Some optimization problems for nonlinear elastic membranes.

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The problem

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$$\mathcal{J}(f) := \int_{\partial\Omega} f(x) u \, dS \longrightarrow \max$$

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$$\begin{cases} -\Delta_p u + |u|^{p-2} u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = f & \text{on } \partial\Omega. \end{cases}$$

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Observe that u is given by

$$\int_{\partial\Omega} fu \, dS = \sup_{v \in W^{1,p}(\Omega)} \frac{1}{p-1} \Big\{ p \int_{\partial\Omega} fv \, dS - \int_{\Omega} |\nabla v|^p + |v|^p \, dx \Big\}.$$

History of the problem

- Related problem $p=2$ $\rightarrow$ [Cherkaev - Cherkaeva, Adv. Math. Appl. Sci. 1999],
- Dirichlet BC and particular class $\mathcal{A}$ $\rightarrow$ [Cuccu, Emamizadeh and Porru, EJDE 2006],

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Known results:

- Existence of optimal load f_*.
- Euler-Lagrange type equation for f_*.

Applications

- Quasi-regular and quasi-conformal mappings in Riemannian manifolds with boundary
- Non-Newtonian fluids.
- Reaction–diffusion problems.
- Flow through porus media.
- Nonlinear elasticity.
- Glaciology.

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Remark

Our results are new even in the linear case $p = 2$.

Admissible classes

- The class of rearrangements of a given function f_0.

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- For the first case $\rightarrow$ [Cuccu, Emamizadeh and Porru, EJDE 2006].
- The second case $\rightarrow$ EASY!
- The last case $\rightarrow$ more interesting...

The class of rearrangements

Definition

We say that f *is a rearrangement of* f_0 *if*

$$\mathcal{H}^{N-1}(f \geq t) = \mathcal{H}^{N-1}(f_0 \geq t) \qquad t \geq 0.$$

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Lemma (Burton - Mc Leod, Proc. Roy. Soc. Edin. 1991)

Let $f_0 \in L^q_+(\partial\Omega)$ *and* $v \in L^{q'}_+(\partial\Omega)$. *Then there exists* $\hat{f} \in \mathcal{A}$ *such that*

$$\int_{\partial\Omega} \hat{f} v \, dS = \sup_{f \in \mathcal{A}} \int_{\partial\Omega} fv \, dS.$$

The class of rearrangements

Theorem
Let $q > \frac{p'}{N'}$. *There exists* $\hat{f} \in \mathcal{A}$ *such that*

$$\mathcal{J}(\hat{f}) = \int_{\partial\Omega} \hat{f}\,\hat{u}\,dS = \sup_{f\in\mathcal{A}} \mathcal{J}(f),$$

where $\hat{u}$ *is the solution to*

$$\begin{cases} -\Delta_p \hat{u} + |\hat{u}|^{p-2}\hat{u} = 0 & \text{in } \Omega, \\ |\nabla \hat{u}|^{p-2}\frac{\partial \hat{u}}{\partial \nu} = \hat{f} & \text{on } \partial\Omega. \end{cases}$$

The class of rearrangements

Proof.

Let

$$I = \sup_{f \in \mathcal{A}} \int_{\partial\Omega} f u_f \, dS,$$

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First, it is easy to see that I is finite. Then we take a maximizing sequence $\{f_i\}_{i \geq 1}$ and let $u_i = u_{f_i}$.

Now, u_i is bounded in $W^{1,p}$, f_i is bounded in $L^q(\partial\Omega)$ and so

$$\begin{array}{lcll}
u_i & \rightharpoonup & u & \text{weakly in } W^{1,p}(\Omega), \\
u_i & \rightarrow & u & \text{strongly in } L^p(\Omega), \\
u_i & \rightarrow & u & \text{strongly in } L^r(\partial\Omega), \ r < p_* = \dfrac{(N-1)p}{N-p}, \\
f_i & \rightharpoonup & f & \text{weakly in } L^q(\partial\Omega).
\end{array}$$

The class of rearrangements

Proof.(cont.)

Then

$$\begin{aligned} I &= \lim_{i\to\infty} \int_{\partial\Omega} f_i u_i \, dS \\ &= \frac{1}{p-1} \lim_{i\to\infty} \left\{ p \int_{\partial\Omega} f_i u_i \, dS - \int_{\Omega} |\nabla u_i|^p + |u_i|^p \, dx \right\} \\ &\leq \frac{1}{p-1} \left\{ p \int_{\partial\Omega} fu \, dS - \int_{\Omega} |\nabla u|^p + |u|^p \, dx \right\}. \end{aligned}$$

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Proof.(cont.)

