Numerical analysis of stochastic differential equations with explosions*

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January 2005

*Joint work with J. Dávila, P. Groisman, J.D. Rossi and M. Sued

The Problem:

$$\begin{cases} dx = b(x)dt + \sigma(x)dW \\ x(0) = z \in \mathbb{R}_{>0} \end{cases}$$

- W is a one dimensional Wiener process

- b and σ are smooth and positive.

If b is not globally Lipschitz, solutions to this problem may explode in finite time with positive probability.

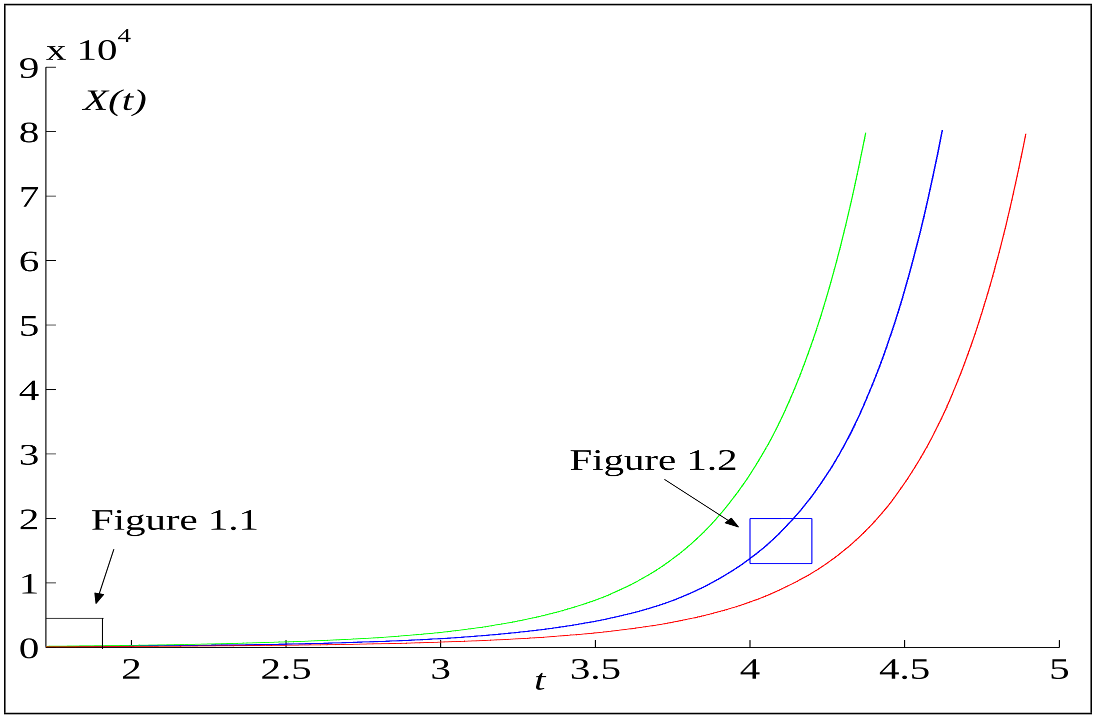

Figure 1

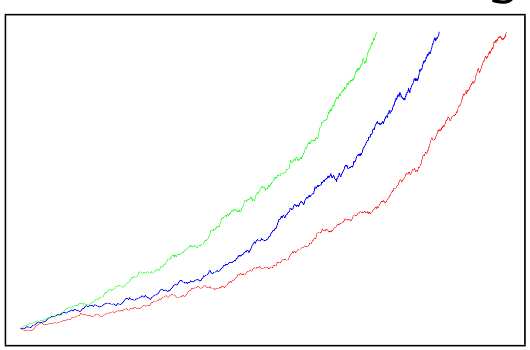

Figure 1.1

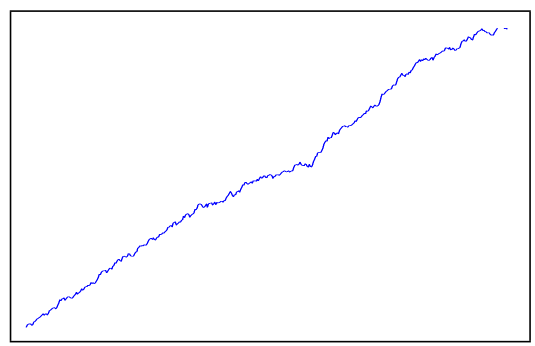

Figure 1.2

There exists a stopping time $T = T(\omega)$ such that $x(\omega, t)$ is defined in $[0, T(\omega))$, but

$$x(\omega, t) \nearrow +\infty \qquad \text{as} \quad t \nearrow T(\omega).$$

and

$$\mathbb{P}(T < \infty) > 0.$$

The *Feller Test for Explosions* provides a precise criteria to determine, in terms of b and σ, whether solutions explode with probability zero, positive or one.

