^{n+sp}}dy.$$

Observe now that for the first term we have

$$\int_{(R^N\setminus\Omega)\cap B_r(x)} \frac{1}{|x-y|^{n+sp}}dy \geq \frac{\left|(\mathbb{R}^N \setminus \Omega) \cap B_r(x)\right|}{r^{n+sp}} = \frac{\left|\Omega \cap (\mathbb{R}^N \setminus B_r(x))\right|}{r^{n+sp}} \geq \int_{\Omega\cap(\mathbb{R}^N\setminus B_r(x))} \frac{1}{|x-y|^{n+sp}}dy.$$

Therefore,

$$\rho_\Omega(x) \geq \int_{\mathbb{R}^N\setminus B_r(x)} \frac{1}{|x-y|^{n+sp}}dy = \int_{|z|\geq r} \frac{1}{|z|^{n+sp}}dy = \frac{n\omega_n}{spr^{sp}} = \frac{n\omega_n^{1+\frac{sp}{n}}}{sp|\Omega|^{\frac{sp}{n}}}.$$

So we can conclude

$$[u]_{s,p}^p \geq \int_{R^N} |u(x)|^p \,\rho_\Omega(x)dx \geq \frac{n\omega_n^{1+\frac{sp}{n}}}{sp\,|\Omega|^{\frac{sp}{n}}}\|u\|_p.$$

Taking $c = \left(\frac{n\omega_n^{1+\frac{sp}{n}}}{sp|\Omega|^{\frac{sp}{n}}}\right)^{-1}$, the proof is finished.

□

We now present an immediate corollary of Proposition 3.41.

Corollary 3.42. *Let $\Omega \subset \mathbb{R}^N$ be an open set of finite measure. Then, $[\,\cdot\,]_{s,p}$ and $\|\cdot\|_{s,p}$ define equivalent norms in $W_0^{s,p}(\Omega)$.*

We will use the notation

$$\mathcal{A}(D) := \{\Omega \subset D\colon \text{ open}\}$$

and therefore this space has a natural structure of a metric space with metric d^H.

Remark 3.43. Is a well known fact that the space $(\mathcal{A}(D), d^H)$ is a compact metric space when D is compact.

For the proof of the following proposition, we refer to the book [28]. We include it for thesis to be selfcontained.

Proposition 3.44. *If $\Omega_k \xrightarrow{H} \Omega$, then for every $\phi \in C_c^\infty(\Omega)$ there is an integer k_0 such that $\phi \in C_c^\infty(\Omega_k)$ for $k \geq k_0$.*

3.2.2 Fractional Capacity

In this subsection, we recall some definitions of the $(s,p)-$capacity and relative capacity that can be found, for instance, in [50].

For a detailed analysis of the $(s,p)-$capacity, we refer to the above mentioned article [50].

We start with the definition of the $(s,p)-$capacity and the relative $(s,p)-$fractional capacity.

Definition 3.45. Let $E \subset \mathbb{R}^N$ be an arbitrary set. We define the $(s,p)-$*fractional capacity* of the set E as

$$\operatorname{cap}_{s,p}(E) := \inf\{[u]_{s,p}^p \colon u \in C_c^\infty(\mathbb{R}^N), u \geq 1 \text{ in } E\} \tag{3.6}$$

Given $\Omega \subset \mathbb{R}^N$ an open and bounded set and $E \subset \Omega$, we can define the capacity of the set E relative to the set Ω as follows.

Definition 3.46.

$$\operatorname{cap}_{s,p}(E;\Omega) = \inf\{[u]_{s,p}^p \colon u \in W_0^{s,p}(\Omega), u \geq 1 \text{ in an open neighborhood of } E\}.$$

Remark 3.47. It is an immediate consequence of the above definitions that $\operatorname{cap}_{s,p}(E) \leq \operatorname{cap}_{s,p}(E;\Omega)$.

Now, when we deal with pointwise properties of Sobolev functions we must change the concept of *almost everywhere* for *quasi everywhere*. The following definition expresses such idea.

Definition 3.48. We say that an property is valid $(s,p)-$quasi everywhere if it is valid except in a set of null $(s,p)-$capacity. We note this fact writing $(s,p)-$q.e.

Definition 3.49. Let $D \subset \mathbb{R}^N$ be an open and bounded set, we say that $\Omega \subset D$ is $(s,p)-$*quasi open* if there is a decreasing sequence $\{\omega_k\}_{k\in\mathbb{N}}$ of open sets such that $\operatorname{cap}_{s,p}(\omega_k, D) \to 0$ and $\Omega \cup \omega_k$ is an open set for each $k \in \mathbb{N}$.

Definition 3.50. A function $u\colon \Omega \to \mathbb{R}$ is called an $(s,p)-$*quasi continuous* function if for every $\varepsilon > 0$, there is an open set U such that $\operatorname{cap}_{s,p}(U,\Omega) < \varepsilon$ and $u|_{\Omega\setminus U}$ is continuous.

The next results, which proofs can be found in [50] will be needed in the course of the proof of our main result. Their proofs follow the same ideas found in the $p(x)$-capacity section so we chose not to include them but give precise references for the interesed reader.

Theorem 3.51 (Theorem 3.7 in [50])**.** *For each $u \in W^{s,p}(\mathbb{R}^N)$ there exists a $(s,p)-$quasicontinuous function $v \in W^{s,p}(\mathbb{R}^N)$ such that $u = v$ a.e. in $\mathbb{R}^N$.*

Remark 3.52. It is easy to see that two $(s,p)-$quasicontinuous representatives of a given function $u \in W^{s,p}(\mathbb{R}^N)$ can only differ in a set of zero $(s,p)-$capacity. Therefore, the unique $(s,p)-$quasicontinuous representative (defined $(s,p)-$q.e.) of u will be denoted by $\tilde{u}$.

Proposition 3.53 (Lemma 3.8 in [50])**.** *Let $0 < s < 1 < p < \infty$. and let $\{v_k\}_{k\in\mathbb{N}} \subset W^{s,p}(\mathbb{R}^N)$ be such that $v_k \to v$ in $W^{s,p}(\mathbb{R}^N)$ for some $v \in W^{s,p}(\mathbb{R}^N)$. Then there is a subsequence $\{v_{k_j}\}_{j\in\mathbb{N}} \subset \{v_k\}_{k\in\mathbb{N}}$ such that $\tilde{v}_{k_j} \to \tilde{v}$ $(s,p)-$q.e.*

Theorem 3.54 (Theorem 4.5 in [50])**.** *Let $D \subset \mathbb{R}^N$ be an open set and $\Omega \subset D$ an open subset. Then,*

$$u \in W_0^{s,p}(\Omega) \Leftrightarrow u \in W_0^{s,p}(D) \text{ and } \tilde{u} = 0 \ (s,p)-\text{q.e. in } D \setminus \Omega.$$

Finalmente, presentamos una noción de convergencia de dominios que será esencial para nuestros resultados, la convergencia en la topología complementaria de Hausdorff.

Definition 3.55. Sea $D \subset \mathbb{R}^N$ compacto. Sean $K_1, K_2 \subset D$ subconjuntos compactos, definimos la distancia Hausdorff d_H como

$$d_H(K_1,K_2) := \max\left\{\sup_{x\in K_1}\inf_{y\in K_2}\|x-y\|, \sup_{x\in K_2}\inf_{y\in K_1}\|x-y\|\right\}.$$

Sean $\Omega_1, \Omega_2 \subset D$ abiertos, definimos la distancia complementaria de Hausdorff d^H como

$$d^H(\Omega_1,\Omega_2) := d_H(D\setminus\Omega_1, D\setminus\Omega_2).$$

Decimos que $\{\Omega_k\}_{k\in\mathbb{N}}$ converge a Ω en el sentido de la topología complementaria de Hausdorff, y lo notamos $\Omega_k \xrightarrow{H} \Omega$, si $d^H(\Omega_k,\Omega) \to 0$.

Resumen del capítulo.

En este capítulo presentamos las definiciones y propiedades de los espacios en los que trabajaremos a lo largo de la tesis.

Este capítulo está dividido en dos partes.

En la primera parte, definimos los espacios de Lebesgue y de Sobolev asociados al $p(x)$-laplaciano.

En este sentido, presentamos las propiedades básicas de estos espacios, que serán una herramienta esencial para los resultados de los capítulos siguientes.

Entre ellas, la desigualdad de Hölder, el teorema de Rellich-Kondrachov, la desigualdad de Poincaré y la noción de continuidad log-Holder, que nos permitirá otorgar a nuestros exponentes variables la regularidad suficiente para asegurar la densidad de las funciones test, cuestión clave en nuestros desarrollos posteriores.

También definimos los espacios duales de los espacios de Sobolev con exponente variable, de particular interés ya que a ellos pertenecerán las funciones fuente de las ecuaciones que analizaremos.

Definimos a continuación las nociones de $p(x)$-capacidad y $p(x)$-capacidad relativa, que resultarán esenciales a la hora de dar condiciones necesarias para la estabilidad de las soluciones del problema de Dirichlet para el $p(x)$-laplaciano con respecto a perturbaciones del dominio.

Presentamos el concepto de $p(x)$-cuasi continuidad, analizamos la existencia y unicidad de un representante $p(x)$-cuasi continuo de cada función en $W^{1,p(x)}(\Omega)$ bajo la hipótesis de log-Hölder continuidad del exponente, cuestión que nos permitirá contar con un teorema de caracterización del espacio $W^{1,p(x)}(\Omega)$. Dicha caracterización será una herramienta sumamente útil para nuestros resultados principales.

En la segunda parte, definimos los espacios de Sobolev asociados al laplaciano fraccionario y presentamos sus propiedades básicas.

Entre ellas, la desigualdad de Poincaré y una equivalencia de normas en el espacio que resultará valiosa para simplificar nuestros cálculos posteriores.

Definimos a continuación las nociones de capacidad fraccionaria y capacidad fraccionaria relativa, que resultarán esenciales a la hora de dar condiciones necesarias para la estabilidad de las soluciones del problema de Dirichlet para el laplaciano fraccionario con respecto a perturbaciones del dominio.

Presentamos el concepto de cuasi continuidad fraccionaria, analizamos la existencia y unicidad de un representante cuasi continuo fraccionario de cada función en $W^{s,p}(\mathbb{R}^N)$, cuestión que nos permitirá contar con un teorema de caracterización del espacio $W^{s,p}(\Omega)$. Dicha caracterización será una herramienta sumamente útil para nuestros resultados principales.

Finalmente, presentamos una noción de convergencia de dominios que será esencial para nuestros resultados, la convergencia en la topología complementaria de Hausdorff.

Chapter 4

The variable exponent case.

4.1 The Dirichlet problem for the $p(x)$−laplacian.

We define the $p(x)$−laplacian as

$$\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u).$$

Observe that when $p(x) = 2$ this operator agrees with the classical Laplace operator, and when $p(x) = p$ is constant is the well-known p−laplacian.

The Dirichlet problem for the $p(x)$−laplacian consists of finding $u \in W_0^{1,p(x)}(\Omega)$ such that

$$\begin{cases} -\Delta_{p(x)} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $f \in L^{p'(x)}(\Omega)$ or, more generally, $f \in W^{-1,p'(x)}(\Omega)$. Recall that $p'(x) = p(x)/(p(x) - 1)$ is the conjugate exponent of $p(x)$.

In its weak formulation, this problem consists of finding $u \in W_0^{1,p(x)}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v \, dx = \langle f, v \rangle \text{ for every } v \in W_0^{1,p(x)}(\Omega).$$

Setting

$$I(v) := \int_\Omega \frac{1}{p(x)} |\nabla v|^{p(x)} \, dx - \langle f, v \rangle,$$

the problem can be reformulated as finding $u \in W_0^{1,p(x)}(\Omega)$ such that

$$I(u) = \min\{I(v) \colon v \in W_0^{1,p(x)}(\Omega)\}.$$

By standard methods, we obtain the following result

Theorem 4.1. *Assume $p_- > 1$. Then there exists a unique minimizer of $I(v)$ in $W_0^{1,p(x)}(\Omega)$ and a unique weak solution of* (5.1) $u \in W_0^{1,p(x)}(\Omega)$.

Proof. The proof is standard and uses the direct method of the calculus of variations. We omit the details. □

Remark 4.2. The unique weak solution of (5.1) will be denoted by $u_\Omega^{f,p(x)}$. When there is no ambiguity with respect to the exponent $p(x)$ we will denote $u_\Omega^{f,p(x)} = u_\Omega^f$. So, since for the following results the variable exponent $p(x)$ is fixed we use the lighter notation u_Ω^f.

Remark 4.3. There is, by now, a well developed regularity theory for weak solutions of (5.1). We refer the reader to the works [2, 3, 17].

Proposition 4.4. *Assume that $p \in \mathcal{P}^{\log}(\Omega)$ with $p_- > 1$. Let $f \in W^{-1,p'(x)}(\Omega)$ and let $\mathcal{A} > 0$ be such that $\|f\|_{-1,p'(x)} \le \mathcal{A}$. Then, there exists a constant C depending only on $\mathcal{A}$, p_- and p_+ such that*

$$\|\nabla u_\Omega^f\|_{p(x)} \le C.$$

Proof. Let us assume that $\|\nabla u_\Omega^f\|_{p(x)} > 1$ (otherwise, the result is clear). By Theorem 3.14 and Proposition 3.1,

$$\begin{aligned}\int_\Omega |\nabla u_\Omega^f|^{p(x)} &= \langle f, u_\Omega^f\rangle \\ &\le \|f\|_{-1,p'(x)}\|u_\Omega^f\|_{1,p(x)} \\ &\le c\|f\|_{-1,p'(x)}\|\nabla u_\Omega^f\|_{p(x)} \\ &\le c\|f\|_{-1,p'(x)}(\rho_{p(x)}(|\nabla u_\Omega^f|))^{\frac{1}{p_-}}.\end{aligned}$$

Therefore,

$$\int_\Omega |\nabla u_\Omega^f|^{p(x)} \le (c\|f\|_{-1,p'(x)})^{\frac{p_-}{p_- -1}},$$

which completes the proof. □

In what follows, the monontonicity of the $p(x)$−laplacian is crucial. This fact is a consequence of the following well-known lemma that is proved in [46, p.p. 210].

Lemma 4.5. *There is a constant $c_1 > 0$ such that for every $a, b \in \mathbb{R}^N$,*

$$(|b|^{p-2}b - |a|^{p-2}a)\cdot(b-a) \ge \begin{cases} c_1|b-a|^p & \text{if } p \ge 2, \\ c_1\frac{|b-a|^2}{(|b|+|a|)^{2-p}} & \text{if } p \le 2. \end{cases}$$

Remark 4.6. Observe that if $u \in W_0^{1,p(x)}(\Omega)$, then $-\Delta_{p(x)}u \in W^{-1,p'(x)}(\Omega)$. In fact,

$$\langle -\Delta_{p(x)}u, v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v\,dx.$$

Definition 4.7. Let $f \in W^{-1,p'(x)}(\Omega)$. We say that $f \ge 0$ if $\langle f, v\rangle \ge 0$ for every $v \in W_0^{1,p(x)}(\Omega)$ such that $v \ge 0$.

Let $f, g \in W^{-1,p'(x)}(\Omega)$. We say that $g \le f$ if $f - g \ge 0$.

We now prove the comparison principle for (5.1)

Lemma 4.8 (Comparison Principle)**.** *Let $u, v \in W_0^{1,p(x)}(D)$ be such that*

$$\begin{cases} -\Delta_{p(x)} u \leq -\Delta_{p(x)} v & \text{in } D, \\ u \leq v & \text{on } \partial D. \end{cases}$$

Then, $u \leq v$ in D.

Proof. Let us call $g := -\Delta_{p(x)} u$ and $f := -\Delta_{p(x)} v$. Then, by Remark 4.6, we obtain that, given $\varphi \in W_0^{1,p(x)}(D)$,

$$\int_D (|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v) \cdot \nabla \varphi(x)\, dx = \langle g - f, \varphi \rangle.$$

In particular, taking $\varphi = (u - v)^+ \in W_0^{1,p(x)}(D)$, since $g \leq f$ we have that

$$\int_D (|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v) \cdot \nabla (u - v)^+\, dx = \langle g - f, (u - v)^+ \rangle \leq 0.$$

Taking into account that $\nabla (u - v)^+ = (\nabla u - \nabla v)\chi_{\{u>v\}}$, we conclude that

$$\int_{\{u>v\}} (|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v) \cdot (\nabla u - \nabla v)\, dx \leq 0.$$

Now, let us define $\Omega_1' := \{x \in D \colon p(x) \geq 2\}$ and $\Omega_1'' := \{x \in D \colon p(x) < 2\}$. Therefore, $D = \Omega_1' \cup \Omega_1''$ (disjoint union).

Now, by Lemma 4.5, there is a constant $c > 0$ such that

$$\begin{aligned} &\int_{\{u \geq v\}} (|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v) \cdot (\nabla u - \nabla v)\, dx \\ &\geq c \int_{\{u \geq v\} \cap \Omega_1'} |\nabla u - \nabla v|^{p(x)}\, dx + c \int_{\{u \geq v\} \cap \Omega_1''} \frac{|\nabla u - \nabla v|^2}{(|\nabla u| + |\nabla v|)^{2-p(x)}}\, dx. \end{aligned}$$

Therefore, since $\nabla (u - v)^+ = (\nabla u - \nabla v)\chi_{u>v}$, we conclude that

$$0 \geq \int_{\Omega_1'} |\nabla (u - v)^+|^{p(x)}\, dx + \int_{\Omega_1''} \frac{|\nabla (u - v)^+|^2}{(|\nabla u| + |\nabla v|)^{2-p(x)}}\, dx.$$

Then, $\nabla (u - v)^+ = 0$ in D. So $(u - v)^+$ is constant in D. Since $(u - v)^+ \in W_0^{1,p(x)}(D)$, we have that $(u - v)^+ = 0$. Therefore $u - v \leq 0$, which completes the proof. □

Corollary 4.9 (Weak maximum principle)**.** *Let $f \in W^{-1,p'(x)}(\Omega)$ be such that $f \geq 0$. Then $u_\Omega^f \geq 0$.*

Proof. Just apply Lemma 4.8 with $u = 0$ and $v = u_\Omega^f$. □

The following proposition gives the monotonicity property of the solution with respect to the domain. The proof follows the ideas of [28, Theorem 3.2.5.] where the linear case $p(x) = 2$ is treated. Nevertheless, since the $p(x)$−laplacian is nonlinear, the monotonicity property of this operator comes into play replacing linearity in the argument.

