On the Sobolev trace constant

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Sobolev inequalities:

$$S\|u\|^p_{L^q(\partial\Omega)} \leq \|u\|^p_{W^{1,p}(\Omega)}, \quad 1 \leq q \leq p_* = \frac{(N-1)p}{N-p} \quad - \text{TRACE}$$

$$\bar{S}\|u\|^p_{L^r(\Omega)} \leq \|u\|^p_{W_0^{1,p}(\Omega)}, \quad 1 \leq r \leq p^* = \frac{Np}{N-p} \quad - \text{IMMERSION}$$

The best constants for these inequalities are

$$S_{p,q}(\Omega) = \inf_{u \in W^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p + |u|^p \, dx}{(\int_{\partial\Omega} |u|^q \, dS)^{p/q}}$$

and

$$\bar{S}_{p,r}(\Omega) = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p \, dx}{(\int_\Omega |u|^r \, dx)^{p/r}}$$

One of the main differences between these two quantities is the fact that the first one is not homogeneous under dilations of the domain while the second one is:

$$S_{p,q}(\mu\Omega) = \mu^{\beta} \inf_{v\in W^{1,p}(\Omega)} \frac{\int_\Omega \mu^{-p}|\nabla v|^p + |v|^p\,dx}{(\int_{\partial\Omega} |v|^q\,dS)^{p/q}}$$

where $\beta = (Nq - Np + p)/q$

but

$$\bar{S}_{p,r}(\mu\Omega) = \mu^{\alpha}\bar{S}_{p,r}(\Omega)$$

where $\alpha = (pr + Nr - pN)/r$

For $1 \leq q < p_*$ and $1 \leq r < p^*$ the inclusions are compact, so *extremals* exists. These extremals are weak solutions of

$$\begin{cases} \Delta_p u = |u|^{p-2} u & \text{in } \Omega \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2} u & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta_p u = \lambda |u|^{r-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

respectively, where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p-$Laplacian.

Problem: To study the dependance of the best Sobolev trace constant and extremals with respect to the domain.

Consider the family of domains

$$\Omega_\mu = \mu\Omega = \{\mu x \mid x \in \Omega\}$$

Flores – del Pino proved (Comm. PDEs, 2001) that for expanding domains ($\mu \to \infty$)

$$S_{2,q}(\Omega_\mu) \to S_{2,q}(\mathbb{R}^N_+) \qquad \text{as } \mu \to \infty, \quad q > 2.$$

For contracting domains ($\mu \to 0$), FB – Rossi (CPAA, 2002) showed

$$\frac{S_{p,q}(\Omega_\mu)}{\mu^\beta} \to \frac{|\Omega|}{|\partial\Omega|^{p/q}} \qquad \text{as } \mu \to 0$$

Behavior of extremals:

Flores - del Pino: for expanding domains the extremals develop a single peak near a point where the mean curvature of the boundary maximizes ($q > p = 2$).

FB - Rossi: for contracting domains the extremals, when rescaled to the original domain as $v(x) = u(\mu x)$ and properly normalized, converge to a constant in $W^{1,p}(\Omega)$.

Consider now a different family of domains. Let $\Omega \subset \mathbb{R}^N = \mathbb{R}^{n+k}$ and set

$$\Omega_\mu = \{(\mu x, y) \mid (x, y) \in \Omega,\ x \in \mathbb{R}^n,\ y \in \mathbb{R}^k\}.$$

Now look for the dependance for $S_{p,q}(\Omega_\mu)$ on μ.

We are specially interested in the case $\mu \to 0$ (thin domains).

Let us define the projection

$$P : \mathbb{R}^{n+k} \to \mathbb{R}^k, \qquad P(x, y) = y$$

and consider the immersion

$$W^{1,p}(P(\Omega), \alpha) \hookrightarrow L^q(P(\Omega), \beta)$$

with best constant $\bar{S}_{p,q}(P(\Omega), \alpha, \beta)$, where $\alpha, \beta \in L^\infty(P(\Omega))$ are nonnegative weight functions.

We have

Theorem 1 (FB – Martinez – Rossi) *There exists two non-negative weights $\alpha, \beta \in L^\infty(P(\Omega))$ such that*

$$\lim_{\mu \to 0+} \frac{S_{p,q}(\Omega_\mu)}{\mu^{(nq-np+p)/q}} = \bar{S}_{p,q}(P(\Omega), \alpha, \beta)$$

and the extremals u_μ of $S_{p,q}(\Omega)$, properly rescaled and normalized converges strongly in $W^{1,p}(\Omega)$, to an extremal for $\bar{S}_{p,q}(P(\Omega), \alpha, \beta)$.

<u>Remark:</u> The weights α and β are given "explicitly" in terms of the geometry of Ω.

Case $p = q$. The eigenvalue problem.

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{on } \partial\Omega \end{cases}$$

For $p = 2$ (linear case) this problem is known as the *Steklov problem*.

We will call this problem as the *nonlinear Steklov problem*.

<u>Observation</u>: The first eigenvalue λ_1 agrees with the best Sobolev trace constant (and the extremals with the respective eigenfunctions).