We assume

- $0 < C_1 < \sigma^2(s) < C_2 b(s)$.

- $b(s)$ is nondecreasing for large s and $\int^{\infty} \frac{1}{b(s)}\, ds < +\infty$.

Under these conditions explosion occur with probability one.

Example:

$$dx = (1 + |x|^2)dt + dW.$$

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- Convergence of the numerical explosion times to the continuous one.

The Euler-Maruyama method (for bounded solutions)

$$X_i \approx x(t_i)$$

$$h = t_{i+1} - t_i, \qquad \Delta W_i = W(t_{i+1}) - W(t_i)$$

$$\begin{cases} X_{i+1} = X_i + b(X_i)h + \sigma(X_i)\Delta W_i \\ X_0 = x(0) = z \end{cases}$$

The E-M method: $X_{i+1} = X_i + b(X_i)h + \sigma(X_i)\Delta W_i$

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Our scheme reads: $X_{i+1} = X_i + h + \sigma(X_i)\Delta W_i$

The numerical solution

$$X(t_i) = X_i,$$

$$X(t) = X_i + b(X_i)(t - t_i) + \sigma(X_i)(W(t) - W(t_i)), \qquad \text{for } t \in [t_i, t_{i+1}),$$

is a well define process up to time

$$T_h := \sum_{i=1}^{\infty} h_i = \sum_{i=1}^{\infty} \frac{h}{b(X_i)}.$$

We say that the numerical solution explode in finite time T_h if

$$\mathbb{P}(T_h < \infty) > 0.$$

Mean Square Convergence (while solutions are bounded)

Theorem 1. Fix a time $S > 0$ and a constant $M > 0$. Consider the stopping times given by

$$R^M := \inf\{t > 0 \ : \ x(t) = M\}, \quad R_h^M := \inf\{t > 0 \ : \ X(t) = M\},$$

$$\tau_h := \min\{R^M, R_h^{2M}, S\}.$$

Then

$$\lim_{h \to 0} \mathbb{E}\left[\sup_{0 \le t \le \tau_h} |x(t) - X(t)|^2\right] = 0.$$

Explosions in the numerical scheme

Theorem 2.

1. $X(\cdot)$ explodes in finite time with probability one.

2. For every $h > 0$ we have

$$\lim_{i \to +\infty} \frac{X(t_i)}{hi} = 1, \quad \text{i.e. } X(t_i) \sim hi$$

This is the asymptotic behavior for the numerical scheme.

For $b(s) \sim s^p$ at infinity, we get

$$X(t_i)(T_h - t_i)^{1/(p-1)} \to \left(\frac{1}{p-1}\right)^{1/(p-1)} \quad \text{as } t_i \to T_h \ (i \to +\infty).$$

Idea of the Proof.

Rewriting our scheme, we get

$$X_i = z + hi + \sum_{j=1}^{i} \sigma(X_j)\Delta W_j$$

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$$\begin{aligned} X_i &= z + hi + \sum_{j=1}^{i} \sigma(X_j)\Delta W_j \\ &= z + hi + \sum_{j=1}^{i} \sigma(X_j)\sqrt{h_j}\frac{\Delta W_j}{\sqrt{h_j}} \end{aligned}$$

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&\simeq z + hi + \sum_{j=1}^{i} \sqrt{h}\frac{\sigma(X_j)}{\sqrt{b(X_j)}} N(0,1) \\
&\simeq z + hi + C \sum_{j=1}^{i} N(0,1)
\end{aligned}$$

So, we get

$$\frac{X_i}{hi} \simeq \frac{z}{hi} + 1 + C\frac{1}{i}\sum_{j=1}^{i} N(0,1) \to 1 \text{ a.s.}$$

by the *Strong Law of the Large Numbers*.

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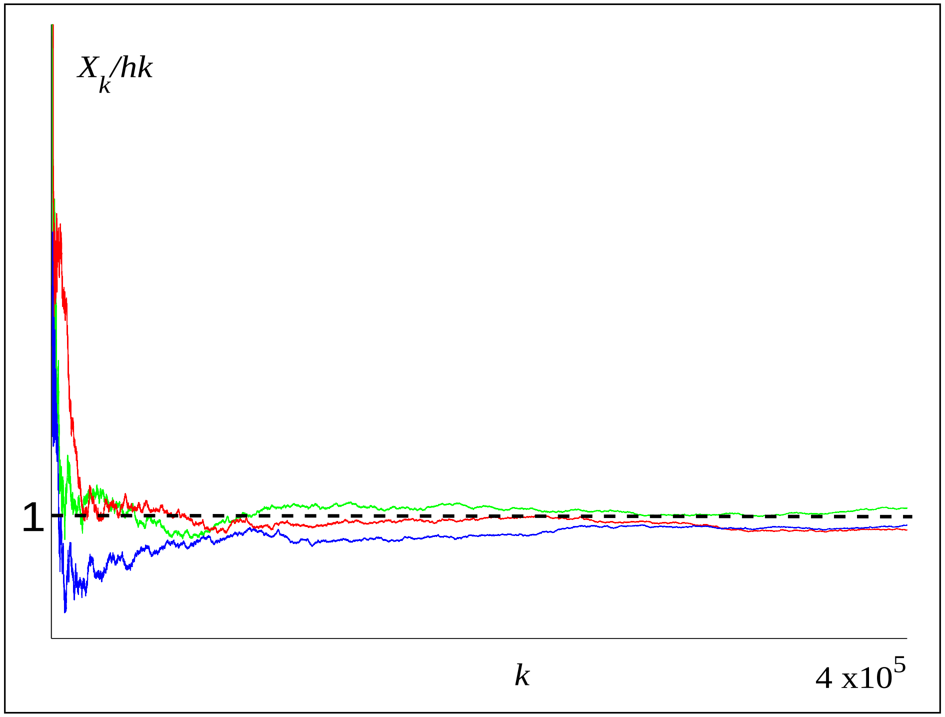

