Estimates for eigenvalues of quasilinear elliptic systems.

Julián Fernández Bonder[1] and Juan Pablo Pinasco [2]

[1]Universidad de Buenos Aires

[2]Universidad Nacional de General Sarmiento.

http://mate.dm.uba.ar/~jfbonder

Second Workshop on Elliptic and Parabolic PDE's
September 2007, PUC – Chile

The problem

In this work we analyze the problem

$$\begin{cases} -\Delta_p u = \lambda r(x)\alpha|u|^{\alpha-2}u|v|^{\beta} \\ -\Delta_q v = \lambda r(x)\beta|u|^{\alpha}|v|^{\beta-2}v \end{cases}$$

In $\Omega \subset \mathbb{R}^N$ smooth and bounded, with homogeneous Dirichlet boundary conditions.

The problem

In this work we analyze the problem

$$\begin{cases} -\Delta_p u = \lambda r(x)\alpha|u|^{\alpha-2}u|v|^{\beta} \\ -\Delta_q v = \lambda r(x)\beta|u|^{\alpha}|v|^{\beta-2}v \end{cases}$$

In $\Omega \subset \mathbb{R}^N$ smooth and bounded, with homogeneous Dirichlet boundary conditions.

$r \in L^\infty$ and bounded away from 0.

$\lambda \in \mathbb{R}$ is the eigenvalue parameter.

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1.$$

History of the problem

[Boccardo - de Figueiredo, NoDEA 2002],
[Fleckinger et al., Adv. Diff. Eq. 1997],
[Manasevich - Mawhin, Adv. Diff. Eq. 2000],
[Allegretto - Huang, Nonlinear Anal. 1996],
.... (many others)

History of the problem

[Boccardo - de Figueiredo, NoDEA 2002],
[Fleckinger et al., Adv. Diff. Eq. 1997],
[Manasevich - Mawhin, Adv. Diff. Eq. 2000],
[Allegretto - Huang, Nonlinear Anal. 1996],
.... (many others)

Known results:

- Existence of a principal eigenvalue.
- Simplicity of the principal eigenvalue.
- Positivity of associated eigenfunction.
- Isolation of principal eigenvalue.

History of the problem
(cont.)

- Existence of a sequence of eigenvalues $\{\lambda_k\} \longrightarrow$ [De Napoli - Mariani, AAA 2002]
- Existence of generalized eigenvalues $\longrightarrow$ [De Napoli - Pinasco, JDE 2006].

History of the problem
(cont.)

- Existence of a sequence of eigenvalues $\{\lambda_k\}$ $\longrightarrow$ [De Napoli - Mariani, AAA 2002]
- Existence of generalized eigenvalues $\longrightarrow$ [De Napoli - Pinasco, JDE 2006].
- Upper bounds for eigenvalues$\longrightarrow$ [De Napoli - Pinasco, JDE 2006].

$$\lambda_k \leq \frac{\Lambda_{p,k}}{p}\left[1 + \left(\frac{p}{q}\right)^{q+1}[(\inf r)\Lambda_{p,k}]^{(q-p)/p}\right]$$

where $\Lambda_{p,k}$ is the k^{th} (variational) eigenvalue of the $p-$laplacian with Dirichlet BC.

History of the problem
(cont.)

Moreover, in 1D, using the asymptotic bound

$$\Lambda_{p,k} \sim \left(\frac{\pi_p}{\int_\Omega r^{1/p}} \right)^p k^p$$

One can obtain

$$\lambda_k \le \left(\frac{\pi_p}{\int_\Omega r^{1/p}} \right)^p \frac{k^p}{p}.$$

for large enough k.

Objective of the work

Our objective is

Find *explicit* and *asymptotic* lower bounds for the k^{th} (variational) eigenvalue of the system in $\mathbb{R}^N$.

Objective of the work

Our objective is

Find *explicit* and *asymptotic* lower bounds for the k^{th} (variational) eigenvalue of the system in $\mathbb{R}^N$.

