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Book of Abstracts

Lectures

Continuity of the temperature in boundary heat control problems

Ioannis Athanasopoulos

In a joint work with L.A. Caffarelli, motivated by the boundary heat control problems formulated in the book of Duvaut and Lions, we study a boundary Stefan problem and a boundary porous media problem. We prove continuity of the solution with the appropriate modulus. We also extend the results to the fractional order case and to the anomalous diffusion problems.

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Semi-linear equations and principal eigenvalues in unbounded domains

Henri Berestycki

Semi-linear elliptic equations of the KPP type describe for instance stationary solutions in populations ecology. This talk is about existence as well as uniqueness results for such equations in unbounded domains and for general heterogeneous settings. These results allow one to describe the long time dynamics of solutions of the associated evolution equations. They hinge on conditions involving the principal eigenvalue of a linearized operator. I will discuss several extensions of this notion for general elliptic operator to the framework of unbounded domains. Properties of these generalized eigenvalues will be presented along the way. I report here on joint works with Luca Rossi as well as with Franois Hamel and Grgoire Nadin.

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Front propagation and phase transitions for fractional diffusion equations

Xavier Cabré

Long-range or "anomalous" diffusions, such as diffusions given by the fractional powers $(-\Delta)^\alpha$ of the Laplacian, attract lately great interest in Physics, Biology, and Finance. They appear in diffusions in plasma, dislocations in crystals, in finance (American options modelled with jump processes), in geophysical fluid dynamics (the quasi-geostrophic equation), in certain reaction fronts and flames, and in population dynamics.
The fractional powers of the Laplacian are the infinitesimal generators of the symmetric stable Lévy diffusion processes. These —also called Lévy flights— are diffusion processes that combine Brownian motion together with a jump process. From the mathematical point of view, nonlinear analysis for fractional diffusions has been mostly developed in the last years.
In this talk, I will mainly describe recent results concerning front propagation for the nonlinear fractional heat equation, as well as phase transitions for the fractional elliptic Allen-Cahn equation.
In collaboration with J.-M. Roquejoffre, we study the propagation of fronts for the fractional KPP equation

$$\partial_t u + (-\Delta)^\alpha u = u(1-u) \quad \text{in } (0,\infty)\times\mathbb{R}^n, \qquad 0 \le u \le 1,$$

with $\alpha \in (0,1)$. In [3, 4], by heuristic considerations it was predicted that fronts should propagate at exponential speed —in contrast with the classical case $\alpha = 1$ for which there is propagation at a constant KPP speed. In particular, no traveling wave should exist when $\alpha < 1$. We establish mathematically these results. For instance, given an initial condition with compact support in $\mathbb{R}^n$, we prove that every level set of u is located at time t, up to an error, near $\{|x| = \exp(\mu^* t)\}$, where $\mu^* = f'(0)/(n+2\alpha)$ and $f(u)$ is equal to $u(1-u)$ or to another concave monostable nonlinearity. Such exponential speed originates from the fact that the fundamental solution of the fractional heat equation has a power decaying tail at infinity —instead of the exponential tail of the Gaussian corresponding to $\alpha = 1$.
In another work in collaboration with Y. Sire, we are concerned with the equation

$$(-\Delta)^\alpha u = f(u) \qquad \text{in } \mathbb{R}^n$$

with $\alpha \in (0,1)$. The case $\alpha = 1/2$ was studied in [1]. Crucial to our analysis for $\alpha \in (0,1)$ is a result of [2] which allows to realize this nonlocal equation as a degenerate elliptic equation posed in $\mathbb{R}^{n+1}_+$ together with a nonlinear Neumann boundary condition on $\mathbb{R}^n = \partial\mathbb{R}^{n+1}_+$. We characterize the nonlinearities f for which there exists a "layer" solution —meaning, essentially, a solution increasing in one direction. We establish several properties of these solutions, such as their uniqueness in $\mathbb{R}$, minimality, symmetry in certain dimensions, and decay at infinity. In collabortation with E. Cinti, we find sharp energy estimates for these and other solutions (such as "saddle-shaped" solutions). These estimates allow to improve the 1D symmetry results of De Giorgi type for the nonlocal equation.
Finally, we will describe results in collaboration with Jinggang Tan on the problem

