Optimal design problems for the first Steklov eigenvalue *

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Sobolev Trace Theorem

$$H^1(\Omega) \hookrightarrow L^q(\partial\Omega), \qquad 1 \le q \le 2_* := \frac{2(N-1)}{N-2}.$$

- Ω is a bounded smooth domain in $\mathbb{R}^N$

Sobolev Trace Constant

$$S_q := \inf \left\{ \frac{\int_\Omega |\nabla v|^2 + v^2, dx}{\left(\int_{\partial\Omega} |v|^q \, dS \right)^{2/q}} \; : \; v \in H^1(\Omega) \setminus H_0^1(\Omega) \right\}$$

Is the optimal constant in the Sobolev inequality

$$S\|v\|^2_{L^q(\partial\Omega)} \le \|v\|^2_{H^1(\Omega)}.$$

If $1 \leq q < 2_*$ (subcritical), the immersion

$$H^1(\Omega) \hookrightarrow L^q(\partial\Omega) \quad \text{is compact}$$

Therefore, there exists extremals. These extremals are weak solutions to

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2} u & \text{on } \partial\Omega \end{cases}$$

(λ = Lagrange multiplier).

If u is such that $\|u\|_{L^q(\partial\Omega)} = 1$ then $\lambda = S_q$.

In the *linear case* ($q = 2$) this problem is known as the

Steklov eigenvalue problem

Problem with Holes

Let $A \subset \Omega$ be measurable. We define

$$S_q(A) := \inf \left\{ \frac{\int_\Omega |\nabla v|^2 + v^2, dx}{\left(\int_{\partial\Omega} |v|^q \, dS \right)^{2/q}} \ : \ v \in H^1(\Omega) \text{ such that } v\,|_A \equiv 0 \right\}$$

If A is *closed*, then extremals are weak solutions to

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \setminus A \\ \dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2} u & \text{on } \partial\Omega \setminus A \\ u = 0 & \text{in } A \end{cases}$$

PROBLEM

There exists $A^* \subset \Omega,\ |A^*| = \alpha$ such that $S_q(A^*) = \inf\limits_{\substack{A \subset \Omega \\ |A| = \alpha}} S_q(A)$?

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And other questions like

- Location
- Topology
- Symmetry
- Regularity
- Computation

Recall that when $q = 2$ the problem becomes a linear eigenvalue problem $\longrightarrow$ Steklov eigenvalue problem.

Our results are new, even in this classical setting.

Similar questions for other eigenvalue problems: A. Henrot, J. Denzler, S. Chanillo et al., R. Pedroza,...

Optimal design problems in general: N. Aguilera–H.W. Alt–L.A. Caffarelli, B. Kawohl, C. Lederman,...

Proof of existence (sketch)

Let $S(\alpha) := \inf_{\substack{A\subset\Omega \\ |A|=\alpha}} S_q(A) = \inf_{\substack{A\subset\Omega \\ |A|\geq\alpha}} S_q(A)$. Then

$$S(\alpha) = \inf \Big\{ \int_\Omega |\nabla v|^2 + v^2 \, dx \mid v \in H^1(\Omega),\ v \geq 0, \\ \|v\|_{L^q(\partial\Omega)} = 1,\ |\{v = 0\}| \geq \alpha \Big\}$$

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Let now $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence.

Then $\|u_n\|_{H^1(\Omega)} \leq C$ and hence...

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(\Omega)$$
$$u_n \to u \quad \text{strongly in } L^2(\Omega)$$
$$u_n \to u \quad \text{strongly in } L^q(\partial\Omega)$$

Then $u \in H^1(\Omega)$, $\|u\|_{L^q(\partial\Omega)} = 1$ and $u \geq 0$.

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Then $u \in H^1(\Omega)$, $\|u\|_{L^q(\partial\Omega)} = 1$ and $u \geq 0$.

Also, if we call $A_n := \{u_n = 0\}$, then there exists $\phi \in L^\infty(\Omega)$, $0 \leq \phi \leq 1$ s.t. $\chi_{A_n} \rightharpoonup \phi$ weakly in $L^2(\Omega)$.

