Symmetry properties for the extremals of the Sobolev trace embedding

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Sobolev inequalities:

\[ S\|u\|_{L^q(\partial \Omega)}^2 \leq \|u\|_{H^1(\Omega)}^2, \quad 1 \leq q \leq 2^* = \frac{2(N-1)}{N-2} \]

**Sobolev trace Theorem**

\[ \bar{S}\|u\|_{L^p(\Omega)}^2 \leq \|u\|_{H^1_0(\Omega)}^2, \quad 1 \leq p \leq 2^* = \frac{2N}{N-2} \]

**Sobolev immersion Theorem**

The best constants for these inequalities are

\[ S_q(\Omega) = \inf_{u \in H^1(\Omega)} \frac{\int_{\Omega} |\nabla u|^2 + |u|^2 \, dx}{(\int_{\partial \Omega} |u|^q \, d\sigma)^2/q} \]

and

\[ \bar{S}_p(\Omega) = \inf_{u \in H^1_0(\Omega)} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{(\int_{\Omega} |u|^p \, dx)^{2/p}} \]
One of the main differences between these two quantities is the fact that the first one is not homogeneous under dilations of the domain while the second one is:

\[ S_q(\mu\Omega) = \mu^\beta \inf_{v \in H^1(\Omega)} \frac{\int_{\Omega} \mu^{-2} |\nabla v|^2 + |v|^2 \, dx}{(\int_{\partial\Omega} |v|^q \, d\sigma)^{2/q}} \]

where \( \beta = (Nq - 2N + 2)/q \)

but

\[ \bar{S}_p(\mu\Omega) = \mu^\alpha \bar{S}_p(\Omega) \]

where \( \alpha = (2p + Np - 2N)/p \)
For $1 \leq q < 2^*$ and $1 \leq p < 2^*$ the inclusions are compact, so *extremals* exists. These extremals are weak solutions of

\[
\begin{cases}
  \Delta u = u & \text{in } \Omega \\
  \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial \Omega
\end{cases}
\]

and

\[
\begin{cases}
  -\Delta u = \lambda |u|^{p-2}u & \text{in } \Omega \\
  u = 0 & \text{on } \partial \Omega
\end{cases}
\]

respectively.
Problem: To study the dependance of the best Sobolev trace constant and extremals with respect to the domain.

Consider the family of domains

$$\Omega_\mu = \mu \Omega = \{\mu x \mid x \in \Omega\}$$

Flores - del Pino proved (Comm. PDE’s, 2001) that for expanding domains ($\mu \to \infty$)

$$S_q(\Omega_\mu) \to S_q(\mathbb{R}^N_+) \quad \text{as } \mu \to \infty$$

For contracting domains ($\mu \to 0$), FB - Rossi (CPAA, 2002) showed

$$\frac{S_q(\Omega_\mu)}{\mu^\beta} \to \frac{|\Omega|}{|\partial \Omega|^{2/q}} \quad \text{as } \mu \to 0$$
Behavior of extremals:

Flores - del Pino: for expanding domains the extremals develop a single peak near a point where the mean curvature of the boundary maximizes.

FB - Rossi: for contracting domains the extremals, when rescaled to the original domain as $v(x) = u(\mu x)$ and properly normalized, converge to a constant in $H^1(\Omega)$. 
Another big difference between the Sobolev trace Theorem and the Sobolev immersion Theorem arises in the behavior of the extremals:

Assume that $\Omega$ is a ball, $\Omega = B(0, \mu)$.

- Extremals of $\bar{S}_p(B(0, \mu))$ are radial functions.

- Extremals of $S_q(B(0, \mu))$ are not radial, at least for large values of $\mu$ (this fact is a consequence of Flores - del Pino)
Question: Is it true that for small balls the extremals for $S_q(B(0, \mu))$ are radial functions?

Answer: Yes! This is a corollary of

**Theorem 1** There exists $\mu_0 > 0$ such that for every $\mu < \mu_0$ there exists a unique positive extremal $u$ for the embedding $H^1(\Omega_\mu) \hookrightarrow L^q(\partial \Omega_\mu)$ (after normalization).

*Proof:* The proof is based on the fact that the extremals are nearly constant for $\mu$ small (FB - Rossi) and the Implicit Function Theorem. ✽
Yet another difference between these problems is the role of the critical exponent.

It is well known that
\[
\begin{cases}
-\Delta u = \lambda |u|^{p-2}u & \text{in } B(0, \mu) \\
u = 0 & \text{on } \partial B(0, \mu)
\end{cases}
\]
has no solution if \( p \geq 2^* \).

However, for
\[
\begin{cases}
\Delta u = u & \text{in } B(0, \mu) \\
\frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial B(0, \mu)
\end{cases}
\]
radial solutions can be explicitly computed for any \( q \).
A first step in understanding the role of the critical exponent $2_* = 2(N - 1)/(N - 2)$ is:

**Theorem 2** There exists $\mu_1 > \mu_2 > 0$ such that, for any $\mu < \mu_2$ there exists a radial extremal for the immersion

$$H^1(B(0, \mu)) \hookrightarrow L^{2*}(\partial B(0, \mu))$$

and for $\mu > \mu_1$ there is no extremals for the immersion.

- The second part of the Theorem, holds for any domain $\Omega$.

**Question:** Is the first part of the Theorem also true for any domain $\Omega$?

**Answer:** Yes. (Work in progress)