Precise asymtotic of eigenvalues of resonant quasilinear systems.

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Joint work with Juan Pablo Pinasco

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Introduction

Recall the $p-$Laplacian

$$\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u).$$

Used to model *Non-Newtonian Fluids*

- $1 < p < 2 \longrightarrow$ pseudo-plastic fluids.
- $p > 2 \longrightarrow$ dilatant fluids.
- $p = 2 \longrightarrow$ Newtonian fluids ($\Delta_2 = \Delta$).

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Important feature: HOMOGENEITY

$$\Delta_p(tu) = t^{p-1}\Delta_p u, \qquad \forall\ t > 0.$$

Introduction - Eigenvalues

The eigenvalue problem for the $p-$Laplacian:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ smooth and bounded.
We denote

$$\Sigma := \{\lambda \in \mathbb{R} : \text{ there exists a nontrivial solution}\}$$

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What can be said about the set Σ?

Introduction - Eigenvalues

We define

$$\lambda_1 := \inf_{v \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p \, dx}{\int_\Omega |v|^p \, dx}.$$

Then λ_1 is the first (lowest) eigenvalue and is a principal eigenvalue (i.e. every eigenfunction associated to λ_1 has constant sign).

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That means:

- λ_1 is simple $\leftrightarrow$ if u and v are eigenfunctions associated to λ_1 then

$$u = cv \qquad \text{for some } c \in \mathbb{R}.$$

- λ_1 is isolated $\leftrightarrow$ there exists $\delta > 0$ such that

$$(\lambda_1, \lambda_1 + \delta) \cap \Sigma = \emptyset.$$

Introduction - Eigenvalues

What about other eigenvalues? $\longrightarrow$ Many constructions

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$$\mathcal{C} := \{C \subset W_0^{1,p}(\Omega) \colon C \text{ is compact}, C = -C\}$$
$$\gamma(C) := \inf\{k \in \mathbb{N} \colon \exists \phi \colon C \to \mathbb{R}^k \setminus \{0\} \text{ odd and continuous}\}$$
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[Garcia Azorero-Peral, 1988] $\longrightarrow \{\lambda_k\}_{k \in \mathbb{N}} \subset \Sigma$

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In dimension > 1 is known:

- Σ is closed.
- $\bar{\lambda}_2 := \text{mín}\{\lambda \in \Sigma \colon \lambda > \lambda_1\}$ then

$$\bar{\lambda}_2 = \lambda_2 \to \text{[Anane]–[Cuesta-De Figueiredo-Gossez, 1999]}$$

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We denote

$$\Sigma_{var} := \{\lambda_k\}_{k\in\mathbb{N}}.$$

Variational Eigenvalues

What can be said about the variational eigenvalues?

- dimension 1 $\longrightarrow$ [Drabek-Manasevich, 1999]

$$\Omega = (0, L), \quad \lambda_k = (p-1)\Big(\frac{\pi_p k}{L}\Big)^p, \quad u_k(x) = \sin_p(\pi_p \lambda_k x/L),$$

where π_p and $\sin_p$ are suitable generalizations of π and sin respectively.

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- dimension > 1 $\longrightarrow$ [Garcia Azorero-Peral, 1988]–[Friedlander, 1989]

  $$\lambda_k \sim \Big(\frac{k}{|\Omega|}\Big)^{p/N}$$

Systems

We consider the following elliptic systems

$$\begin{cases} -\Delta_p u = \lambda f(x,u,v) \\ -\Delta_q v = \lambda g(x,u,v) \end{cases}$$

In $\Omega \subset \mathbb{R}^N$ smooth and bounded, with homogeneous Dirichlet boundary conditions.

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We want to recover *homogeneity*.

Since Δ_p is $(p-1)$-homogeneous and Δ_q is $(q-1)$-homogeneous is natural tu assume that the system must have the following homogeneity:

$$(u,v) \text{ is a solution } \leftrightarrow (t^{1/p}u, t^{1/q}v) \text{ is a solution } \forall t > 0.$$

Systems

Moreover, we want our system to be *variational*. That is

$$f(x,u,v) = \frac{\partial F}{\partial u}(x,u,v) \quad \text{and} \quad g(x,u,v) = \frac{\partial F}{\partial v}(x,u,v)$$

so that solutions to the system are critical points of

$$\Phi(u,v) := \int_\Omega \frac{|\nabla u|^p}{p} + \frac{|\nabla v|^q}{q}\,dx - \int_\Omega \lambda F(x,u,v)\,dx.$$

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So the potential function F must be

$$F(x,u,v) = a(x)|u|^p + b(x)|v|^q + c(x)|u|^\alpha |v|^\beta$$

with

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1$$

Systems

For simplicity

$$a(x) = b(x) = 0, \qquad c(x) = 1$$

So the system reads

$$\begin{cases} -\Delta_p u = \lambda\alpha|u|^{\alpha-2}u|v|^{\beta} \\ -\Delta_q v = \lambda|u|^{\alpha}\beta|v|^{\beta-2}v \end{cases}$$

In $\Omega \subset \mathbb{R}^N$ smooth and bounded, with homogeneous Dirichlet boundary conditions.

