Un Problema de Frontera Libre en Teoría de Combustión

Resumen

En esta Tesis consideramos el siguiente problema de perturbación singular que se presenta en teoría de combustión

$$\begin{aligned} \Delta u^\varepsilon - u^\varepsilon_t &= Y^\varepsilon f_\varepsilon(u^\varepsilon) \quad \text{en } \mathcal{D}, \\ \Delta Y^\varepsilon - Y^\varepsilon_t &= Y^\varepsilon f_\varepsilon(u^\varepsilon) \quad \text{en } \mathcal{D}, \end{aligned}$$

donde $\mathcal{D} \subset \mathbb{R}^{N+1}$, $f_\varepsilon(s) = \frac{1}{\varepsilon^2} f(\frac{s}{\varepsilon})$ con f una función Lipschitz soportada en $(-\infty, 1]$.

En este sistema Y^ε es la fracción de masa de algún reactante, u^ε la temperatura rescalada de la mezcla y ε es esencialmente el inverso de la energía de activación. Este modelo es derivado en el contexto de la teoría de llamas premezcladas equidifusionales para número de Lewis 1.

Probamos que, bajo hipótesis adecuadas sobre las funciones u^ε e Y^ε, podemos pasar al límite ($\varepsilon \to 0$) – llamado *límite de alta energía de activación* – y que la función límite $u = \lim u^\varepsilon = \lim Y^\varepsilon$ es una solución del siguiente problema de frontera libre

$$(P) \qquad \begin{aligned} &\Delta u - u_t = 0 && \text{en } \{u > 0\}, \\ &|\nabla u| = \sqrt{2M(x,t)} && \text{en } \partial\{u > 0\}, \end{aligned}$$

en un sentido puntual en los puntos regulares de la frontera libre y en el sentido de la viscosidad. En (P), $M(x,t) = \int_{-w_0(x,t)}^1 (s + w_0(x,t)) f(s) ds$ y $-1 < w_0 = \lim_{\varepsilon \to 0} \frac{Y^\varepsilon - u^\varepsilon}{\varepsilon}$.

Como $Y^\varepsilon - u^\varepsilon$ es una solución de la ecuación del calor, queda completamente determinada por sus datos iniciales y de contorno. En particular, la condición de frontera libre depende fuertemente de las aproximaciones de esos datos.

También probamos que, bajo condiciones más débiles sobre los datos, la función límite u (que llamaremos *solución límite*) es una supersolución clásica del problema de frontera libre. Más aún, si $\mathcal{D} \cap \partial\{u > 0\}$ es una superficie Lipschitz, u resulta una solución clsica de (P).

Finalmente probamos, bajo hipótesis geométricas adecuadas sobre los datos, la unicidad de solución límite para el problema (P).

Palabras clave: Sistemas parabólicos, reacción-difusión, combustión, estimaciones uniformes, problemas de frontera libre, solución viscosa, solución límite, solución clásica.

A Free bounday Problem in Combustion Theory

Abstract

In this work we consider the following problem arising in combustion theory

$$\begin{aligned} \Delta u^\varepsilon - u^\varepsilon_t &= Y^\varepsilon f_\varepsilon(u^\varepsilon) \quad \text{in } \mathcal{D}, \\ \Delta Y^\varepsilon - Y^\varepsilon_t &= Y^\varepsilon f_\varepsilon(u^\varepsilon) \quad \text{in } \mathcal{D}, \end{aligned}$$

where $\mathcal{D} \subset \mathbb{R}^{N+1}$, $f_\varepsilon(s) = \frac{1}{\varepsilon^2} f(\frac{s}{\varepsilon})$ with f a Lipschitz continuous function with support in $(-\infty, 1]$.

Here Y^ε is the mass fraction of some reactant, u^ε the rescaled temperature of the mixture and ε is essentially the inverse of the activation energy. This model is derived in the framework of the theory of equidiffusional premixed flames for Lewis number 1.

We prove that, under suitable assumptions on the functions u^ε and Y^ε, we can pass to the limit ($\varepsilon \to 0$) – the so called *high activation energy limit* – and that the limit function $u = \lim u^\varepsilon = \lim Y^\varepsilon$ is a solution of the following free bounday problem

$$(P) \qquad \begin{aligned} &\Delta u - u_t = 0 && \text{in } \{u > 0\}, \\ &|\nabla u| = \sqrt{2M(x,t)} && \text{on } \partial\{u > 0\}, \end{aligned}$$

in a pointwise sense at regular free bounday points and in a viscosity sense. Here $M(x,t) = \int_{-w_0(x,t)}^{1} (s + w_0(x,t)) f(s) ds$ and $-1 < w_0 = \lim_{\varepsilon\to 0} \frac{v^\varepsilon - u^\varepsilon}{\varepsilon}$.

Since $Y^\varepsilon - u^\varepsilon$ is a solution of the heat equation it is fully determined by its initial-boundary datum. In particular, the free bounday condition only (but strongly) depends on the approximation of the initial-boundary datum.

Also we prove that, under weaker assumptions on the data, the limit function u (that we call *limit solution*) is a classical supersolution of the free bounday problem. Moreover, if $\mathcal{D} \cap \partial\{u > 0\}$ is a Lipschitz surface, u is a classical solution to (P).

Finally we prove, under adequate geometric assumptions on the data, the uniqueness of limit solutions for problem (P).

Keywords: Parabolic systems, reaction-diffusion, combustion, uniform estimates, free bounday problems, viscosity solution, limit solution, classical solution.

Contents

Introducción

1. Descripción del modelo

El trabajo de esta Tesis es una contribución al análisis matemático de un modelo termo-difusivo que aparece en teoría de combustión.

Este modelo aparece en el análisis de la propagación de llamas curvas. Para una reacción elemental de orden uno, del tipo

Reactante → Producto,

el problema general de propagación de llamas se reduce a resolver el sistema:

$$\rho_t - \operatorname{div}(\rho \mathbf{v}) = 0, \tag{0.1.1}$$

$$\rho \mathbf{v}_t + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} - \mu \Delta \mathbf{v} - \lambda \nabla(\nabla \cdot \mathbf{v}) + \nabla p = 0, \tag{0.1.2}$$

$$\rho T_t + \rho(\mathbf{v} \cdot \nabla)T - K\Delta T = \frac{Q}{c_p}\omega, \tag{0.1.3}$$

$$\rho y_t + \rho(\mathbf{v} \cdot \nabla)y - K_1 \Delta y = -m_y \omega, \tag{0.1.4}$$

$$p = \rho R T, \tag{0.1.5}$$

donde las incógnitas son la densidad ρ, la velocidad $\mathbf{v}$, la presión p, la temperatura T y la concentración del reactante y. Las ecuaciones (0.1.1) y (0.1.2) son las ecuaciones de conservación de masa y la de Navier-Stokes; la ecuación (0.1.5) es la ecuación de estado para un gas perfecto; y las ecuaciones (0.1.3) y (0.1.4) son las ecuaciones de la cinética química para la que adoptamos la *ley de Arrhenius*:

$$\omega = \rho_b B(T_b) \frac{y}{m_y} \exp\left(-\frac{E}{RT}\right). \tag{0.1.6}$$

Suponemos que las cantidades μ, λ, c_p, m_y, Q, R, K y K_1 son constantes positivas. Más aún, ρ_b y T_b representan la densidad y la temperatura del gas quemado, y E es la energía de activación.

Este último parámetro juega un rol importante debido a la dependencia exponencial en el término de reacción para la temperatura ω;

Esta dependencia se incrementa cuando la energía de activación se incrementa. Más aún, es la base de los métodos de análisis asintóticos comúnmente usados por los físicos. Es también la base para la identificación de diferentes zonas caracterizadas por la importancia relativa de los términos que aparecen en las ecuaciones. Cuando E tiene a infinito, aparece un *problema de frontera libre* (ver [**8, 37, 38**]).

Con esta generalidad, el problema es demasiado complejo. El modelo termo-difusivo, consiste en una simplificación de este problema por medio de dos suposiciones que son clásicas en teoría de combustión. La primera, es la suposición de número de match pequeño, es decir considerar a la propagación de la llama como un proceso isobárico. La segunda consiste en considerar la densidad de la mezcla constante. Estas dos hipótesis permiten desacoplar el sistema en el conjunto de ecuaciones que modelan el proceso hidrodinámico del gas, y las ecuaciones que contienen el proceso de combustión. Este modelo está físicamente justificado (cf. [**29**]) para altas energías de activación

$$\frac{E}{RT_b} \gg 1,$$

bajo la hipótesis de cuasi-equidifusividad:

$$\frac{E}{RT_b}\frac{T_b - T_c}{T_b}\left(1 - \frac{1}{Le}\right) = O(1), \tag{0.1.7}$$

donde $Le = K/K_1$ es el número de Lewis y T_c es la temperatura del gas frío.

Este modelo se adapta bien a la descripción del fenómeno de combustión donde la dinámica del gas juega un rol secundario en comparación con los efectos difusivos y reactivos. Este es el caso, por ejemplo, en el fenómeno de inestabilidad celular [**29, 33, 34**].