$$\begin{cases} (-\Delta_{\mathrm{Dir}})^{1/2} u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where $(-\Delta_{\mathrm{Dir}})^{1/2}$ stands for the unique positive square root of the Laplacian in a bounded domain $\Omega \subset \mathbb{R}^n$ with zero Dirichlet boundary conditions. Using also a local realization of the problem in the cylinder $\Omega \times [0,\infty)$, we establish existence and regularity results of positive solutions as well as a priori estimates of Gidas-Spruck type for subcritical powers, Liouville type theorems in a half space, and a symmetry result of Gidas-Ni-Nirenberg type.

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Almost global solutions of Eikonal equations and distance functions

Michael Crandall

Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is continuous, differentiable at each point of a dense open subset $\mathbb{R}^n \setminus \mathcal{S}$ of $\mathbb{R}^n$ and satisfies

$$\|Du(x)\| = 1 \quad \text{for} \quad x \in \mathbb{R}^n \setminus \mathcal{S}.$$

at x. Here $\|\cdot\|$ is a norm on $\mathbb{R}^n$. The closed set $\mathcal{S}$ consists of potential singularities of u where $\|Du(x)\| = 1$ might fail to hold; for example, u might not be differentiable at a point of $\mathcal{S}$.
Suppose $\|\cdot\|$ is the Euclidean norm. Then for each unit vector p and $a \in \mathbb{R}$,
$u(x) = a + \langle x, p\rangle$ satisfies $\|Du\| = \|p\| = 1$ everywhere. If $z \in \mathbb{R}^n$ and $a \in \mathbb{R}$, then

$$u(x) = a \pm \|x - z\|$$

satisfies our conditions with $\mathcal{S} = \{z\}$. We call these functions "cone functions". If u is the distance to a closed bounded line segment L, then it satisfies our assumptions with $\mathcal{S} = L$. The question arises if there are any cases in which the set of genuine singularities is larger than a singleton and smaller than a segment. The answer, in the Euclidean case, is "no"; if the Hausdroff 1-measure of $\mathcal{S}$ is 0, then u is either affine or a cone function.
With suitable cone functions, this result is true in greater generality, and, in particular, it holds for all the p-norms

$$\|x\| = \|x\|_p = \left(|x_1|^p + \cdots + |x_n|^p\right)^{1/p}, 1 < p < \infty.$$

The proof hinges on establishing the "distance formula"

$$u(x) = r + \mathrm{dist}(x, L_r) \quad \text{if} \quad u(x) \geq r,$$
$$u(x) = r - \mathrm{dist}(x, L_r) \quad \text{if} \quad u(x) \leq r,$$

where $L_r = \{x : u(x) = r\}$ is a nonempty level set of u and $\mathrm{dist}(x, L_r)$ is the distance from x to L_r measured in the norm $\|\cdot\|_*$ dual to $\|\cdot\|$. To establish the distance formula requires a study of certain properties of a mapping defined at intermediate points of "rays" of the distance function. This seems to be a new chapter in Minkowski geometry, and finding suitable results for the cases $p \neq 2$ of the p-norms was demanding. One may well regard our proofs of the distance formula in various cases as the main contributions of the work, and statements such as in the first paragraph as applications of these.
This is a joint work with **Luis Caffarelli**, who has not approved this message. :)

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Poincaré and Korn type inequalities and decomposition of functions

Ricardo Durán

Theoretical and numerical analysis of the differential equations arising in mechanics are based on several classic inequalities (Poincaré, Korn, etc.).
In this talk we analyze the relation between these inequalities and prove that all of them are valid in a bounded domain Ω if a so-called improved Poincaré inequality holds in Ω. Our arguments are based on Whitney decompositions and a related decomposition for functions with vanishing average.
Moreover, we show that this basic improved Poincaré inequality can be proved by elementary arguments for the so-called John domains, which is a very general class containing in particular the Lipschitz domains.
The results presented are joint work with M. A. Muschietti, E. Russ, and P. Tchamitchian.