Proposition 4.10 (Property of monotonicity with respect to the domain.)**.** *Let* $\Omega_1 \subset \Omega_2$ *and* $f \in W^{-1,p'(x)}(\Omega_2)$ *be such that* $f \geq 0$. *Then,* $u^f_{\Omega_1} \leq u^f_{\Omega_2}$.

Proof. We will denote $u_1 = u^f_{\Omega_1}$ and $u_2 = u^f_{\Omega_2}$.

Given $v \in W^{1,p(x)}_0(\Omega_1) \subset W^{1,p(x)}_0(\Omega_2)$,

$$\int_{\Omega_i} |\nabla u_i|^{p(x)-2}\nabla u_i \cdot \nabla v\, dx = \langle f, v\rangle, \quad i = 1, 2. \tag{4.2}$$

Therefore,

$$\int_{\Omega_1} (|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla u_2|^{p(x)-2}\nabla u_2) \cdot \nabla v\, dx = 0, \tag{4.3}$$

for every $v \in W^{1,p(x)}_0(\Omega_1)$.

Since $f \geq 0$, we have that $u_2 \geq 0$. Then, $(u_1 - u_2)^+ \leq u_1^+ \in W^{1,p(x)}_0(\Omega_1)$ and hence, by Lemma 3.39, $(u_1 - u_2)^+ \in W^{1,p(x)}_0(\Omega_1)$. Therefore

$$\int_{\Omega_1} (|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla u_2|^{p(x)-2}\nabla u_2) \cdot \nabla (u_1 - u_2)^+\, dx = 0.$$

Now, let us define $\Omega_1' := \{x \in \Omega_1 : p(x) \geq 2\}$ and $\Omega_1'' := \{x \in \Omega_1 : p(x) < 2\}$. Therefore, $\Omega_1 = \Omega_1' \cup \Omega_1''$ (disjoint union).

Now, by Lemma 4.5, there is a constant $c > 0$ such that

$$\begin{aligned} 0 &= \int_{\{u_1 \geq u_2\}\cap\Omega_1} (|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla u_2|^{p(x)-2}\nabla u_2) \cdot (\nabla u_1 - \nabla u_2)\, dx \\ &\geq c \int_{\{u_1 \geq u_2\}\cap\Omega_1'} |\nabla u - \nabla v|^{p(x)}\, dx + c \int_{\{u_1 \geq u_2\}\cap\Omega_1''} \frac{|\nabla u - \nabla v|^2}{(|\nabla u| + |\nabla v|)^{2-p(x)}}\, dx. \end{aligned}$$

Therefore, since $\nabla (u - v)^+ = (\nabla u - \nabla v)\chi_{\{u>v\}}$, where χ_A denotes de characteristic function of the set A, we conclude that

$$0 \geq \int_{\Omega_1'} |\nabla (u - v)^+|^{p(x)}\, dx + \int_{\Omega_1''} \frac{|\nabla (u - v)^+|^2}{(|\nabla u| + |\nabla v|)^{2-p(x)}}\, dx.$$

Then, $\nabla (u_1 - u_2)^+ = 0$ in Ω_1. Hence, $(u_1 - u_2)^+$ is constant in Ω_1. Since $(u_1 - u_2)^+ \in W^{1,p(x)}_0(\Omega_1)$, we have that $(u_1 - u_2)^+ = 0$. Therefore $u_1 - u_2 \leq 0$, which completes the proof. □

We now end this section with a stability result for solutions of the Dirichlet problem

Theorem 4.11. *Let $D \subset \mathbb{R}^N$ be open, and let $p \in \mathcal{P}^{\log}(D)$ with $p_- > 1$ and $f_i \in W^{-1,p'(x)}(D)$, $i = 1, 2$. There exists a constant $C > 0$ depending only on p_-, p_+ and $\max\{\|f_i\|_{-1,p'(x)}\}$ such that, if $\Omega \subset D$,*

$$\int_D |\nabla u_\Omega^{f_1} - \nabla u_\Omega^{f_2}|^{p(x)}\, dx \le C(\|f_1 - f_2\|_{-1,p'(x)} + \|f_1 - f_2\|^\beta_{-1,p'(x)}),$$

where the constant $\beta > 0$ depends only on p_- and p_+.

Theorem 4.11 immediately implies the following Corollary.

Corollary 4.12. *Under the same assumptions of Theorem 4.11, let $f_n, f \in W^{-1,p'(x)}(\Omega)$ be such that $\|f_n - f\|_{-1,p'(x)} \to 0$. Then*

$$\|\nabla u_\Omega^{f_n} - \nabla u_\Omega^{f}\|_{p(x)} \to 0.$$

Now we proceed with the proof of the Theorem.

Proof of Theorem 4.11. Let us denote $u_i = u_\Omega^{f_i}$.

Given $\varphi \in W_0^{1,p(x)}(\Omega)$, we have that

$$\int_\Omega |\nabla u_i|^{p(x)-2}\nabla u_i \cdot \nabla\varphi\, dx = \langle f_i, \varphi\rangle, \quad i = 1, 2. \tag{4.4}$$

In particular, considering $\varphi = u_1 - u_2 \in W_0^{1,p(x)}(\Omega)$ and subtracting, we obtain

$$\begin{aligned}\int_\Omega (|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla u_2|^{p(x)-2}\nabla u_2)\cdot(\nabla u_1 - \nabla u_2)\, dx \\ &= \langle f_1 - f_2, u_1 - u_2\rangle \\ &\le \|f_1 - f_2\|_{-1,p'(x)}\|\nabla u_1 - \nabla u_2\|_{p(x)} \\ &\le \|f_1 - f_2\|_{-1,p'(x)}(\|\nabla u_1\|_{p(x)} + \|\nabla u_2\|_{p(x)}) \\ &\le C\|f_1 - f_2\|_{-1,p'(x)},\end{aligned}$$

where we have used Proposition 4.4 in the last inequality.

On the other hand, naming $\Omega_1 = \Omega \cap \{p(x) \ge 2\}$ and $\Omega_2 = \Omega \cap \{p(x) < 2\}$, we have that

$$\begin{aligned}&\int_\Omega (|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla u_2|^{p(x)-2}\nabla u_2)\cdot(\nabla u_1 - \nabla u_2)\, dx \\ &\quad = \sum_{i=1}^2 \int_{\Omega_i} (|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla u_2|^{p(x)-2}\nabla u_2)\cdot(\nabla u_1 - \nabla u_2)\, dx.\end{aligned}$$

Let us study each of these integrals. By Lemma 4.5,

$$\int_{\Omega_1} (|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla u_2|^{p(x)-2}\nabla u_2)\cdot(\nabla u_1 - \nabla u_2)\, dx \ge c\int_{\Omega_1} |\nabla(u_1 - u_2)|^{p(x)}\, dx.$$

Let us now analyze the integral over Ω_2.

$$\begin{aligned}\int_{\Omega_2} |\nabla(u_1-u_2)|^{p(x)}dx &= \int_{\Omega_2} (|\nabla u_1|+|\nabla u_2|)^{\frac{(2-p(x))p(x)}{2}} \left(\frac{|\nabla(u_1-u_2)|}{(|\nabla u_1|+|\nabla u_2|)^{\frac{2-p(x)}{2}}}\right)^{p(x)} dx \\ &\le 2\|(|\nabla u_1|+|\nabla u_2|)^{\frac{(2-p(x))p(x)}{2}}\|_{\frac{2}{2-p(x)}} \left\|\left(\frac{|\nabla(u_1-u_2)|}{(|\nabla u_1|+|\nabla u_2|)^{\frac{2-p(x)}{2}}}\right)^{p(x)}\right\|_{\frac{2}{p(x)}} \\ &\le 2\left(\int_{\Omega_2}(|\nabla u_1|+|\nabla u_2|)^{p(x)}dx\right)^{\alpha}\left(\int_{\Omega_2}\frac{|\nabla(u_1-u_2)|^2}{(|\nabla u_1|+|\nabla u_2|)^{2-p(x)}}\,dx\right)^{\beta}\end{aligned}$$

for some constants α and β depending only on p_- and p_+. Let us observe that for the first inequality we took into account Hölder's inequality and for the second one, Remark 3.2.

Let us now find a bound for the first factor. In fact, by Proposition 4.4.

$$\int_{\Omega_2}(|\nabla u_1|+|\nabla u_2|)^{p(x)}\,dx \le 2^{p_+-1}\int_{\Omega_2}(|\nabla u_1|^{p(x)}+|\nabla u_2|^{p(x)})\,dx \le C.$$

Observe that, by Lemma 4.5, we are able to find a bound for the second factor.

$$\int_{\Omega_2}\frac{|\nabla(u_1-u_2)|^2}{(|\nabla u_1|+|\nabla u_2|)^{2-p(x)}}\,dx \le C\int_{\Omega_2}(|\nabla u_1|^{p(x)-2}\nabla u_1-|\nabla u_2|^{p(x)-2}\nabla u_2)\cdot(\nabla u_1-\nabla u_2)\,dx.$$

Then,

$$\begin{aligned}\int_{\Omega_2}|\nabla(u_1-u_2)|^{p(x)}\,dx &\le C\left(\int_{\Omega_2}(|\nabla u_1|^{p(x)-2}\nabla u_1-|\nabla u_2|^{p(x)-2}\nabla u_2)\cdot(\nabla u_1-\nabla u_2)\,dx\right)^{\beta} \\ &\le C\left(\int_{\Omega}(|\nabla u_1|^{p(x)-2}\nabla u_1-|\nabla u_2|^{p(x)-2}\nabla u_2)\cdot(\nabla u_1-\nabla u_2)\,dx\right)^{\beta} \\ &\le C\|f_1-f_2\|_{-1,p'(x)}^{\beta}.\end{aligned}$$

So we can conclude that

$$\int_{\Omega}|\nabla(u_1-u_2)|^{p(x)}\,dx \le C(\|f_1-f_2\|_{-1,p'(x)}+\|f_1-f_2\|_{-1,p'(x)}^{\beta}).$$

This finishes the proof. □

4.2 Continuity of the Dirichlet problem with respect to perturbations on the domain.

In this section we investigate the dependence of the solutions of the Dirichlet problem u_{Ω}^{f} with respect to perturbations on the domain. We will analyze a rather general problem considering a sequence of uniformly bounded domains Ω_n converging to a limit domain Ω in the Hausdorff complementary topology. Then we study whether $u_{\Omega_n}^{f}$ converges to u_{Ω}^{f} or not.

For this purpose, in the remaining of the section Ω will be a bounded open set of $\mathbb{R}^N$.

Now we state a couple of immediate corollaries of Proposition 4.4 that will be most useful.

Corollary 4.13. *Let $D \subset \mathbb{R}^N$ be an open bounded set and let $\Omega_n \subset D$ be a sequence of open domains. Let $p \in \mathcal{P}^{\log}(D)$ such that $p_- > 1$. Then, $\{u^f_{\Omega_n}\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p(x)}_0(D)$.*

Corollary 4.14. *Under the same assumptions as in Corollary 4.13, we have that the sequence $\{|\nabla u^f_{\Omega_n}|^{p(x)-2}\nabla u^f_{\Omega_n}\}_{n\in\mathbb{N}}$ is bounded in $L^{p'(x)}(D)$.*

We now extend to variable exponent spaces Proposition 3.7 in [7].

Theorem 4.15. *Let $D \subset \mathbb{R}^N$ be an open and bounded. Let $p \in \mathcal{P}^{\log}(D)$ with $p_- > 1$. Let $\Omega_n \subset D$, $n \in \mathbb{N}$ and let $f \in W^{-1,p'(x)}(D)$.*

Let us denote $u_n = u^f_{\Omega_n}$. Assume that $u_n \rightharpoonup u^*$ weakly in $W^{1,p(x)}_0(D)$. Let $\Omega \subset D$ be such that for every compact subset $K \subset \Omega$, there is an integer n_0 such that $K \subset \Omega_n$ for every $n \geq n_0$. Then, there holds that

$$-\Delta_{p(x)} u^* = f \text{ in } \Omega.$$

Remark 4.16. Observe that in order to conclude that $u^* = u^f_\Omega$ it remains to see that $u^* \in W^{1,p(x)}_0(\Omega)$.

Proof. As $p \in \mathcal{P}^{\log}(D)$, we need to verify that, given $\varphi \in C^\infty_c(\Omega)$, the following equality is valid:

$$\int_\Omega |\nabla u^*|^{p(x)-2}\nabla u^* \cdot \nabla\varphi\, dx = \langle f, \varphi\rangle.$$

Let $\varphi \in C^\infty_c(\Omega)$. Since $\operatorname{supp}(\varphi) \subset \Omega$ is compact, there is an integer n_0 such that $\operatorname{supp}(\varphi) \subset \Omega_n$ for every $n \geq n_0$. Therefore, $\varphi \in C^\infty_c(\Omega_n)$ for every $n \geq n_0$.

Set $K = \operatorname{supp}(\varphi)$ and $K^\varepsilon = \{x \in \mathbb{R}^N : d(x, K) < \varepsilon\}$ with ε sufficiently small to make sure that $K^\varepsilon \subset\subset \Omega_n \cap \Omega$ for every $n \geq n_1$.

We will, from now on, work with $n \geq \max\{n_0, n_1\}$.

Let $\eta \in C^\infty_c(\Omega)$ be such that $\eta = 1$ in $K^{\frac{\varepsilon}{2}}$, $\eta = 0$ in $(K^\varepsilon)^c$ and $0 \leq \eta \leq 1$.

Consider $\phi_n = \eta(u_n - u^*)$ and since $\phi_n \in W^{1,p(x)}_0(\Omega_n)$ we have

$$\int_D |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\phi_n\, dx = \int_{\Omega_n} |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\phi_n\, dx = \langle f, \phi_n\rangle.$$

Standard computations now give us

$$\int_D \eta|\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla(u_n - u^*)\, dx \leq \langle f, \phi_n\rangle - \int_D |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\eta(u_n - u^*)\, dx.$$

Since $u_n \rightharpoonup u^*$ in $W^{1,p(x)}_0(D)$, $\phi_n \rightharpoonup 0$ in $W^{1,p(x)}_0(\Omega)$ and so $\langle f, \phi_n\rangle \to 0$.

On the one hand, by the compactness of the embedding $W^{1,p(x)}_0(D) \subset L^{p(x)}(D)$, we have that $u_n \to u^*$ in $L^{p(x)}(D)$, and so, by Hölder's inequality,

$$\int_D |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\eta(u_n - u^*)\, dx \leq 2\|\nabla\eta\|_\infty \||\nabla u_n|^{p(x)-2}\nabla u_n\|_{p'(x)}\|u_n - u\|_{p(x)} \to 0,$$

by Corollary 4.14. Then we can conclude that

$$\limsup \int_D \eta |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla(u_n - u^*)\, dx \leq 0.$$

Since $\eta = 0$ on $(K^\varepsilon)^c$,

$$\limsup \int_{K^\varepsilon} \eta |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla(u_n - u^*)\, dx \leq 0 \tag{4.5}$$

On the other hand, since $\nabla u_n \rightharpoonup \nabla u^*$ in $L^{p(x)}(K^\varepsilon)$,

$$\int_{K^\varepsilon} \eta |\nabla u^*|^{p(x)-2} \nabla u^* \cdot \nabla(u_n - u^*)\, dx \to 0 \tag{4.6}$$

By (4.5) and (4.6) we have that

$$\limsup \int_{K^\varepsilon} \eta (|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u^*|^{p(x)-2} \nabla u^*) \cdot \nabla(u_n - u^*)\, dx \leq 0.$$

Since $K^{\frac{\varepsilon}{2}} \subset K^\varepsilon$, by Lemma 4.5, we can conclude that

$$\lim \int_{K^{\frac{\varepsilon}{2}}} (|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u^*|^{p(x)-2} \nabla u^*) \cdot \nabla(u_n - u^*)\, dx = 0.$$

Again, by Lemma 4.5, it follows that $(|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u^*|^{p(x)-2} \nabla u^*) \cdot \nabla(u_n - u^*) \to 0$ in $L^1(K^{\frac{\varepsilon}{2}})$ and therefore a.e. in $K^{\frac{\varepsilon}{2}}$.

From these facts, it easily follows that

$$\nabla u_n \to \nabla u^* \text{ a.e. in } K^{\frac{\varepsilon}{2}}. \tag{4.7}$$

Finally, by Corollary 4.14, there exists $\xi \in L^{p'(x)}(K^{\frac{\varepsilon}{2}})$ such that $|\nabla u_n|^{p(x)-2} \nabla u_n \rightharpoonup \xi$ in $L^{p'(x)}(K^{\frac{\varepsilon}{2}})$.

From (4.7), we can conclude that $\xi = |\nabla u^*|^{p(x)-2} \nabla u^*$ in $K^{\frac{\varepsilon}{2}}$ and that

$$\int_{K^{\frac{\varepsilon}{2}}} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \varphi\, dx \to \int_{K^{\frac{\varepsilon}{2}}} |\nabla u^*|^{p(x)-2} \nabla u^* \cdot \nabla \varphi\, dx.$$

Since $\operatorname{supp}(\nabla \varphi) \subset K \subset K^{\frac{\varepsilon}{2}} \subset K^\varepsilon \subset \Omega_n \cap \Omega$,

$$\int_{\Omega_n} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \varphi\, dx \to \int_{\Omega} |\nabla u^*|^{p(x)-2} \nabla u^* \cdot \nabla \varphi\, dx.$$

This finishes the proof. □

As we mentioned in Remark 4.16, in order to obtain the continuity of solutions with respect to the domain, we need to provide for conditions that ensure $u^* \in W_0^{1,p(x)}(\Omega)$. This is the content of the next theorem.

Theorem 4.17. *Let $D \subset \mathbb{R}^N$ be an open bounded set and let $\Omega_n, \Omega \subset D$ be open for every n. Let $p \in \mathcal{P}^{\log}(D)$ with $p_- > 1$. If $\Omega_n \overset{H}{\to} \Omega$ and $\operatorname{cap}_{p(x)}(\Omega_n \setminus \Omega, D) \to 0$, then $u^f_{\Omega_n} \rightharpoonup u^f_\Omega$ weakly in $W_0^{1,p(x)}(D)$.*

Proof. As before, we denote $u_n = u^f_{\Omega_n}$. By Corollary 4.13, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(D)$, therefore, we can assume that $u_n \rightharpoonup u^*$ weakly in $W_0^{1,p(x)}(D)$.