El límite $E \to +\infty$ es, por sí mismo, de poco interés dado que el término de reacción ω dado en (0.1.6) tiende a cero. Para preservar la reacción, es necesario que el término $B(T_b)$ tienda a infinito; i.e. debemos considerar el límite distinguido caracterizado esencialmente por

$$B(T_b) \sim e^{\frac{E}{RT_b}} \tag{0.1.8}$$

Para $T < T_b$ el término de reacción ω tiende a cero exponencialmente; esto es conocido como el límite frío. Para $T > T_b$, (0.1.4) y (0.1.6) implican que – al menos formalmente – $y \to 0$ exponencialmente y de nuevo ω tiende a cero exponencialmente. Luego, el primer paso para hacer que este método funcione, consiste en asumir que la temperatura T_f en el frente de la combustión verifica una estimación de la forma:

$$\frac{E}{RT_b^2}(T_f - T_b) = O(1). \tag{0.1.9}$$

El análisis asintótico del sistema cuando $\frac{E}{RT_b} \to +\infty$, conduce – por lo menos formalmente – a un problema de frontera libre (ver [**20, 35**]).

En esta Tesis, nos enfocamos en el análisis matemático riguroso de este modelo y, más precisamente, en el estudio del análisis asintótico para grandes energías de activación. Consideraremos la mezcla de gas en reposo (i.e. $\mathbf{v} = 0$). Luego de adimensionalizar las ecuaciones, el problema (0.1.1)-(0.1.4) es reducido a resolver el sistema

$$\Delta u - u_t = \omega(u, Y), \tag{0.1.10}$$

$$\frac{1}{Le}\Delta Y - Y_t = \omega(u, Y), \tag{0.1.11}$$

donde $u = \frac{1}{T_f - T_c}(T_f - T)$ es la temperatura rescalada (o menos la temperatura) e Y es la fracción de masa rescalada del reactante. El término $\omega(u, Y)$ posee propiedades precisas que describimos más adelante.

Para una deducción más detallada del modelo, referimos a [**8**].

El modelo termo-difusivo descripto, ha sido estudiado por muchos autores: existencia de ondas estacionarias (por ejemplo [**4, 7, 36**]), soluciones de problemas elípticos (ver [**3, 4, 6**]), el problema parabólico ([**26**]), estabilidad de ondas viajeras ([**5, 31, 32**]), etc.

El análisis asintótico para grandes energías de activación ha sido estudiado para ondas estacionarias por [**7, 18, 28**] entre otros. Para problemas elípticos y parabólicos, ha sido estudiado en el caso $Le = 1$ y $u = Y$ (que es una suposición natural en el caso de ondas viajeras). Citamos los trabajos [**2, 25**] para ondas viajeras y la ecuación elíptica y [**10, 11, 13**] para el problema parabólico.

También queremos hacer mención del trabajo [**27**] donde el sistema (0.1.10)-(0.1.11) es estudiado en el caso $Le \sim 1$ y se obtienen resultados similares a los de [**13**] en dimensiones $N = 1, 2, 3$.

2. Descripción del problema matemático

En esta Tesis consideramos el problema (0.1.10)-(0.1.11) en el caso equidifusional (i.e. $Le = 1$). Haremos las siguientes suposiciones naturales sobre el término no lineal $\omega(u, Y)$: Llamamos ε al inverso de la energía de activación rescalada, $\varepsilon^{-1} = \frac{E}{RT_b^2}(T_b - T_c)$. Entonces, por (0.1.6), $\omega = \omega_\varepsilon$ viene dado por

$$\omega_\varepsilon(u, Y) = Y f_\varepsilon(u).$$

Para evitar la llamada *dificultad del borde frío*, es usual en la literatura fijar f_ε como cero en las zonas donde es exponencialmente pequeña, es decir, asumimos de aquí en más que $f_\varepsilon(s) = 0$ si $s \geq \varepsilon$. Para una discusión más detallada sobre la dificultad del borde frío, ver [**8**].

Debido a (0.1.8), es fácil verificar que las funciones f_ε verifican que

$$\int_0^\varepsilon s f_\varepsilon(s)\, ds \to M_0 > 0, \qquad \varepsilon \to 0.$$

Esta constante M_0 juega un rol esencial en el análisis asintótico del modelo cuando $\varepsilon \to 0$. Una forma usual – y conveniente – de simplificar el análisis, es cambiar las funciones f_ε asumiendo que están dadas en términos de una única función f en la forma

$$f_\varepsilon(s) = \frac{1}{\varepsilon^2} f\left(\frac{s}{\varepsilon}\right),$$

con lo cual, la integral $\int_0^\varepsilon s f_\varepsilon(s)\, ds$ resulta independiente de ε.

Estas funciones f_ε todavía capturan las características esenciales de (0.1.6). Luego, sobre f, asumimos que es una función no negativa, Lipschitz continua, que es positiva en el intervalo $(-\infty, 1)$ y cero en el complemento (i.e., la reacción sólo ocurre cuando $T > T_f - \varepsilon(T_f - T_c)$).

A partir de ahora, haremos explícita la dependencia en ε de la temperatura rescalada u y de la fracción de masa del reactante Y, con lo cual el sistema a considerar será

$$\left\{ \begin{array}{rcll} \Delta u^\varepsilon - u_t^\varepsilon & = & Y^\varepsilon f_\varepsilon(u^\varepsilon) & \text{en } \mathcal{D}, \\ \Delta Y^\varepsilon - Y_t^\varepsilon & = & Y^\varepsilon f_\varepsilon(u^\varepsilon) & \text{en } \mathcal{D}, \end{array} \right. \tag{0.2.1}$$

donde $\mathcal{D} \subset \mathbb{R}^{N+1}$.

El estudio del límite cuando $\varepsilon \to 0$ fue propuesto en la década del 30 por Zeldovich y Frank-Kamenetski [**38**] y ha sido muy discutido en la literatura de combustión.

En el caso $u^\varepsilon = Y^\varepsilon$ el término de reacción $u^\varepsilon f_\varepsilon(u^\varepsilon)$ tiende a una delta de Dirac, $M_0 \delta(u)$ donde $M_0 = \int_0^1 s f(s) ds$. De esta manera, la zona de reacción donde $u^\varepsilon f_\varepsilon(u^\varepsilon)$ actúa se ve reducida a una superficie, el frente de la llama, y aparece el problema de frontera libre. El hecho que $M_0 > 0$ asegura que un proceso de combustión no trivial tiene lugar con lo cual aparece una frontera libre no vacía.

Si bien la convergencia de las formas más relevantes de propagación, i.e. las ondas viajeras, fue ya discutido por Zeldovich y Frank-Kamenetski, y un gran progreso se ha hecho en esa dirección, una investigación matemática rigurosa sobre la convergencia de soluciones generales se encuentra todavía en curso. Berestycki y sus colaboradores han estudiado rigurosamente el problema de la convergencia para ondas viajeras y, más generalmente, el caso elíptico estacionario – cf. [**2**] y sus referencias. Ver también [**25**]. El estudio del límite en el caso general de evolución para la ecuación del calor fue realizado en [**13**] para el caso de una fase (esto es, con $u^\varepsilon \geq 0$) y en [**10, 11**] para el caso de dos fases, donde no se impone ninguna restricción en el signo de u^ε.

En [**13**] los autores muestran que, bajo ciertas hipótesis sobre los datos iniciales y sus aproximaciones, para toda suceción $\varepsilon_n \to 0$ existe una subsucesión ε_{n_k} y una función límite $u = \lim u^{\varepsilon_{n_k}}$ que resuelve el siguiente problema de frontera libre

$$\begin{cases} \Delta u - u_t = 0 & \text{en } \mathcal{D} \cap \{u > 0\}, \\ |\nabla u^+| = \sqrt{2M_0} & \text{en } \mathcal{D} \cap \partial\{u > 0\}, \end{cases} \tag{0.2.2}$$

en un sentido débil integral. Acá $M_0 = \int_0^1 sf(s)ds$.

En [**10, 11**] los autores muestran que la condición de frontera libre para el caso de dos fases (asumiendo que no ocurre ninguna reacción si $u^\varepsilon \leq 0$) es

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 2M_0$$

y que la función límite es una solución del problema de frontera libre en un sentido puntual en los puntos regulares de la frontera libre cuando $\{u = 0\}$ tiene "densidad parabólica" cero y en el sentido de la viscosidad en la ausencia de una fase nula (i.e. cuando $\{u = 0\}^\circ \cap \mathcal{D} = \emptyset$)

Una pregunta natural es: ¿Será cierto que si uno tiene una sucesión de soluciones uniformemente acotadas $(u^\varepsilon, Y^\varepsilon)$ de (0.2.1) con $(Y^\varepsilon - u^\varepsilon) \to 0$ cuando $\varepsilon \to 0$ entonces u^ε (o una subsucesión) converge a una solución del problema de frontera libre (0.2.2)? Es decir, ¿Será el límite asintótico para energía de activación tendiendo a infinito, en el caso que $(Y^\varepsilon - u^\varepsilon) \to 0$ pero $u^\varepsilon \neq Y^\varepsilon$, una solución del mismo problema de frontera libre que en el caso $u^\varepsilon = Y^\varepsilon$?

Observemos que en el caso en consideración, cuando el número de Lewis es 1, la función $w^\varepsilon = Y^\varepsilon - u^\varepsilon$ es una solución de la ecuación del calor. Luego está completamente determinada por sus valores iniciales y de contorno. Más aún, el sistema (0.2.1) puede ser reescrito como una única ecuación para u^ε,

$$(P_\varepsilon) \qquad \Delta u^\varepsilon - u^\varepsilon_t = (u^\varepsilon + w^\varepsilon) f_\varepsilon(u^\varepsilon).$$

En esta Tesis consideramos el caso que $w^\varepsilon/\varepsilon$ converge a una cierta función w_0 (o sea que, en particular, $Y^\varepsilon - u^\varepsilon \to 0$). Más precisamente, asumimos que los datos iniciales Y^ε_0 y u^ε_0 verifican

$$(0.2.3) \qquad \frac{Y^\varepsilon_0(x) - u^\varepsilon_0(x)}{\varepsilon} \to w_0(x) \quad \text{uniformemente en } \mathbb{R}^N,$$

con $w_0 > -1$. Luego, la función $w^\varepsilon(x,t)$ es la solución de la ecuación del calor con dato inicial $Y^\varepsilon_0(x) - u^\varepsilon_0(x)$ y por (0.2.3), satisface que existe el límite

$$(0.2.4) \qquad \lim_{\varepsilon\to 0} \frac{w^\varepsilon(x,t)}{\varepsilon} = w_0(x,t),$$

donde $w_0(x,t)$ es la solución de la ecuación del calor con dato inicial $w_0(x)$.