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&\leq \frac{1}{p-1} \left\{ p \int_{\partial\Omega} f u \, dS - \int_{\Omega} |\nabla u|^p + |u|^p \, dx \right\}.
\end{aligned}
$$

Now, by the Lemma of Burton – Mc Leod,

$$
\int_{\partial\Omega} f u \, dS \leq \int_{\partial\Omega} \hat{f} u \, dS \qquad \text{with } \hat{f} \in \mathcal{A}.
$$

The class of rearrangements

Proof.(the end)

Finally, denoting $\hat{u} := u_{\hat{f}}$, we get

$$\begin{aligned} I &\leq \frac{1}{p-1}\left\{p\int_{\partial\Omega} fu\,dS - \int_\Omega |\nabla u|^p + |u|^p\,dx\right\} \\ &\leq \frac{1}{p-1}\left\{p\int_{\partial\Omega} \hat{f}\,u\,dS - \int_\Omega |\nabla u|^p + |u|^p\,dx\right\} \\ &\leq \frac{1}{p-1}\left\{p\int_{\partial\Omega} \hat{f}\,\hat{u}\,dS - \int_\Omega |\nabla \hat{u}|^p + |\hat{u}|^p\,dx\right\} \\ &= \int_{\partial\Omega} \hat{f}\,\hat{u}\,dS \\ &\leq I. \end{aligned}$$

This completes the proof. $\square$

The unit ball of L^q

In this section, we consider the class

$$\mathcal{A} := \{f \in L^q(\partial\Omega) \colon \|f\|_{L^q(\partial\Omega)} = 1\}.$$

Again, in order for $\mathcal{J}$ to make sense, we need $q > \frac{p'}{N'}$.

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This is the easiest case. The optimal load is given in terms of the extremal of the Sobolev trace constant

$$S_{q'} = \min_{v \in W^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p + |v|^p \, dx}{\left(\int_{\partial\Omega} |v|^{q'} \, dS \right)^{p/q'}}$$

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Observe that the condition

$$q > \frac{p'}{N'} \leftrightarrow q' < p_* = \frac{(N-1)p}{N-p}$$

The unit ball of L^q

More precisely,

$$\hat{f} = v_{q'}^{q'-1}, \qquad \hat{u} = u_{\hat{f}} = \frac{1}{S_{q'}^{1/p-1}} v_{q'},$$

where $v_{q'}$ is a nonnegative, normalized extremal.

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where $v_{q'}$ is a nonnegative, normalized extremal.
In the linear setting, $p = q = 2$, v_2 is the first Steklov eigenfunction

$$\begin{cases} -\Delta v + v = 0 & \text{in } \Omega \\ \frac{\partial v}{\partial \nu} = \lambda v & \text{on } \partial\Omega. \end{cases}$$

Characteristic functions

We consider the class

$$\mathcal{A} = \{\chi_D \colon D \subset \partial\Omega,\ \mathcal{H}^{N-1}(D) = A\}$$

with $0 < A < \mathcal{H}^{N-1}(\partial\Omega)$ fixed.

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For technical reasons, it is better to work in the *relaxed* class

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$$\mathcal{B} = \{\phi \in L^\infty(\partial\Omega) \colon 0 \le \phi(x) \le 1,\ \int_{\partial\Omega} \phi\, dS = A\}.$$

It is well known that $\mathcal{B}$ is the closure of $\mathcal{A}$ with respect to the weak* convergence in L^∞.

Characteristic functions

Lemma

There exists $\hat{\phi} \in \mathcal{B}$ *such that*

$$\mathcal{J}(\hat{\phi}) = \sup_{\phi \in \mathcal{B}} \mathcal{J}(\phi).$$

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We like now to show that $\hat{\phi}$ belongs to the class $\mathcal{A}$, i.e.

$$\hat{\phi} = \chi_{\hat{D}}, \quad \mathcal{H}^{N-1}(\hat{D}) = A.$$

Characteristic functions

We need this result

Lemma (Bathtub principle)

For any $u \in L^p(\partial\Omega)$, *the maximization problem*

$$I = \sup_{\phi \in \mathcal{B}} \int_{\partial\Omega} u(x)\phi(x)\, dS$$

is solved by

$$\hat{\phi}(x) = \chi_{\hat{D}}(x), \quad \{u > s\} \subset \hat{D} \subset \{u \geq s\}$$

where

$$s = \inf\{t : \mathcal{H}^{N-1}(\{u \geq t\}) \leq A\}.$$

The maximizer given is unique ($\mathcal{H}^{N-1}$*–a.e.) if* $\mathcal{H}^{N-1}(\{u = s\}) = 0$.

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$$\begin{aligned}
\sup_{\mathcal{A}} \mathcal{J}(\phi) &\leq \sup_{\mathcal{B}} \mathcal{J}(\phi) = \mathcal{J}(\hat{\phi}) \\
&= \frac{1}{p-1}\left\{p \int_{\partial\Omega} \hat{\phi} u_{\hat{\phi}}\, dS - \int_{\Omega} |\nabla u_{\hat{\phi}}|^p + |u_{\hat{\phi}}|^p\, dx\right\} \\
&\leq \frac{1}{p-1}\left\{p \int_{\partial\Omega} \chi_{\hat{D}} u_{\hat{\phi}}\, dS - \int_{\Omega} |\nabla u_{\hat{\phi}}|^p + |u_{\hat{\phi}}|^p\, dx\right\} \\
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&\leq \sup_{\mathcal{A}} \mathcal{J}(\phi).
\end{aligned}
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This completes the proof of existence. □

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First step: Shape derivative approach.