This problem presents some similar facts with the following

$$\begin{aligned} -\Delta_p u = \lambda |u|^{p-2} u \quad & \text{in } \Omega \\ u = 0 \qquad\qquad & \text{on } \partial\Omega \end{aligned}$$

that was studied by many authors (Anane - Cuesta - de Figueiredo - Gossez - Lindqvist - etc.)

When applying the Ljusternik–Schnirelman theory on the non-linear Steklov problem, we obtain:

- There exists a sequence of *variational* eigenvalues $\{\lambda_k\}$ with $\lambda_k \nearrow \infty$ (FB – Rossi, JMAA '01).

- The first eigenvalue λ_1 is isolated and simple (Martínez – Rossi, Abst. Appl. Anal. '02).

- The second eigenvalue $\lambda_2 = \inf\{\lambda > \lambda_1\}$ agrees with the second variational eigenvalue (FB – Rossi, Pub. Mat. '02).

We are interested in the behavior of the eigenvalues and eigenfunctions of the nonlinear Steklov problem for thin domains. We have:

Theorem 2 (FB – Martínez – Rossi) *Every eigenvalue (variational or not) and eigenfunction of the nonlinear Steklov problem converges (when properly "normalized") to an eigenvalue and an eigenfunction of*

$$-\mathit{div}(\alpha|\nabla u|^{p-2}\nabla u) + \alpha|u|^{p-2}u = \bar{\lambda}_k \beta |u|^{p-2}u \quad \text{in } P(\Omega)$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial P(\Omega)$$

where $P(\Omega)$ is the projection of Ω over the y variable and α, β are the same weights as before.

Behavior of Extremals.

Another big difference between the Sobolev trace Theorem and the Sobolev immersion Theorem arises in the behavior of the extremals:

Assume $p = 2$ and that Ω is a ball, $\Omega = B(0, \mu)$.

- Extremals for $\bar{S}_{2,r}(B(0, \mu))$ are radial functions.

- Extremals for $S_{2,q}(B(0, \mu))$ are not radial, at least for large values of μ (this fact is a consequence of Flores – del Pino)

Question: Is it true that for small balls the extremals for $S_{2,q}(B(0,\mu))$ are radial functions?

Answer: Yes! This is a corollary of

Theorem 3 (FB – Lami-Dozo – Rossi) *There exists $\mu_0 > 0$ such that for every $\mu < \mu_0$ there exists a unique positive extremal u for the embedding $H^1(\Omega_\mu) \hookrightarrow L^q(\partial\Omega_\mu)$ (after normalization).*

Proof: The proof is based on the fact that the extremals are nearly constant for μ small (FB – Rossi) and the Implicit Function Theorem. $\diamondsuit$

As a consequence of this result we have

Theorem 4 (FB – Lami-Dozo – Rossi) *There exists* $\mu_0 > 0$ *such that, for any* $\mu < \mu_0$ *there exists a* <u>*radial*</u> *extremal for the immersion*

$$H^1(B(0,\mu)) \hookrightarrow L^{2_*}(\partial B(0,\mu)).$$

The existence of extremals in the critical Sobolev case, for general domains Ω (under very mild geometric assumptions) has been proved by Adimurthi – Yadava (Comm. PDEs, 1991).

<u>Question:</u> There exists extremals for the Sobolev trace immersion in the case $p \neq 2$, $q = 2_*$?

Answer: yes! but we do not have a result as general as Adimurthi – Yadava for $p = 2$.

Theorem 5 (FB – Rossi) *Let Ω be a bounded smooth domain in $\mathbb{R}^N$ such that*

$$\frac{|\Omega|}{|\partial\Omega|^{p/p_*}} < K(N,p),$$

where

$$K(N,p)^p = \inf_{\nabla w \in L^p(\mathbb{R}^N_+),\ w \in L^{p_*}(\partial\mathbb{R}^N_+)} \frac{\displaystyle\int_{\mathbb{R}^N_+} |\nabla w|^p \, dx}{\left(\displaystyle\int_{\partial\mathbb{R}^N_+} |w|^{p_*} \, dx' \right)^{p/p_*}}.$$

Then there exists an extremal for the immersion $W^{1,p}(\Omega) \to L^{p_*}(\partial\Omega)$.

Remark: Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and let

$$\Omega_\mu = \mu\Omega = \{\mu x \mid x \in \Omega\},$$

where $\mu > 0$. Then, if μ is small,

$$\mu < K(N,p)^{1/p} \frac{|\partial\Omega|^{1/p_*}}{|\Omega|^{1/p}},$$

then Ω_μ verifies the hypotheses of the Theorem and hence there is an extremal for the immersion $W^{1,p}(\Omega_\mu) \to L^{p_*}(\partial\Omega_\mu)$.

Remark: Observe that from the proof of the Theorem, we obtain the existence of extremals for every domain Ω that satisfies

$$S_{p,p_*}(\Omega) < K(N,p). \tag{1}$$

The condition in the Theorem is the simplest geometric condition that ensures (1).