De esta manera, por lo menos formalmente, el término de reacción todavía converge a una función delta y aparece un problema de frontera libre. *Pero en este trabajo probamos que la condición de frontera libre depende fuertemente de la función límite* w_0*, o sea que es diferente para diferentes aproximaciones de los datos iniciales y de contorno de* u*.*

En efecto, probamos que para cada sucesión $\varepsilon_n \to 0$ existe una subsucesión ε_{n_k} y una función límite $u = \lim u^{\varepsilon_{n_k}}$ que es una solución del siguiente problema de frontera libre

$$(P) \qquad \begin{cases} \Delta u - u_t = 0 & \text{en } \mathcal{D} \cap \{u > 0\}, \\ |\nabla u^+| = \sqrt{2M(x,t)} & \text{en } \mathcal{D} \cap \partial\{u > 0\}, \end{cases}$$

donde $M(x,t) = \int_{-w_0(x,t)}^1 \big(s + w_0(x,t)\big) f(s) ds$.

La presencia de la función w_0 en el límite de integración, garantiza la positividad de la función $M(x,t)$.

En conclusión, *el problema de combustión es muy inestable en el sentido que el límite asintótico para energía de activación tendiendo a infinito depende de perturbaciones de orden* ε *de los datos iniciales y de contorno.*

En esta Tesis probamos que la función límite es una solución "viscosa" de (P), con lo cual, como consecuencia de nuestros resultados y de los resultados de regularidad para soluciones viscosas de (P) en [**17**], deducimos que, cuando la frontera libre de una función límite u

viene dada por $x_1 = g(x', t)$, $x = (x_1, x')$ con g Lipschitz continua, u es una solución clásica.

Queremos remarcar que, debido a nuestra suposición $Y^\varepsilon - u^\varepsilon \to 0$ y dado que $Y^\varepsilon \geq 0$, la función límite u debe ser no negativa, luego el hecho de que u sea una solución viscosa de (P) es novedoso, aún en el caso $u^\varepsilon = Y^\varepsilon$.

En particular, como consecuencia de nuestros resultados vemos que funciones límite u con $u^\varepsilon(x, 0)$ construidas como en [**13**], e $Y^\varepsilon(x, 0)$ pequeñas perturbaciones de $u^\varepsilon(x, 0)$ son soluciones viscosas de (P). En esta construcción, w_0 es cualquier constante tal que $w_0 \geq -\eta$ donde $\eta > 0$ es suficientemente pequeño.

Finalmente, estudiamos la unicidad del límite $u = \lim u^{\varepsilon_{n_k}}$ de (P), puesto que es una pregunta natural averiguar si la única condición que determina la función límite u es la condición (0.2.3).

El propósito del último capítulo de esta Tesis es probar que este es el caso, por lo menos bajo ciertas hipótesis de monotonía sobre el dato inicial u_0. Estas hipótesis de monotonía son similares a las utilizadas para probar unicidad del límite en el caso $u^\varepsilon = Y^\varepsilon$ en [**30**].

Nuestros resultados pueden ser resumidos en, *bajo ciertas hipótesis sobre el dominio y sobre el dato inicial* u_0*, existe a lo sumo una solución límite del problema de frontera libre (P) cuyo gradiente no se anula cerca de su frontera libre, siempre y cuando las aproximaciones de los datos iniciales – que convergen uniformemente a* u_0 *con soportes que convergen al soporte de* u_0 *– satisfagan (0.2.3).*

Más aún, bajo las mismas hipótesis geométricas, *si existe una solución clásica de (P), entonces ella es el único límite de soluciones de (P_ε) con datos iniciales que satisfacen las condiciones antes mencionadas. En particular, es la única solución clásica de (P).*

Queremos remarcar que la unicidad del límite resulta independiente de la aproximación del dato inicial u_0 y de la aproximación de la función constitutiva w_0. Más precisamente, tomemos $u_0^{\varepsilon_j}, \tilde{u}_0^{\varepsilon_k}$ distintas aproximaciones del dato inicial u_0 y $w^{\varepsilon_j}/\varepsilon_j, \tilde{w}^{\varepsilon_k}/\varepsilon_k$ distintas aproximaciones de w_0, sean u^{ε_j} (resp. $\tilde{u}^{\varepsilon_k}$) la solución de (P_{ε_j}) con función w^{ε_j} y dato inicial $u_0^{\varepsilon_j}$ (resp. solución de (P_{ε_k}) con función $\tilde{w}^{\varepsilon_k}$ y dato inicial $\tilde{u}_0^{\varepsilon_k}$). Sean $u = \lim u^{\varepsilon_j}$ y $\tilde{u} = \lim \tilde{u}^{\varepsilon_k}$. Entonces, bajo las condiciones antes mencionadas, $u = \tilde{u}$.

Como ya hemos mencionado, en el caso $u^\varepsilon = Y^\varepsilon$, resultados de unicidad para soluciones límite bajo hipótesis geométricas similares a las hechas en este trabajo pueden ser encontrados en [**30**]. Las técnicas utilizadas en este trabajo difieren de las de [**30**] ya que éstas últimas requerirían en nuestro caso, hipótesis suplementarias sobre la función f.

En [**22**] los autores estudian la unicidad y coincidencia entre diferentes conceptos de soluciones del problema (P) (nuevamente en el caso $u^\varepsilon = Y^\varepsilon$) bajo la suposición de la existencia de una solución clásica y bajo condiciones geométricas diferentes. Ver también [**23**] para un resultado similar en el caso de dos fases. Usamos algunas de las ideas de esos trabajos en el estudio de nuestro problema.

3. Notación

A lo largo de esta Tesis N denotara a la dimensión espacial y, además, la siguiente notación será usada:

Para cualquier $x_0 \in \mathbb{R}^N,\ t_0 \in \mathbb{R}$ y $\tau > 0$

$$\begin{aligned}
B_\tau(x_0) &\equiv \{x \in \mathbb{R}^N / \ |x - x_0| < \tau\},\\
B_\tau(x_0, t_0) &\equiv \{(x,t) \in \mathbb{R}^{N+1} / \ |x - x_0|^2 + |t - t_0|^2 < \tau^2\},\\
Q_\tau(x_0, t_0) &\equiv B_\tau(x_0) \times (t_0 - \tau^2, t_0 + \tau^2),\\
Q_\tau^-(x_0, t_0) &\equiv B_\tau(x_0) \times (t_0 - \tau^2, t_0],
\end{aligned}$$

y para cualquier conjunto $K \subset \mathbb{R}^{N+1}$

$$\begin{aligned}
\mathcal{N}_\tau(K) &\equiv \bigcup_{(x_0,t_0)\in K} Q_\tau(x_0, t_0),\\
\mathcal{N}_\tau^-(K) &\equiv \bigcup_{(x_0,t_0)\in K} Q_\tau^-(x_0, t_0).
\end{aligned}$$

De ser necesario, notaremos a los puntos en $\mathbb{R}^N$ por $x = (x_1, x')$, con $x' \in \mathbb{R}^{N-1}$. Además, $\langle\cdot,\cdot\rangle$ denotará el producto escalar usual en $\mathbb{R}^N$. Dada una función v, notaremos $v^+ = \max(v, 0),\ v^- = \max(-v, 0)$.

También, los símbolos Δ y ∇ notarán los correspondiente operadores en las variables espaciales; el símbolo ∂_p notará el borde parabólico.

Diremos que una función v pertenece a la clase $Lip_{\mathrm{loc}}(1, \frac{1}{2})$ en un dominio $\mathcal{D} \subset \mathbb{R}^{N+1}$, si para cada $\mathcal{D}' \subset\subset \mathcal{D}$, existe una constante $L = L(\mathcal{D}')$ tal que

$$|v(x,t) - v(y,s)| \leq L(|x - y| + |t - s|^{1/2})$$

para todo $(x,t),(y,s) \in \mathcal{D}'$. Si la constante L no depende del conjunto $\mathcal{D}'$, diremos que $v \in Lip(1,\frac{1}{2})$ en $\mathcal{D}$.

Finalmente, diremos que u es supercalórica si $\Delta u - u_t \leq 0$, y u es subcalórica si $\Delta u - u_t \geq 0$.

4. Hipótesis y estructura de la Tesis

Para la existencia de una función límite para una subsucesión $u^{\varepsilon_{n_k}}$ sólo necesitamos la condición más débil que para cada compacto $K \subset \mathcal{N}_\tau^-(K) \subset \mathcal{D}$,

$$\|Y^\varepsilon - u^\varepsilon\|_{L^\infty(\mathcal{N}_\tau^-(K))} = \mathrm{O}(\varepsilon). \tag{0.4.5}$$

Entonces, tenemos (ver [**21**])

$$\|Y^\varepsilon - u^\varepsilon\|_{C^{2,1}(K)} = \mathrm{O}(\varepsilon). \tag{0.4.6}$$

Bajo esta suposición, somos capaces de aplicar los resultados de [**9**] y obtener las estimaciones Lipschitz uniformes necesarias para pasar al límite en (0.2.1). Esto está realizado en el Capítulo 1 donde tambiés se prueban algunos lemas técnicos que son usados a lo largo de la Tesis.