Applications:

- Bifurcation problems
- anti-maximum principles
- existence / non-existence results

([Azizieh - Clément, JDE 2002], [Drabek et al., Diff. Int. Eq. 2003], [Stavrakakis - Zographopoulos, EJDE 1999], etc.)

The Spectral Counting Function

We introduce the *Spectral Counting Function* as

$$N(\lambda) = \#\{k \colon \lambda_k \leq \lambda\}$$

The Spectral Counting Function

We introduce the *Spectral Counting Function* as

$$N(\lambda) = \#\{k \colon \lambda_k \leq \lambda\}$$

Lower and upper bounds on eigenvalues can be translated in lower and upper bounds on $N(\lambda)$. For instance:

$$ck^b \leq \lambda_k \leq Ck^a \quad \rightleftarrows \quad (C^{-1}\lambda)^{1/a} \leq N(\lambda) \leq (c^{-1}\lambda)^{1/b}$$

The 1D case

Theorem

Let $\Omega = (0,1)$ *and* $N(\lambda)$ *be the Spectral Counting Function. Then, as* $\lambda \to \infty$,

1. *If* $q < p$,

$$c_1 \lambda^{1/p} \leq N(\lambda) \leq C_1 \lambda^{1/q} + C_2 \lambda^{1/p}.$$

2. *If* $q = p$,

$$c_1 \lambda^{1/p} \leq N(\lambda) \leq (C_1 + C_2) \lambda^{1/p}.$$

3. *If* $q = p$ *and* $\alpha = \beta$,

$$N(\lambda) \sim c_2 \lambda^{1/p}.$$

The 1D case
(cont.)

Remarks:

- In the 1D case, the variational eigenvalues exhaust the hole spectrum (see, for instance, [JFB - Pinasco, Ark. Math. 2003])

The 1D case
(cont.)

Remarks:

- In the 1D case, the variational eigenvalues exhaust the hole spectrum (see, for instance, [JFB - Pinasco, Ark. Math. 2003])
- The constants c_1, c_2, C_1 and C_2 are given explicitly in terms of p, q, α, β and the weight $r(x)$.

The 1D case
(cont.)

Remarks:

- In the 1D case, the variational eigenvalues exhaust the hole spectrum (see, for instance, [JFB - Pinasco, Ark. Math. 2003])
- The constants c_1, c_2, C_1 and C_2 are given explicitly in terms of p, q, α, β and the weight $r(x)$.
- A more precise lower bound can be given. In fact, let $S_p = \{\Lambda_{p,k}/\alpha\}$, $S_q = \{\Lambda_{q,k}/\beta\}$ and

$$S = S_p \cup S_q = \{\mu_k\}$$

Then $\mu_k \leq \lambda_k$.

The N-dimensional case

Theorem

Let $\Omega \subset \mathbb{R}^N$ *open and bdd. Then, as* $\lambda \to \infty$,

1. *If* $q \leq p$,

$$\bar{c}_1 \lambda^{N/p} \leq N(\lambda) \leq \bar{C}_1 \lambda^{N/q} + \bar{C}_2 \lambda^{N/p}.$$

2. *If* $q = p$,

$$\bar{c}_1 \lambda^{N/p} \leq N(\lambda) \leq (\bar{C}_1 + \bar{C}_2) \lambda^{N/p}.$$

The N-dimensional case
(cont.)

Remarks:

- Again, the constants $\bar{c}_1, \bar{c}_2, \bar{C}_1$ and C_2 can be given explicitly in terms of p, q, α, β and $r(x)$.
- In the N-dimensional case, it os not known (even for a single equation) that the variational eigenvalues exhaust the hole spectrum.
- The analogous item 3. of the previous Theorem (i.e. $p = q$ and $\alpha = \beta$) we can only prove it for the linear system $p = q = 2$ and $\alpha = \beta = 1$, that correspond to the eigenvalues of the bi-laplacian with Navier BC.

$$\begin{cases} \Delta^2 u = \lambda u & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma

Let $\Lambda_{p,k}$ be the k^{th} eigenvalue of the $p-$laplacian. Then there exists c_p, C_p such that

$$c_p k^{p/N} \leq \Lambda_{p,k} \leq C_p k^{p/N}.$$

Lemma

Let $\Lambda_{p,k}$ be the k^{th} eigenvalue of the $p-$laplacian. Then there exists c_p, C_p such that

$$c_p k^{p/N} \leq \Lambda_{p,k} \leq C_p k^{p/N}.$$

Proof: (case $r(x) \equiv 1$)
Let $Q_1 \subset \Omega \subset Q_2$ be two cubes. Then,

$$\Lambda_{p,k}(Q_1) \leq \Lambda_{p,k}(\Omega) \leq \Lambda_{p,k}(Q_2)$$

So we need to bound the eigenvalues of a cube.