By Theorem 4.15 and Remark 4.16 the proof will be finished if we can prove that $u^* \in W_0^{1,p(x)}(\Omega)$. By Theorem 3.54, it is enough to prove that $\tilde{u^*} = 0$ $p(x)-$q.e. in Ω^c.

Consider $\tilde{\Omega}_j = \cup_{n\geq j}\Omega_n$ and $E = \cap_{j\geq 1}\tilde{\Omega}_j$.

Since $u_n \rightharpoonup u^*$ in $W_0^{1,p(x)}(D)$, by Mazur's Lemma (see for instance [20]), there is a sequence $v_j = \sum_{n=j}^{k_j} a_{n_j} u_n$ such that $a_{n_j} \geq 0$, $\sum_{n=j}^{k_j} a_{n_j} = 1$ and $v_j \to u^*$ in $W_0^{1,p(x)}(D)$.

Since $u_n \in W_0^{1,p(x)}(\Omega_n)$, by Theorem 3.54, $\tilde{u}_n = 0$ $p(x)-$q.e. in Ω_n^c. Therefore, $\tilde{v}_j = \sum_{n=j}^{k_j} a_{n_j}\tilde{u}_n = 0$ $p(x)-$q.e. in $\cap_{n=j}^{k_j}\Omega_n^c \supset \tilde{\Omega}_j^c$ for every $j \geq 1$. Then, $\tilde{v}_j = 0$ $p(x)-$q.e. in $\tilde{\Omega}_j^c$ for every $j \geq 1$. As a consequence, $\tilde{v}_j = 0$ $p(x)-$q.e. $\cup_{j\geq 1}\tilde{\Omega}_j^c = E^c$.

On the other hand, since $v_j \to u^*$ in $W_0^{1,p(x)}(D)$, by Proposition 3.37 $\tilde{v}_{j_k} \to \tilde{u^*}$ $p(x)-$q.e. Then we conclude that $\tilde{u^*} = 0$ $p(x)-$q.e. in E^c.

Since $\operatorname{cap}_{p(x)}(\Omega_n \setminus \Omega, D)$ goes to zero, passing to a subsequence, if necessary, we can assume that $\operatorname{cap}_{p(x)}(\Omega_n \setminus \Omega, D) \leq \frac{1}{2^n}$. Therefore,

$$\begin{aligned}\operatorname{cap}_{p(x)}(\tilde{\Omega}_j \setminus \Omega, D) &= \operatorname{cap}_{p(x)}(\cup_{n\geq j}\Omega_n \setminus \Omega, D)\\ &\leq \sum_{n\geq j} \operatorname{cap}_{p(x)}(\Omega_n \setminus \Omega, D)\\ &\leq \sum_{n\geq j} \frac{1}{2^n} = \frac{1}{2^{j-1}}.\end{aligned}$$

Since $E \subset \tilde{\Omega}_j$, we have that $E \setminus \Omega \subset \tilde{\Omega}_j \setminus \Omega$ for every $j \geq 1$ and so,

$$\operatorname{cap}_{p(x)}(E \setminus \Omega, D) \leq \operatorname{cap}_{p(x)}(\tilde{\Omega}_j \setminus \Omega, D) \leq \frac{1}{2^{j-1}} \text{ for every } j \geq 1.$$

Taking the limit $j \to \infty$, we have that $\operatorname{cap}_{p(x)}(E \setminus \Omega, D) = \operatorname{cap}_{p(x)}(\Omega^c \setminus E^c, D) = 0$. So we can conclude that $\tilde{u^*} = 0$ $p(x)-$q.e. in Ω^c, which completes the proof. □

The next result shows that the continuity of the solutions of the Dirichlet problem for the $p(x)-$laplacian with respect to the domain is independent of the second member f.

For constant exponents, this result was obtained in [7, Lemma 4.1]. The proof that we present here, in the non-constant exponent case, follows closely the one in [28, Theorem 3.2.5] where the linear case $p(x) \equiv 2$ is studied.

Theorem 4.18 (Independence with respect to the second member)**.** *Let $D \subset \mathbb{R}^N$ be a bounded, open set. Let $p \in \mathcal{P}^{\log}(D)$ with $p_- > 1$ and let $\Omega, \Omega_n \subset D$ be such that for every compact* *subset $K \subset \Omega$, there is an integer n_0 such that $K \subset \Omega_n$ for every $n \geq n_0$. Assume moreover that $u^M_{\Omega_n} \to u^M_\Omega$ in $L^{p(x)}(D)$ for every $M > 0$. Then $u^f_{\Omega_n} \rightharpoonup u^f_\Omega$ in $W^{1,p(x)}_0(D)$ for every $f \in W^{-1,p'(x)}(D)$.*

Proof. Let us assume first that $f \in L^\infty(D)$. Therefore, there is a constant $M > 0$ such that $-M \leq f \leq M$ a.e.

We will name $u^f_n = u^f_{\Omega_n}$ and $u^f = u^f_\Omega$.

Now, observe that $u^{-M}_n = -u^M_n$ and so by Proposition 4.10, we obtain that

$$-u^M_n \leq u^f_n \leq u^M_n. \tag{4.8}$$

By Corollary 4.13, $\{u^f_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p(x)}_0(D)$. Then, by Alaoglu's Theorem, there is a subsequence, which will remaine denoted by $\{u^f_n\}_{n\in\mathbb{N}}$, such that $u^f_n \rightharpoonup u^*$ in $W^{1,p(x)}_0(D)$.

Since, by Rellich-Kondrachov's Theorem, we know that $W^{1,p(x)}_0(D)$ is compactly embedded in $L^{p(x)}(D)$, we have that $u^f_n \to u^*$ in $L^{p(x)}(D)$.

Then, taking into account the convergence in $L^{p(x)}(D)$ in (4.8), we have that

$$-u^M \leq u^* \leq u^M.$$

Therefore, $|u^*| \leq u^M$ and, since $u^M \in W^{1,p(x)}_0(\Omega)$ we can conclude that $u^* \in W^{1,p(x)}_0(\Omega)$.

Let us assume now that $f \in W^{-1,p'(x)}(D)$. By density, there is a sequence $\{f_j\}_{j\in\mathbb{N}} \subset L^\infty(D)$ such that $f_j \to f$ in $W^{-1,p'(x)}(D)$.

Given $\varphi \in W^{-1,p'(x)}(D)$,

$$\langle \varphi, u^f_n - u^f \rangle = \langle \varphi, u^f_n - u^{f_j}_n \rangle + \langle \varphi, u^{f_j}_n - u^{f_j} \rangle + \langle \varphi, u^{f_j} - u^f \rangle.$$

Now, by Theorem 4.11, given $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that

$$\|\nabla u^f_n - \nabla u^{f_j}_n\|_{p(x)} \leq \varepsilon \quad \text{and} \quad \|\nabla u^f - \nabla u^{f_j}\|_{p(x)} \leq \varepsilon,$$

uniformly in $n \in \mathbb{N}$ for every $j \geq j_0$. By the first part of the proof,

$$\langle \varphi, u^{f_{j_0}}_n - u^{f_{j_0}} \rangle \to 0 \quad \text{as } n \to \infty.$$

This completes the proof. □

4.3 Extension of a result of Šverák.

In this section, we apply our results to prove the extension of the theorems of Šverák discussed in the introduction. Our main result being Theorem 4.28.

We begin by establishing some capacity estimate from below for compact connected sets. This was obtained for $p(x) \equiv 2$ by Šverák in [47]. See the book [28] for a proof. For general constant exponents, this estimate was obtained in [7]. Our extension to variable exponents will rely on Bucur and Trebeschi's result [7]. In fact, we use the following proposition.

Lemma 4.19 ([7], Lemma 5.1). *Let us consider a curve $\gamma_{[x,\xi]} \subseteq B(x,r)$, such that $\xi \in \partial B(x,r)$. Then*

$$\text{cap}_p(\gamma_{[x,\xi]}, B(x,2r)) \geq \text{cap}_p([x,\xi], B(x,2r)),$$

where $[x,\xi]$ denotes the segment with extrems x and ξ.

Proof. For some $\epsilon > 0$, let us consider a function $\varphi \in C_0^\infty(B(x,2r), \mathbb{R}^+)$ such that

$$\int_{B(x,2r)} |\varphi\delta|^p \, dx \leq \text{cap}_p(\gamma_{[x,\xi]}, B(x,2r)) + \epsilon$$

and $\varphi \geq 1$ on a neighbourhood U of $\gamma_{[x,\xi]}$.

Then we denote by φ^* the Steiner symmetrisation of φ with respect to the line $x\xi$. Then $\varphi^* \in H_0^{1,p}(B(x,2r))$ and $\varphi^* \geq 1$ on $U^* \supset [x,\xi]$ and

$$\int_{B(x,2r)} |\nabla^*\varphi|^p \, dx \leq \int_{B(x,2r)} |\nabla\varphi|^p \, dx.$$

Since

$$\text{cap}_p(\gamma_{[x,\xi]}, B(x,2r)) \leq \int_{B(x,2r)} |\nabla^*\varphi|^p \, dx$$

, making $\epsilon \to 0$, we conclude the proof. □

Proposition 4.20 ([7], Lemma 5.2). *Let $p > N-1$ be constant and let $K \subset \mathbb{R}^N$ be compact and connected. Assume that there exists a constant $a > 0$ such that $2a < diam\, K$. Then, for every $x \in K$ and $a \leq r < \frac{diam\, K}{2}$, we have the following inequality:*

$$\text{cap}_p(K \cap \overline{B(x,r)}, B(x,2r)) \geq c,$$

for some constant $c > 0$ depending only on p and a.

Proof. Let us consider $x \in K$ and the set $K^\delta =: \{x \in \mathbb{R}^N : d(x,K) < \delta\}$. Then, $K^\delta \subseteq K^{\delta+\epsilon}$ for all $\epsilon > 0$ and the property of the capacity on decreasing sequences of compact sets together with their monotonicity give

$$\text{cap}_p(K \cap \overline{B(x,r)}, B(x,2r)) = \lim_{\delta\to 0} \text{cap}_p(K^\delta \cap \overline{B(x,r)}, B(x,2r)).$$

The set K^δ is open, contains K and so it is connected by curves. Since K^δ is not contained in $\overline{B(x,r)}$ (since $r < \frac{1}{2}diam(K)$), there exists a point $\xi \in \partial B(x,r) \cap K^\delta$ and a continous curve $\gamma_{[x,\xi]}$ which links x and ξ and lies in $\overline{B(x,r)} \cap K^\delta$. To conclude the proof, it is sufficient to use Lemma 4.19 and the behaviour of the capacity on homothetic sets. Indeed, we get

$$\text{cap}_p(K^\delta \cap \overline{B(x,r)}, B(x,2r)) > \text{cap}_p(\gamma_{[x,\xi]} \cap \overline{B(x,r)}, B(x,2r)) \geq \text{cap}_p([x,\xi] \cap \overline{B(x,r)}, B(x,2r)).$$

The, making $\delta \to 0$ we get

$$\text{cap}_p(K \cap \overline{B(x,r)}, B(x,2r)) \geq \text{cap}_p([x,\xi] \cap \overline{B(x,r)}, B(x,2r)).$$

Since

$$\frac{\mathrm{cap}_p([x,\xi]\cap\overline{B(x,r)},B(x,2r))}{\mathrm{cap}_p(\overline{B(x,r)},B(x,2r))}=\frac{\mathrm{cap}_p([0,1]\times\{0\}^{N-1},B(0,2))}{\mathrm{cap}_p(\overline{B(0,1)},B(0,2))},$$

which completes the proof. □

The next proposition relates the relative capacity of a set for constant exponents with the one with variable exponents.

Proposition 4.21. *Let $p\in\mathcal{P}^{\log}(D)$. Then,*

$$\mathrm{cap}_{p_-}(E,D)\le C\max\{\mathrm{cap}_{p(x)}(E,D),(\mathrm{cap}_{p(x)}(E,D))^{\frac{p_-}{p_+}}\},$$

where $C>0$ depends on $|D|$, p_+ and p_-.

Proof. Given $\varphi\in W_0^{1,p(x)}(D)$, by Hölder's inequality and Proposition 3.1, we obtain

$$\int_D|\nabla\varphi|^{p_-}\,dx\le C\max\left\{\int_D|\nabla\varphi|^{p(x)}\,dx,\left(\int_D|\nabla\varphi|^{p(x)}\,dx\right)^{\frac{p_-}{p_+}}\right\}.$$

Let us denote by $h(t)=\max\{t,t^{\frac{p_-}{p_+}}\}$ for $t>0$, so we conclude

$$\begin{aligned}\inf_{\varphi\in S_{p(x)}(E,D)}\int_D|\nabla\varphi|^{p_-}\,dx&\le C\inf_{\varphi\in S_{p(x)}(E,D)}h\left(\int_D|\nabla\varphi|^{p(x)}\,dx\right)\\&=Ch\left(\inf_{\varphi\in S_{p(x)}(E,D)}\int_D|\nabla\varphi|^{p(x)}\,dx\right),\end{aligned}$$

since h is increasing.

On the other hand, since $W_0^{1,p(x)}(D)\subset W_0^{1,p_-}(D)$,

$$\inf_{\varphi\in S_{p_-}(E,D)}\int_D|\nabla\varphi|^{p_-}\,dx\le Ch\left(\inf_{\varphi\in S_{p(x)}(E,D)}\int_D|\nabla\varphi|^{p(x)}\,dx\right).$$

We can conclude that $\mathrm{cap}_{p_-}(E,D)\le Ch(\mathrm{cap}_{p(x)}(E,D))$. □

From Proposition 4.20 and Proposition 4.21 we obtain the following corollary.

Corollary 4.22. *Given $K\subset D\subset\mathbb{R}^N$ compact and connected and $p\in\mathcal{P}^{\log}(B(x,2r))$ such that $p_->N-1$. Then, for every $x\in K$ and $a\le r<\frac{\operatorname{diam}K}{2}$ for some positive constant a,*

$$\mathrm{cap}_{p(x)}(K\cap\overline{B(x,r)},B(x,2r))\ge\kappa,$$

for some constant $\kappa>0$ depending on $|D|$, $\operatorname{diam}D$, p_+ and p_-.

Proof. Just apply Proposition 4.21 to the sets $K\cap\overline{B(x,r)}$ and $B(x.2r)$, and observe that $2r<\operatorname{diam}K\le\operatorname{diam}D$. Then apply Proposition 4.20. □

Now we look for an extension of Theorem 4.17 in the sense that instead of requiring some capacity condition on the differences of the approximating domains with the limiting domain, we require a uniform boundary regularity in terms of capacity.

Definition 4.23. We say that Ω verifies the condition $(p(x), \alpha, r)$ if

$$\mathrm{cap}_{p(x)}(\Omega^c \cap B(x,r), B(x,2r)) \geq \alpha, \quad x \in \partial\Omega.$$

Set $\mathcal{O}_{\alpha,r_0}(D) = \{\Omega \subset D \text{ open} \colon \Omega \text{ verifies the condition } (p(x),\alpha,r) \text{ for every } 0 < r < r_0\}$.

From now on we will need a result on uniform continuity with respect to $\Omega \in \mathcal{O}_{\alpha,r_0}(D)$ for the solutions of the Dirichlet problem, u^f_Ω with f sufficiently integrable.

This result for $p(x) \equiv 2$ is classic and can be found, for instance, in [28, Lemma 3.4.11 and Theorem 3.4.12, p.p. 109]. The key for its proof is to obtain the *Wiener conditions*, see [26].

The extension for $1 < p < N$ constant can be found in the articles [24, 33, 39]. Consult the book [37], Theorem 4.22. The result for $p(x)$ variable was recently obtained in [36].

Lemma 4.24 ([36], Theorem 4.4)**.** *Given $\Omega \in \mathcal{O}_{\alpha,r_0}(D)$, $f \in L^r(D)$, $r > N$. Then, there are constants $K > 0$ and $0 < \delta < 1$ such that $|u^f_\Omega(x) - u^f_\Omega(y)| \leq K|x-y|^\delta$.*

With this result we are able to prove the analogous of Theorem 4.17 for domains in $\mathcal{O}_{\alpha,r}$.

Theorem 4.25. *Given $\{\Omega_n\}_{n\in\mathbb{N}} \subset \mathcal{O}_{\alpha,r_0}(D)$ such that $\Omega_n \xrightarrow{H} \Omega$. Then, $u^f_{\Omega_n} \rightharpoonup u^f_\Omega$ in $W^{1,p(x)}_0(D)$.*

Proof. By Theorem 4.18, we can assume that $f = M$ with $M > 0$ and $u^M_{\Omega_n} \rightharpoonup u^*$ in $W^{1,p(x)}_0(D)$.

In order to see that $u^* = u^M_\Omega$, by Theorem 4.15, it is enough to verify that $u^* \in W^{1,p(x)}_0(\Omega)$. By Theorem 3.54, it is enough to prove that $\tilde{u^*} = 0$ $p(x)$–q.e. in Ω^c.

As a direct consequence of Lemma 4.8, $u^M_D \geq 0$ and $u^M_{\Omega_n} \geq 0$.

By Lemma 4.24, given $y \in \partial D$, for every $x \notin \Omega$ we have

$$u^M_D(x) = |u^M_D(x) - u^M_D(y)| \leq K|x-y|^\delta \leq K(\mathrm{diam}\, D)^\delta.$$

By Lemma 4.8, $0 \leq u^M_{\Omega_n} \leq u^M_D \leq K(\mathrm{diam}\, D)^\delta$. Therefore, $\{u^M_{\Omega_n}\}_{n\in\mathbb{N}}$ is uniformly bounded for every $M > 0$.

By Lemma 4.24, $\{u^M_{\Omega_n}\}_{n\in\mathbb{N}}$ is uniformly equicontinuous for every $M > 0$. Therefore, $\{u^M_{\Omega_n}\}_{n\in\mathbb{N}}$ converges uniformly to u^*.

Given $x \in \Omega^c$, since $\Omega_n \xrightarrow{H} \Omega$, there is a sequence $x_n \in \Omega_n^c$ such that x_n converges to x. By uniform convergence, we have that $u^M_{\Omega_n}(x_n)$ converges to $u^*(x)$. Since $\mathrm{supp}\, u^M_{\Omega_n} \subset \bar{\Omega}_n$, we obtain that $u^M_{\Omega_n}(x_n) = 0$ for every n and, therefore, $u^*(x) = 0$, which completes the proof. □

Remark 4.26. If $p_- > N$, the same proof can be applied. It is enough to observe that, by Morrey's estimates, $W^{1,p(x)}_0(D) \subset W^{1,p_-}_0(D) \subset C^\alpha(D)$ with $\alpha = 1 - N/p_-$.