En el Capítulo 2 asumimos que $u^\varepsilon \to 0$ en $\{u = 0\}$ suficientemente rápido. Esta es una condición esencial que ya fue considerada en [**13**]. Esta suposición es natural en aplicaciones, significa que la temperatura de la mezcla alcanza la temperatura de la llama sólo si alguna combustión esta siendo llevada a cabo.

También asumimos que existe $\lim_{\varepsilon\to 0}(Y^\varepsilon - u^\varepsilon)/\varepsilon =: w_0$ y, como consecuencia de la hipótesis $u^\varepsilon \to 0$ en $\{u = 0\}$ suficientemente rápido, mostramos que necesariamente $w_0 > -1$ en $\{u \equiv 0\}^\circ$. Luego, en el Capítulo 2, asumimos que para cada $K \subset \mathcal{N}_\tau^-(K) \subset \mathcal{D}$ compacto

$$\frac{Y^\varepsilon - u^\varepsilon}{\varepsilon} \to w_0 \qquad \text{uniformemente en } N_\tau^-(K). \tag{0.4.7}$$

Entonces,

$$\left\| \frac{Y^\varepsilon - u^\varepsilon}{\varepsilon} - w_0 \right\|_{C^{2,1}(K)} \to 0. \tag{0.4.8}$$

Y, para simplificar en análisis, asumimos que $w_0 > -1$ en $\mathcal{D}$. También, en el Capítulo 2, mostramos que la función límite u es una solución del problema de frontera libre (P) en un sentido puntual, y finalmente probamos que la función límite u es de hecho una solución viscosa del problema de frontera libre (P) bajo una hipótesis de nodegeneración de la función límite u. Además probamos algunos resultados que garantizan la nodegeneración de u.

Nuestra presentación en este capítulo es de una naturaleza local, con lo cual nuestras hipótesis están enunciadas en términos de la solución $(u^\varepsilon, Y^\varepsilon)$. Como puede verse en el ejemplo tratado en el Corolario 2.3.8 es posible deducir nuestras hipótesis sobre $(u^\varepsilon, Y^\varepsilon)$ a partir de condiciones sobre los datos iniciales y de contorno.

En el Capítulo 3, nos enfrentamos con el problema de unicidad para funciones límite de (P), bajo ciertas hipótesis geométricas adicionales que ya han sido consideradas en el caso $w^\varepsilon = 0$ [**22, 23, 30**]. Más precisamente, asumimos que el dato inicial u_0 es estrellado con respecto a algún punto. Esta hipótesis de monotonía nos permite aproximar una supersolución clásica de (P) por una familia de supersoluciones estrictas de (P_ε). Probamos que el límite de una sucesión de soluciones de (P_ε) es independiente de la sucesión siempre y cuando el límite de sus datos iniciales y de $w^\varepsilon/\varepsilon$ sea fijo.

Introduction

1. Description of the model

The work in this Thesis is a contribution to the mathematical analysis of a thermal-diffusive model that appears in combustion theory in the analysis of the propagation of curved flames.

For an elementary reaction of order one, of type

$$\textbf{Reactant} \rightarrow \textbf{Product},$$

the general problem of propagation of flames is reduced to solving the system:

$$\rho_t - \operatorname{div}(\rho \mathbf{v}) = 0, \tag{0.1.1}$$

$$\rho \mathbf{v}_t + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} - \mu \Delta \mathbf{v} - \lambda \nabla(\nabla \cdot \mathbf{v}) + \nabla p = 0, \tag{0.1.2}$$

$$\rho T_t + \rho(\mathbf{v} \cdot \nabla)T - K \Delta T = \frac{Q}{c_p}\omega, \tag{0.1.3}$$

$$\rho y_t + \rho(\mathbf{v} \cdot \nabla)y - K_1 \Delta y = -m_y \omega, \tag{0.1.4}$$

$$p = \rho R T, \tag{0.1.5}$$

where the unknowns are the density ρ, the velocity $\mathbf{v}$, the pressure p, the temperature T and the concentration of the reactant y. Equations (0.1.1) and (0.1.2) are the conservation of mass and Navier-Stokes equations; equation (0.1.5) is the equation of state for a perfect gas; and equations (0.1.3) and (0.1.4) are the equations of the chemical cinetic for which we adopt the *Arrhenius law*:

$$\omega = \rho_b B(T_b) \frac{y}{m_y} \exp\left(-\frac{E}{RT}\right). \tag{0.1.6}$$

We make the assumption that the quantities μ, λ, c_p, m_y, Q, R, K and K_1 are positive constants. Moreover, ρ_b and T_b represent the density and the temperature of the burned gas, and E is the activation energy.

This last parameter plays an important role because of the exponential dependence in the temperature of the reaction rate ω; this dependance is increased as the activation energy is increased. Moreover, it is the basis of the asymptotic analysis commonly performed by physicists. Also it is the basis for the identification of different zones characterized by the relative importance of the terms that appear in the equations. As E tends to infinity, a *free boundary problem* appears (see [**8, 37, 38**]).

With this generality, the problem is too complex. The thermal-diffusive model, consists in a simplification of this problem by means of two basic assumptions that are classical in combustion theory. The first one is the assumption of low match number, this is to consider the propagation of the flame as an isobaric process. The second one consists in considering the density of the mixture as constant. These two hypotheses allow us to decouple the system into the set of equations that model the hydrodynamic process of the gas, and the equations that describe the combustion process.

This model is physically justified (cf. [**29**]) for large activation energies

$$\frac{E}{RT_b} \gg 1,$$

under the hypothesis of almost-equidiffusion:

$$\frac{E}{RT_b}\frac{T_b - T_c}{T_b}\left(1 - \frac{1}{Le}\right) = O(1), \tag{0.1.7}$$

where $Le = K/K_1$ is the Lewis number and T_c is the temperature of the cold gas.

This model adapts well to the description of the phenomenon of combustion when the dynamic of the gas plays a secondary role in terms of the diffusive and reactive effects. This is the case, for instance, in the phenomenon of cellular instability [**29, 33, 34**].

The limit $E \to +\infty$ is, by itself, of little interest since the reaction term ω given in (0.1.6) vanishes. To preserve the reaction, it is necessary for the term $B(T_b)$ to become unboundedly large; i.e. we must consider a distinguished limit characterized essentially by

$$B(T_b) \sim e^{\frac{E}{RT_b}} \tag{0.1.8}$$

For $T < T_b$ the reaction term ω vanishes exponentially; this is known as the frozen limit. For $T > T_b$, (0.1.4) and (0.1.6) imply that – at least formally – $y \to 0$ exponentially and again ω vanishes exponentially. So the first step for making this method work, consists in assuming that the temperature T_f on the front of combustion verifies an estimate of the form:

$$\frac{E}{RT_b^2}(T_f - T_b) = O(1). \tag{0.1.9}$$

The asymptotic analysis when $\frac{E}{RT_b} \to +\infty$ of the system, leads – at least formally – to a free boundary problem (see [**20, 35**]).

In this Thesis, we will focus on the rigorous mathematical analysis of this model, and more precisely, on the study of its asymptotic analysis for large activation energies. We will consider the mixture of a gas in repose (i.e. $\mathbf{v} = 0$). After adimensionalization of the equations, the problem (0.1.1)-(0.1.4) is reduce to solving the system

$$\Delta u - u_t = \omega(u, Y), \tag{0.1.10}$$

$$\frac{1}{Le}\Delta Y - Y_t = \omega(u, Y), \tag{0.1.11}$$

where $u = \frac{1}{T_f - T_c}(T_f - T)$ is the rescaled temperature (or minus the temperature) and Y is the rescaled mass fraction of the reactant. The term $\omega(u, Y)$ has some precise properties that will be described below.

For a more precise description of the model, we refer to [**8**].

The thermal-diffusive model described above, has been studied by many authors: existence of stationary waves (for example [**4, 7, 36**]), solution of elliptic problems (see [**3, 4, 6**]), the parabolic problem ([**26**]), stability of traveling waves ([**5, 31, 32**]).

The asymptotic analysis for large activation energies has been studied for stationary waves in [**7, 18, 28**] among others. For elliptic and parabolic problems, it has been studied in the case $Le = 1$ and $u = Y$ (which is a natural assumption in the case of traveling waves). We cite the works [**2, 25**] for the traveling waves and the elliptic equation and [**10, 11, 13**] for the parabolic problem.

We also mention the work [**27**] where the system (0.1.10)-(0.1.11) is studied in the case $Le \sim 1$ and results similar to those in [**13**] are obtained in dimensions $N = 1, 2, 3$.

2. Description of the mathematical problem

In this Thesis we consider the problem (0.1.10)-(0.1.11) in the equidiffusional case (i.e. $Le = 1$). We make the following natural assumptions on the nonlinear term $\omega(u,Y)$: We call ε the inverse of the rescaled activation energy, $\varepsilon^{-1} = \frac{E}{RT_b^2}(T_b - T_c)$, then, by (0.1.6), $\omega = \omega_\varepsilon$ is given by

$$\omega_\varepsilon(u,Y) = Y f_\varepsilon(u).$$

To avoid what is called *the cold boundary difficulty*, it is usual in the literature to set f_ε to be zero wherever is exponentially small, that is, we will assume in what follows that $f_\varepsilon(s) = 0$ if $s \geq \varepsilon$. For a more detailed discussion about the cold boundary difficulty, see [**8**].