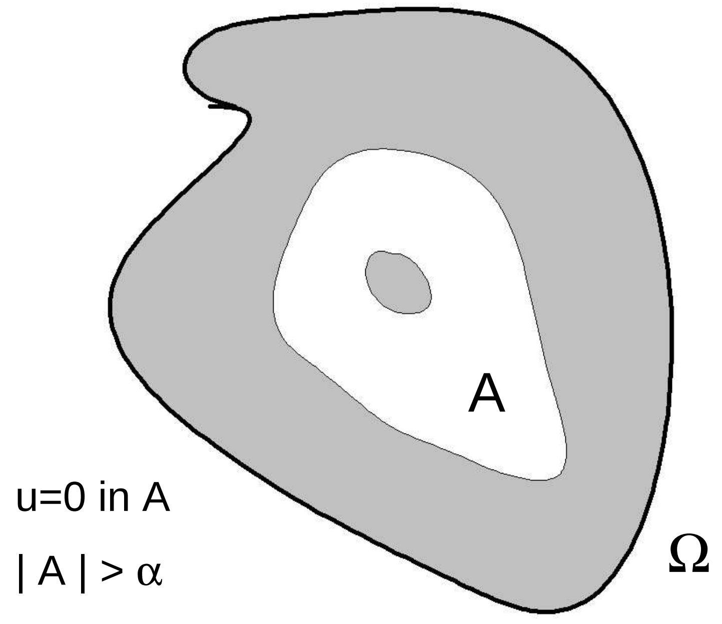
Hence, if $A := \{\phi = 0\}$, we have

$$
|A| \geq \int_\Omega \phi \, dx = \lim_{n\to\infty} \int_\Omega \chi_{A_n} \, dx = \lim_{n\to\infty} |A_n| = \alpha.
$$

On the other hand, $\int_\Omega u\phi\,dx = \int_\Omega u_n \chi_{A_n}\,dx = 0$.

Therefore, $u = 0$ a.e. $A = \{\phi > 0\}$.

It remains to see that $|\{u = 0\}| = \alpha$. Assume that $|\{u = 0\}| > \alpha$



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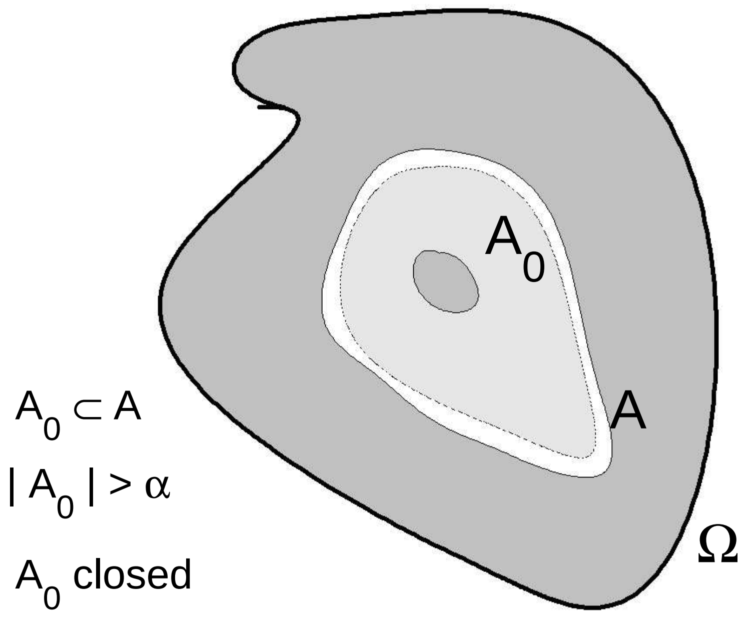