Having presented the previous results, the proof of the extension is similar to the one given by Šverák for $p = 2$. We include it for the reader's convenience.

Definition 4.27. Given $l \in \mathbb{N}$ and $\Omega \subset D$, set $\#\Omega$ the number of connected components of $D \setminus \Omega$.

Set $\mathcal{O}_l(D) = \{\Omega \subset D \text{ open}\colon \#\Omega \le l\}$.

Theorem 4.28. *Given* $p \in \mathcal{P}^{\log}(D)$ *such that* $N - 1 < p_-$ *and* $\{\Omega_n\}_{n\in\mathbb{N}} \subset \mathcal{O}_l(D)$ *such that* $\Omega_n \overset{H}{\to} \Omega$. *Then* $u^f_{\Omega_n} \rightharpoonup u^f_\Omega$ *in* $W^{1,p(x)}_0(D)$.

Proof. By Remark 4.26, we only have to consider the case $N - 1 < p_- \le N$.

By Theorem 4.18, we can assume that $f = M$ with $M > 0$ and $u_n := u^M_{\Omega_n} \rightharpoonup u^*$ in $W^{1,p(x)}_0(D)$.

In order to see that $u^* = u^M_\Omega$, by Theorem 4.15, it is sufficient to verify that $u^* \in W^{1,p(x)}_0(\Omega)$.

Set $\bar{D} \setminus \Omega_n = F_n = F^1_n \cup F^2_n \cup ... \cup F^l_n$ where each F^i_n is compact and connected. Assume that $F^j_n \overset{H}{\to} F^j$ for every $1 \le j \le l$.

Let us analyze each of the three possibilities. We will find that it is possible to disregard the first two.

(1) If $F^j = \emptyset$, then $F^j_n = \emptyset$ for every $n \ge n_0$. Set $J_0 = \{j \in \{1, \dots, l\}\colon F^j_n = \emptyset \text{ for } n \text{ large}\}$.

(2) If $F^j = \{x_j\}$, set $J_1 = \{j \in \{1, \dots, l\}\colon F^j = \{x_j\} \text{ and } p(x_j) \le N\}$. Now consider the set $\Omega^* = \Omega \setminus \cup_{i\in J_1}\{x_i\}$. Since $\text{cap}_{p(x)}(\{x_i\}, D) = 0$, we have that $\text{cap}_{p(x)}(\Omega^*, D) = \text{cap}_{p(x)}(\Omega, D)$. Then, by Theorem 3.54, $W^{1,p(x)}_0(\Omega^*) = W^{1,p(x)}_0(\Omega)$. It is enough therefore to verify that $u^* \in W^{1,p(x)}_0(\Omega^*)$.

Set $I = \{1, \dots, l\} \setminus (J_0 \cup J_1)$ and consider $\Omega^*_n = D \setminus \cup_{j\in I} F^j_n \overset{H}{\to} \Omega^*$.

(3) If, for $j \in I$, F^j contains al least two points, let a_j be the distance between them. These points are limits of points from F^j_n which we may assume to have a distance at least of $\frac{a_j}{2}$ between them for n large enough.

Given $x \in \partial\Omega^*_n$ and $j = j(x) \in I$ such that $x \in F^j_n$, by Corollary 4.22, if $a \le r < \frac{a_j}{4}$ for some positive constant a, then there is a universal constant κ that verifies the following inequality:

$$\text{cap}_{p(x)}((\Omega^*_n)^c \cap \overline{B(x, r)}, B(x, 2r)) \ge \text{cap}_{p(x)}(F^j_n \cap \overline{B(x, r)}, B(x, 2r)) \ge \kappa > 0.$$

This shows that the open sets Ω^*_n belong to $\mathcal{O}_{\alpha,r_0}$ with $\alpha = \kappa$ and $r_0 = \frac{1}{4}\min\{a_j\colon j \in I\}$.

Since $\Omega^*_n \overset{H}{\to} \Omega$, by Theorem 4.25, we have that $u^M_{\Omega^*_n} \rightharpoonup u^M_\Omega$ in $W^{1,p(x)}_0(D)$.

On the other hand, since $\Omega_n \subset \Omega^*_n$, by a direct consequence of Lemma 4.8 and Proposition 4.10, we have that $0 \le u^M_{\Omega_n} \le u^M_{\Omega^*_n}$. Passing to the limit $n \to \infty$, $0 \le u^* \le u^M_\Omega$. We conclude then, by Lemma 3.39, that $u^* \in W^{1,p(x)}_0(\Omega)$

If F^j contains exactly one point x_0, then $p(x_0) > N$ and so $\{x_0\}$ has positive $p(x)$−capacity, the bound from below will be its capacity, which completes the proof. □

Resumen del capítulo.

En este capítulo se presentan propiedades de comparación y de monotonía para el $p(x)$-laplaciano (esta última respecto del dominio) que son claves para el tratamiento de la cuestión de la continuidad para el problema de Dirichlet respecto de la fuente y del dominio. Se arriba luego al primer resultado importante de la tesis que garantiza la convergencia de los minimizantes bajo convergencia de los dominios (en términos adecuados que involucran la distancia complementaria de Haussdorff y la $p(x)$- capacidad). En otro resultado relevante de este capítulo se generaliza un famoso teorema de Šverák para el caso de exponente variable dando un enunciado geométrico en términos del nḿero de componentes conexas complementarias de los dominios aproximantes.

Este capítulo está dividido en dos partes.

En la primera, definimos el planteo del problema de Dirichlet asociado al $p(x)$-laplaciano junto con su formulación variacional para luego presentar la unicidad y existencia de solución débil de dicho problema.

A continuación presentamos el Principio de comparación, el Principio del máximo débil y la propiedad de monotonía de las soluciones con respecto al dominio.

Finalmente, luego de algunos resultados previos, obtenemos nuestro primer resultado importante de esta parte de la tesis: damos condiciones capacitarias sobre la diferencia simétrica de los dominios que garantizan la convergencia de las soluciones en el problema del Dirichlet asociado al laplaciano correspondiente al caso de exponente variable. Más precisamente,

Sea $D \subset \mathbb{R}^N$ un conjunto abierto y acotado y sean $\Omega_n, \Omega \subset D$ abiertos para todo n. Sea $p \in \mathcal{P}^{\log}(D)$ con $p_- > 1$. Si $\Omega_n \overset{H}{\to} \Omega$ y $\text{cap}_{p(x)}(\Omega_n \setminus \Omega, D) \to 0$, entonces $u^f_{\Omega_n} \rightharpoonup u^f_{\Omega}$ débil en $W^{1,p(x)}_0(D)$.

A continuación, mostramos que la continuidad de las soluciones del problema de Dirichlet para el $p(x)-$laplaciano con respecto del dominio es independiente de f. Con mayor precisión,

Sean $D \subset \mathbb{R}^N$ un conjunto abierto acotado, $p \in \mathcal{P}^{\log}(D)$ con $p_- > 1$ y $\Omega, \Omega_n \subset D$ tales que para cada subconjunto compacto $K \subset \Omega$, existe n_0 tal que $K \subset \Omega_n$ para cada $n \geq n_0$. Si $u^M_{\Omega_n} \to u^M_{\Omega}$ en $L^{p(x)}(D)$ para cada $M > 0$, entonces $u^f_{\Omega_n} \rightharpoonup u^f_{\Omega}$ en $W^{1,p(x)}_0(D)$ para cada $f \in (W^{1,p(x)}(D))'$.

En la segunda parte, aplicamos nuestros resultados para probar la extensión de un teorema de Šverák que asegura que, dado un dominio acotado $D \subset \mathbb{R}^2$ y una sucesión de dominios $\Omega_n \subset D$ tales que $\Omega_n \to \Omega$ en el sentido de la topología complementaria de Hausdorff, la condición que garantiza la convergencia de las soluciones en Ω_n a la solución en Ω es que el número de componentes conexas de $D \setminus \Omega_n$ sea acotado.

La razón por la cual el resultado de Šverák vale en dimensión 2 es que la capacidad de las curvas en dimensión 2 es positiva, mientras que en dimensiones más altas las curvas tienen capacidad nula.

En nuestro caso, fue necesario dar condiciones geométricas sobre la sucesión de dominios aproximantes que garantizan la convergencia de las soluciones para el caso $p(x)$-laplaciano. Más precisamente,

Sea f suficientemente integrable. Dados $l \in \mathbb{N}$ y $\Omega \subset D$, sea $\#\Omega$ el número de componentes conexas de $D \setminus \Omega$. Consideramos $\mathcal{O}_l(D) = \{\Omega \subset D \text{ abierto}: \#\Omega \leq l\}$.

Dados $p \in \mathcal{P}^{\log}(D)$ tal que $N - 1 < p_-$ y $\{\Omega_n\}_{n\in\mathbb{N}} \subset \mathcal{O}_l(D)$ tales que $\Omega_n \stackrel{H}{\to} \Omega$. Entonces $u^f_{\Omega_n} \rightharpoonup u^f_\Omega$ en $W^{1,p(x)}_0(D)$.

Chapter 5

The fractional diffusion case.

5.1 Continuity of the problems with respect to variable domains

Throughout this section we consider $0 < s < 1 < p < \infty$ to be fixed.

Let $D \subset \mathbb{R}^N$ be a bounded, open set and let $\Omega \subset D$ be an open set. The Dirichlet problem for the $(s, p)-$laplacian consists of finding $u \in W_0^{s,p}(\Omega)$ such that

$$\begin{cases} (-\Delta)_p^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Omega^c := \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.1}$$

where $f \in W^{-s,p'}(D) := [W_0^{s,p}(D)]'$.

In its weak formulation, this problem consists of finding $u \in W_0^{s,p}(\Omega)$ such that

$$\langle (-\Delta)_p^s u, v\rangle = \langle f, v\rangle \text{ for every } v \in W_0^{s,p}(\Omega).$$

That is, for every $v \in W_0^{s,p}(\Omega)$, the following equality holds

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}\,(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+sp}}\, dx\, dy = \langle f, v\rangle.$$

Lemma 5.1. *Let $f \in W^{-s,p'}(D)$ and $\Omega \in \mathcal{A}(D)$. Then there exists a unique $u \in W_0^{s,p}(\Omega)$, which we will denote u_Ω^f, solution of* (5.1).

Proof. It is enough to consider $\mathfrak{I}\colon W_0^{s,p}(\Omega) \to \mathbb{R}$ defined by $\mathfrak{I}(v) := \frac{1}{p}[v]_{s,p}^p - \langle f, v\rangle$ and observe that u is solution of (5.1) if and only if u is a minimizer for $\mathfrak{I}$. Since $\mathfrak{I}$ has a unique minimizer (observe that $\mathfrak{I}$ is strictly convex), this completes the proof. □

Now we observe that these solutions u_Ω^f are bounded independently of Ω.

Lemma 5.2. *There is a constant $C = C(\|f\|_{-s,p'}, s, p, n, |D|)$ such that $\|u_\Omega^f\|_{s,p} \le C$ for every $\Omega \in \mathcal{A}(D)$.*

Proof. Let us observe that

$$[u^f_\Omega]^p_{s,p} = \langle (-\Delta)^s_p u^f_\Omega, u^f_\Omega\rangle = \langle f, u^f_\Omega\rangle \leq \|f\|_{-s,p'}\|u^f_\Omega\|_{s,p}.$$

Combining this inequality with Theorem 3.41, there exists a constant $C > 0$ such that

$$\|u^f_\Omega\|^p_{s,p} \leq C\|f\|_{-s,p'}\|u^f_\Omega\|_{s,p},$$

from where the conclusion of the lemma follows. □

As an immediate corollary, we have the following result.

Corollary 5.3. *Let $\{\Omega_k\}_{k\in\mathbb{N}} \subset \mathcal{A}(D)$. Then, $\{u^f_{\Omega_k}\}_{k\in\mathbb{N}}$ is bounded in $W^{s,p}_0(D)$ and, therefore, there exists $u^* \in W^{s,p}_0(D)$ and a subsequence $\{u^f_{\Omega_{k_j}}\}_{j\in\mathbb{N}} \subset \{u^f_{\Omega_k}\}_{k\in\mathbb{N}}$ such that $u^f_{\Omega_{k_j}} \rightharpoonup u^*$ weakly in $W^{s,p}_0(D)$.*

The next result is a first step in proving the continuity result.

Theorem 5.4. *Let $\{\Omega_k\}_{k\in\mathbb{N}} \subset \mathcal{A}(D)$ and $\Omega \in \mathcal{A}(D)$ be such that $\Omega_k \xrightarrow{H} \Omega$. Assume that $u^f_{\Omega_k} \rightharpoonup u^*$ weakly in $W^{s,p}_0(D)$ for some $u^* \in W^{s,p}_0(D)$ when $k \to \infty$. Then*

$$(-\Delta)^s_p u^* = f \text{ in } \Omega,$$

in the sense of distributions. That is

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u^*(x) - u^*(y)|^{p-2}(u^*(x) - u^*(y))(\phi(x) - \phi(y))}{|x - y|^{n+sp}}\, dx\, dy = \langle f, \phi\rangle, \tag{5.2}$$

for every $\phi \in C^\infty_c(\Omega)$.

Proof. We denote $u_k = u^f_{\Omega_k}$, and also denote

$$\xi_k(x, y) = \frac{|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))}{|x - y|^{\frac{n+sp}{p'}}}.$$

Then, $\xi_k \in L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ and

$$\|\xi_k\|^{p'}_{L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)} = [u_k]^p_{s,p}.$$

Therefore, from Lemma 5.2, we get that $\{\xi_k\}_{k\in\mathbb{N}}$ is bounded in $L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$. So, up to some subsequence, there exists a function $\xi \in L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ such that

$$\xi_k \rightharpoonup \xi \text{ weakly in } L^{p'}(\mathbb{R}^N \times \mathbb{R}^N).$$

Therefore,

$$\begin{aligned}\lim_{k\to\infty} \langle (-\Delta_p)^s u_k, \phi\rangle &= \lim_{k\to\infty} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \xi_k(x, y)\frac{\phi(x) - \phi(y)}{|x - y|^{\frac{n}{p}+s}}\, dx\, dy\\ &= \iint_{\mathbb{R}^N\times\mathbb{R}^N} \xi(x, y)\frac{\phi(x) - \phi(y)}{|x - y|^{\frac{n+sp}{p}}}\, dx\, dy,\end{aligned} \tag{5.3}$$

for all $\phi \in W^{s,p}(\mathbb{R}^N)$. In particular, (5.3) holds for every $\phi \in C_c^\infty(\Omega)$. Moreover, by the compactness of the immersion $W_0^{s,p}(D) \subset L^p(D)$ (see [16]), since $u_k \rightharpoonup u^*$ weakly in $W_0^{s,p}(D)$ we can conclude that $u_k \to u^*$ a.e. in $\mathbb{R}^N$, then

$$\xi_k(x,y) \to \frac{|u^*(x) - u^*(y)|^{p-2}(u^*(x) - u^*(y))}{|x-y|^{\frac{n+sp}{p'}}},$$

a.e. in $\mathbb{R}^N \times \mathbb{R}^N$, from where it follows that

$$\xi(x,y) = \frac{|u^*(x) - u^*(y)|^{p-2}(u^*(x) - u^*(y))}{|x-y|^{\frac{n+sp}{p'}}}. \tag{5.4}$$

Finally, observe that if $\phi \in C_c^\infty(\Omega)$ then $\phi \in C_c^\infty(\Omega_k)$ for every k sufficiently large. Therefore, from (5.3) we conclude that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \xi(x,y)\frac{\phi(x)-\phi(y)}{|x-y|^{\frac{n}{p}+s}}\,dx\,dy = \langle f\phi\rangle.$$

The proof is then completed by combining this last equality with (5.4). □

Remark 5.5. In order to show that $u^* = u_\Omega^f$, what remains is to show that $u^* = 0$ on Ω^c. This is the hard part and is where some geometric hypotheses on the nature of the convergence of the domains needs to be made.

Theorem 5.6. *Assume that the hypotheses of Theorem 5.4 are satisfied. If, in addition,*

$$\text{cap}_{s,p}(\Omega_k \setminus \Omega, D) \to 0, \tag{5.5}$$

then $u_{\Omega_k}^f \rightharpoonup u_\Omega^f$ weakly in $W_0^{s,p}(D)$.

Proof. As before, we denote $u_k = u_{\Omega_k}^f$. By Corollary 5.3, $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W_0^{s,p}(D)$ and therefore we can assume that $u_k \rightharpoonup u^*$ weakly in $W_0^{s,p}(D)$.

By Theorem 5.4 the proof will be finished if we can prove that $u^* \in W_0^{s,p}(\Omega)$, and by Theorem 3.54, it is enough to prove that $\tilde{u}^* = 0$ $(s,p)-$q.e. in Ω^c.

Consider $\tilde{\Omega}_j = \bigcup_{k\geq j}\Omega_k$ and $E = \bigcap_{j\geq 1}\tilde{\Omega}_j$.

Since $u_k \rightharpoonup u^*$ in $W_0^{s,p}(D)$, by Mazur's Lemma (see for instance [20]), there is a sequence $v_j = \sum_{k=j}^{N_j} a_k^j u_k$ such that $a_k^j \geq 0$, $\sum_{k=j}^{N_j} a_k^j = 1$ and $v_j \to u^*$ strongly in $W_0^{s,p}(D)$.

Since $u_k \in W_0^{s,p}(\Omega_k)$, by Theorem 3.54, $\tilde{u}_k = 0$ $(s,p)-$q.e. in Ω_k^c. Therefore, $\tilde{v}_j = \sum_{k=j}^{N_j} a_k^j \tilde{u}_k = 0$ $(s,p)-$q.e. in $\cap_{k=j}^{N_j}\Omega_k^c \supset \tilde{\Omega}_j^c$ for every $j \in \mathbb{N}$.

Then, $\tilde{v}_j = 0$ $(s,p)-$q.e. in $\tilde{\Omega}_j^c$ for every $j \in \mathbb{N}$ and, since $\tilde{\Omega}_j^c \subset \tilde{\Omega}_{j+1}^c$, we conclude that $\tilde{v}_j = 0$ $(s,p)-$q.e. $\tilde{\Omega}_i^c$ for every $i \leq j$.