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Critical non-minimal points in periodic media

Rafael de la Llave

We consider models of materials with a periodic structure described by an energy with local interaction and a double well. For example, the Allen-Cahn models in metallurgy. We show that, for every orientation there is minimizing solution whose interface remains at a bounded distance from a plane. Furthermore, if there is not a continuous family of such solutions, there are critical points of the energy which are not minimizers (metastable states). This alternative happens if and only iff a number vanishes. In the one dimensional case, this number is called the Peierls-Nabarro barrier. This is joint work with E. Valdinoci.

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Breaking of symmetry in some nonlinear variational problems

Enrique Lami Dozo

Minimizers in a special Sobolev trace inequality are solutions of a nonlinear elliptic problem. In the case of a ball B_ρ in $\mathbb{R}^n$, we study this problem in perspective of a breaking of symmetry which appears when the radius ρ and the Lebesgue exponent in $L^q(\partial B_\rho)$ vary. A representation formula gives some new insight to this subject.

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Parabolic method to eigenvalue problems of non-local equations

Ki-Ahm Lee

In this talk, we are going to consider parabolic approach for the eigenvalue problems of non-local equations. The eigenvalue problems for nonlocal equations has been approximated by degenerate nonlocal equations of porous medium type or fast diffusion type. Long time behavior of non-local degenerate equation will be discussed.

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Classical convolution inequalities and Boltzmann equations for integrable angular section

Irene Martínez-Gamba

We study the integrability properties of the of the Boltzmann collision operator using radial symmetrization techniques from harmonic analysis to show Young's inequality in the case of hard potentials and Hardy-Littlewood-Sobolev inequality for soft potentials. The constants are given by exact formulas depending on the angular cross section. By applying these estimates, we can revisit and obtain new results for existence, uniqueness and L^p stability to the corresponding space inhomogeneous equations with special initial data, which are, in addition, classical solutions.
This is work partly in collaboration with E. Carneiro and R. Alonso

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A near field refraction problem in geometric optics

Cristian Gutierrez

Let Ω be a domain in the sphere S^{n-1} and let $D \subset \mathbb{R}^n$ be a domain contained in an $n-1$ dimensional surface called the target domain or screen to be illuminated. Let n_1 and n_2 be the indexes of refraction of two homogeneous and isotropic media I and II, respectively, for example, glass and air, and suppose that from a point O surrounded by medium I, light emanates with intensity $f(x)$ for $x \in \Omega$, and D is surrounded by media II. We prove the existence of an optical surface $\mathcal{R}$ parameterized by $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, interface between media I and II, such that all rays refracted by $\mathcal{R}$ into medium II illuminate the object D, and the prescribed illumination intensity received at each point $P \in D$ is $g(P)$. This yields the existence of a lens refracting light in a prescribed way.
This is joint work with Qingbo Huang.

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Lagrangian formulations to solve fluid-structure interactions with free-surfaces problems

Sergio Idelshon

A general formulation for treating elastic solids and fully incompressible fluids in a unified form is presented. The formulation allows treating solid and fluid sub-domains indistinctly in fluid-structure interaction (FSI) situations including free surface flows.
The resulting differential equations are solved via the Particle Finite Element Method (PFEM). The PFEM is an effective technique for modeling complex interactions between floating and submerged bodies including free surface flows, accounting for splashing of waves, large motions of the bodies and frictional contact conditions.
The main goal is to solve the equations for both the fluid and solid domains using the same lagrangian formulation. This basically means that the analysis domain, containing both: incompressible fluid and elastic solid sub-domains which interact with each other, is seen as a single continuum domain with different material properties assigned to each of the interacting sub domains (i.e. the fluid and solid regions). This allows making no distinction between fluids and solids for the numerical solution and a single computer code can be used for solving the coupled problem. The pressure unknown for the incompressible domain is treated in a segregated way, allowing solving a Laplace type equation separately from the unified domain.
Nevertheless, the most important conclusion of this lecture is to stand out the topology of the matrices involved in the coupled problem of FSI. The formulation shows clearly the need to include an interface laplacian type matrix on the boundary between the incompressible flow and elastic boundary. In several cases this matrix is essential to obtain a convergent procedure for the pressure segregated problem.
The concluding remark concerning the interface laplacian type matrix may be also applied to FSI problem solved in a staggered way

