Symmetry properties for the extremals of the Sobolev trace embedding

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Sobolev inequalities:

$$S\|u\|^2_{L^q(\partial\Omega)} \leq \|u\|^2_{H^1(\Omega)}, \qquad 1 \leq q \leq 2_* = \frac{2(N-1)}{N-2}$$

Sobolev trace Theorem

$$\bar{S}\|u\|^2_{L^p(\Omega)} \leq \|u\|^2_{H^1_0(\Omega)}, \qquad 1 \leq p \leq 2^* = \frac{2N}{N-2}$$

Sobolev immersion Theorem

The best constants for these inequalities are

$$S_q(\Omega) = \inf_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 + |u|^2 \, dx}{\left(\int_{\partial\Omega} |u|^q \, d\sigma\right)^{2/q}}$$

and

$$\bar{S}_p(\Omega) = \inf_{u \in H^1_0(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, dx}{\left(\int_\Omega |u|^p \, dx\right)^{2/p}}$$

One of the main differences between these two quantities is the fact that the first one is not homogeneous under dilations of the domain while the second one is:

$$S_q(\mu\Omega) = \mu^\beta \inf_{v \in H^1(\Omega)} \frac{\int_\Omega \mu^{-2}|\nabla v|^2 + |v|^2 \, dx}{(\int_{\partial\Omega} |v|^q \, d\sigma)^{2/q}}$$

where $\beta = (Nq - 2N + 2)/q$

but

$$\bar{S}_p(\mu\Omega) = \mu^\alpha \bar{S}_p(\Omega)$$

where $\alpha = (2p + Np - 2N)/p$

For $1 \leq q < 2_*$ and $1 \leq p < 2^*$ the inclusions are compact, so *extremals* exists. These extremals are weak solutions of

$$\begin{cases} \Delta u = u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2} u & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

respectively.

Problem: To study the dependance of the best Sobolev trace constant and extremals with respect to the domain.

Consider the family of domains

$$\Omega_\mu = \mu\Omega = \{\mu x \mid x \in \Omega\}$$

Flores - del Pino proved (Comm. PDE's, 2001) that for expanding domains ($\mu \to \infty$)

$$S_q(\Omega_\mu) \to S_q(\mathbb{R}^N_+) \qquad \text{as } \mu \to \infty$$

For contracting domains ($\mu \to 0$), FB - Rossi (CPAA, 2002) showed

$$\frac{S_q(\Omega_\mu)}{\mu^\beta} \to \frac{|\Omega|}{|\partial\Omega|^{2/q}} \qquad \text{as } \mu \to 0$$

Behavior of extremals:

Flores - del Pino: for expanding domains the extremals develop a single peak near a point where the mean curvature of the boundary maximizes.

FB - Rossi: for contracting domains the extremals, when rescaled to the original domain as $v(x) = u(\mu x)$ and properly normalized, converge to a constant in $H^1(\Omega)$.

Another big difference between the Sobolev trace Theorem and the Sobolev immersion Theorem arises in the behavior of the extremals:

Assume that Ω is a ball, $\Omega = B(0, \mu)$.

- Extremals of $\bar{S}_p(B(0, \mu))$ are radial functions.

- Extremals of $S_q(B(0, \mu))$ are not radial, at least for large values of μ (this fact is a consequence of Flores - del Pino)

Question: Is it true that for small balls the extremals for $S_q(B(0,\mu))$ are radial functions?

Answer: Yes! This is a corollary of

Theorem 1 *There exists $\mu_0 > 0$ such that for every $\mu < \mu_0$ there exists a unique positive extremal u for the embedding $H^1(\Omega_\mu) \hookrightarrow L^q(\partial\Omega_\mu)$ (after normalization).*

Proof: The proof is based on the fact that the extremals are nearly constant for μ small (FB - Rossi) and the Implicit Function Theorem. $\diamondsuit$

Yet another difference between these problems is the role of the critical exponent.

It is well known that

$$\begin{cases} -\Delta u = \lambda |u|^{p-2} u & \text{in } B(0,\mu) \\ u = 0 & \text{on } \partial B(0,\mu) \end{cases}$$

has no solution if $p \geq 2^*$.

However, for

$$\begin{cases} \Delta u = u & \text{in } B(0,\mu) \\ \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2} u & \text{on } \partial B(0,\mu) \end{cases}$$

radial solutions can be explicitly computed for any q.

A first step in understanding the role of the critical exponent $2_* = 2(N-1)/(N-2)$ is:

Theorem 2 *There exists* $\mu_1 > \mu_2 > 0$ *such that, for any* $\mu < \mu_2$ *there exists a radial extremal for the immersion*

$$H^1(B(0,\mu)) \hookrightarrow L^{2_*}(\partial B(0,\mu))$$

and for $\mu > \mu_1$ *there is no extremals for the immersion.*

- The second part of the Theorem, holds for any domain Ω.

<u>Question:</u> Is the first part of the Theorem also true for any domain Ω?

<u>Answer:</u> Yes. (Work in progress)