Auxiliary Results

Estimation of $\Lambda_{p,k}$ cont.

We now define $\nu_{p,k}(Q)$ the eigenvalues of the pseudo $p-$laplacian in the cube Q,

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i} u|^{p-2}\partial_{x_i} u\right) = \nu|u|^{p-2}u & \text{on } Q \\ u = 0 & \text{on } \partial Q \end{cases}$$

Auxiliary Results

Estimation of $\Lambda_{p,k}$ cont.

We now define $\nu_{p,k}(Q)$ the eigenvalues of the pseudo $p-$laplacian in the cube Q,

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i} u|^{p-2}\partial_{x_i} u\right) = \nu|u|^{p-2}u & \text{on } Q \\ u = 0 & \text{on } \partial Q \end{cases}$$

and observe that this eigenvalues $\nu_{p,k}$ can be computed by separation of variables.
In fact

$$u_{p,1}(x) = \sin_p(\pi_p x_1/L)\cdots\sin_p(\pi_p x_N/L), \qquad \nu_{p,1} = \frac{\pi_p^p N}{L^p}$$

L being the length of Q

Auxiliary Results

Estimation of $\Lambda_{p,k}$ cont.

Now, the result follows from comparison of the Rayleigh quotients, since

$$\nu_{p,k} = \min\max \frac{\||\nabla u|_p\|_p^p}{\|u\|_p^p}$$

$$\Lambda_{p,k} = \min\max \frac{\||\nabla u|_2\|_p^p}{\|u\|_p^p}$$

and the norms in $\mathbb{R}^N$, $|\cdot|_p$ and $|\cdot|_2$ being equivalent. $\square$

Auxiliary Results

Related problem

Lemma

Let $S_p = \{\Lambda_{p,k}/\alpha\}$, $S_q = \{\Lambda_{q,k}/\beta\}$ *and* $S = S_p \cup S_q = \{\mu_k\}$.
Then, S consists exactly of the variational eigenvalues of the (uncoupled) system

$$\begin{cases} -\Delta_p u = \lambda r(x)|u|^{p-2}u & \text{on } \Omega \\ -\Delta_q v = \lambda r(x)|v|^{q-2}v & \text{on } \Omega \end{cases}$$

with Dirichlet BC.

Auxiliary Results

Related problem

Lemma

Let $S_p = \{\Lambda_{p,k}/\alpha\}$, $S_q = \{\Lambda_{q,k}/\beta\}$ *and* $S = S_p \cup S_q = \{\mu_k\}$. *Then, S consists exactly of the variational eigenvalues of the (uncoupled) system*

$$\begin{cases} -\Delta_p u = \lambda r(x)|u|^{p-2}u & \text{on } \Omega \\ -\Delta_q v = \lambda r(x)|v|^{q-2}v & \text{on } \Omega \end{cases}$$

with Dirichlet BC.

Proof.

Easy. □

Proof of the main results

Upper bound

We will show the upper bound for $N(\lambda)$ which corresponds to the lower bound for λ_k. Also we consider the case $r(x) \equiv 1$.