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A gradient bound for free boundary graphs

David Jerison

We prove the analogue for a free boundary problem of the classical gradient bound due to Bombieri, De Giorgi, and Giusti for minimal graphs in terms of the oscillation of the graph. As in the classical case, this leads to full regularity of free boundaries. The analogy with the minimal surface result reveals a connection between the Harnack inequality on area minimizing surfaces and the boundary Harnack inequality in non-tangentially accessible domains. This is joint work with Daniela De Silva.

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The global behavior of solutions to critical nonlinear dispersive and wave equations

Carlos Kenig

In this lecture we will describe a method (which I call the concentration-compactness/rigidity theorem method) which Frank Merle and I have developed to study global well-posedness and scattering for critical non-linear dispersive and wave equations. Such problems are natural extensions of non-linear elliptic problems which were studied earlier, for instance in the context of the Yamabe problem and of harmonic maps. We will ilustrate the method with some concrete examples and also mention other applications of these ideas.

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Remarks on fully nonlinear elliptic partial differential equations

Louis Nirenberg

Some properties of singular solutions of fully nonlinear elliptic equations are presented. Symmetry and monotonicity results for them will be described.

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AFEM for geometric biomembranes and fluid-membrane interaction

Ricardo Nochetto

We study two models for biomembranes. The first one is purely geometric since the equilibrium shapes are the minimizers of the Willmore energy under area and volume constraints. We present a novel method based on ideas from shape differential calculus. The second model incorporates the effect of the inside (bulk) viscous incompressible fluid and leads to more physical dynamics.
We use a parametric approach, which gives rise to fourth order highly nonlinear PDEs on surfaces and involves large domain deformations. We discretize these PDEs in space with an adaptive finite element method (AFEM), with either piecewise linear or quadratic polynomials, and a semi-implicit time stepping scheme. We employ the Taylor-Hood element for the Navier-Stokes equations together with iso-parametric elements, the latter being crucial for the correct approximation of curvature. We discuss several computational tools such as space-time adaptivity and mesh smoothing.
We also discuss a method to execute refinement, coarsening, and smoothing of meshes on manifolds with incomplete information about their geometry and yet preserve position and curvature accuracy. This is a new paradigm in adaptivity.
This work is joint with Andrea Bonito and M. Sebastian Pauletti.

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Non-local minimal surfaces

Jean-Michel Roquejoffre

In the de Giorgi theory for minimal surfaces, one studies the boundary of a set whose indicator function minimises, locally, the BV norm. We switch here from the BV norm to the H^α ($0 < \alpha < \frac{1}{2}$) and investigate what the new minimising sets (that we call α-minimal) look like.
Similarly to minimal surfaces having - at least at regular points - zero mean curvature, an α-minimal surface forces an integral curvature operator to vanish. We will discuss in this talk a de Giorgi-type result: if a piece of α-minimal surface is flat enough, then it is smooth.

Joint work with L. Caffarelli and O. Savin.

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Regularity of the solution in two phase parabolic problems

Sandro Salsa

We present recent results on the regularity for two phase free boundary problems for parabolic operators and discuss some open questions.

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The obstacle type problems, an overview and some new directions

Henrik Shahgholian

I will give some overview of the obstacle type problems, referring to

$$\Delta u = f(u)$$

with $f(t)$ having a discontinuity at $t = 0$.
Recent developments concerning both two phase cases, as well as unstable case will be discussed.

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Results in nonlocal elliptic equations

Luis Silvestre

We will discuss some recent progress in the study of fully nonlinear problems involving integro-differential equations. Integro-differential equations appear in many contexts and share many qualities with elliptic PDE. They arise very naturally from the study of jump stochastic processes. The optimal stopping time problem with a jump process corresponds to the obstacle problem for an integro-differential equation. General problems in stochastic control correspond general Bellman integro-differential equations. We obtain regularity results for nonlocal equations that can be seen as generalizations of regularity results for fully nonlinear elliptic PDE, like the Krylov-Safonov theorem and the Evans-Krylov theorem.