On the other hand, since $v_j \to u^*$ strongly in $W_0^{s,p}(D)$, by Proposition 3.53, $\tilde{v}_{j_k} \to \tilde{u}^*$ $(s,p)-$q.e. Then we conclude that $\tilde{u}^* = 0$ $(s,p)-$q.e. in E^c.

In order to finish the proof of the theorem, we show that the capacitary condition (5.5) implies that $\Omega^c \subset E^c$ up to some set of zero $(s,p)-$capacity.

In fact, since $\text{cap}_{s,p}(\Omega_k \setminus \Omega) \to 0$, passing to a subsequence, if necessary, we can assume that $\text{cap}_{s,p}(\Omega_k \setminus \Omega) \leq \frac{1}{2^k}$. Therefore,

$$\text{cap}_{s,p}(\tilde{\Omega}_j \setminus \Omega) = \text{cap}_{s,p}(\cup_{k\geq j}\Omega_k \setminus \Omega) \leq \sum_{k\geq j} \text{cap}_{s,p}(\Omega_k \setminus \Omega) \leq \sum_{k\geq j} \frac{1}{2^k} = \frac{1}{2^{j-1}}.$$

Recall now that $E \subset \tilde{\Omega}_j$ for every $j \in \mathbb{N}$, then we have that

$$\text{cap}_{s,p}(E \setminus \Omega) \leq \text{cap}_{s,p}(\tilde{\Omega}_j \setminus \Omega) \leq \frac{1}{2^{j-1}} \text{ for every } j \in \mathbb{N}.$$

Taking the limit $j \to \infty$, we have that $\text{cap}_{s,p}(E \setminus \Omega) = \text{cap}_{s,p}(\Omega^c \setminus E^c) = 0$ and the proof is finished. □

As a simple corollary, we can show that the convergence of the solutions in Theorem 5.6 is actually strong.

Corollary 5.7. *Under the assumptions of Theorem 5.6 we have that* $u^f_{\Omega_m} \to u^f_\Omega$ *strongly in* $W^{s,p}_0(D)$.

Proof. The proof is simple. Just observe that from the weak convergence $u^f_{\Omega_m} \rightharpoonup u^f_\Omega$ given by Theorem 5.6, we get

$$[u^f_{\Omega_m}]^p_{s,p} = \langle f, u^f_{\Omega_m} \rangle \to \langle f, u^f_\Omega \rangle = [u^f_\Omega]^p_{s,p}.$$

Since $W^{s,p}_0(D)$ is a uniformly convex Banach space, the result follows. □

Resumen del capítulo.

En este capítulo estudiamos problemas similares a los tratados en el capítulo anterior pero para el caso del laplaciano fraccionario.

Más precisamente, estudiamos el problema de Dirichlet asociado al operador $(s,p)-$laplaciano fraccionario.

El operador $(s,p)-$laplaciano se define como

$$(-\Delta_p)^s u(x) := 2 \text{ p.v.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\,dy,$$

salvo una constante de normalización.

El término p.v. debe leerse como *valor principal*, es decir,

$$(-\Delta_p)^s u(x) = 2 \lim_{\epsilon\searrow 0} \int_{\mathbb{R}^N\setminus B(x,\epsilon)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\,dy$$

para todo $x \in \mathbb{R}^N$.

En este sentido probamos que, dados Ω y f, el problema

$$\begin{cases} (-\Delta_p)^s u = f & \text{en } \Omega \\ u = 0 & \text{en } \mathbb{R}^N \setminus \Omega, \end{cases}$$

tiene una única solución.

A continuación extendemos los resultados de contiuidad con respecto del domino para estos operadores no locales.

Asumiendo que la sucesión de dominios $\{\Omega_k\}_{k\in\mathbb{N}}$ es tal que $\Omega_k \to \Omega$ en una noción de convergencia de conjuntos apropiada, nos interesa poder responder: Entonces es cierto que $u^f_{\Omega_k} \to u^f_\Omega$ en algún sentido? O, más generalmente, dar condiciones necesarias y/o suficientes para que ello ocurra.

Luego, nuestro objetivo en este trabajo fue hallar condiciones capacitarias sobre las diferencias simétricas $\Omega_k \triangle \Omega$ para poder obtener la convergencia de las soluciones $u^f_{\Omega_k} \to u^f_\Omega$.

Arribamos a un resultado de continuidad respecto de los dominios, una vez más en términos de la diferencia complementaria de Hausdorff, pero en esta ocasión involucrando la capacidad fraccionaria como concepto clave:

Sea $\{\Omega_k\}_{k\in\mathbb{N}} \subset \{\Omega \subset D : \Omega \text{ es abierto}\}$ y $\Omega \in \mathcal{A}(D)$.

Si $\Omega_k \overset{H}{\to} \Omega$ y $u^f_{\Omega_k} \rightharpoonup u^*$ débil en $W^{s,p}_0(D)$ para algún $u^* \in W^{s,p}_0(D)$, entonces vale que $(-\Delta)^s_p u^* = f$ in Ω en el sentido de las distribuciones:

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u^*(x)-u^*(y)|^{p-2}(u^*(x)-u^*(y))(\phi(x)-\phi(y))}{|x-y|^{n+sp}}\,dx\,dy = \langle f, \phi\rangle,$$

para todo $\phi \in C^\infty_c(\Omega)$.

Si, además, $\text{cap}_{s,p}(\Omega_k \setminus \Omega, D) \to 0$, entonces $u^f_{\Omega_k} \rightharpoonup u^f_\Omega$ débil en $W^{s,p}_0(D)$.

Chapter 6

Hadamard derivative in image restoration.

6.1 Introduction

In this section, we describe the Hadamard shape derivative. For further reading on the subject, see [28].

Let $\mathcal{A}$ be a class of admissible domains and assume that associated to each $\Omega \in \mathcal{A}$ we have a well-defined cost $J(\Omega)$, that is we have a cost functional $J\colon \mathcal{A} \to \mathbb{R}$.

The main objective in shape optimization is to find $\Omega^* \in \mathcal{A}$ that minimizes the cost J among all $\Omega \in \mathcal{A}$. To this end, a first step is to find necessary conditions on any optimal configuration Ω^*.

The main idea is to try to compute a sort of *derivative* of the cost functional J and that derivative on the optimal configuration Ω^* must be 0.

In order to compute a derivative of J we proceed as follows. Let $V\colon \mathbb{R}^N \to \mathbb{R}^N$ be a Lipschitz deformation field. Now, we consider the flow associated to the field V, $\{\Phi_t\}_{t\in\mathbb{R}}$, $\Phi_t\colon \mathbb{R}^N \to \mathbb{R}^N$ given by

$$\dot{\Phi}_t(x) = V(\Phi_t(x)), \qquad \Phi_0(x) = x.$$

Observe that the flow is well-defined since V is Lipschitz. See [11].

Let $\Omega \subset \mathbb{R}^N$ be a domain and consider the family of perturbed domains Ω_t given by $\Omega_t := \Phi_t(\Omega)$.

Finally, we define $j(t) = J(\Omega_t) = J(\Phi_t(\Omega))$ the cost associated to each Ω_t.

Therefore, if any deformation Ω_t belongs to the admissible class $\mathcal{A}$ and if Ω is an optimal configuration, one must have $j'(0) = 0$.

The explicit form $j'(t)$ in $t = 0$ is particulary interesing because of two main applications. First, what neccesary conditions must the optimal domains verify? (in a minimun, the derivative is 0) and secondly, how to deform a given domain Ω to minimize its cost? (we mean to obtain a derivative as negative as possible).

For applications of Hadamard derivative see [28, 41, 42].

6.2 Application to image restoration

In this section we deal with the problem of obtaining an image which is modeled by a function $u\colon \Omega \to \mathbb{R}$, where $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, given a distorted image $I\colon \Omega \to \mathbb{R}$. We assume that the introduced error, $e = u - I$, is small and the objective is to recover u from I without making any further assumptions on the error e.

As we detailed in the Introduction, we will now present a method that approximates the one created by Chambolle and Lions [8] preserving the good functional properties given by the one presented by Chen, Levine and Rao [9].

As we have explained in the Introduction, we start by dividing the region Ω into two sub regions D_1 and D_2 such that for $i = 1, 2$,

$$D_i \subset \Omega \text{ is open}, \ \overset{\circ}{\overline{D_i}} = D_i, \ D_1 \cap D_2 = \emptyset, \text{ and } \overline{\Omega} = \overline{D_1} \cup \overline{D_2}. \tag{6.1}$$

By this division, we make sure that D_1 contains the regions with boundaries of the image and D_2 its complement. One way of creating this division is the following:

$$D_1 = \{x \in \Omega\colon |\nabla G_\sigma * I| > \beta\}, \quad D_2 = \{x \in \Omega\colon |\nabla G_\sigma * I| < \beta\}.$$

Here $\sigma > 0$ is a small parameter and G_σ is the gaussian distribution with 0 mean and σ^2 variance.

We define an exponent $p\colon \Omega \to \mathbb{R}$ given by

$$p(x) = \begin{cases} 1 + \epsilon & \text{if } x \in D_1 \\ 2 & \text{if } x \in D_2. \end{cases}$$

Then we compute u by minimizing the functional

$$\mathbb{J}(v) = \frac{1}{2\beta} \int_\Omega |\nabla v|^{p(x)} \, dx + \frac{\beta}{2} \int_\Omega (v - I)^2 \, dx,$$

where $\beta > 0$ is a parameter to be determined by the operator of the method.

Once this problem is solved, we want to improve the image found. In order to do that, we then apply an iterative *steepest descent–type method*.

To this end, observe that the infimum u of $\mathbb{J}$ depends on the chosen domain D_1. Hence, we can define the cost functional $J(D_1) = \mathbb{J}(u)$ and now we want to compute the Hadamard derivative of J.

We will assume that the deformation vector field V verifies that $\operatorname{supp}(V) \subset \Omega$, so that $\Phi_t(\Omega) = \Omega$ for every $t \in \mathbb{R}$.

Then, the regions D_i are deformed by Φ_t and we obtain a family of partitions $D_i^t = \Phi_t(D_i), \quad i = 1, 2$ that verify (6.10) and we define

$$p_t(x) = \begin{cases} 1 + \epsilon & \text{si } x \in D_1^t \\ 2 & \text{si } x \in D_2^t. \end{cases}$$

Observe that $p_t = p \circ \Phi_{-t}$.

Then, for each $t \in \mathbb{R}$ we define the following functional

$$J_t(v) = \frac{1}{2\beta} \int_\Omega |\nabla v|^{p_t(x)}\, dx + \frac{\beta}{2} \int_\Omega (v - I)^2\, dx,$$

Let u_t be the minimizer of J_t. We can consider the function $j\colon \mathbb{R} \to \mathbb{R}$ given by $j(t) = J_t(u_t)$.

So, the shape (or Hadamard) derivative consists then in computing $j'(0)$.

One important problem in this cases is to obtain an expression as simple as possible for this shape derivative, that allows us to see for what deformation fields V we obtain the *steepest descent* and also to obtain good necessary conditions for optimal partitions.

Then, by finding a good expression for such derivative, we will be able to compute the deformations field V which makes it as negative as possible and so choose the optimal deformation field to then iterate

$$D_i^{\Delta t} \simeq (id + \Delta t V)(D_i).$$

Remark 6.1. Note that, by Taylor expansion, we have

$$\Phi_t(x) = x + V(x)t + o(t).$$

And so we have the following asymptotic formulas:

$$D\Phi_t(x) = Id + tDV(x) + o(t) = Id + O(t),$$

$$J\Phi_t(x) = 1 + t \operatorname{div} V(x) + o(t) = 1 + O(t),$$

for all $x \in \mathbb{R}^N$, where $J\Phi_t$ is the Jacobian of Φ_t.

The following proposition, though elementary, will be useful in the sequel and shows that any diffeomorphism $\Phi\colon \mathbb{R}^N \to \mathbb{R}^N$, induces a bounded linear isomorphism between Sobolev spaces.

Proposition 6.2. *Let $\Phi\colon \Omega_1 \to \Omega_2$ be a diffeomorphism and $p \in \mathcal{P}(\Omega_1)$ be a bounded exponent.*

Then, Φ induces a bounded linear isomorphism

$$\mathcal{F}\colon W^{1,p}(\Omega_1) \to W^{1,q}(\Omega_2),$$

where $q\colon \Omega_2 \to [1, +\infty)$ is given by $q(x) := p(\Phi^{-1}(x))$, by the expression

$$\mathcal{F}(u) := u \circ \Phi^{-1}.$$

Proof. We first observe that $\mathcal{F}$ is clearly a linear isomorphism with inverse given by

$$\mathcal{F}^{-1}\colon W^{1,q}(\Omega_2) \to W^{1,p}(\Omega_1), \qquad \mathcal{F}^{-1}(v) := v \circ \Phi.$$

Let us now see that it is also bounded.

Let us consider $\lambda > 0$ and, for simplicity, let us denote $v = \mathcal{F}(u)$. Then, by changing variables $y = \Phi^{-1}(x)$,

$$\begin{aligned}\int_{\Omega_2} \left|\frac{v(x)}{\lambda}\right|^{q(x)} dx &= \int_{\Omega_2} \left|\frac{u(\Phi^{-1}(x))}{\lambda}\right|^{p(\Phi^{-1}(x))} dx \\ &= \int_{\Omega_1} \left|\frac{u(y)}{\lambda}\right|^{p(y)} J\Phi(y)\, dy \\ &\le \|J\Phi\|_\infty \int_{\Omega_1} \left|\frac{u(y)}{\lambda}\right|^{p(y)} dy\end{aligned}$$

Let us observe that, if $C := \|J\Phi\|_\infty \le 1$, clearly we have

$$\|u\|_{p,\Omega_1} = \inf\{\lambda > 0\colon \int_{\Omega_1} \left|\frac{u(y)}{\lambda}\right|^{p(y)} dy \le 1\} \ge \inf\{\lambda > 0\colon \int_{\Omega_2} \left|\frac{v(y)}{\lambda}\right|^{q(y)} dy \le 1\} = \|v\|_{q,\Omega_2}$$

Let us now assume that $C > 1$. Then, since

$$\begin{aligned}\left\{\lambda > 0\colon \int_{\Omega_1} \left|\frac{C^{\frac{1}{p_-}} u(y)}{\lambda}\right|^{p(y)} dy \le 1\right\} &\subset \left\{\lambda > 0\colon \int_{\Omega_1} \left|\frac{u(y)}{\lambda}\right|^{p(y)} dy \le \frac{1}{C}\right\} \\ &\subset \left\{\lambda > 0\colon \int_{\Omega_2} \left|\frac{v(x)}{\lambda}\right|^{q(x)} dx \le 1\right\},\end{aligned}$$

taking infimum, we conclude that

$$C^{\frac{1}{p_-}} \|u\|_{p,\Omega_1} = \|C^{\frac{1}{p_-}} u\|_{p,\Omega_1} \ge \inf\{\lambda > 0\colon \int_{\Omega_1} \left|\frac{u(y)}{\lambda}\right|^{p(y)} dy \le \frac{1}{C}\} \ge \|v\|_{q,\Omega_2} = \|\mathcal{F}(u)\|_{q,\Omega_2}.$$

Analogously,

$$\begin{aligned}\int_{\Omega_2} \left|\frac{\nabla v(x)}{\lambda}\right|^{q(x)} dx &= \int_{\Omega_2} \left|\frac{\nabla(u \circ \Phi^{-1})(x)}{\lambda}\right|^{q(x)} dx \\ &= \int_{\Omega_2} \left|\frac{\nabla u(\Phi^{-1}(x)) D\Phi^{-1}(x)}{\lambda}\right|^{p(\Phi^{-1}(x))} dx \\ &= \int_{\Omega_1} \left|\frac{\nabla u(y) D\Phi^{-1}(\Phi(y))}{\lambda}\right|^{p(y)} J\Phi(y)\, dy \\ &\le \max\{1, \|D\Phi^{-1}\|_\infty\}^{p_+} \|J\Phi\|_\infty \int_{\Omega_1} \left|\frac{\nabla u(y)}{\lambda}\right|^{p(y)} dy.\end{aligned}$$

Therefore, $\|\nabla \mathcal{F}(u)\|_{q,\Omega_2} \le C \|\nabla u\|_{p,\Omega_1}$, which completes the proof. □

Remark 6.3. In the previous proof, given $A\colon \Omega \to \mathbb{R}^{N\times N}$, we considered the norm $\|A\|_\infty := \sup_{x\in\Omega} \|A(x)\|$ and, given $B \in \mathbb{R}^{N\times N}$, we considered the norm $\|B\| := \sup_{\xi\neq 0} \frac{|B\xi|}{|\xi|}$.

Observe that, since $\mathrm{supp}(V) \subset\subset \Omega$, it follows that $\Phi_t(\Omega) = \Omega$ for every $t \in \mathbb{R}$ and that if $p = p_1\chi_{D_1} + p_2\chi_{D_2}$ then $p_t := p \circ \Phi_{-t} = p_1\chi_{D_1^t} + p_2\chi_{D_2^t}$, where $D_i^t = \Phi_t(D_i)$, $i = 1, 2$.

Therefore, in view of Proposition 6.2, we have that

$$\mathcal{F}_t \colon W^{1,p}(\Omega) \to W^{1,p_t}(\Omega), \qquad u \mapsto u \circ \Phi_{-t}$$

is a bounded linear isomorphism.

Let us consider the space $X_t := W^{1,p_t}(\Omega) \cap L^2(\Omega)$ equipped with the norm

$$\|\cdot\|_{X_t} := \|\cdot\|_{W^{1,p_t}(\Omega)} + \|\cdot\|_{L^2(\Omega)}$$

and the space $X := W^{1,p}(\Omega) \cap L^2(\Omega)$ equipped with the norm

$$\|\cdot\|_X := \|\cdot\|_{W^{1,p}(\Omega)} + \|\cdot\|_{L^2(\Omega)}.$$

It is clear that $\mathcal{F}_t \colon X \to X_t$ is still a bounded linear isomorphism.

Given $f \in L^2(\Omega)$, we define the quantity

$$\tilde{s}(t) := \inf_{v\in X_t} \int_\Omega \frac{|\nabla v|^{p_t}}{p_t}\,dx + \int_\Omega \frac{|v-f|^2}{2}\,dx$$

which is clearly equivalent to

$$s(t) := \inf_{v\in X_t} \int_\Omega \frac{|\nabla v|^{p_t}}{p_t}\,dx + \int_\Omega \frac{|v|^2}{2}\,dx - \int_\Omega vf\,dx. \tag{6.2}$$

In fact, $\tilde{s}(t) = s(t) + \|f\|_2^2$.

