Behavior of the best Sobolev trace constant on domains with holes

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Sobolev Trace Inequality:

$$S\|u\|^p_{L^q(\partial\Omega)} \le \|u\|^p_{W^{1,p}(\Omega)}, \qquad 1 \le q \le p_* = \frac{p(N-1)}{N-p}$$

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The best constant for this inequality is

$$S = \inf_{u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p + |u|^p \, dx}{\left(\int_{\partial\Omega} |u|^q \, dS\right)^{p/q}}$$

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For $1 \le q < p_*$ the inclusion is compact, so *extremals* exist. These extremals are weak solutions of

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega \end{cases}$$

Let $A \subset \Omega$, $|A| = \alpha > 0$ and consider

$$S_A = \inf \left\{ \frac{\int_\Omega |\nabla u|^p + |u|^p \, dx}{\left(\int_{\partial\Omega} |u|^q \, dS\right)^{p/q}} \;\middle|\; u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega), \; u = 0 \text{ a.e. in } A \right\}$$

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Extremals for S_A are weak solutions of

$$\begin{cases} \Delta_p u = |u|^{p-2} u & \text{in } \Omega \setminus A, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2} u & \text{on } \partial\Omega, \\ u = 0 & \text{in } A. \end{cases}$$

Observe that if $A = \emptyset$ or if $|A| = 0$ then, $S_A = S$.

Problem 1: For fixed $0 < \alpha < |\Omega|$, find $A_0 \subset \Omega$, $|A_0| = \alpha$ such that

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Study properties of this optimal (minimal) set (existence, symmetry, regularity, etc.)

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Problem 2: How can we give sense to S_A if $|A| = 0$?

Special case: $p = q \longrightarrow$ Nonlinear eigenvalue problem

When $A = \emptyset$ was studied by J.D. Rossi, S. Martinez, J.F.B.

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Questions similar to Problem 1 for other eigenvalue problems: A. Henrot, J. Denzler, S. Chanillo et al., R. Pedroza,...

Optimal design problems in general: N. Aguilera – H.W. Alt – L.A. Caffarelli, B. Kawohl, C. Lederman,...

Problem 1

Theorem (Semicontinuity) *Let* $A_n, A \subset \Omega$ *such that*

$$\chi_{A_n} \overset{*}{\rightharpoonup} \chi_A \qquad \text{in } L^\infty(\Omega).$$

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Remark The continuity does not hold. Take, for instance, $A_n = B_1(x^0) \cup B_{1/n}(x^1)$, $A = B_1(x^0)$. Then $\chi_{A_n} \overset{*}{\rightharpoonup} \chi_A$ in $L^\infty(\Omega)$, but $S_A < \liminf_{n\to\infty} S_{A_n}$ (if $p > N$). A similar construction works for $1 < p \leq N$.

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Remark The semicontinuity result does not give the existence of minimal *holes* because sets of fixed positive measure are not compact with respect to the topology of the Theorem.

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Theorem (Existence of minimal holes) *Given* $0 < \alpha < |\Omega|$ *we define* $S(\alpha) := \inf_{A \subset \Omega,\ |A| = \alpha} S_A$. *Then there exists* $A_0 \subset \Omega$, $|A_0| = \alpha$ *such that* $S_{A_0} = S(\alpha)$.

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Proof: The idea is:

Step 1: minimize $J(u) = \int_\Omega |\nabla u|^p + |u|^p \, dx$ over the class

$$E(\Omega,\alpha) = \Big\{ (u,\phi) \in W^{1,p}(\Omega) \times L^\infty(\Omega) \ \Big| \ \|u\|_{L^q(\partial\Omega)} = 1,\ 0 \le \phi \le 1, \\ \int_\Omega \phi \ge \alpha,\ u.\phi = 0 \text{ a.e. in } \Omega \Big\}$$

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$$\left. \int_\Omega \phi \ge \alpha,\ u.\phi = 0 \text{ a.e. in } \Omega \right\}$$

Step 2: Show that the minimizer (u,ϕ) verifies that ϕ is a characteristic function and $\int_\Omega \phi = \alpha$.

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Corollary *The strict monotonicity of $S(\alpha)$ implies that*

$$|\{u = 0\}| = \alpha.$$

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Proof: Take a "strip" of measure α,

$$A_\varepsilon = \{x \in \Omega \mid \varepsilon < \mathrm{dist}(x, \partial\Omega) < \delta(\varepsilon)\}$$

and observe that the extremals for S_{A_ε} verify $u_\varepsilon = 0$ in $\Omega_\varepsilon = \{x \in \Omega \mid \varepsilon < dist(x, \partial\Omega)\}$.

If S_{A_ε} were bounded, up to a subsequence, $u_\varepsilon \to u$ and $u = 0$ in Ω, but $\|u\|_{L^q(\partial\Omega)} = 1$. $\diamondsuit$

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Definition *Given* $A \subset \mathbb{R}^N$ *the spherical symmetrization* A^* *of* A *is defined as follows: for each* $r > 0$, *take* $A \cap \partial B_r(0)$ *and replace it by the spherical cap of the same area and center* re_N. *The union of these caps is* A^*.

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Theorem *Let* $\Omega = B_1(0)$ *and* $0 < \alpha < |B_1(0)|$. *Then there exists an optimal hole that is spherically symmetric. Moreover, if* $p = 2$, *then every optimal hole is spherically symmetric.*

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Define

$$\mathbf{S}_A = \inf_{W_A^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p + |u|^p \, dx}{\left(\displaystyle\int_{\partial\Omega} |u|^q \, dS\right)^{p/q}}.$$

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Remark *If A is a regular surface of dimension k, there exists a trace $T : W^{1,p}(\Omega) \to L^p(A)$ if $k > N - p$. This relates to our Theorem by the fact that* $\dim_H(A) \leq N - p \Rightarrow \mathrm{Cap}_p(A) = 0$.

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In the general case, one is led to consider the "precise" representative of the set A*. There exists a "good" representative of* A *such that* $S_A = \mathbf{S}_A$*? We don't know.*