By (0.1.8), it is easy to check that the functions f_ε verify that

$$\int_0^\varepsilon s f_\varepsilon(s)\, ds \to M_0 > 0, \qquad \varepsilon \to 0.$$

This constant M_0 plays a crucial role in the asymptotic analysis of the model as $\varepsilon \to 0$. A usual – and convenient – way of simplifying the analysis, is to change the functions f_ε by assuming that they are given in terms of a single function f in the form

$$f_\varepsilon(s) = \frac{1}{\varepsilon^2} f\left(\frac{s}{\varepsilon}\right),$$

and so the integral $\int_0^\varepsilon s f_\varepsilon(s)\, ds$ is independent of ε.

These functions f_ε still capture the essential features of (0.1.6). Then, on f we assume that it is a nonnegative Lipschitz continuous function which is positive in the interval $(-\infty, 1)$ and vanishes otherwise (i.e., reaction occurs only when $T > T_f - \varepsilon(T_f - T_c)$).

From now on, we will make explicit the dependance on ε of the rescaled temperature u and the mass fraction of the reactant Y, so the system under consideration will be

$$\left\{ \begin{array}{rcll} \Delta u^\varepsilon - u_t^\varepsilon & = & Y^\varepsilon f_\varepsilon(u^\varepsilon) & \text{in } \mathcal{D}, \\ \Delta Y^\varepsilon - Y_t^\varepsilon & = & Y^\varepsilon f_\varepsilon(u^\varepsilon) & \text{in } \mathcal{D}, \end{array} \right. \tag{0.2.1}$$

where $\mathcal{D} \subset \mathbb{R}^{N+1}$.

The study of the limit as $\varepsilon \to 0$ was proposed in the 30's by Zeldovich and Frank-Kamenetski [**38**] and has been much discussed in the combustion literature.

In the case $u^\varepsilon = Y^\varepsilon$ the reaction function $u^\varepsilon f_\varepsilon(u^\varepsilon)$ tends to a Dirac delta, $M_0\delta(u)$ where $M_0 = \int_0^1 s f(s) ds$. In this way the reaction zone where $u^\varepsilon f_\varepsilon(u^\varepsilon)$ acts is reduced to a surface, the flame front, and a free boundary problem arises. The fact that $M_0 > 0$ ensures that a nontrivial combustion process takes place so that a non-empty free boundary actually appears.

Although the convergence of the most relevant propagation modes, i.e. the traveling waves, was already discussed by Zeldovich and Frank-Kamenetski, and an enormous progress in this direction has been made, a rigorous mathematical investigation of the convergence of general solutions is still in progress. Berestycki and his collaborators have rigorously studied the convergence problem for traveling waves and, more generally in the elliptic stationary case, cf. [**2**] and its references. See also [**25**]. The study of the limit in the general evolution case for the heat operator has been performed in [**13**] for the one phase case (this is, with $u^\varepsilon \geq 0$) and in [**9, 10, 11**] for the two-phase case, where no sign restriction on u^ε is made.

In [**13**] the authors show that, under certain assumptions on the initial datum and its approximations, for every sequence $\varepsilon_n \to 0$ there exists a subsequence ε_{n_k} and a limit function $u = \lim u^{\varepsilon_{n_k}}$ which solves the following free boundary problem

$$\begin{cases} \Delta u - u_t = 0 & \text{in } \mathcal{D} \cap \{u > 0\}, \\ |\nabla u^+| = \sqrt{2M_0} & \text{on } \mathcal{D} \cap \partial\{u > 0\}, \end{cases} \tag{0.2.2}$$

in a weak integral sense. Here $M_0 = \int_0^1 sf(s)ds$.

In [**10**] and [**11**] the authors show that the free boundary condition for the two phase case (when it is assumed that no reaction takes place if $u^\varepsilon \leq 0$) is

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 2M_0$$

and that the limit function is a solution of the free boundary problem in a pointwise sense at regular free boundary points when $\{u = 0\}$ has zero "parabolic density" and in a viscosity sense in the absence of a zero phase (i.e. when $\{u = 0\}^\circ \cap \mathcal{D} = \emptyset$)

So that a natural question is: Will a sequence of uniformly bounded solutions $(u^\varepsilon, Y^\varepsilon)$ of (0.2.1) with $(Y^\varepsilon - u^\varepsilon) \to 0$ as $\varepsilon \to 0$ be such that u^ε converges to a solution of the free boundary problem (0.2.2)? This is, will the asymptotic limit for activation energy going to infinity, in the case in which $(Y^\varepsilon - u^\varepsilon) \to 0$ but $u^\varepsilon \neq Y^\varepsilon$, be a solution of the same free boundary problem as in the case in which $u^\varepsilon = Y^\varepsilon$?

Let us point out that in the case under consideration this is, when Lewis number is 1, the function $w^\varepsilon = Y^\varepsilon - u^\varepsilon$ is a solution of the heat equation. So that it is fully determined by its initial-boundary datum.

Moreover, the system (0.2.1) may be rewritten as a single equation for u^ε, namely

$$(P_\varepsilon) \qquad \Delta u^\varepsilon - u_t^\varepsilon = (u^\varepsilon + w^\varepsilon) f_\varepsilon(u^\varepsilon).$$

In this thesis we consider the case in which $w^\varepsilon/\varepsilon$ converges to a function w_0 (so that in particular, $Y^\varepsilon - u^\varepsilon \to 0$). More precisely, we assume that the initial data Y_0^ε and u_0^ε verify

$$(0.2.3) \qquad \frac{Y_0^\varepsilon(x) - u_0^\varepsilon(x)}{\varepsilon} \to w_0(x) \quad \text{uniformly in } \mathbb{R}^N,$$

with $w_0 > -1$. Therefore, the function $w^\varepsilon(x,t)$ is the solution of the heat equation with initial datum $Y_0^\varepsilon(x) - u_0^\varepsilon(x)$ and by (0.2.3), satisfies that there exists the limit

$$(0.2.4) \qquad \lim_{\varepsilon \to 0} \frac{w^\varepsilon(x,t)}{\varepsilon} = w_0(x,t)$$

and $w_0(x,t)$ is the solution of the heat equation with initial datum $w_0(x)$.

In this way, at least formally, the reaction term still converges to a delta function and a free boundary problem appears. *But we prove in this work that the free boundary condition strongly depends on the limit function w_0, so that it is different for different approximations of the initial-boundary datum of u.*

In fact, we prove that for every sequence $\varepsilon_n \to 0$ there exists a subsequence ε_{n_k} and a limit function $u = \lim u^{\varepsilon_{n_k}}$ which is a solution of the following free boundary problem

$$(P) \qquad \begin{cases} \Delta u - u_t = 0 & \text{in } \mathcal{D} \cap \{u > 0\}, \\ |\nabla u^+| = \sqrt{2M(x,t)} & \text{on } \mathcal{D} \cap \partial\{u > 0\}, \end{cases}$$

where $M(x,t) = \int_{-w_0(x,t)}^1 \big(s + w_0(x,t)\big) f(s)ds$.

The presence of the function w_0 in the limit of integration gives the necessary positive sign of the function $M(x,t)$.

In conclusion, *the combustion problem is very unstable in the sense that the asymptotic limit for activation energy going to infinity depends on order ε perturbations of the initial-boundary data.*

In this Thesis we prove that the limit function u is a “viscosity” solution to (P), so that, as a consequence of our results and of the regularity results for viscosity solutions to (P) of [**17**], we deduce that, when the free boundary of a limit function u is given by $x_1 = g(x', t)$, $x = (x_1, x')$ with g Lipschitz continuous, u is a classical solution.

We want to stress, that because of our assumption that $Y^\varepsilon - u^\varepsilon \to 0$ and since $Y^\varepsilon \geq 0$, the limit function u must be nonnegative, so our result that u is a viscosity solution to (P) is new, even in the case $u^\varepsilon = Y^\varepsilon$.

In particular, as a consequence of our results we see that limit functions u with $u^\varepsilon(x, 0)$ constructed as in [**13**], and $Y^\varepsilon(x, 0)$ small perturbations of $u^\varepsilon(x, 0)$ are viscosity solutions to (P). In this construction, w_0 is any constant such that $w_0 \geq -\eta$ where $\eta > 0$ is small enough.

Finally, we study the uniqueness of the limit functions $u = \lim u^{\varepsilon_{n_k}}$, since it is therefore natural to wonder whether the only condition that determines the limit function u is condition (0.2.3).

The purpose of the last chapter of this Thesis is to prove that this is indeed the case, at least under some monotonicity assumption on the initial value u_0. This monotonicity assumption is similar to that used to prove uniqueness of the limit for the case $u^\varepsilon = Y^\varepsilon$ in [**30**].

Our result can be summarized as saying that, *under suitable assumptions on the domain and on the initial datum u_0, there exists at most one limit solution to the free boundary problem (P) with non-vanishing gradient near its free boundary, as long as the approximate initial data – converging uniformly to u_0 with supports that converge to the support of u_0 – satisfy (0.2.3).*

Moreover, under the same geometric assumptions, *if there exists a classical solution to (P), this is the only limit of solutions to (P_ε) with initial data satisfying the conditions above. In particular, it is the only classical solution to (P).*

We want to stress that the uniqueness of the limit turns out to be independent of the approximation of the initial datum u_0 and the approximation of the constitutive function w_0. More precisely, let us take $u_0^{\varepsilon_j}, \tilde{u}_0^{\varepsilon_k}$ different approximations of the initial datum u_0 and $w^{\varepsilon_j}/\varepsilon_j, \tilde{w}^{\varepsilon_k}/\varepsilon_k$ different approximations of w_0, let u^{ε_j} (resp. $\tilde{u}^{\varepsilon_k}$) be the solution of (P_{ε_j}) with function w^{ε_j} and initial datum $u_0^{\varepsilon_j}$ (resp. the solution of (P_{ε_k}) with function $\tilde{w}^{\varepsilon_k}$ and initial datum $\tilde{u}_0^{\varepsilon_k}$). Let $u = \lim u^{\varepsilon_j}$ and $\tilde{u} = \lim \tilde{u}^{\varepsilon_k}$. Then, under the same conditions stated before, $u = \tilde{u}$.