Observe that, since $\mathcal{F}_t$ is an isomorphism, one actually has

$$s(t) = \inf_{u\in X} \int_\Omega \frac{|\nabla(u\circ\Phi_{-t})|^{p_t}}{p_t}\,dx + \int_\Omega \frac{|u\circ\Phi_{-t}|^2}{2}\,dx - \int_\Omega (u\circ\Phi_{-t})f\,dx.$$

So, in view ot our previous discussions, our primary goal is to find an expression for $\frac{ds}{dt}(0)$.

Remark 6.4. Let us observe that, by changing variables $y = \Phi_{-t}(x)$,

$$s(t) = \inf_{u\in X} \int_\Omega \frac{|\nabla u D\Phi_{-t}\circ\Phi_t|^p}{p} J\Phi_t\,dy + \int_\Omega \frac{|u|^2}{2} J\Phi_t\,dy - \int_\Omega uf\circ\Phi_t J\Phi_t\,dy.$$

Let us call

$$\mathbb{J}_t u := \int_\Omega \frac{|\nabla u D\Phi_{-t}\circ\Phi_t|^p}{p} J\Phi_t\,dy + \int_\Omega \frac{|u|^2}{2} J\Phi_t\,dy - \int_\Omega uf\circ\Phi_t J\Phi_t\,dy$$

and

$$\mathbb{J}u := \int_\Omega \frac{|\nabla u|^p}{p}\,dy + \int_\Omega \frac{|u|^2}{2}\,dy - \int_\Omega uf\,dy.$$

Lemma 6.5. *There exists $\delta > 0$ such that the functionals $\{\mathbb{J}_t\}_{|t|<\delta}$ are uniformly coercive with respect to the weak topology of X. That is, for any $\lambda \in \mathbb{R}$, there exists a weakly compact set $K \subset X$ such that*

$$\{\mathbb{J}_t \leq \lambda\} \subset K, \quad \textit{for every } |t| < \delta.$$

Proof. Take $\delta > 0$ such that $\frac{1}{2} \leq J\Phi_t \leq 2$. Therefore,

$$\mathbb{J}_t u \geq \frac{1}{2}\int_\Omega \frac{|\nabla u D\Phi_{-t} \circ \Phi_t|^p}{p}\, dy + \frac{1}{2}\int_\Omega \frac{|u|^2}{2}\, dy - 2\int_\Omega |f||u|\, dy. \tag{6.3}$$

By Young inequality with $\epsilon = \frac{1}{8}$,

$$2\int_\Omega |f||u|\, dy \leq \frac{1}{8}\int_\Omega |u|^2\, dy + 8\int_\Omega |f|^2\, dy. \tag{6.4}$$

As $D\Phi_{-t} \rightrightarrows Id$ uniformly on Ω, it follows that $\|D\Phi_t\|_\infty$ is bounded away from zero and infinity for every $|t| < \delta$, so

$$\int_\Omega \frac{|\nabla u D\Phi_{-t} \circ \Phi_t|^p}{p}\, dy \geq c\int_\Omega |\nabla u|^p\, dy. \tag{6.5}$$

So, combining (6.3), (6.4) and (6.5), we get

$$\mathbb{J}_t u \geq c\int_\Omega |\nabla u|^p\, dy + \frac{1}{8}\int_\Omega |u|^2\, dy - 8\|f\|_2^2.$$

By Proposition 3.2 we easily conclude that there exists a radius $R = R(\lambda)$ such that $\{\mathbb{J}_t \leq \lambda\} \subset B_X(0, R)$.

Therefore, if we denote $K := \{\|u\|_X < R\}$, satisfies our requirements. This finishes the proof of the lemma. □

The next lemma is stated for future reference, its proof is standard.

Lemma 6.6. *There exists a unique extremal for $s(t)$ and $s(0)$.*

Proof. The proof is an immediate consequence of the fact that both $\mathbb{J}_t$ and $\mathbb{J}$ are strictly convex and sequentially weakly lower semicontinuous on $W^{1,p}(\Omega)$. □

Our first result shows that $s(t)$ is continuous with respect to t at $t = 0$.

Theorem 6.7. *With the previous notation,*

$$\lim_{t\to 0^+} s(t) = s(0). \tag{6.6}$$

Moreover, if u_t and u are the extremals associated to $s(t)$ and $s(0)$ respectively, then $u_t \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$. Finally, if $p^* := \frac{pN}{N-p} > 2$ then $u_t \to u$ strongly in $W^{1,p}(\Omega)$.

Remark 6.8. The hypothesis $p^* > 2$ is needed in order to secure the compact embedding $W^{1,p}(\Omega) \subset L^2(\Omega)$ for any dimension N.

For the case $N = 2$, one has $p^* > 2$ for any $p > 1$ so no extra hypothesis is needed.

Proof. Since, by Lemma 6.5, we know that the functionals $\mathbb{J}_t$ are uniformly coercive, the proof of (6.6) will follow from Remark A.2 if we show that $\mathbb{J}_t \rightrightarrows \mathbb{J}$ uniformly on bounded sets of X. Observe that since the minimizers are unique, we will then have that the whole sequence of minimizers is weakly convergent.

Let us consider now $B \subset X$ a bounded subset and $u \in B$. By Remark 6.1,

$$\begin{aligned}\mathbb{J}_t u = &\int_\Omega \frac{|\nabla u(Id + O(t))|^p}{p}(1 + O(t))\,dy + \int_\Omega \frac{|u|^2}{2}(1 + O(t))\,dy - \int_\Omega u(f \circ \Phi_t)(1 + O(t))\,dy\\ =&(1 + O(t))\left\{\int_\Omega \frac{|\nabla u(Id + O(t))|^p}{p}\,dy + \int_\Omega \frac{|u|^2}{2}\,dy - \int_\Omega u(f \circ \Phi_t)\,dy\right\}.\end{aligned}$$

Again by Remark 6.1, and by Taylor expansion formula, we get

$$\int_\Omega \frac{|\nabla u(Id + O(t))|^p}{p}\,dy = \int_\Omega \frac{|\nabla u|^p}{p}\,dy + O(t),$$

uniformly in B.

Assume for a moment that f is a continuous function with compact support. Then, since $\Phi_t \to id$ uniformly as $t \to 0$, we have that $f \circ \Phi_t \to f$ uniformly as $t \to 0$ and therefore,

$$\|f \circ \Phi_t - f\|_2^2 = \int_\Omega |f \circ \Phi_t - f|^2\,dx \le \|f \circ \Phi_t - f\|_\infty^2 |\Omega| \to 0,\ (t \to 0).$$

And so we have that $\|f \circ \Phi_t - f\|_2 \to 0$, $(t \to 0)$.

Now, by a standard density argument, it is easy to see that the same result holds for any $f \in L^2(\Omega)$.

Then, by Hölder inequality and since $u \in B$, there is a constant C, independent of u, such that

$$\Big|\int_\Omega u(f \circ \Phi_t - f)\Big| \le C\|f \circ \Phi_t - f\|_2 \to 0$$

as $t \to 0^+$.

Assume now that $p^* > 2$. It remains to see the strong convergence of u_t to u in $W^{1,p}(\Omega)$.

Let us observe that in order to see the strong convergence it is enough to show the convergence of the modulars (see [18]).

Let us now recall that

$$\begin{aligned}\int_\Omega \frac{|\nabla u_t|^p}{p}\,dy + \int_\Omega \frac{|u_t|^2}{2}\,dy =&s(t) + \int_\Omega \frac{|\nabla u_t|^p}{p}\,dy - \int_\Omega \frac{|\nabla u_t D\Phi_{-t} \circ \Phi_t|^p}{p} J\Phi_t\,dy\\ &+ \int_\Omega \frac{|u_t|^2}{2}(1 - J\Phi_t)\,dy + \int_\Omega u_t(f \circ \Phi_t)J\Phi_t\,dy.\end{aligned}$$

By Remark 6.1,

$$\int_\Omega \frac{|\nabla u_t D\Phi_{-t} \circ \Phi_t|^p}{p} J\Phi_t\, dy = \int_\Omega \frac{|\nabla u_t - t\nabla u_t DV + o(t)|^p}{p}(1 + t\,\text{div}\, V + o(t))\, dy.$$

Using the following Taylor expansion,

$$|\nabla u_t - t\nabla u_t DV + o(t)|^p = |\nabla u_t|^p - pt|\nabla u_t|^{p-2}\nabla u_t \cdot \nabla u_t DV + o(t),$$

we find that

$$\int_\Omega \frac{|\nabla u_t D\Phi_{-t} \circ \Phi_t|^p}{p} J\Phi_t\, dy = \int_\Omega \frac{|\nabla u_t|^p + t(|\nabla u_t|^p\,\text{div}\, V - p|\nabla u_t|^{p-2}\nabla u_t \cdot \nabla u_t DV)}{p}\, dy + o(t).$$

And so we have

$$\int_\Omega \frac{|\nabla u_t|^p}{p}\, dy - \int_\Omega \frac{|\nabla u_t D\Phi_{-t} \circ \Phi_t|^p}{p} J\Phi_t\, dy = -\int_\Omega \frac{t(|\nabla u_t|^p\,\text{div}\, V - p|\nabla u_t|^{p-2}\nabla u_t \cdot \nabla u_t DV)}{p}\, dy + o(t)$$

Now, for our fourth term, we only need to observe that $\frac{|u_t|^2}{2}$ is bounded and $1 - J\Phi_t \to 0$ uniformly.

Then, since $s(t) \to s(0)$ and

$$\int_\Omega u_t(f \circ \Phi_t) J\Phi_t\, dy \to \int_\Omega uf,$$

we can conclude that

$$\int_\Omega \frac{|\nabla u_t|^p}{p}\, dy + \int_\Omega \frac{|u_t|^2}{2}\, dy \to \int_\Omega \frac{|\nabla u|^p}{p}\, dy + \int_\Omega \frac{|u|^2}{2}\, dy,$$

which completes the proof. □

Now we prove the differentiability of the cost functional $s(t)$. For this result we will need the function f to be of class C^1.

Theorem 6.9. *$s(t)$ is differentiable at $t = 0$ and*

$$\frac{ds}{dt}(0) = R(u) - \int_\Omega uf\, div\, V\, dy - \int_\Omega u\nabla f \cdot V\, dy,$$

where

$$R(u) := \int_\Omega \frac{|\nabla u|^p}{p}\, div\, V - |\nabla u|^{p-2}\nabla u \cdot \nabla u DV + div\, V \frac{|u|^2}{2}\, dy$$

and u is the extremal of $s(0)$.

Proof. By Lemma 6.6, we can consider u the extremal of $s(0)$. Then, by Remark 6.4,

$$s(t) = \inf_X \mathbb{J}_t \leq \mathbb{J}_t(u) = \int_\Omega \frac{|\nabla u D\Phi_{-t} \circ \Phi_t|^p}{p} J\Phi_t \, dy + \int_\Omega \frac{|u|^2}{2} J\Phi_t \, dy - \int_\Omega u f \circ \Phi_t J\Phi_t \, dy.$$

Now, by Remark 6.1, as in the proof of Theorem 6.7 we find that

$$\int_\Omega \frac{|\nabla u D\Phi_{-t} \circ \Phi_t|^p}{p} J\Phi_t \, dy = \int_\Omega \frac{|\nabla u|^p + t(|\nabla u|^p \operatorname{div} V - p|\nabla u|^{p-2}\nabla u \cdot \nabla u DV)}{p} \, dy + o(t).$$

On the other hand, again by Remark 6.1,

$$\begin{aligned}\int_\Omega \frac{|u|^2}{2} J\Phi_t \, dy &= \int_\Omega \frac{|u|^2}{2}(1 + t \operatorname{div} V + o(t)) \, dy \\ &= \int_\Omega \frac{|u|^2}{2} \, dy + t \int_\Omega \operatorname{div} V \frac{|u|^2}{2} \, dy + o(t).\end{aligned}$$

Therefore, setting

$$R(u) := \int_\Omega \frac{|\nabla u|^p}{p} \operatorname{div} V - |\nabla u|^{p-2}\nabla u \cdot \nabla u DV + \operatorname{div} V \frac{|u|^2}{2} \, dy,$$

we can conclude that

$$s(t) \leq \int_\Omega \frac{|\nabla u|^p}{p} \, dy + \int_\Omega \frac{|u|^2}{2} \, dy + tR(u) + o(t) - \int_\Omega u(f \circ \Phi_t)(1 + t \operatorname{div} V + o(t)) \, dy.$$

Recall that

$$s(0) = \int_\Omega \frac{|\nabla u|^p}{p} \, dy + \int_\Omega \frac{u^2}{2} \, dy - \int_\Omega u f \, dy.$$

Therefore,

$$\frac{s(t) - s(0)}{t} \leq R(u) + \frac{o(t)}{t} - \int_\Omega u(f \circ \Phi_t) \operatorname{div} V \, dy - \int_\Omega u \frac{(f \circ \Phi_t) - f}{t} \, dy.$$

Taking the limit $t \to 0^+$, we get

$$\limsup_{t \to 0+} \frac{s(t) - s(0)}{t} \leq R(u) - \int_\Omega u f \operatorname{div} V \, dy - \int_\Omega u \nabla f \cdot V \, dy,$$

where we have used the fact that $\Phi_0 = id$ and $\dot{\Phi}_t = V \circ \Phi_t$.

Let us consider now $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to 0^+$ and

$$\liminf_{t \to 0^+} \frac{s(t) - s(0)}{t} = \lim_{n \to \infty} \frac{s(t_n) - s(0)}{t_n}.$$

Let $u_n := u_{t_n} \in X_{t_n}$ be the extremal associated to $s(t_n)$. By Remark 6.4,

$$s(t_n) = \int_\Omega \frac{|\nabla u_n D\Phi_{-t_n} \circ \Phi_{t_n}|^p}{p} J\Phi_{t_n} \, dy + \int_\Omega \frac{|u_n|^2}{2} J\Phi_{t_n} \, dy - \int_\Omega u_{t_n} f \circ \Phi_{t_n} J\Phi_{t_n} \, dy$$

Arguing as in the previous case, we have that

$$\begin{aligned}\frac{s(t_n)-s(0)}{t_n} \geq & \int_\Omega \frac{|\nabla u_n|^p}{p}\operatorname{div} V - |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u_n DV + \operatorname{div} V \frac{|u_n|^2}{2}\,dy \\ & + \frac{o(t_n)}{t_n} - \int_\Omega u_n(f\circ\Phi_{t_n})\operatorname{div} V\,dy - \int_\Omega u_n \frac{(f\circ\Phi_{t_n})-f}{t_n}\,dy \\ =& R(u_n) + \frac{o(t_n)}{t_n} - \int_\Omega u_n(f\circ\Phi_{t_n})\operatorname{div} V\,dy - \int_\Omega u_n \frac{(f\circ\Phi_{t_n})-f}{t_n}\,dy.\end{aligned}$$

Since $R(u_n) \to R(u)$ when $n \to \infty$ (just observe that R is continuous with respect to the strong topology and $u_n \to u$ in $W^{1,p}(\Omega)$ by Theorem 6.7), we have

$$\liminf_{t\to 0^+} \frac{s(t)-s(0)}{t} \geq R(u) - \int_\Omega uf \operatorname{div} V\,dy - \int_\Omega u\nabla f\cdot V\,dy.$$

And so we can conclude that $s(t)$ is differentiable at $t=0$ and

$$\frac{ds}{dt}(0) = R(u) - \int_\Omega uf \operatorname{div} V\,dy - \int_\Omega u\nabla f\cdot V\,dy,$$

where $u \in X$ is the extremal of $s(0)$. This completes the proof. □

6.3 Improvement of the formula.

Now we try to find a more explicit formula for $s'(0)$.

In the following study, we will need the solution u to

$$\begin{cases} -\Delta_{p(x)}u + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.7}$$

to be $C^2_{\text{loc}}(D_1) \cap C^2_{\text{loc}}(D_2)$ in order for our computations to work. However, this is not true since the optimal regularity is known to be $C^{1,\alpha}_{\text{loc}}(D_1) \cap C^{1,\alpha}_{\text{loc}}(D_2)$. See [49].

In order to overcome such difficulty, we will proceed as follows.

6.3.1 Formal computations.

Now we try to find a more explicit formula for $s'(0)$. To make the ideas more transparent we first assume the solutions are smooth.

We consider the solution u to the following equation

$$\begin{cases} -\Delta_{p(x)}u + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.8}$$

Remember that $\Delta_{p(x)}u = \operatorname{div}((|\nabla u|^{p(x)-1})$.

Now, since

$$\operatorname{div}(|u|^2 V) = 2u\nabla u \cdot V + |u|^2 \operatorname{div} V,$$
$$\operatorname{div}(|\nabla u|^q V) = q|\nabla u|^{q-2}\nabla u D^2 u \cdot V + |\nabla u|^q \operatorname{div} V$$

for any constant exponent q. Therefore, if p equals p_1 in D_1 and p_2 in D_2, we have that

$$\frac{ds}{dt}(0) = \frac{1}{p_1}\int_{D_1} \operatorname{div}(|\nabla u|^{p_1} V)\,dy - \int_{D_1} |\nabla u|^{p_1-2}\nabla u D^2 u \cdot V\,dy$$
$$+\frac{1}{p_2}\int_{D_2} \operatorname{div}(|\nabla u|^{p_2} V)\,dy - \int_{D_2} |\nabla u|^{p_2-2}\nabla u D^2 u \cdot V\,dy - \int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla u DV\,dy$$
$$+\frac{1}{2}\int_{\Omega} \operatorname{div}(|u|^2 V)\,dy - \int_{\Omega} u\nabla u \cdot V\,dy - \int_{\Omega} uf \operatorname{div} V\,dy - \int_{\Omega} u\nabla f \cdot V\,dy.$$

Let us call ν_1 the exterior unit normal vector to ∂D_1. Analogosuly, ν_2 will denote the exterior unit normal vector to ∂D_2.