Results for Systems

[Boccardo - de Figueiredo, 2002],
[Fleckinger et al., 1997],
[Manasevich - Mawhin, 2000],
[Allegretto - Huang, 1996],
[Stavrakakis - Zographopoulos, 2003],
[De Napoli - Mariani, 2002],
[De Napoli - Pinasco, 2006],
[FP - Pinasco, 2008],
... (many others)

- Existence of a principal eigenvalue.
  The first eigenvalue is given by

$$\lambda_1 := \inf_{(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)} \frac{\int_\Omega \frac{1}{p}|\nabla u|^p + \frac{1}{q}|\nabla v|^q \, dx}{\int_\Omega |u|^\alpha |v|^\beta \, dx}$$

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- Positivity of associated eigenfunction.
- Isolation of principal eigenvalue.

- Existence of a sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$
  Eigenvalues are defined as

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What about the asymptotic behavior of these eigenvalues?

Asymptotic behavior

Previous results

[De Napoli - Pinasco, 2006] $\longrightarrow$ dimension 1

$$\lambda_k \leq \frac{\Lambda_{p,k}}{p}\left[1 + \left(\frac{p}{q}\right)^{q+1} \Lambda_{p,k}^{(q-p)/p}\right], \quad (q \leq p)$$

where $\Lambda_{p,k}$ is the k^{th} (variational) eigenvalue of the $p-$laplacian with Dirichlet BC.

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[FB - Pinasco, 2008] $\longrightarrow$ dimension ≥ 1
Consider the *spectral counting function*

$$N(\lambda) := \#\{k \in \mathbb{N} \colon \lambda_k \leq \lambda\}$$

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$$ck^b \leq \lambda_k \leq Ck^a \iff (C^{-1}\lambda)^{1/a} \leq N(\lambda) \leq (c^{-1}\lambda)^{1/b}$$

Asymptotic behavior

Previous results

[FB - Pinasco, 2008] obtained

$$c\lambda^{n/p} \leq N(\lambda) \leq C_1\lambda^{n/q} + C_2\lambda^{n/p}, \quad (q \leq p)$$

(In dimension $n = 1$ one can get sharper constants)

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Remark: The coupling parameters α and β are not reflected in these bounds.

Objective of our work

Improve the existing bounds looking for

- obtain the same exponent in both the upper and the lower bound for $N(\lambda)$.
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Theorem

There exists $c, C > 0$ depending on p, q, α, β and Ω such that

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Equivalently,

$$ck^{(\alpha+\beta)/n} \leq \lambda_k \leq Ck^{(\alpha+\beta)/n}.$$

Remarks and Applications

- Observe that when $p = q$, we have that $\alpha + \beta = q$.
- This is the case for semilinear systems ($p = q = 2$).
- In these cases the coupling plays no role in the asymptotic behaviour of the $N(\lambda)$ and we believe that is why this phenomenum was not observe before.

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Applications:

- Bifurcation problems
- anti-maximum principles
- existence / non-existence results

[Azizieh-Clément, 2002] – [Drabek et al., 2003],
[Stavrakakis-Zographopoulos, 1999], etc.

Let Q_t be the cube of side t centered at the origin.

Lemma

Let λ_1^t be the first eigenvalue of the system in Q_t, i.e.

$$\lambda_1^t = \inf_{(u,v)\in W_0^{1,p}(Q_t)\times W_0^{1,q}(Q_t)} \frac{\int_{Q_t} \frac{1}{p}|\nabla u|^p + \frac{1}{q}|\nabla v|^q \, dx}{\int_{Q_t} |u|^\alpha |v|^\beta \, dx}$$