Proof of the main results

Upper bound

We will show the upper bound for $N(\lambda)$ which corresponds to the lower bound for λ_k. Also we consider the case $r(x) \equiv 1$.
First, by Young's inequality,

$$\int_\Omega |u|^\alpha |v|^\beta \, dx \leq \frac{\alpha}{p} \int_\Omega |u|^p \, dx + \frac{\beta}{q} \int_\Omega |v|^q \, dx$$

Proof of the main results

Upper bound

We will show the upper bound for $N(\lambda)$ which corresponds to the lower bound for λ_k. Also we consider the case $r(x) \equiv 1$.
First, by Young's inequality,

$$\int_\Omega |u|^\alpha |v|^\beta \, dx \le \frac{\alpha}{p} \int_\Omega |u|^p \, dx + \frac{\beta}{q} \int_\Omega |v|^q \, dx$$

Therefore

$$\frac{\frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{q}\int_\Omega |\nabla v|^q \, dx}{\int_\Omega |u|^\alpha |v|^\beta \, dx} \ge \frac{\frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{q}\int_\Omega |\nabla v|^q \, dx}{\frac{\alpha}{p}\int_\Omega |u|^p \, dx + \frac{\beta}{q}\int_\Omega |v|^q \, dx}$$

We will show the upper bound for $N(\lambda)$ which corresponds to the lower bound for λ_k. Also we consider the case $r(x) \equiv 1$.
First, by Young's inequality,

$$\int_\Omega |u|^\alpha |v|^\beta \, dx \leq \frac{\alpha}{p} \int_\Omega |u|^p \, dx + \frac{\beta}{q} \int_\Omega |v|^q \, dx$$

Therefore

$$\frac{\frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{q} \int_\Omega |\nabla v|^q \, dx}{\int_\Omega |u|^\alpha |v|^\beta \, dx} \geq \frac{\frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{q} \int_\Omega |\nabla v|^q \, dx}{\frac{\alpha}{p} \int_\Omega |u|^p \, dx + \frac{\beta}{q} \int_\Omega |v|^q \, dx}$$

So $\lambda_k \geq \mu_k$

Proof of the main results

Upper bound (cont.)

Then

$$N(\lambda) = \#\{k \colon \lambda_k \le \lambda\} \le \#\{k \colon \mu_k \le \lambda\}$$

Proof of the main results

Upper bound (cont.)

Then

$$N(\lambda) = \#\{k\colon \lambda_k \le \lambda\} \le \#\{k\colon \mu_k \le \lambda\}$$

$$= \#\{k\colon \Lambda_{p,k}/\alpha \le \lambda\} + \#\{k\colon \Lambda_{q,k}/\beta \le \lambda\}$$

Proof of the main results

Upper bound (cont.)

Then

$$N(\lambda) = \#\{k \colon \lambda_k \leq \lambda\} \leq \#\{k \colon \mu_k \leq \lambda\}$$

$$= \#\{k \colon \Lambda_{p,k}/\alpha \leq \lambda\} + \#\{k \colon \Lambda_{q,k}/\beta \leq \lambda\}$$

$$= N_p(\alpha\lambda) + N_q(\beta\lambda)$$

Proof of the main results

Upper bound (cont.)

Then

$$N(\lambda) = \#\{k\colon \lambda_k \leq \lambda\} \leq \#\{k\colon \mu_k \leq \lambda\}$$

$$= \#\{k\colon \Lambda_{p,k}/\alpha \leq \lambda\} + \#\{k\colon \Lambda_{q,k}/\beta \leq \lambda\}$$

$$= N_p(\alpha\lambda) + N_q(\beta\lambda)$$

On the other hand,

$$\Lambda_{p,k} \geq c_p k^{p/N}$$

Proof of the main results

Upper bound (cont.)

Then

$$N(\lambda) = \#\{k\colon \lambda_k \le \lambda\} \le \#\{k\colon \mu_k \le \lambda\}$$

$$= \#\{k\colon \Lambda_{p,k}/\alpha \le \lambda\} + \#\{k\colon \Lambda_{q,k}/\beta \le \lambda\}$$

$$= N_p(\alpha\lambda) + N_q(\beta\lambda)$$

On the other hand,

$$\Lambda_{p,k} \ge c_p k^{p/N}$$

Therefore

$$N_p(\alpha\lambda) \le \left(\frac{\alpha\lambda}{c_p}\right)^{N/p} = \bar{c}_p \lambda^{N/p}.$$

Proof of the main results

Upper bound (cont.)