Since D_1 and D_2 are disjoint sets, we have that

$$\int_{D_1\cup D_2} \operatorname{div}(|u|^2 V)\,dy = \int_{D_1} \operatorname{div}(|u|^2 V)\,dy + \int_{D_2} \operatorname{div}(|u|^2 V)\,dy$$
$$= \int_{\partial D_1} |u|^2 V \cdot \nu_1^{\epsilon}\,dS + \int_{\partial D_2} |u|^2 V \cdot \nu_2\,dS$$
$$= \int_{\partial D_1\setminus\Gamma_1} |u|^2 V \cdot \nu_1\,dS + \int_{\Gamma_1} |u|^2 V \cdot \nu_1\,dS + \int_{\partial D_2\setminus\Gamma_2} |u|^2 V \cdot \nu_2\,dS + \int_{\Gamma_2} |u|^2 V \cdot \nu_2\,dS.$$

Since $\operatorname{supp} V \subset\subset \Omega$,

$$\int_{\partial D_1\setminus\Gamma_1} |u|^2 V \cdot \nu_1\,dS = \int_{\partial D_1\setminus\Gamma_2} |u|^2 V \cdot \nu_2\,dS = 0.$$

Therefore, taking into account that $\nabla u D^2 u \cdot V = \nabla u \cdot D^2 u V^T$,

$$\frac{ds}{dt}(0) = \frac{1}{p_1}\int_{\Gamma_1} |\nabla u|^{p_1} V \cdot \nu_1\,dS + \frac{1}{p_2}\int_{\Gamma_2} |\nabla u|^{p_2} V \cdot \nu_2\,dS$$
$$-\int_{D_1\cup D_2} |\nabla u|^{p-2}\nabla u \cdot (D^2 u V^T + \nabla u DV)\,dy + \int_{\Gamma_1} |u|^2 V \cdot \nu_1\,dS + \int_{\Gamma_2} |u|^2 V \cdot \nu_2\,dS$$
$$-\int_{D_1\cup D_2} u\nabla u \cdot V\,dy - \int_{D_1\cup D_2} uf \operatorname{div} V\,dy - \int_{D_1\cup D_2} u\nabla f \cdot V\,dy.$$

Since u is a weak solution of our equation, we can conclude that

$$\frac{ds}{dt}(0) = \frac{1}{p_1}\int_{\Gamma_1} |\nabla u|^{p_1} V \cdot \nu_1\,dS + \frac{1}{p_2}\int_{\Gamma_2} |\nabla u|^{p_2} V \cdot \nu_2\,dS$$
$$+\int_{\Gamma_1} |u|^2 V \cdot \nu_1\,dS + \int_{\Gamma_2} |u|^2 V \cdot \nu_2\,dS - \int_{D_1\cup D_2} (f\nabla u \cdot V + uf \operatorname{div} V + u\nabla f \cdot V)\,dy,$$

Observe that

$$\int_{D_1^\epsilon \cup D_2^\epsilon} (f\nabla u \cdot V + uf \operatorname{div} V + u\nabla f \cdot V)\, dy = \int_{D_1^\epsilon \cup D_2^\epsilon} \operatorname{div}(ufV)\, dy$$

$$= \int_{D_1^\epsilon} \operatorname{div}(ufV)\, dy + \int_{D_2^\epsilon} \operatorname{div}(ufV)\, dy = \int_{\partial D_1^\epsilon} ufV \cdot \nu_1^\epsilon\, dS + \int_{\partial D_2^\epsilon} ufV \cdot \nu_2^\epsilon\, dS = 0$$

Since $\nu_1 = -\nu_2$,

$$\int_{\Gamma_1} |u|^2 V \cdot \nu_1\, dS + \int_{\Gamma_2} |u|^2 V \cdot \nu_2\, dS = 0.$$

Therefore we obtain the following expression for the shape derivative

$$\frac{ds}{dt}(0) = \frac{1}{p_1} \int_{\Gamma_1} |\nabla u|^{p_1} V \cdot \nu_1\, dS + \frac{1}{p_2} \int_{\Gamma_2} |\nabla u|^{p_2} V \cdot \nu_2\, dS.$$

If $\Gamma = \overline{D_1} \cap \overline{D_2}$ and we denote $[|\nabla u|^p] := |\nabla u|^{p_1} - |\nabla u|^{p_2}$, we can conclude that

$$\frac{ds}{dt}(0) = \int_\Gamma \frac{1}{p} [|\nabla u|^p] V \cdot \nu_1\, dS,$$

which is a more explicit formula for $\frac{ds}{dt}(0)$.

6.3.2 Domain regularization.

Let us first define $D_i(t) := \Phi_t(D_i)$.

Now given a fixed $\delta > 0$, we define the following sets

$$D_i^\delta := \{x \in D_i : \operatorname{dist}(x, D_j) > \delta\},\ i \neq j$$

and consider $D_i^\delta(t) := \Phi_t(D_i^\delta)$. And now consider the sets

$$\Gamma_i^\delta(t) := \partial D_i^\delta(t) \cap \Omega.$$

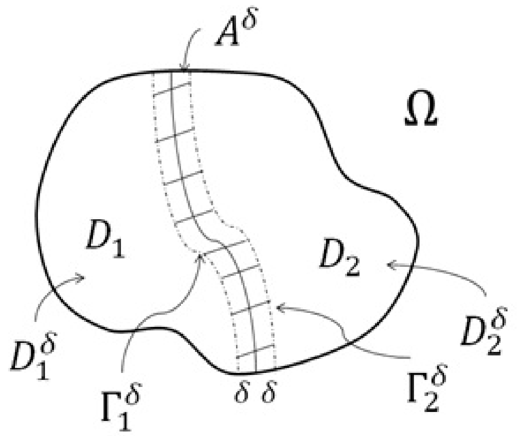
Let us observe that, in each D_i^δ, the exponent $p(x) = p_i$ is constant so we can apply the classic regularity results. See for instance [49].

Now we define the sets $A_\delta := \Omega \setminus (D_1^\delta \cup D_2^\delta)$ and observe that

$$\partial A_\delta \cap \Omega = \Gamma_1^\delta \cup \Gamma_2^\delta$$

$$\Omega = D_1^\delta \cup D_2^\delta \cup A^\delta.$$

Figure 6.1: Partition of Ω



6.3.3 Operator regularization.

Now, for $\epsilon \geq 0$, we consider the regularized problems

$$\begin{cases} -\operatorname{div}((|\nabla v|^2+\epsilon^2)^{\frac{p_i-2}{2}}\nabla v)+v=f^\epsilon & \text{in } D_i^\delta(t), \\ v=0 & \text{on } \partial\Omega\cap(\overline{D_1^\delta(t)}\cup\overline{D_2^\delta(t)}), \\ v=u(0)\circ\Phi_t^{-1} & \text{on } \Gamma_i^\delta(t), \end{cases} \tag{6.9}$$

with $f^\epsilon \in C^\infty$ such that $f^\epsilon \to f$ in $L^{p'}$.

Remark 6.10. Applying classical estimates (see for instance [26] it is possible to see that the solution of (6.9) is $C^{2,\alpha}_{\text{loc}}(D_i^\delta)\cap C^1(\overline{D_i^\delta})$ if $\epsilon>0$, since $u(0)$ is $C^1(\overline{D_i^\delta})$ and Φ_t is the identity map in a neighborhood of $\partial\Omega$. See also [49] for regularity estimates in Sobolev spaces.

Let us define the following sets

$$X_i^\delta := \{v\in W^{1,p_i}(D_i^\delta) \text{ such that } v=0 \text{ in } \partial\Omega\cap\overline{D_i^\delta} \text{ and } v=u \text{ in } \Gamma_i^\delta\}.$$

$$X_i^\delta(t) := \{v\in W^{1,p_i}(D_i^\delta(t)) \text{ such that } v=0 \text{ in } \partial\Omega\cap\overline{D_i^\delta(t)} \text{ and } v=u(0)\circ\Phi_t^{-1} \text{ in } \Gamma_i^\delta(t)\}.$$

Let us also consider the functionals $\tilde{\mathbb{J}}_{t,i}^{\epsilon,\delta} : X_i^\delta(t)\to\mathbb{R}$ defined by

$$\tilde{\mathbb{J}}_{t,i}^{\epsilon,\delta}(v) := \int_{D_i^\delta(t)} \frac{(|\nabla v|^2+\epsilon^2)^{\frac{p_i}{2}}}{p_i}\,dy + \int_{D_i^\delta(t)} \frac{|v|^2}{2}\,dy - \int_{D_i^\delta(t)} v f^\epsilon\,dy.$$

Remark 6.11. $X_i^\delta(t)$ is strongly closed and convex, therefore it is weakly closed.

Remark 6.12. The solutions of (6.9) are the minimums of the functionals $\tilde{\mathbb{J}}^{\epsilon,\delta}_{t,i}$ in $X^\delta_i(t)$.

Since the functional $\tilde{\mathbb{J}}^{\epsilon,\delta}_{t,i}$ is continuous for the strong topology, strictly convex and coercive, it has a unique minimum in $X^\delta_i(t)$ and, therefore, (6.9) has a unique weak solution.

We will denote $\tilde{u}^{\epsilon,\delta}_i(t)$ as the function where the minimum is attained.

Remark 6.13. Observe that $\psi_t\colon X^\delta_i \to X^\delta_i(t)$ defined by $v \mapsto v \circ \Phi_t^{-1}$ is a biyection between X^δ_i and $X^\delta_i(t)$ and the following equality holds

$$\mathbb{J}^{\epsilon,\delta}_{t,i} = \tilde{\mathbb{J}}^{\epsilon,\delta}_{t,i} \circ \psi_t^{-1}.$$

By changing variables as in the previous section we get the functional $\mathbb{J}^{\epsilon,\delta}_{t,i}\colon X^\delta_i \to \mathbb{R}$ given by

$$\mathbb{J}^{\epsilon,\delta}_{t,i}(v) := \int_{D^\delta_i} \frac{(|\nabla v D\Phi_{-t} \circ \Phi_t|^2 + \epsilon^2)^{\frac{p_i}{2}}}{p_i} J\Phi_t \, dy + \int_{D^\delta_i} \frac{|v|^2}{2} J\Phi_t \, dy - \int_{D^\delta_i} v f^\epsilon \circ \Phi_t J\Phi_t \, dy.$$

and define

$$s^{\epsilon,\delta}_i(t) = \inf_{v \in X^\delta_i} \mathbb{J}^{\epsilon,\delta}_{t,i}(v) = \inf_{v \in X^\delta_i(t)} \tilde{\mathbb{J}}^{\epsilon,\delta}_{t,i}(v).$$

We will denote $u^{\epsilon,\delta}_i(t) \in X^\delta_i$ as the function where the minimum of $\mathbb{J}^{\epsilon,\delta}_{t,i}$ is attained.

Observe that $u^{\epsilon,\delta}_i(t)(x) = \tilde{u}^{\epsilon,\delta}_i(t)(\Phi_t(x))$.

In order to make the notation lighter, we will focus on the needed parameter in each step. First, u_i, then u^ϵ and finally, u^δ.

Let us now define $s^{\epsilon,\delta}$ as $s^{\epsilon,\delta} := s^{\epsilon,\delta}_1 + s^{\epsilon,\delta}_2$.

Proposition 6.14. *If* $2 < p_i^*$, *then* $u^{\epsilon,\delta}_i(0)$ *converges to* $u^{0,\delta}_i(0)(= u^\delta_i)$ *strongly in* $W^{1,p_i}(D^\delta_i)$ *and* $s^{\epsilon,\delta}_i(0)$ *converges to* $s^{0,\delta}_i(0)(= s^\delta_i(0))$ *when* $\epsilon \to 0$.

Proof. Let us begin by observing that

$$\mathbb{J}^{\epsilon,\delta}_{0,i}(v) := \int_{D^\delta_i} \frac{(|\nabla v|^2 + \epsilon^2)^{\frac{p_i}{2}}}{p_i} \, dy + \int_{D^\delta_i} \frac{|v|^2}{2} \, dy - \int_{D^\delta_i} v f^\epsilon \, dy.$$

Now let us denote

$$\mathbb{J}^\delta_i(v) := \int_{D^\delta_i} \frac{|\nabla v|^2}{p_i} \, dy + \int_{D^\delta_i} \frac{|v|^2}{2} \, dy - \int_{D^\delta_i} v f \, dy.$$

Observe that $\mathbb{J}^{\epsilon,\delta}_{0,i}, \mathbb{J}^\delta_i(v) : X^\delta_i \to \mathbb{R}$. By Theorem A.1, it is enough to prove that $\mathbb{J}^{\epsilon,\delta}_{0,i}$ Γ- converges to $\mathbb{J}^\delta_i$ in $W^{1,p_i}(D^\delta_i)$ for the weak topology.

First, let $v^\epsilon \rightharpoonup v$ weakly in $W^{1,p_i}(D^\delta_i)$. Let us observe that $v \in X^\delta_i$ since X^δ_i is weakly closed.

Observe that the first and second terms in $\mathbb{J}^\delta_i$ are convex and strongly continuous, therefore weakly lower semicontinuous. And the third term is linear and continuous, therefore weakly continuous.

Therefore,

$$\mathbb{J}_i^\delta(v) \le \lim \mathbb{J}_i^\delta(v^\epsilon) \le \liminf \mathbb{J}_{0,i}^{\epsilon,\delta}(v^\epsilon) + \int_{D_i^\delta} v^\epsilon(f - f^\epsilon).$$

Applying Hölder's inequality for the last term above, we have that

$$\int_{D_i^\delta} v^\epsilon(f - f^\epsilon) \le \|v^\epsilon\|_{p_i}\|f - f^\epsilon\|_{p_i'}.$$

Since $\|v^\epsilon\|_{p_i}$ is bounded (because of the weak convergence) and $f^\epsilon \to f$ in $L^{p_i'}$, the last term goes to 0.

Now, taking $\{v^\epsilon\} = v$ as recovery sequence, we have that $\mathbb{J}_{0,i}^{\epsilon,\delta}(v) \to \mathbb{J}^{\epsilon,\delta}(v)$, which completes the proof. □

Performing analogous computations as in the previous section, we can see that $s_i^{\epsilon,\delta}(t)$ is differentiable at $t = 0$ and

$$\frac{ds_i^{\epsilon,\delta}(0)}{dt} = R_i^{\epsilon,\delta}(u_i^{\epsilon,\delta}) - \int_{D_i^\delta} u_i^{\epsilon,\delta} f \operatorname{div} V\, dy - \int_{D_i^\delta} u_i^{\epsilon,\delta} \nabla f^\epsilon \cdot V\, dy$$

where

$$R_i^{\epsilon,\delta}(v) := \int_{D_i^\delta} \frac{(|\nabla v|^2 + \epsilon^2)^{\frac{p_i}{2}}}{p_i} \operatorname{div} V - (|\nabla v|^2 + \epsilon^2)^{\frac{p_i}{2}-1} \nabla v \cdot \nabla v DV + \operatorname{div} V \frac{|v|^2}{2}\, dy.$$

Since the expression of $\frac{ds_i^{\epsilon,\delta}(0)}{dt}$ given above only involves first derivatives, we can conclude the following result from Corollary 6.14.

Proposition 6.15. *$\frac{ds_i^{\epsilon,\delta}(0)}{dt}$ converges to $\frac{ds_i^{0,\delta}(0)}{dt}$ when $\epsilon \to 0$.*

Observe that, by Propositions 6.14 and 6.15, if we find an expression for the shape derivative of the regularized operator, we will have found one for the original operator.

6.3.4 Improvement of the formula for the regularized operator.

Our main concern in this part of our work will be to find a formula for the shape derivative that does not involve second order derivatives. Therefore, we will be able to pass to the limit when ϵ goes to 0. And so, by Proposition 6.15, we will have found an expression for the shape derivative of the original operator.

We start with some preliminaries computations in which we will see the need to have C^2 regularity for our solutions. Since

$$\begin{aligned}\operatorname{div}((|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p_i}{2}} \cdot V) &= \frac{p_i}{2}(|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p_i}{2}-1} D(|\nabla u^\epsilon|^2 + \epsilon^2) \cdot V + (|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p_i}{2}} \operatorname{div} V \\ &= p_i(|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p_i}{2}-1} \nabla u^\epsilon D^2 u^\epsilon \cdot V + (|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p_i}{2}} \operatorname{div} V,\end{aligned}$$

we have that

$$\frac{1}{p_i}\int_{D_i^\delta}(|\nabla u^\epsilon|^2+\epsilon^2)^{\frac{p_i}{2}}\operatorname{div}V=\frac{1}{p_i}\int_{D_i^\delta}\operatorname{div}((|\nabla u^\epsilon|^2+\epsilon^2)^{\frac{p_i}{2}}V)-\int_{D_i^\delta}(|\nabla u^\epsilon|^2+\epsilon^2)^{\frac{p_i}{2}-1}\nabla u^\epsilon D^2u^\epsilon\cdot V$$

Therefore,

$$\begin{aligned}\frac{ds_i^{\epsilon,\delta}}{dt}(0)=&\frac{1}{p_i}\int_{D_i^\delta}\operatorname{div}((|\nabla u^\epsilon|^2+\epsilon^2)^{\frac{p_i}{2}}V)-\int_{D_i^\delta}(|\nabla u^\epsilon|^2+\epsilon^2)^{\frac{p_i}{2}-1}\nabla u^\epsilon D^2u^\epsilon\cdot V\\&-\int_{D_i^\delta}(|\nabla u^\epsilon|^2+\epsilon^2)^{\frac{p_i}{2}-1}\nabla u^\epsilon\cdot\nabla u^\epsilon DV\,dy+\frac{1}{2}\int_{D_i^\delta}\operatorname{div}(|u^\epsilon|^2V)\,dy\\&-\int_{D_i^\delta}u^\epsilon\nabla u^\epsilon\cdot V\,dy-\int_{D_i^\delta}u^\epsilon f^\epsilon\operatorname{div}V\,dy-\int_{D_i^\delta}u^\epsilon\nabla f^\epsilon\cdot V\,dy.\end{aligned}$$

Let us call ν_i^δ the exterior unit normal vector to ∂D_i^δ and observe that, since $\operatorname{supp}V\subset\subset\Omega$,

$$\int_{D_i^\delta}\operatorname{div}(|u^\epsilon|^2V)\,dy=\int_{\Gamma_i^\delta}|u^\epsilon|^2V\cdot\nu_i^\delta\,dS.$$

Since u^ϵ is a weak solution of our equation, for every test function φ we have

$$\int_{D_i^\delta}(|\nabla u^\epsilon|^2+\epsilon^2)^{\frac{p_i-2}{2}}\nabla u^\epsilon\nabla\varphi+\int_{D_i^\delta}u^\epsilon\varphi=\int_{D_i^\delta}f^\epsilon\varphi.$$