Then

$$\lambda_1^t = \frac{\lambda_1^1}{t^{\alpha+\beta}}.$$

Proof of the Theorem

Lemma 1

Proof. The proof is just a scaling argument.
If $(u, v) \in W_0^{1,p}(Q_1) \times W_0^{1,q}(Q_1)$, then

$$u_t(x) = \frac{1}{t} u(tx), \qquad v_t(x) = \frac{1}{t} v(tx)$$

verify that $(u_t, v_t) \in W_0^{1,p}(Q_t) \times W_0^{1,q}(Q_t)$

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Now changing variables, we get that

$$\frac{\int_{Q_t} \frac{1}{p} |\nabla u_t|^p + \frac{1}{q} |\nabla v_t|^q \, dx}{\int_{Q_t} |u_t|^\alpha |v_t|^\beta \, dx} = \frac{1}{t^{\alpha+\beta}} \frac{\int_{Q_1} \frac{1}{p} |\nabla u|^p + \frac{1}{q} |\nabla v|^q \, dx}{\int_{Q_1} |u|^\alpha |v|^\beta \, dx}$$

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From where it follows

$$\lambda_1^t = \frac{1}{t^{\alpha+\beta}} \lambda_1^1.$$

□

Analogously, we have

Lemma

Let μ_2^t be the first non-zero eigenvalue of the system with homogeneous Neumann boundary condition, i.e.

$$\mu_2^t = \inf_{C \in \mathcal{C}_2} \sup_{(u,v)\in C} \frac{\int_{Q_t} \frac{1}{p}|\nabla u|^p + \frac{1}{q}|\nabla v|^q \, dx}{\int_{Q_t} |u|^\alpha |v|^\beta \, dx}$$

Then

$$\mu_2^t = \frac{\mu_2^1}{t^{\alpha+\beta}}$$

Remark

In this case, since is a Neumann problem, the spaces $W_0^{1,p}$ and $W_0^{1,q}$ are replaced with $W^{1,p}$ and $W^{1,q}$ respectively

The following is a generalization of the Dirichlet-Neumann bracketing of Courant made in [FB-Pinasco, 2003]

Theorem (Dirichlet-Neumann bracketing)

Let $U_1, U_2 \subset \Omega$ *be open disjoint sets such that* $(\overline{U_1 \cup U_2})^\circ = \Omega$ *and* $|\Omega \setminus (U_1 \cup U_2)| = 0$. *Then,*

$$\begin{aligned} N^D(\lambda, U_1) + N^D(\lambda, U_2) &= N^D(\lambda, U_1 \cup U_2) \\ &\leq N^D(\lambda, \Omega) \leq N^N(\lambda, \Omega) \\ &\leq N^N(\lambda, U_1 \cup U_2) = N^N(\lambda, U_1) + N^N(\lambda, U_2) \end{aligned}$$

Proof of the Theorem

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- Lower bound for $N(\lambda)$ (i.e. upper bound for λ_k)

Take t such that $\lambda_1^t = \lambda$, i.e.

$$t = \left(\frac{\lambda}{\lambda_1^1}\right)^{-\frac{1}{\alpha+\beta}} \qquad \text{(Lemma 1)}$$

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Hence, $N^D(\lambda, Q_t) = 1$.

Proof of the Theorem

By the Dirichlet-Neumann bracketing

$$N^D(\lambda, \Omega) \geq \sum_{i=1}^{K} N^D(\lambda, Q_t^i) = K$$

where K is the number of cubes of the lattice contained in Ω.

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$$t^n K \to |\Omega| \qquad \text{as } t \to 0+$$

Then, as $\lambda \to \infty$

$$N^D(\lambda, \Omega) \geq K \sim \frac{|\Omega|}{t^n} = |\Omega| \Big(\frac{\lambda}{\lambda_1^1} \Big)^{\frac{n}{\alpha+\beta}}$$

Proof of the Theorem

- Upper bound for $N(\lambda)$ (i.e. lower bound for λ_k)

Now take t such that $\mu_2^t = \lambda$, i.e.

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$$N^D(\lambda,\Omega) \leq N^N(\lambda,\Omega) \leq \sum_{i=1}^{K} N^N(\lambda, Q_t^i) = 2K$$

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$$t = \left(\frac{\lambda}{\mu_2^1}\right)^{-\frac{1}{\alpha+\beta}} \qquad \text{(Lemma 2)}$$

Hence, $N^N(\lambda, Q_t) = 2$.Again, by the Dirichlet-Neumann bracketing

$$N^D(\lambda, \Omega) \leq N^N(\lambda, \Omega) \leq \sum_{i=1}^{K} N^N(\lambda, Q_t^i) = 2K$$

and as $t^n K \to |\Omega|$ as $t \to 0+$

$$N^D(\lambda, \Omega) \leq 2\frac{|\Omega|}{t^n} = 2|\Omega|\left(\frac{\lambda}{\mu_2^1}\right)^{\frac{n}{\alpha+\beta}}$$

Improvements

We wuould like to have more explicit constants for the above inequalities.