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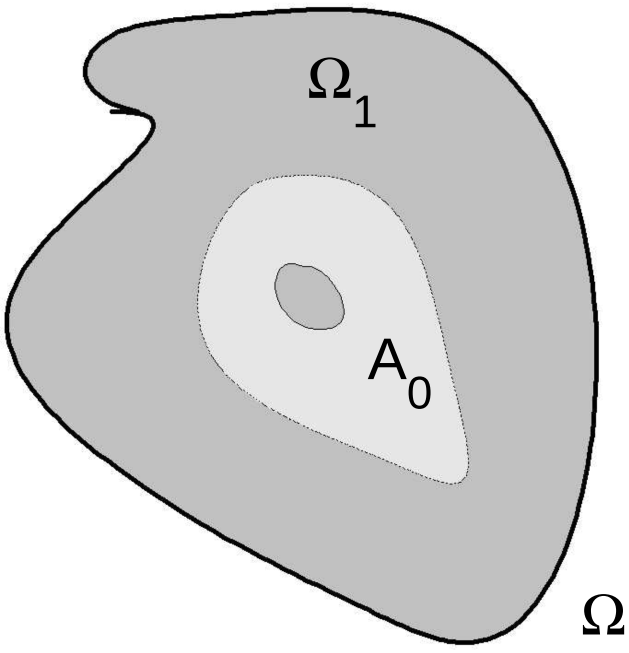

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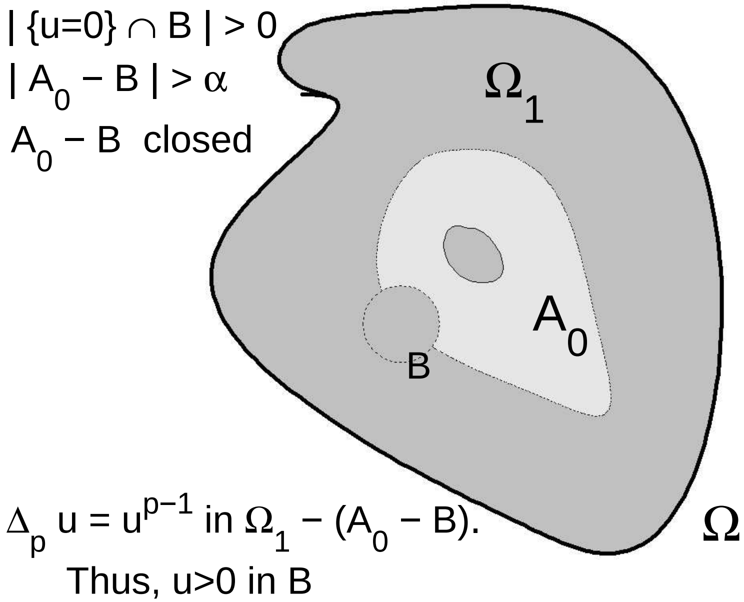

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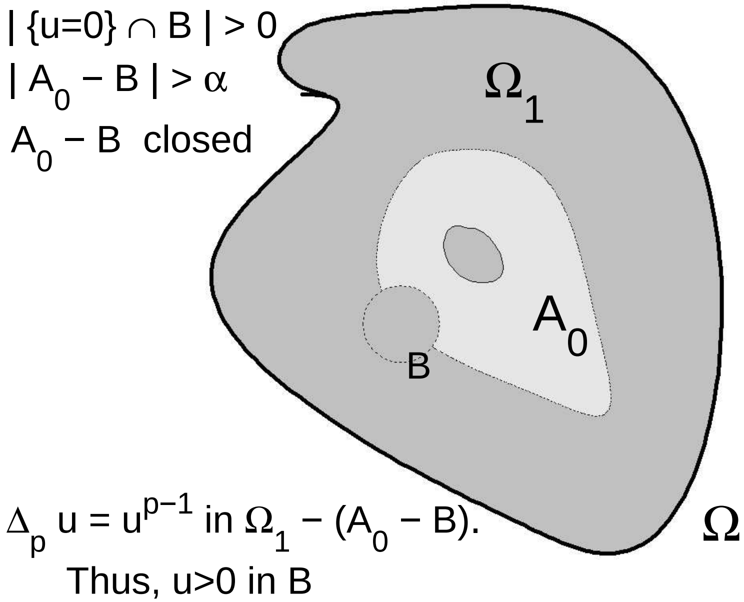

CONTRADICTION! $\square$

Properties of *Optimal Holes*

- Topology: We only know that the complement of the hole $\Omega \setminus A^* = \{u > 0\}$ is (measure–theoretic) connected.