Then

$$N(\lambda) = \#\{k\colon \lambda_k \le \lambda\} \le \#\{k\colon \mu_k \le \lambda\}$$

$$= \#\{k\colon \Lambda_{p,k}/\alpha \le \lambda\} + \#\{k\colon \Lambda_{q,k}/\beta \le \lambda\}$$

$$= N_p(\alpha\lambda) + N_q(\beta\lambda)$$

On the other hand,

$$\Lambda_{p,k} \ge c_p k^{p/N}$$

Therefore

$$N_p(\alpha\lambda) \le \left(\frac{\alpha\lambda}{c_p}\right)^{N/p} = \bar{c}_p \lambda^{N/p}.$$

So

$$N(\lambda) \le \bar{c}_p \lambda^{N/p} + \bar{c}_q \lambda^{N/q}.$$

Proof of the main results

Lower bound

Now we show how to obtain the lower bound for $N(\lambda)$, which gives upper bounds for the eigenvalues λ_k.

Proof of the main results

Lower bound

Now we show how to obtain the lower bound for $N(\lambda)$, which gives upper bounds for the eigenvalues λ_k.
First, we cover Ω by a union of non-overlapping cubes Q_i with sides of length L

$$\Omega \subset \bigcup_{i=1}^{J} Q_i \qquad \ell(Q_i) = L.$$

Proof of the main results

Lower bound

Now we show how to obtain the lower bound for $N(\lambda)$, which gives upper bounds for the eigenvalues λ_k.
First, we cover Ω by a union of non-overlapping cubes Q_i with sides of length L

$$\Omega \subset \bigcup_{i=1}^{J} Q_i \qquad \ell(Q_i) = L.$$

Now, it is easy to see that

$$N(\lambda) \geq \sum_{i=1}^{J} N(\lambda, Q_i)$$

Proof of the main results

Lower bound (cont.)

Then we need to estimate $N(\lambda, Q)$ for any cube Q with $\ell(Q) = L$

Proof of the main results

Lower bound (cont.)

Then we need to estimate $N(\lambda, Q)$ for any cube Q with $\ell(Q) = L$
We recall the following result:

$$\lambda_1 \leq \frac{\Lambda_{p,1}}{p}\left[1 + \left(\frac{p}{q}\right)^{q+1} \Lambda_{p,1}^{(p-q)/p}\right]$$

[de Napoli - Pinasco, JDE 2006]
But now, we observe that

$$\Lambda_{p,1}(Q) \leq \nu_{p,1} = \frac{\pi_p^p N}{L^p}.$$

Proof of the main results

Lower bound (cont.)

Then we need to estimate $N(\lambda, Q)$ for any cube Q with $\ell(Q) = L$
We recall the following result:

$$\lambda_1 \leq \frac{\Lambda_{p,1}}{p}\left[1 + \left(\frac{p}{q}\right)^{q+1} \Lambda_{p,1}^{(p-q)/p}\right]$$

[de Napoli - Pinasco, JDE 2006]
But now, we observe that

$$\Lambda_{p,1}(Q) \leq \nu_{p,1} = \frac{\pi_p^p N}{L^p}.$$

Combining these, we can choose $L = L(\lambda)$ such that

$$N(\lambda, Q) = 1.$$

In fact $L = \pi_p (N/\lambda)^{1/p}$.

Proof of the main results

Lower bound (cont.)

So

$$N(\lambda) \geq \sum_{i=1}^{J} N(\lambda, Q_i) = J$$

Proof of the main results

Lower bound (cont.)

So

$$N(\lambda) \geq \sum_{i=1}^{J} N(\lambda, Q_i) = J$$

Finally, it is easy to see that, for $\lambda \to \infty$,

$$J \sim c\lambda^{N/p}.$$

Proof of the main results

Lower bound (cont.)

So

$$N(\lambda) \geq \sum_{i=1}^{J} N(\lambda, Q_i) = J$$

Finally, it is easy to see that, for $\lambda \to \infty$,

$$J \sim c\lambda^{N/p}.$$

That is,

$$N(\lambda) \geq c\lambda^{N/p}.$$