This require an explicit upper bound for λ_1^1 and an explicit lower bound for μ_2^1.

Using the results of [Drabek-Manasevich, 1999] on the one dimensional problem it is posible to find an explicit upper bound for λ_1^1.

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Using the results of [Drabek-Manasevich, 1999] on the one dimensional problem it is posible to find an explicit upper bound for λ_1^1.
In order to do this, we introduce the pseudo $p-$laplace operator

$$\hat{\Delta}_p u = \sum_{i=i}^{n} (|u_{x_i}|^{p-2} u_{x_i})_{x_i}$$

Improvements

Consider the eigenvalue problem associated to $\hat{\Delta}_p$

$$\nu_1 = \inf_{(u,v)\in W_0^{1,p}(Q_1)\times W_0^{1,q}(Q_1)} \frac{\int_{Q_1} \frac{1}{p}|\nabla u|_p^p + \frac{1}{q}|\nabla v|_q^q \, dx}{\int_{Q_1} |u|^\alpha |v|^\beta \, dx}$$

where $|x|_p^p = \sum |x_i|^p$.

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Advantadge. The first eigenfunction of

$$-\hat{\Delta}_s u = \rho |u|^{s-2} u \qquad \text{in } Q_1$$

with Dirichlet BC can be computed explicitly by separation of variables!

Improvements

Let $\phi_s(x) = \sin_s(\pi_s x)$ be the first eigenfunction of the one dimensional $s-$laplacian, in the interval $(0,1)$ then

$$w_s(x) = \prod_{i=1}^{n} \phi_s(x_i)$$

is the first eigenfunction of the n dimensional pseudo $s-$laplacian in the cube Q_1.

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Now, we use (w_s, w_s), with $s = \alpha + \beta$ as a test function for ν_1 to get

$$\nu_1 \leq \frac{\frac{n}{p}\left(\int_0^1 |\phi_s(t)|^p\,dt\right)^{n-1} \int_0^1 |\phi_s'(t)|^p\,dt}{\left(\int_0^1 |\phi_s(t)|^s\,dt\right)^n} + \frac{\frac{n}{q}\left(\int_0^1 |\phi_s(t)|^q\,dt\right)^{n-1} \int_0^1 |\phi_s'(t)|^q\,dt}{\left(\int_0^1 |\phi_s(t)|^s\,dt\right)^n}$$

Improvements

Now, we make use of the Pytagorian-like identity

$$|\sin_s(t)|^s + |\sin_s'(t)|^s = 1$$

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So

$$\begin{aligned}\nu_1 &\leq \frac{n}{p}\pi_s^{p-s}(s-1)\pi_s^s + \frac{\frac{n}{q}\left(\int_0^1 |\phi_s(t)|^q\,dt\right)^{n-1}\int_0^1 |\phi_s'(t)|^q\,dt}{\left(\int_0^1 |\phi_s(t)|^s\,dt\right)^n} \\ &= \frac{n(s-1)}{p}\pi_s^p + \frac{\frac{n}{q}\left(\int_0^1 |\phi_s(t)|^q\,dt\right)^{n-1}\int_0^1 |\phi_s'(t)|^q\,dt}{\left(\int_0^1 |\phi_s(t)|^s\,dt\right)^n}\end{aligned}$$

Improvements

To bound the other term, we use Hölder's inequality to get

$$\left(\int_0^1 |\phi_s(t)|^q \, dt\right)^{n-1} \leq \left(\int_0^1 |\phi_s(t)|^s \, dt\right)^{(n-1)q/s}$$

$$\begin{aligned}\int_0^1 |\phi_s'(t)|^q \, dt &\leq \left(\int_0^1 |\phi_s'(t)|^s \, dt\right)^{q/s} \\ &= \left((s-1)\pi_s^s \int_0^1 |\phi_s(t)|^s \, dt\right)^{q/s}\end{aligned}$$

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So

$$\nu_1 \leq \frac{n(s-1)}{p}\pi_s^p + (s-1)^{q/s}\pi_s^q \left(\int_0^1 |\phi_s(t)|^s \, dt\right)^{-n(1-q/s)}$$

which is an explicit bound for ν_1.

Thanks for your attention