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If $\Omega = B_1(0)$, then A^* is *spherically symmetric*.

Is A^* radially symmetric? NO!

Properties of *Optimal Holes*

• Regularity: Different approach. Let $\beta := |\Omega| - \alpha$ and

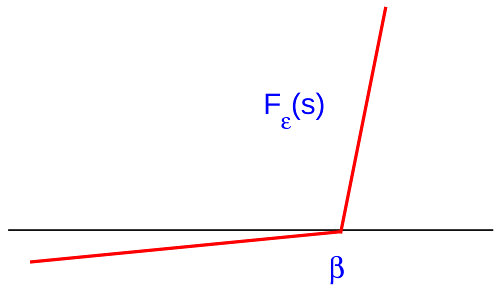
$\mathcal{A}_\beta := \{v \in H^1(\Omega) \mid \|v\|_{L^q(\partial\Omega)} = 1, \ |\{v > 0\}| = \beta\}$. Then

$$S(\alpha) = \inf_{v \in \mathcal{A}_\beta} \int_\Omega |\nabla v|^2 + v^2 \, dx$$

The idea is to penalize and minimize without the measure restriction.

Let $F_\varepsilon(s)$ be

$$F_\varepsilon(s) := \begin{cases} \frac{1}{\varepsilon}(s-\beta) & s \geq \beta \\ \varepsilon(s-\beta) & s < \beta \end{cases}$$



Then, we minimize the penalized functional

$$\mathcal{J}_\varepsilon(v) := \int_\Omega |\nabla v|^2 + v^2\, dx + F_\varepsilon(|\{v > 0\}|)$$

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The main feature of the method is: For each $\varepsilon > 0$ fixed, there exists u_ε minimizer of $\mathcal{J}_\varepsilon$ that is locally Lipschitz and the *free boundary* $\partial\{u_\varepsilon > 0\}$ is a $C^{1,\gamma}$ surface ([Alt–Caffarelli], 1981.)

Then, for ε_0 small (but fixed!), we have

$$|\{u_{\varepsilon_0} > 0\}| = \beta.$$

So we recover a solution of our original optimization problem.

Computation of Optimal holes,

We consider:

- $\mathcal{T}_h$ a regular triangulation of Ω,

- $\mathcal{V}_h \subset H^1(\Omega)$ the subspace of continuous piecewise linear functions,

- $\mathcal{A}_h := \{A \subset \Omega \mid A = \cup_{i=1}^{n} T_{h_i},\ |A| \geq \alpha$ and $\exists T_{h_i}$ s.t. $|A \setminus T_{h_i}| < \alpha\}$ the class of admissible *numerical holes*.

Then:

$$S_q^h(A) := \inf \left\{ \int_\Omega |\nabla v|^2 + v^2 \, dx \mid v \in \mathcal{V}_h, \ \|v\|_{L^q(\partial\Omega)} = 1, \ v|_A \equiv 0 \right\}$$

and

$$S_q^h(\alpha) := \inf_{A \in \mathcal{A}_h} S_q^h(A) = \min_{A \in \mathcal{A}_h} S_q^h(A) = S_q^h(A_h^*).$$

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We have the following result:

Theorem There holds:

1. $S_q^h(\alpha) \to S_q(\alpha)$ as $h \to 0$.

2. The extremals u_h for $S_q^h(\alpha)$ converges in $H^1(\Omega)$, along subsequences, to an extremal u of $S_q(\alpha)$.

3. Again, along subsequences, $\chi_{A_h^*} \to \chi_{A^*}$ in $L^1(\Omega)$.

How to compute the *discrete optimal hole*?

• Optimality criteria: At the free boundary $\partial\{u > 0\}$ the extremal u verifies $\partial u / \partial \nu = constant$.