As already stated, in the case $u^\varepsilon = Y^\varepsilon$, uniqueness results for limit solutions under geometric hypotheses similar to the ones made here can be found in [**30**]. Nevertheless, in our work we use a different technique since, in our situation, the method used in [**30**] would require several additional hypotheses on f.

Also in [**22**] the authors study the uniqueness and agreement between different concepts of solutions of problem (P) (again in the case $u^\varepsilon = Y^\varepsilon$) under the assumption of the existence of a classical solution and under different geometric assumptions. See also [**23**] for a similar result in the two-phase case. We use some of the ideas in these works for the study of our present situation.

3. Notation

Throughout this Thesis N will denote the spatial dimension and, in addition, the following notation will be used:

For any $x_0 \in \mathbb{R}^N,\ t_0 \in \mathbb{R}$ and $\tau > 0$

$$\begin{aligned}
B_\tau(x_0) &\equiv \{x \in \mathbb{R}^N / \ |x - x_0| < \tau\},\\
B_\tau(x_0, t_0) &\equiv \{(x,t) \in \mathbb{R}^{N+1} / \ |x - x_0|^2 + |t - t_0|^2 < \tau^2\},\\
Q_\tau(x_0, t_0) &\equiv B_\tau(x_0) \times (t_0 - \tau^2, t_0 + \tau^2),\\
Q_\tau^-(x_0, t_0) &\equiv B_\tau(x_0) \times (t_0 - \tau^2, t_0],
\end{aligned}$$

and for any set $K \subset \mathbb{R}^{N+1}$

$$\begin{aligned}
\mathcal{N}_\tau(K) &\equiv \bigcup_{(x_0,t_0)\in K} Q_\tau(x_0,t_0),\\
\mathcal{N}_\tau^-(K) &\equiv \bigcup_{(x_0,t_0)\in K} Q_\tau^-(x_0,t_0).
\end{aligned}$$

When necessary, we will denote points in $\mathbb{R}^N$ by $x = (x_1, x')$, with $x' \in \mathbb{R}^{N-1}$. Also, $\langle \cdot, \cdot \rangle$ will mean the usual scalar product in $\mathbb{R}^N$. Given a function v, we will denote $v^+ = \max(v, 0),\ v^- = \max(-v, 0)$.

In addition, the symbols Δ and ∇ will denote the corresponding operators in the space variables; the symbol ∂_p will denote parabolic boundary.

We will say that a function v is in the class $Lip_{\mathrm{loc}}(1, \frac{1}{2})$ in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$, if for every $\mathcal{D}' \subset\subset \mathcal{D}$, there exists a constant $L = L(\mathcal{D}')$ such that

$$|v(x,t) - v(y,s)| \le L(|x-y| + |t-s|^{1/2})$$

for every $(x,t), (y,s) \in \mathcal{D}'$. If the constant L does not depend on the set $\mathcal{D}'$, we will say that $v \in Lip(1, \frac{1}{2})$ in $\mathcal{D}$.

Finally, we will say that u is supercaloric if $\Delta u - u_t \le 0$, and u is subcaloric if $\Delta u - u_t \ge 0$.

4. Hypotheses and outline of the Thesis

For the existence of a limit function for a subsequence $u^{\varepsilon_{n_k}}$ we only need the weaker condition that for every compact $K \subset \mathcal{N}_\tau^-(K) \subset \mathcal{D}$,

$$\|Y^\varepsilon - u^\varepsilon\|_{L^\infty(\mathcal{N}_\tau^-(K))} = \mathrm{O}(\varepsilon). \tag{0.4.5}$$

Then, we have (see [**21**])

$$\|Y^\varepsilon - u^\varepsilon\|_{C^{2,1}(K)} = \mathrm{O}(\varepsilon). \tag{0.4.6}$$

Under this assumption, we are able to apply the results of [**9**] and get the uniform Lipschitz estimates needed to pass to the limit in (0.2.1). This is done in Chapter 1 where we also prove some technical lemmas that are used throughout the thesis.

In Chapter 2 we assume that $u^\varepsilon \to 0$ in $\{u = 0\}$ fast enough. This is an essential condition that was already considered in [**13**]. This assumption is a natural one in applications, roughly speaking it means that the mixture temperature reaches the flame temperature only if some combustion is taking place.

We also assume that there exists $\lim_{\varepsilon\to 0}(Y^\varepsilon - u^\varepsilon)/\varepsilon =: w_0$ and, as a consequence of the hypothesis that $u^\varepsilon \to 0$ in $\{u = 0\}$ fast enough, we show that necessarily $w_0 > -1$ in $\{u \equiv 0\}^\circ$. So that, in Chapter 2 we assume that for every $K \subset \mathcal{N}_\tau^-(K) \subset \mathcal{D}$ compact

$$\frac{Y^\varepsilon - u^\varepsilon}{\varepsilon} \to w_0 \qquad \text{uniformly in } N_\tau^-(K). \tag{0.4.7}$$

Thus,

$$\left\| \frac{Y^\varepsilon - u^\varepsilon}{\varepsilon} - w_0 \right\|_{C^{2,1}(K)} \to 0. \tag{0.4.8}$$

And, for the sake of simplicity, we assume that $w_0 > -1$ in $\mathcal{D}$. Also, in Chapter 2, we show that the limit function u is a solution to the free boundary problem (P) in a pointwise sense, and finally we prove that the limit function u is in fact a viscosity solution of the free boundary problem (P) under a nondegeneracy assumption on the limit function u. We also prove some results that give the necessary nondegeneracy of u.

Our presentation is of a local nature, so that our hypotheses are stated in terms of the solution $(u^\varepsilon, Y^\varepsilon)$. As can be seen in the example treated in Corollary 2.3.8 it is possible to deduce our hypotheses on $(u^\varepsilon, Y^\varepsilon)$ from conditions on its initial-boundary datum.

In Chapter 3, we deal with the uniqueness nature of the limit of solutions to (P_ε), under some additional geometric assumptions that were already considered in the case $w^\varepsilon = 0$ [**22, 23, 30**]. More precisely, we assume that the initial datum u_0 is starshaped with respect to some point. This monotonicity assumption allows us to approximate a classical supersolution to (P) by a family of strict supersolutions to (P_ε). We prove that the limit of a sequence of solutions of (P_ε) is independent of the sequence as long as the limit of their initial values and of $w^\varepsilon/\varepsilon$ is fixed.

CHAPTER 1

Uniform Estimates

In this chapter we consider a family $(u^\varepsilon, Y^\varepsilon)$ of solutions to

$$\begin{aligned} \Delta u^\varepsilon - u^\varepsilon_t &= Y^\varepsilon f_\varepsilon(u^\varepsilon), \\ \Delta Y^\varepsilon - Y^\varepsilon_t &= Y^\varepsilon f_\varepsilon(u^\varepsilon), \end{aligned} \tag{1.0.1}$$

in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$ which are uniformly bounded in L^∞ norm in $\mathcal{D}$ and satisfy that for every compact $K \subset \mathcal{N}_\tau^-(K) \subset \mathcal{D}$,

$$\|Y^\varepsilon - u^\varepsilon\|_{L^\infty(\mathcal{N}_\tau^-(K))} = \mathrm{O}(\varepsilon). \tag{1.0.2}$$

Then, we have (see [**21**])

$$\|Y^\varepsilon - u^\varepsilon\|_{C^{2,1}(K)} = \mathrm{O}(\varepsilon). \tag{1.0.3}$$

In Section 1, we show that the functions $u^\varepsilon, Y^\varepsilon$ are locally uniformly bounded in the seminorm $Lip(1, \frac{1}{2})$. Then, in Section 2, we get further local uniform estimates and pass to the limit as $\varepsilon \to 0$. We also show that the limit function u is a solution to the free boundary problem (P) in a very weak sense. In Section 3, we prove an approximation lemma that will be used throughout the rest of the work.

In Sections 4 and 5, we further assume that for every $K \subset \mathcal{N}_\tau^-(K) \subset \mathcal{D}$ compact there exists a function w_0 such that

$$\frac{Y^\varepsilon - u^\varepsilon}{\varepsilon} \to w_0 \qquad \text{uniformly in } N_\tau^-(K). \tag{1.0.4}$$

Thus,

$$\left\| \frac{Y^\varepsilon - u^\varepsilon}{\varepsilon} - w_0 \right\|_{C^{2,1}(K)} \to 0. \tag{1.0.5}$$

We will see that is natural to impose that $w_0 > -1$ in $\mathcal{D}$. We observe that, as $Y^\varepsilon - u^\varepsilon$ is a solution of the heat equation, condition (1.0.4) (as well as condition (1.0.2)) can be deduce from initial-boundary data.

In Section 4 we prove some lemmas concerning particular limit functions in the particular case where w_0 is constant. These lemmas will be useful in the next chapters. Finally, in Section 5, we begin our study of the limit functions and prove that every limit function u is a supersolution to the free boundary problem (P).