Shape Derivative

Assume that we fix a set $D \subset \partial\Omega$ that is an open subset with regular boundary (i.e. its boundary relative to $\partial\Omega$ is a smooth $N-2$ dimensional surface).

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We perturb D as follows:

- Take $V : \mathbb{R}^N \to \mathbb{R}^N$ smooth, globally Lipschitz, with support in a neighborhood of $\partial\Omega$ such that $\langle V(x), \nu(x)\rangle = 0, \ \forall x \in \partial\Omega$.

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$$\frac{d}{dt}\psi_t(x) = V(\psi_t(x)), \quad \psi_0(x) = x.$$

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- Define the flow

$$\frac{d}{dt}\psi_t(x) = V(\psi_t(x)), \quad \psi_0(x) = x.$$

- Define the perturbed domain D_t along the field V as

$$D_t = \psi_t(D)$$

(Observe that $\psi_t(\Omega) = \Omega$ and $\psi_t(\partial\Omega) = \partial\Omega$)

Shape Derivative

We denote by u_t the state of the system associated to the set D_t, i.e.

$$\begin{cases} -\Delta_p u_t + |u_t|^{p-2} u_t = 0 & \text{in } \Omega \\ |\nabla u_t|^{p-2} \frac{\partial u_t}{\partial \nu} = \chi_{D_t} & \text{on } \partial\Omega \end{cases}$$

and define

$$j(t) = \mathcal{J}(\chi_{D_t}) = \int_{\partial\Omega} u_t \chi_{D_t} \, dS$$

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We want to compute

$$j'(0) = \frac{\partial \mathcal{J}}{\partial V}(D)$$

Shape Derivative

Theorem

$j(t)$ is differentiable at $t = 0$ and we have

$$j'(0) = \frac{p}{p-1} \int_{\partial D} u_D \langle V, \nu_\tau \rangle \, d\mathcal{H}^{N-2}$$

where ν_τ stands for the exterior unit normal vector to D along $\partial\Omega$.

Shape Derivative

Proof.

The proof follows from lengthy computations and adequate use of

- The *tangential* Changes of Variables Formula.

$$\int_{\Phi(\Gamma)} f \, dS = \int_{\Gamma} (f \circ \Phi) J_\tau(\Phi) \, dS$$

($J_\tau(\Phi)$ being the *tangential* Jacobian of Φ)

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- The *tangential* Divergence Theorem.

$$\int_D \mathrm{div}_\tau W \, dS = \int_{\partial D} \langle W, \nu_\tau \rangle \, d\mathcal{H}^{N-2} + \int_D H \langle W, \nu \rangle \, dS$$

  ($\mathrm{div}_\tau(W)$ being the *tangential* divergence of W, H is the mean curvature of D)

Shape Derivative

Proof. (cont.)

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To overcome this problem, we work with the solutions to a regularized problem and then pass to the limit

$$\begin{cases} -\mathrm{div}((|\nabla u_t^\varepsilon|^2+\varepsilon^2)^{(p-2)/2}\nabla u_t^\varepsilon)+|u_t^\varepsilon|^{p-2}u_t^\varepsilon=0 & \text{in } \Omega \\ (|\nabla u_t^\varepsilon|^2+\varepsilon^2)^{(p-2)/2}\frac{\partial u_t^\varepsilon}{\partial \nu}=\chi_{D_t} & \text{on } \partial\Omega. \end{cases}$$

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Standard Elliptic Regularity Theory gives $u_t^\varepsilon \in C^{2,\delta}$ and, moreover

$$u_t^\varepsilon \to u_t \quad \text{in } C^1 \ (\varepsilon \to 0)$$

□

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By (much easier) similar computations we also get

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$$\frac{d}{dt}\mathcal{H}^{N-1}(D_t)\mid_{t=0} = \int_D \operatorname{div} V \, dS.$$

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Observe that, as V is tangential to ∂D, we can get

$$\int_D \operatorname{div} V\, dS = \int_D \operatorname{div}_\tau V\, dS = \int_{\partial D} \langle V, \nu_\tau\rangle\, d\mathcal{H}^{N-2}.$$

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Combining these formulas with the Lagrange Multipliers Rule, we finally obtain...

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Let D be a maximizing set and assume that it has smooth $N-2$ dimensional boundary. Then u_D is constant along ∂D.

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Proof.

By our assumptions, we have that D maximizes $j(t)$ along the constrain $\mathcal{H}^{N-1}(D_t) = A$.

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Proof.

By our assumptions, we have that D maximizes $j(t)$ along the constrain $\mathcal{H}^{N-1}(D_t) = A$.
So, by the Lagrange Multipliers Rule, there exists a constant c such that

$$j'(0) = c\frac{d}{dt}\mathcal{H}^{N-1}(D_t)\mid_{t=0}$$

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This is

$$\frac{p}{p-1}\int_{\partial D} u_D \langle V, \nu_\tau\rangle \, d\mathcal{H}^{N-2} = c\int_{\partial D} \langle V, \nu_\tau\rangle \, d\mathcal{H}^{N-2}$$

for every tangential vector field V. So the result follows. $\square$

Applications and open problems

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THANK YOU