The algorithm:

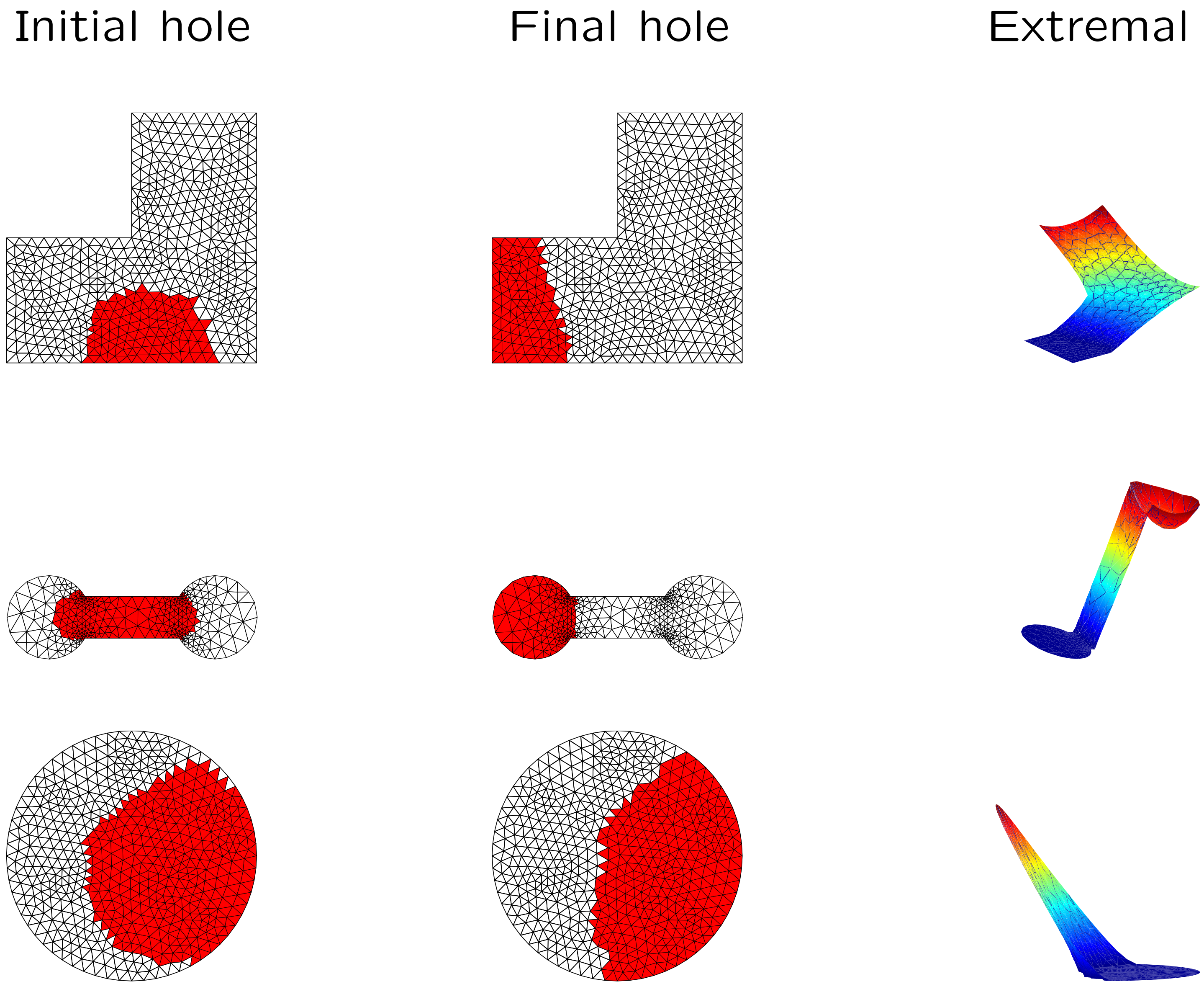
1. Choose an initial hole $A^0 \in \mathcal{A}_h$.

2. Compute $S_q^h(A^0)$ and the extremal u_h^0.

3. Compute $\frac{\partial u_h^0}{\partial \nu}$ at ∂A^0.

4. Remove the triangles with larger normal derivative until the measure of the hole lies below α and add triangles to the hole in regions of the boundary where the normal derivative is small.

5. Update the hole.


Initial hole
Final hole
Extremal

Optimal computed holes for different values of h