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Rates of convergence for monotone approximations of viscosity solutions

Panagiotis Souganidis

In this lecture I review the general theory of convergence of monotone, stable and consistent approxination schemes to visocity solutions. I also discuss recent results about rates of convergence for such schemes as well as homogenization.

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Porous medium flow with nonlocal diffusion effects

Juan Luis Vázquez

We study a model for flow in porous media including nonlocal (long-range) diffusion effects. It is based on Darcy's law and the pressure is related to the density by an inverse fractional Laplacian operator. We prove existence of solutions that propagate with finite speed.
The model has the very interesting property that mass preserving selfsimilar solutions can be found by solving an elliptic obstacle problem with fractional Laplacian for the pair pressure-density. We use entropy methods to show that the asymptotic behaviour is described after renormalization by these solutions which play the role of the Barenblatt profiles of the standard porous medium model.
Joint work with Luis Caffarelli

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A free boundary problem for the $p(x)$–laplacian

Noemi Wolanski

Let $\mathcal{K};=\{v\in W^{1,p(x)}(\Omega)\,,\,v=\phi \text{ on } \partial\Omega\}$ where Ω is a smooth bounded domain, $0\le\phi\in W^{1,p(x)}(\Omega)\cap L^\infty(\Omega)$.
$W^{1,p(x)}(\Omega)$ is the set of functions in $L^{p(x)}(\Omega)$ with first derivatives in $L^{p(x)}(\Omega)$: the set of measurable functions f in Ω such that $\int_\Omega |f(x)|^{p(x)}\,dx<\infty$.
We study the following minimization problem; Find $u\in\mathcal{K}$ that minimizes the functional

$$J(v)=\int_\Omega \frac{|\nabla v(x)|^{p(x)}}{p(x)}+\lambda(x)\chi_{\{v>0\}}\,dx \quad \text{for } v\in\mathcal{K}.$$

This problem may be of interest in the study of electrorheological fluids. These are fluids such that their properties are strongly influenced by the presence of an electromagnetic field.
We assume throughout this work that there exist $1<p_1\le p_2<\infty$ such that $p_1\le p(x)\le p_2$ in Ω, p log-Holder continuous, $0<\lambda_1\le\lambda(x)\le\lambda_2<\infty$, and prove that minimizers do exist and that they are nonnegative and bounded.
Then, we prove that any minimizer u is locally Holder continuous and deduce that

$$\Delta_{p(x)}u;=\operatorname{div}\big(|\nabla u(x)|^{p(x)-2}\nabla u(x)\big)=0 \quad \text{in } \{u>0\}.$$

If, moreover $p\in C^{0,1}_{loc}(\Omega)$ and $p_1\ge 2$, we prove that u is locally Lipschitz continuous.
Still under the assumption that $p\in C^{0,1}_{loc}(\Omega)$, with $p_1>1$ we prove that, if $\lambda\in C(\Omega)$ then, any locally Lipschitz minimizer is a weak solution of the following free boundary problem;

$$\begin{cases}\Delta_{p(x)}u=0 & \text{in } \{u>0\}\\ u=0, \quad |\nabla u(x)|=\lambda^*(x):=\Big(\frac{p(x)}{p(x)-1}\lambda(x)\Big)^{1/p(x)} & \text{on } \partial\{u>0\}\end{cases}$$

in the sense that, for every $\varphi\in C_0^\infty(\Omega)$,

$$-\int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla\varphi(x)\,dx=\int_{\partial_{red}\{u>0\}}(\lambda^*(x))^{p(x)-1}\varphi\,d\mathcal{H}^{N-1}.$$

Moreover, $\mathcal{H}^{N-1}\big(\partial\{u>0\}\setminus\partial_{red}\{u>0\}\big)=0$.

Finally, when λ is Holder continuous, $\partial_{red}\{u>0\}\in C^{1,\alpha}$.

This is a joint work with J. F. Bonder and S. Martinez from the University of Buenos Aires.

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