1. The estimates

In this Section, we show that uniformly bounded solutions to (1.0.1) are locally uniformly bounded in $Lip(1, \frac{1}{2})$ norm. First, we manage to apply the results in [**9**] and obtain a uniform bound on the gradients of $(u^\varepsilon, Y^\varepsilon)$ and then (as usual in parabolic regularity theory) we get the Hölder 1/2 bound on t.

For convenience, let us define the following function

$$w^\varepsilon(x,t) = Y^\varepsilon(x,t) - u^\varepsilon(x,t), \tag{1.1.1}$$

then, w^ε is a caloric function and, by (1.0.3), $\|w^\varepsilon\|_{C^{2,1}(K)} = O(\varepsilon)$ for every compact set $K \subset \mathcal{D}$.

For further references, let us now state the following Theorem proved in [**9**]

THEOREM 1.1.2 ([**9**], Corollary 2). *Let u be a bounded solution in Q_1 of*

$$0 \le \Delta u - u_t \le \frac{C}{\varepsilon}\mathcal{X}_{\{0<u<\varepsilon\}}.$$

Then u is Lipschitz (in space) in $Q_{1/2}$ with bounds independent of ε.

We begin with a proposition (which is a consequence of Theorem 1.1.2) that gives us the uniform control on the gradients of solutions of (1.0.1).

PROPOSITION 1.1.3. *Let $(u^\varepsilon, Y^\varepsilon)$ be solutions of (1.0.1) such that $\|u^\varepsilon\|_\infty \le \mathcal{A}$, $Y^\varepsilon \ge 0$ and verify (1.0.2). Let $K \subset \mathcal{D}$ compact and $\tau > 0$ such that $\mathcal{N}_\tau^-(K) \subset \mathcal{D}$. Then, there exists $L = L(\tau, \mathcal{A})$ such that*

$$|\nabla u^\varepsilon(x,t)| \le L, \quad |\nabla Y^\varepsilon(x,t)| \le L.$$

PROOF. Let us start by making the following observation

$$u^\varepsilon = Y^\varepsilon - w^\varepsilon \ge -w^\varepsilon \ge -C\varepsilon.$$

Then, let $z^\varepsilon = \frac{1}{C+1}(u^\varepsilon + C\varepsilon)$ and define, for $(x_0, t_0) \in K$

$$z_\tau^\varepsilon(x,t) = \frac{1}{\tau} z^\varepsilon(x_0 + \tau x, t_0 + \tau^2 t).$$

In $B_1(0)\times[-1,0]$, z^ε_τ verifies (with $B\ge\|f\|_\infty$)

$$\begin{aligned}0\le\Delta z^\varepsilon_\tau-\frac{\partial z^\varepsilon_\tau}{\partial t}&\le\frac{\tau}{C+1}(C\varepsilon+|u^\varepsilon|)\frac{1}{\varepsilon^2}f(\frac{u^\varepsilon}{\varepsilon})\\&\le B\tau\frac{1}{\varepsilon}\mathcal{X}_{[-C\varepsilon,\varepsilon]}(u^\varepsilon)=\frac{B}{\varepsilon/\tau}\mathcal{X}_{[0,\varepsilon/\tau]}(z^\varepsilon_\tau).\end{aligned}$$

On the other hand

$$|z^\varepsilon_\tau(x,t)|\le\frac{|u^\varepsilon(x,t)|+C}{\tau(1+C)}\le\frac{1}{\tau}\frac{\mathcal{A}+C}{1+C}.$$

Therefore, by Theorem 1.1.2, it follows that

$$|\nabla z^\varepsilon_\tau(x,t)|\le\bar L=\bar L(\tau,\mathcal{A})\qquad\text{in } B_{1/2}(0)\times(-1/2,0].$$

In particular,

$$\begin{aligned}|\nabla u^\varepsilon(x_0,t_0)|&=(C+1)|\nabla z^\varepsilon(x_0,t_0)|=(C+1)|\nabla z^\varepsilon_\tau(0,0)|\le(C+1)\bar L,\\|\nabla Y^\varepsilon(x_0,t_0)|&\le|\nabla u^\varepsilon(x_0,t_0)|+|\nabla w^\varepsilon(x_0,t_0)|\le(C+1)\bar L+C.\end{aligned}$$

The proof is finished □

As is usual in parabolic regularity theory, Lipschitz regularity in space, gives Hölder 1/2 regularity in time. For the proof we need the following result

Proposition 1.1.4 ([**10**], Proposition 2.2). *Let $u\in C(\bar B_1(0)\times[0,1/(4N+\Lambda)])$ be such that $|\Delta u-u_t|\le\Lambda$ in $\{u<0\}\cup\{u>1\}$, for some $\Lambda>0$. Let us assume that $|\nabla u|\le L$, for some $L>0$. Then there exists a constant $C=C(L)$ such that*

$$|u(0,T)-u(0,0)|\le C\qquad\textit{if}\quad 0\le T\le\frac{1}{4N+\Lambda}.$$

Proposition 1.1.5. *Let $(u^\varepsilon,Y^\varepsilon)$ be solutions of (1.0.1) such that $\|u^\varepsilon\|_\infty\le\mathcal{A}$, $Y^\varepsilon\ge0$, and verify (1.0.2). Let $K\subset\mathcal{D}$ compact and $\tau>0$ such that $\mathcal{N}_\tau(K)\subset\mathcal{D}$. Then there exists $C=C(\tau,\mathcal{A})$ such that*

$$|u^\varepsilon(x,t+\Delta t)-u^\varepsilon(x,t)|\le C|\Delta t|^{1/2},\ |Y^\varepsilon(x,t+\Delta t)-Y^\varepsilon(x,t)|\le C|\Delta t|^{1/2},$$

for every $(x,t),(x,t+\Delta t)\in K$.

Proof. As in Proposition 1.1.3 we define $z^\varepsilon=\frac{1}{C+1}(u^\varepsilon+C\varepsilon)$ and

$$z^\varepsilon_\lambda(x,t)=\frac{1}{\lambda}z^\varepsilon(x_0+\lambda x,t_0+\lambda^2t),$$

for $0<\lambda<\tau$ and $(x_0,t_0)\in K$.

By a simple computation we get, as in Proposition 1.1.3

$$0 \leq \Delta z_\lambda^\varepsilon - \frac{\partial z_\lambda^\varepsilon}{\partial t} \leq \frac{B}{\varepsilon/\lambda} \mathcal{X}_{[0,\varepsilon/\lambda]}(z_\lambda^\varepsilon).$$

Now, $z_\lambda^\varepsilon \geq 0$, and in $\{z_\lambda^\varepsilon > 1\}$ we have

$$|\Delta z_\lambda^\varepsilon - \frac{\partial z_\lambda^\varepsilon}{\partial t}| \begin{cases} \leq B & \text{if } \varepsilon/\lambda \geq 1 \\ = 0 & \text{if } \varepsilon/\lambda < 1. \end{cases}$$

Moreover, we have that

$$|\nabla z_\lambda^\varepsilon(x,t)| = \frac{1}{C+1} |\nabla u^\varepsilon(x_0 + \lambda x, t_0 + \lambda^2 t)| \leq \bar{L}$$

in $B_{\tau/\lambda}(0) \times [0, \tau^2/\lambda^2]$. Then, by Proposition 1.1.4, we have

$$|z_\lambda^\varepsilon(0,t) - z_\lambda^\varepsilon(0,0)| \leq C(\bar{L}) \qquad \forall\ 0 \leq t \leq \frac{1}{4N+B}$$

which, in terms of u^ε, is

$$|u^\varepsilon(x_0, t_0 + \lambda^2 t) - u^\varepsilon(x_0, t_0)| \leq C(\bar{L})\lambda.$$

In particular

$$|u^\varepsilon(x_0, t_0 + \frac{\lambda^2}{4N+B}) - u^\varepsilon(x_0, t_0)| \leq C(\bar{L})\lambda.$$

Let $(x_0, t_0 + \Delta t) \in K$. If $0 < \Delta t < \tau^2/(4N+B)$, we take $\lambda = \Delta t^{1/2}\sqrt{4N+B} < \tau$ to get

$$|u^\varepsilon(x_0, t_0 + \Delta t) - u^\varepsilon(x_0, t_0)| \leq C(\bar{L})\sqrt{4N+B}\,\Delta t^{1/2}.$$

If $\Delta t \geq \frac{\tau^2}{4N+B}$, we have

$$|u^\varepsilon(x_0, t_0 + \Delta t) - u^\varepsilon(x_0, t_0)| \leq 2\mathcal{A} \leq \frac{2\mathcal{A}}{\tau}\sqrt{4N+B}\,\Delta t^{1/2}.$$

The analogous inequality for Y^ε is an immediate consequence of (1.0.3). □

Remark 1.1.6. *Under the hypothesis of the previous propositions, we have that*

$$u^\varepsilon \in Lip_{loc}(1, 1/2).$$

2. Passing to the limit

In this Section, we prove further uniform estimates on the solutions of (1.0.1) and pass to the limit. Then we show that the limit function u is a solution to (P) in a very weak form.

PROPOSITION 1.2.1. *Let $(u^\varepsilon, Y^\varepsilon)$ be solutions of (1.0.1) such that $\|u^\varepsilon\|_\infty \le \mathcal{A}$, $Y^\varepsilon \ge 0$ and verify* (1.0.2). *Then, for every sequence $\varepsilon_n \to 0$, there exists $\varepsilon_{n'} \to 0$ a subsequence and $u \in Lip_{loc}(1, 1/2)$ such that*

1. *$u^{\varepsilon_{n'}} \to u$ uniformly on compacts subsets of $\mathcal{D}$.*
2. *$\nabla u^{\varepsilon_{n'}} \to \nabla u$ in L^2_{loc}.*
3. *$\frac{\partial}{\partial t} u^{\varepsilon_{n'}} \to \frac{\partial}{\partial t} u$ weakly in L^2_{loc}.*
4. *$\Delta u - \frac{\partial u}{\partial t} = 0$ in $\{u > 0\}$*
5. *For every compact $K \subset \mathcal{D}$, exists $C_K > 0$ such that*
$$\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^2(K)} \le C_K$$
for every $\varepsilon > 0$.