Figure 2: $X_i/hi \to 1$ a.s.

Convergence of the Numerical Explosion Times

Theorem 3. The stopping times R_h^M defined in Theorem 1, converges to the continuous explosion time T in probability as $h \to 0$ and $M \to +\infty$. That is, for every $\varepsilon > 0$,

$$\lim_{M\to+\infty} \lim_{h\to 0} \mathbb{P}(|R_h^M - T| > \varepsilon) = 0.$$

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For the proof, we use the following Lemma:

Lemma. $\mathbb{P}(R^M \geq R_h^{2M}) \to 0$ and $\mathbb{P}(R_h^M \geq R^{2M}) \to 0$ as $h \to 0$.

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Now, we compute:

$$\mathbb{P}(|R_h^M - T| > \varepsilon) = \mathbb{P}(R_h^M - T > \varepsilon) + \mathbb{P}(R_h^M - T < -\varepsilon)$$

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&\qquad \leq \mathbb{P}(R_h^M \geq R^{2M}) + \mathbb{P}(R^{M/2} - R_h^M > \varepsilon/2) + \mathbb{P}(T - R^{M/2} > \varepsilon/2)
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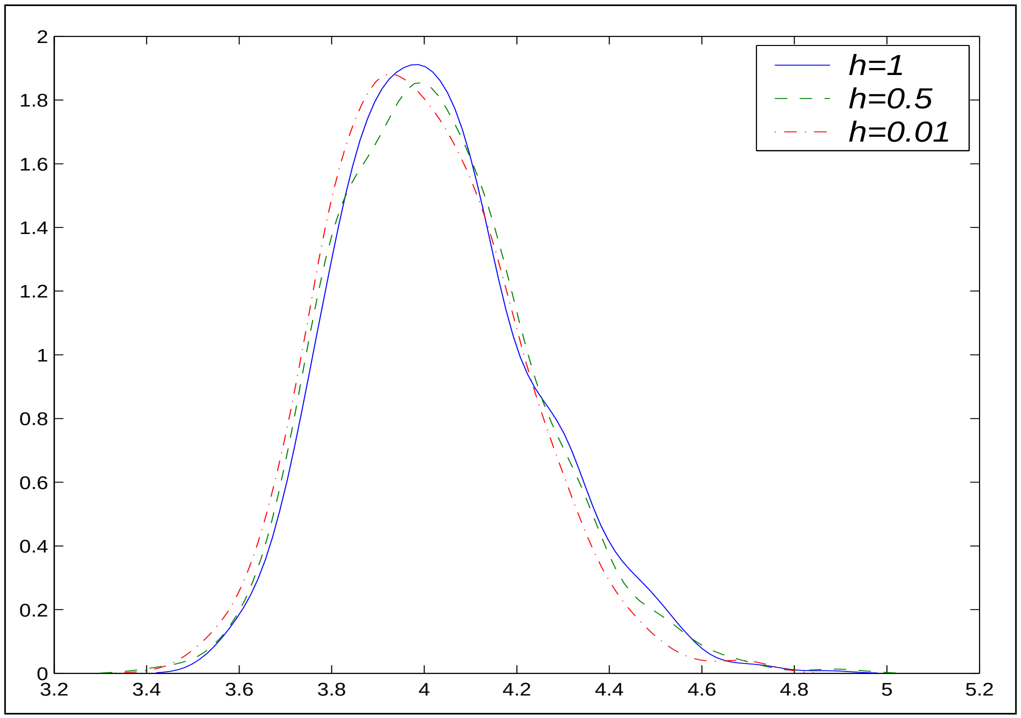
&\qquad \to 0 \qquad \text{by the Lemma and since } R^M \to T \text{ a.s.} \square
\end{aligned}
$$



The kernel density estimator of R_h^M for different values of h.