Asymptotic behavior for the Sobolev trace embedding in thin domains

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Sobolev inequalities:

Sobolev trace Theorem

$$S\|u\|^p_{L^q(\partial\Omega)} \le \|u\|^p_{W^{1,p}(\Omega)}, \qquad 1 \le q \le p_* = \frac{(N-1)p}{N-p}$$

Sobolev immersion Theorem

$$\bar{S}\|u\|^p_{L^r(\Omega)} \le \|u\|^p_{W_0^{1,p}(\Omega)}, \qquad 1 \le r \le p^* = \frac{Np}{N-p}$$

The best constants for these inequalities are

$$S_{p,q}(\Omega) = \inf_{u \in W^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p + |u|^p \, dx}{\left(\int_{\partial\Omega} |u|^q \, d\sigma\right)^{p/q}}$$

and

$$\bar{S}_{p,r}(\Omega) = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p \, dx}{\left(\int_\Omega |u|^r \, dx\right)^{p/r}}$$

One of the main differences between these two quantities is the fact that the first one is not homogeneous under dilations of the domain while the second one is:

$$S_{p,q}(\mu\Omega) = \mu^{\beta} \inf_{v \in W^{1,p}(\Omega)} \frac{\int_\Omega \mu^{-p}|\nabla v|^p + |v|^p\, dx}{\left(\int_{\partial\Omega} |v|^q\, d\sigma\right)^{p/q}}$$

where $\beta = (Nq - Np + p)/q$

but

$$\bar{S}_{p,r}(\mu\Omega) = \mu^{\alpha} \bar{S}_{p,r}(\Omega)$$

where $\alpha = (rp + Nr - Np)/r$

For $1 \leq q < p_*$ and $1 \leq r < p^*$ the inclusions are compact, so *extremals* exist. These extremals are weak solutions of

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

respectively, where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$laplacian and $\frac{\partial}{\partial \nu}$ is the outer normal derivative.

Problem: To study the dependance of the best Sobolev trace constant and extremals with respect to the domain.

Consider the family of domains

$$\mu\Omega = \{\mu x \mid x \in \Omega\}$$

and we are interested first on contracting domains (that is $\mu \to 0$).

FB - Rossi (CPAA, 2002) showed

$$\frac{S_{p,q}(\mu\Omega)}{\mu^{\beta}} \to \frac{|\Omega|}{|\partial\Omega|^{p/q}} \qquad \text{as } \mu \to 0$$

Recall $\beta = (Nq - Np + p)/q$.

Behavior of extremals:

FB - Rossi: the extremals, when rescaled to the original domain as $v(x) = u(\mu x)$ and properly normalized, converge to a constant in $W^{1,p}(\Omega)$.

Consider now a different family of domains. Let $\Omega \subset \mathbb{R}^N = \mathbb{R}^{n+k}$ and set

$$\Omega_\mu = \{(\mu x, y) \mid (x,y) \in \Omega,\ x \in \mathbb{R}^n,\ y \in \mathbb{R}^k\}$$

and look for the dependance on $S_{p,q}(\Omega_\mu)$ on μ.

We are specially interested in the case $\mu \to 0$ (thin domains).

Let us define the projection

$$P(\Omega) = \{y \in \mathbb{R}^k \mid \exists x \in \mathbb{R}^n \text{ with } (x,y) \in \Omega\}.$$

and consider the immersion

$$W^{1,p}(P(\Omega), \alpha) \hookrightarrow L^q(P(\Omega), \beta)$$

with best constant $\bar{S}_{p,q}(P(\Omega), \alpha, \beta)$, where $\alpha, \beta \in L^\infty(P(\Omega))$ are nonnegative weights functions.

We have

Theorem 1 (FB - Martinez - Rossi, 2002) *There exists two nonnegative weights* $\alpha, \beta \in L^\infty(P(\Omega))$ *such that*

$$\lim_{\mu \to 0+} \frac{S_{p,q}(\Omega_\mu)}{\mu^{(nq-np+p)/q}} = \bar{S}_{p,q}(P(\Omega), \alpha, \beta)$$

and the extremals u_μ *of* $S_{p,q}(\Omega)$*, properly rescaled and normalized converges strongly in* $W^{1,p}(\Omega)$*, to an extremal for* $\bar{S}_{p,q}(P(\Omega), \alpha, \beta)$*.*

Remark: The weights α and β are given "explicitly" in terms of the geometry of Ω.

Case $p = q$. The eigenvalue problem.

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{on } \partial\Omega \end{cases}$$

For $p = 2$ (linear case) this problem is known as the *Steklov problem*.

We will call this problem as the *nonlinear Steklov problem*.

Observation: The first eigenvalue λ_1 coincides with the best Sobolev trace constant (and the extremals with the respective eigenfunctions).

This problem presents some similar facts with the following

$$
\begin{aligned}
-\Delta_p u = \lambda |u|^{p-2} u \quad & \text{in } \Omega \\
u = 0 \quad & \text{on } \partial\Omega
\end{aligned}
$$

that was studied by many authors (Anane - Cuesta - de Figueiredo - Gossez - etc.)

When applying the Ljusternik-Schnirelman theory on the nonlinear Steklov problem, we obtain:

- There exists a sequence of *variational* eigenvalues $\{\lambda_k\}$ with $\lambda_k \nearrow \infty$ (FB - Rossi, JMAA '01).

- The first eigenvalue λ_1 is isolated and simple (Martínez - Rossi, Abst. Appl. Anal. '02).

- The second eigenvalue $\lambda_2 = \inf\{\lambda > \lambda_1\}$ coincides with the second variational eigenvalue (FB - Rossi, Pub. Mat. '02).

We are interested in the behavior of the eigenvalues and eigenfunctions of the nonlinear Steklov problem for thin domains. We have:

Theorem 2 (FB - Martínez - Rossi, '02) *Every eigenvalue (variational or not) and eigenfunction of the nonlinear Steklov problem converges (when properly "normalized") to an eigenvalue and an eigenfunction of*

$$\begin{aligned} -\mathit{div}(\alpha|\nabla u|^{p-2}\nabla u) + \alpha|u|^{p-2}u = {}& \bar{\lambda}_k \beta |u|^{p-2}u \\ & \text{in } P(\Omega) \\ \frac{\partial u}{\partial \nu} = {}& 0 \qquad \text{on } \partial P(\Omega) \end{aligned}$$

where $P(\Omega)$ is the projection of Ω over the y variable and α, β are the same weights as before.