Let us consider $\varphi=\nabla u^\epsilon\cdot V$ as a test function. Since $\nabla(\nabla u^\epsilon\cdot V)=D^2u^\epsilon\cdot V^t+\nabla u^\epsilon DV$, we get

$$\int_{D_i^\delta}(|\nabla u^\epsilon|^2+\epsilon^2)^{\frac{p_i-2}{2}}\nabla u^\epsilon(D^2u^\epsilon\cdot V^t+\nabla u^\epsilon DV)=\int_{\Gamma_i^\delta}(|\nabla u^\epsilon|^2+\epsilon^2)^{\frac{p_i-2}{2}}\nabla u^\epsilon\cdot\eta\nabla u^\epsilon\cdot V+\int_{D_i^\delta}(f^\epsilon-u^\epsilon)\nabla u^\epsilon\cdot V.$$

And, since V has compact support in Ω, we arrive at

$$\int_{\partial D_i^\delta}(|\nabla u^\epsilon|^2+\epsilon^2)^{\frac{p_i-2}{2}}\nabla u^\epsilon\cdot\eta\nabla u^\epsilon\cdot V=\int_{\Gamma_i^\delta}(|\nabla u^\epsilon|^2+\epsilon^2)^{\frac{p_i-2}{2}}\nabla u^\epsilon\cdot\nu_i^\delta\nabla u^\epsilon\cdot V.$$

Therefore, taking into account that $\nabla u^\epsilon D^2u^\epsilon\cdot V=\nabla u^\epsilon\cdot D^2u^\epsilon V^T$, we have that

$$\begin{aligned}\frac{ds_i^{\epsilon,\delta}}{dt}(0)=&\frac{1}{p_i}\int_{\Gamma_i^\delta}\left(|\nabla u^\epsilon|^2+\epsilon^2\right)^{\frac{p_i}{2}}V\nu_i^\delta-\int_{\Gamma_i^\delta}\left(|\nabla u^\epsilon|^2+\epsilon^2\right)^{\frac{p_i}{2}-1}\nabla u^\epsilon\nu_i^\delta\nabla u^\epsilon V\\&+\frac{1}{2}\int_{\Gamma_i^\delta}|u^\epsilon|^2V\nu_i^\delta-\int_{D_i^\delta}\Big(\underbrace{f^\epsilon\nabla u^\epsilon\cdot V+u^\epsilon f^\epsilon\operatorname{div}V+u^\epsilon\nabla f^\epsilon\cdot V}_{\operatorname{div}(u^\epsilon f^\epsilon V)}\Big)\,dy.\end{aligned}$$

Again since V has compact support in Ω, we have that

$$\int_{D_i^\delta}\operatorname{div}(u^\epsilon f^\epsilon V)\,dy=\int_{\partial D_i^\delta}u^\epsilon f^\epsilon V\cdot\nu_i^\delta\,dS=\int_{\Gamma_i^\delta}u^\epsilon f^\epsilon V\cdot\nu_i^\delta\,dS.$$

Observe that we arrive at an expression for the shape derivative that does not involve second order derivatives of u^ϵ:

$$\frac{ds_i^{\epsilon,\delta}}{dt}(0) = \frac{1}{p_i}\int_{\Gamma_i^\delta}\left(|\nabla u^\epsilon|^2+\epsilon^2\right)^{\frac{p_i}{2}} V\nu_i^\delta - \int_{\Gamma_i^\delta}\left(|\nabla u^\epsilon|^2+\epsilon^2\right)^{\frac{p_i}{2}-1}\nabla u^\epsilon \nu_i^\delta \nabla u^\epsilon V$$

$$+\frac{1}{2}\int_{\Gamma_i^\delta}|u^\epsilon|^2 V\nu_i^\delta - \int_{\Gamma_i^\delta} u^\epsilon f^\epsilon V\cdot\nu_i^\delta\, dS.$$

6.3.5 Back to the original operator: the limit when ϵ goes to 0.

Now we able to apply Tolksdorf's regularity estimates (see [49]). These estimates give us uniform bounds for $\|u^\epsilon\|_{C^{1,\alpha}}$ so we have $u^\epsilon \to u$ in C^1. And so we can pass to the limit when ϵ goes to 0. Therefore,

$$\frac{ds_i^{0,\delta}}{dt}(0) = \frac{1}{p_i}\int_{\Gamma_i^\delta}|\nabla u|^{p_i} V\nu_i^\delta - \int_{\Gamma_i^\delta}|\nabla u|^{p_i-2}\nabla u\nu_i^\delta\nabla u V + \frac{1}{2}\int_{\Gamma_i^\delta}|u|^2 V\nu_i^\delta - \int_{\Gamma_i^\delta} ufV\cdot\nu_i^\delta\, dS.$$

In conclusion we arrive at

$$\frac{ds^{0,\delta}}{dt}(0) = \frac{ds_1^{0,\delta}}{dt}(0) + \frac{ds_2^{0,\delta}}{dt}(0)$$

$$= \frac{1}{p_1}\int_{\Gamma_1^\delta}|\nabla u|^{p_1}V\nu_1^\delta + \frac{1}{p_2}\int_{\Gamma_2^\delta}|\nabla u|^{p_2}V\nu_2^\delta - \int_{\Gamma_1^\delta}|\nabla u|^{p_1-2}\nabla u\nu_1^\delta\nabla u V - \int_{\Gamma_2^\delta}|\nabla u|^{p_2-2}\nabla u\nu_2^\delta\nabla u V$$

$$+\frac{1}{2}\int_{\Gamma_1^\delta}|u|^2 V\nu_1^\delta - \int_{\Gamma_1^\delta} ufV\cdot\nu_1^\delta\, dS + \frac{1}{2}\int_{\Gamma_2^\delta}|u|^2 V\nu_2^\delta - \int_{\Gamma_2^\delta} ufV\cdot\nu_2^\delta\, dS.$$

Let us now observe that $\nu_1^\delta \to \nu_1$ and $\nu_2^\delta \to \nu_2 = -\nu_1$ when $\delta \to 0$. Therefore, taking limit when $\delta \to 0$, the last four terms in the expression above vanish and so we have proved the following.

Theorem 6.16. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Let $D_1, D_2 \subset \Omega$ be such that* (6.10) *is satisfied, let $p = p_1\chi_{D_1} + p_2\chi_{D_2}$, where $1 < p_1 < p_2$ and $\Gamma = \bar{D}_1 \cap \bar{D}_2$.*

Let $V\colon \mathbb{R}^N \to \mathbb{R}^N$ be a Lipschitz deformation field, such that $spt(V) \subset\subset \Omega$ and let $s(t)$ be defined by (6.2)*. Then, the following formula for the derivative $s'(0)$ holds:*

$$\frac{ds}{dt}(0) = \int_\Gamma \Big[\frac{|\nabla u|^p}{p}\Big] V\cdot\nu\, dS - \int_\Gamma [|\nabla u|^{p-2}](\nabla u\cdot\nu)(\nabla u\cdot V)\, dS,$$

where

$$\int_\Gamma [f]G\cdot\nu\, dS := \lim_{\delta\to 0}\left(\int_{\Gamma_1^\delta} fG\cdot\nu_1\, dS - \int_{\Gamma_2^\delta} fG\cdot\nu_2\, dS\right).$$

Resumen del capítulo.

Sea $\mathcal{A}$ una clase admisible de dominios.

Asumimos que, asociado a cada $\Omega \in \mathcal{A}$, tenemos una función de costo bien definida $J(\Omega)$, es decir, un funcional de costo $J \colon \mathcal{A} \to \mathbb{R}$.

Nos interesa encontrar $\Omega^* \in \mathcal{A}$ que minimice el costo J sobre todos los $\Omega \in \mathcal{A}$.

La principal idea es tratar de computar una especie de *derivada* del funcional del costo J y que esa derivada en la configuración óptima Ω^* sea 0.

Sea $V \colon \mathbb{R}^N \to \mathbb{R}^N$ un campo de deformaciones Lipschitz. Consideramos el flujo asociado al campo V, $\{\Phi_t\}_{t\in\mathbb{R}}$, $\Phi_t \colon \mathbb{R}^N \to \mathbb{R}^N$ dado por

$$\dot{\Phi}_t(x) = V(\Phi_t(x)), \qquad \Phi_0(x) = x.$$

Sea $\Omega \subset \mathbb{R}^N$ un dominio, consideramos $\Omega_t := \Phi_t(\Omega)$.

Finalmente, definimos $j(t) = J(\Omega_t)$ el costo asociado a cada Ω_t.

Por lo tanto, si cualquier deformación Ω_t pertenece a la clase admisible $\mathcal{A}$ y si Ω es una configuración óptima, debe suceder que $j'(0) = 0$.

La forma explícita de $j'(t)$ en $t = 0$ es particularmente interesante por dos aplicaciones principales:

- Qué condiciones necesarias deben verificar los dominios óptimos? (en un mínimo, la derivada es 0)
- Cómo deformar un dominio dado Ω para minimizar su costo? (nos interesa obtener una derivada lo más negativa posible)

Analizamos el problema de obtener una imagen modelada por una función $u \colon \Omega \to \mathbb{R}$, donde $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, dada una imagen distorsionada $I \colon \Omega \to \mathbb{R}$.

Asumimos que el error, $e = u - I$, es pequeño y el objetivo es recuperar u a partir I sin hipótesis adicionales sobre el error e.

Dividimos la región Ω en dos subregiones D_1 y D_2 tales que para $i = 1, 2$,

$$D_i \subset \Omega \text{ es abierto}, \ \overset{\circ}{\overline{D_i}} = D_i, \ D_1 \cap D_2 = \emptyset, \ \text{y } \overline{\Omega} = \overline{D_1} \cup \overline{D_2}. \tag{6.10}$$

Así, D_1 contiene las regiones con bordes de la imagen y D_2 su complemento.

Definimos el exponente $p \colon \Omega \to \mathbb{R}$ dado por

$$p(x) = \begin{cases} 1 + \epsilon & \text{si } x \in D_1 \\ 2 & \text{si } x \in D_2. \end{cases}$$

Luego computamos u minimizando el funcional

$$\mathbb{J}(v) = \frac{1}{2\beta} \int_\Omega |\nabla v|^{p(x)}\, dx + \frac{\beta}{2} \int_\Omega (v - I)^2\, dx,$$

donde $\beta > 0$ es un parámetro que debe ajustar el operador del método para cada imagen.

Una vez resuelto este problema, nos interesa mejorar la imagen encontrada.

Observemos que el ínfimo u de $\mathbb{J}$ depende del dominio elegido D_1. Luego, podemos definir el funcional de costo $J(D_1) = \mathbb{J}(u)$ y ahora nos interesa computar la derivada de J.

Asumiremos que el campo de deformaciones V verifica que $\operatorname{supp}(V) \subset\subset \Omega$, luego $\Phi_t(\Omega) = \Omega$ para cada $t \in \mathbb{R}$.

Obtenemos una familia de particiones $D_i^t = \Phi_t(D_i), \quad i = 1, 2$ que verifica (6.10) y a partir de la cual definimos

$$p_t(x) = \begin{cases} 1 + \epsilon & \text{si } x \in D_1^t \\ 2 & \text{si } x \in D_2^t. \end{cases}$$

Luego, para cada $t \in \mathbb{R}$ definimos el siguiente funcional:

$$J_t(v) = \frac{1}{2\beta} \int_\Omega |\nabla v|^{p_t(x)}\, dx + \frac{\beta}{2} \int_\Omega (v - I)^2\, dx,$$

Sea u_t el minimizante de J_t. Consideramos $j \colon \mathbb{R} \to \mathbb{R}$ dada por $j(t) = J_t(u_t)$.

Luego, la derivada de forma (o de Hadamard) consiste en computar $j'(0)$.

Obteniendo una buena expresión para esa derivada, podremos computar el campo de deformaciones V que la hace lo más negativa posible y elegir el campo de deformaciones óptimo para luego iterar $D_i^{\Delta t} \simeq (id + \Delta t V)(D_i)$.

En este sentido obtuvimos el siguiente teorema:

Consideremos el espacio $X := W^{1,p}(\Omega) \cap L^2(\Omega)$ equipado con la norma

$$\|\cdot\|_X := \|\cdot\|_{W^{1,p}(\Omega)} + \|\cdot\|_{L^2(\Omega)}.$$

Dada $f \in L^2(\Omega)$,

$$s(t) := \inf_{u \in X} \int_\Omega \frac{|\nabla(u \circ \Phi_{-t})|^{p_t}}{p_t}\, dx + \int_\Omega \frac{|u \circ \Phi_{-t}|^2}{2}\, dx - \int_\Omega (u \circ \Phi_{-t}) f\, dx.$$

verifica que

$$\lim_{t \to 0^+} s(t) = s(0).$$

Más aún, si u_t y u son los extremales asociados a $s(t)$ y $s(0)$ respectivamente, entonces $u_t \rightharpoonup u$ débil en $W^{1,p}(\Omega)$.

Finalmente, si $p^* := \frac{pN}{N-p} > 2$ entonces $u_t \to u$ fuerte en $W^{1,p}(\Omega)$.

A continuación probamos la diferenciabilidad del funcional de costo $s(t)$:

Sea f sea de clase C^1, entonces $s(t)$ es diferenciable en $t = 0$ y

$$\frac{ds}{dt}(0) = R(u) - \int_\Omega uf \operatorname{div} V\, dy - \int_\Omega u \nabla f \cdot V\, dy,$$

donde

$$R(u) := \int_\Omega \frac{|\nabla u|^p}{p} \operatorname{div} V - |\nabla u|^{p-2} \nabla u \cdot \nabla u DV + \operatorname{div} V \frac{|u|^2}{2}\, dy$$

y u es el extremal de $s(0)$.

Por último nos interesa dar una fórmula más explícita para $s'(0)$. Para ello, dado $\delta > 0$ fijo, definimos $D_i^\delta := \{x \in D_i : \operatorname{dist}(x, D_j) > \delta\}$, $i \neq j$ y $\Gamma_i^\delta := \partial D_i^\delta \cap \Omega$.

En cada D_i^δ, $p(x) = p_i$ es constante así que podemos aplicar los resultados clásicos de regularidad.

A través de esta regularizacón, pudimos presentar una mejora de la fórmula de la derivada de forma del costo. Más precisamente, Sea $\Omega \subset \mathbb{R}^N$ abierto y acotado. Sean $D_1, D_2 \subset \Omega$ tales que (6.10), sea $p = p_1\chi_{D_1} + p_2\chi_{D_2}$, donde $1 < p_1 < p_2$ y $\Gamma = \bar{D}_1 \cap \bar{D}_2$.

Sea $V : \mathbb{R}^N \to \mathbb{R}^N$ un campo Lipschitz de deformaciones tal que $\operatorname{spt}(V) \subset\subset \Omega$. Luego,

$$\frac{ds}{dt}(0) = \int_\Gamma \Big[\frac{|\nabla u|^p}{p}\Big] V \cdot \nu \, dS - \int_\Gamma [|\nabla u|^{p-2}](\nabla u \cdot \nu)(\nabla u \cdot V)\, dS,$$

donde

$$\int_\Gamma [f] G \cdot \nu \, dS := \lim_{\delta \to 0} \left(\int_{\Gamma_1^\delta} f G \cdot \nu_1 \, dS - \int_{\Gamma_2^\delta} f G \cdot \nu_2 \, dS \right).$$

La obtención de una fórmula que permita el cálculo de dicha derivada de manera efectiva resulta de suma importancia en aplicaciones concretas.

Appendix A

Gamma convergence results

In this appendix we will recall some basic concepts of $\Gamma-$convergence that are needed in the present paper. Although these results are well-known, we decide to include this appendix in order to make the paper self contained. Also, the results presented here are not stated in the most general form, but in a for that will be enough for our work. For a complete presentation of the theory of $\Gamma-$convergence, see the book of Dal Maso [13].

Let ψ_n and ψ defined in a topological space X_τ with T^2 topology. For our applications, X_τ will be a Banach space and we will consider the weak topology. Then, a family of functionals ψ_n Γ-converges to ψ if

- (liminf inequality) $x_n \to_\tau x$ implies that $\psi(x) \leq \liminf \psi_n(x_n)$ and
- (limsup inequality) there exists $y_n \rightharpoonup x$ such that $\psi(x) \geq \limsup \psi_n(y_n)$.

Theorem A.1. *Let X be a Banach space, $C \subset X$ closed and convex. Let $\psi_n, \psi \colon C \to [-\infty, \infty]$ be weakly lower semicontinuous, strictly convex and uniformly coercive functionals (i.e. for every λ, the set $\{x \in C : \psi_n(x) \leq \lambda\} \subset B_r$ for every n), then* $\inf_C \psi_n = \min_C \psi_n \to \inf_C \psi = \min_C \psi$.

And, if $x_n \in C$ is such that $\psi_n(x_n) = \min_C \psi_n$, then (x_n) es precompact and $\psi(x_0) = \min_C \psi$ where $x_0 = \lim x_n$.

Proof. Let us start by observing that, since ψ_n weakly lower semicontinuous, strictly convex and uniformly coercive functionals, for every n there is a unique x_n such that $\psi_n(x_n) = \inf_C \psi_n$ and (x_n) is bounded if $\psi_n(x_n)$ is bounded. Let us consider now the following recovery function: $x \in C$ such that $y_n \rightharpoonup x$. Therefore,

$$\psi_n(x_n) = \inf_C \psi_n \leq \psi_n(y_n).$$

And so for every x we have that

$$\limsup \psi_n(x_n) \leq \limsup \psi_n(y_n) \leq \psi(x).$$

Therefore,

$$\limsup \psi_n(x_n) \leq \inf_C \psi < \infty$$

and we can conclude that $x_n \in \{x \in C : \psi_n(x) \leq \lambda\} \subset B_r$ for every $n \geq n_0$ taking $\lambda = \inf_C \psi + 1$. So (x_n) is bounded and, via subsequences if necessary, $x_n \rightharpoonup x_0 \in C$ (remember that C is convex and closed, therefore weakly closed)).

Finally, observing that

$$\inf_C \psi \leq \psi(x_0) \leq \liminf \psi_n(x_n) \leq \liminf(\inf_C \psi_n),$$

the proof is completed. □

Remark A.2. If $\psi_n \to \psi$ point-wise, the inequality of the inferior limit (it is enough to take y_n equal to x for every n) always holds. Therefore, to obtain the convergence of the functionals it would only be necessary to check the superior limit inequality.

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