PROOF. Let $K \subset \mathcal{D}$ be a compact set, and $\tau > 0$ such that $\mathcal{N}_{3\tau}(K) \subset \mathcal{D}$. Let $L = L(K)$ such that
$$|u^\varepsilon(x,t) - u^\varepsilon(y,s)| \le L\left(|x-y| + |t-s|^{1/2}\right),$$
where $(x,t), (y,s) \in \mathcal{N}_\tau(K)$.

Then, by Arzela-Ascoli's theorem, there exists $\varepsilon_{n'} \to 0$ and $u \in Lip(1, 1/2)$ in $\mathcal{N}_\tau(K)$ such that $u^{\varepsilon_{n'}} \to u$ uniformly in $\mathcal{N}_\tau(K)$. By a standard diagonal argument, (1) follows.

Let us now find uniform bounds for $\frac{\partial u^\varepsilon}{\partial t}$ in $L^2_{\rm loc}(\mathcal{D})$. In fact, u^ε verifies
$$\Delta u^\varepsilon - \frac{\partial u^\varepsilon}{\partial t} = Y^\varepsilon f_\varepsilon(u^\varepsilon).$$

Now, let $(x_0, t_0) \in K$ and let us multiply the equation by $u^\varepsilon_t \psi^2$ where $\psi \ge 0$, $\psi = \psi(x) \in C^\infty_c(B_\tau(x_0))$, $\psi \equiv 1$ in $B_{\tau/2}(x_0)$. Then, integrating by parts, we get

$$\iint_{Q_\tau(x_0,t_0)} (u^\varepsilon_t)^2 \psi^2 \, dxdt + \frac{1}{2} \iint_{Q_\tau(x_0,t_0)} (|\nabla u^\varepsilon|^2)_t \psi^2 \, dxdt$$
$$+ 2 \iint_{Q_\tau(x_0,t_0)} \nabla u^\varepsilon u^\varepsilon_t \psi \nabla \psi \, dxdt = - \iint_{Q_\tau(x_0,t_0)} Y^\varepsilon f_\varepsilon(u^\varepsilon) u^\varepsilon_t \psi^2 \, dxdt.$$

Now we use Young's inequality to obtain

$$\frac{1}{2}\iint_{Q_\tau(x_0,t_0)}(u_t^\varepsilon)^2\psi^2\,dxdt+\frac{1}{2}\int_{B_\tau(x_0)}|\nabla u^\varepsilon(x_0,t_0+\tau^2)|^2\psi^2\,dx\le$$
$$\frac{1}{2}\int_{B_\tau(x_0)}|\nabla u^\varepsilon(x_0,t_0-\tau^2)|^2\psi^2\,dx-\iint_{Q_\tau(x_0,t_0)}Y^\varepsilon f_\varepsilon(u^\varepsilon)u_t^\varepsilon\psi^2\,dxdt$$
$$+C\iint_{Q_\tau(x_0,t_0)}|\nabla u^\varepsilon|^2|\nabla\psi|^2\,dxdt.$$

Then, by Proposition 1.1.3

$$\int_{B_{\tau/2}(x_0)}\int_{t_0-\tau^2}^{t_0+\tau^2}(u_t^\varepsilon)^2\,dxdt\le\int_{B_\tau(x_0)}|\nabla u^\varepsilon(x_0,t_0-\tau^2)|^2\psi^2\,dx$$
$$+2\left|\iint_{Q_\tau(x_0,t_0)}Y^\varepsilon f_\varepsilon(u^\varepsilon)u_t^\varepsilon\psi^2\,dxdt\right|+C\iint_{Q_\tau(x_0,t_0)}|\nabla u^\varepsilon|^2|\nabla\psi|^2\,dxdt$$
$$\le C(\tau)+2\left|\iint_{Q_\tau(x_0,t_0)}Y^\varepsilon f_\varepsilon(u^\varepsilon)u_t^\varepsilon\psi^2\,dxdt\right|.$$

Hence, it only remains to get bounds for

$$\iint_{Q_\tau}\psi^2u_t^\varepsilon Y^\varepsilon f_\varepsilon(u^\varepsilon)dxdt=I.$$

Let

$$\mathcal{G}_\varepsilon(u,x,t)=\int_0^u(w^\varepsilon(x,t)+s)f_\varepsilon(s)ds,$$

then

$$\frac{\partial}{\partial t}\left(\mathcal{G}_\varepsilon(u^\varepsilon,x,t)\right)=\frac{\partial u^\varepsilon}{\partial t}Y^\varepsilon f_\varepsilon(u^\varepsilon)+\frac{\partial\mathcal{G}_\varepsilon}{\partial t}(u^\varepsilon,x,t),$$

so that we get

$$I=\iint_{Q_\tau}\psi^2\frac{\partial}{\partial t}\left(\mathcal{G}_\varepsilon(u^\varepsilon,x,t)\right)\,dxdt-\iint_{Q_\tau}\psi^2\frac{\partial\mathcal{G}_\varepsilon}{\partial t}(u^\varepsilon,x,t)dxdt=A-B.$$

Let us first get bounds on A:

$$\begin{aligned}
A &= \int_{t_0-\tau^2}^{t_0+\tau^2} \int_{B_\tau(x_0)} \psi^2 \frac{\partial}{\partial t} \left(\mathcal{G}_\varepsilon(u^\varepsilon, x, t)\right)\, dxdt \\
&= \int_{B_\tau(x_0)} \psi^2 \left[\int_{t_0-\tau^2}^{t_0+\tau^2} \frac{\partial}{\partial t} \left(\mathcal{G}_\varepsilon(u^\varepsilon, x, t)\right)\, dt \right] dx \\
&= \int_{B_\tau(x_0)} \psi^2 \left[\mathcal{G}_\varepsilon(u^\varepsilon(x, t_0+\tau^2), x, t_0+\tau^2) - \right. \\
&\qquad\qquad \left. \mathcal{G}_\varepsilon(u^\varepsilon(x, t_0-\tau^2), x, t_0-\tau^2)\right]\, dx.
\end{aligned}$$

Since $u^\varepsilon \geq -C\varepsilon$, $f_\varepsilon(s) = 0$ if $s \geq \varepsilon$ and $|w^\varepsilon| = \mathrm{O}(\varepsilon)$, we have

$$|\mathcal{G}_\varepsilon(u^\varepsilon, x, t)| \leq C\varepsilon \int_{-C\varepsilon}^{\varepsilon} f_\varepsilon(s)ds + \int_{-C\varepsilon}^{\varepsilon} s f_\varepsilon(s)ds \leq C,$$

so that

$$|A| \leq C(\tau).$$

It only remains to get bounds on B. For that purpose, let us first make the following observation:

$$\left|\frac{\partial \mathcal{G}_\varepsilon}{\partial t}(u^\varepsilon, x, t)\right| = \left|\frac{\partial w^\varepsilon}{\partial t}(x,t) \int_0^{u^\varepsilon} f_\varepsilon(s)ds\right| \leq \frac{C}{\varepsilon} \left|\frac{\partial w^\varepsilon}{\partial t}(x,t)\right|.$$

By (1.0.3),

$$\left|\frac{\partial w^\varepsilon}{\partial t}\right| \leq C\varepsilon \quad \text{ for } (x,t) \in \mathcal{N}_\tau(K).$$

Therefore, using the fact that $0 \leq \psi \leq 1$, we get

$$B \leq \frac{C}{\varepsilon} \iint_{Q_\tau} \left|\frac{\partial w^\varepsilon}{\partial t}(x,t)\right| dxdt \leq \frac{C}{\varepsilon} |Q_\tau| \left|\frac{\partial w^\varepsilon}{\partial t}\right| \leq C(K, \tau).$$

Thus,

$$\int_{B_{\tau/2}(x_0)} \int_{t_0-\tau^2}^{t_0+\tau^2} (u_t^\varepsilon)^2\, dxdt \leq C,$$

with C independent of ε and $(x_0, t_0) \in K$. Now, as K is compact,

$$\iint_K (u_t^\varepsilon)^2\, dxdt \leq C,$$

so that, for a subsequence, $\frac{\partial}{\partial t} u^{\varepsilon_{n'}} \to \frac{\partial}{\partial t} u$ weakly in $L^2(K)$ and by a standard diagonal argument, (3) follows.

Let us see that u is a solution of the heat equation in $\{u > 0\}$. In fact, from the fact that $u^\varepsilon \to u$ uniformly on compact subsets of $\mathcal{D}$, we deduce that every point $(x_0, t_0) \in \{u > 0\}$ has a neighborhood

V such that $u^\varepsilon(x,t) \geq \lambda > 0$ for some $\lambda > 0$. Therefore, for $\varepsilon < \lambda$, $f_\varepsilon(u^\varepsilon(x,t)) = 0$ in V. Thus u^ε is caloric in V for every $\varepsilon < \lambda$, and then, the same fact holds for u.

Let us finally analyze the convergence of the gradients. We already know that $\|\nabla u^\varepsilon\|_{L^\infty(\mathcal{N}_\tau(K))} \leq L$. So we can assume that $\nabla u^\varepsilon \to \nabla u$ weakly in $L^2(\mathcal{N}_\tau(K))$. In particular

$$\iint_{\mathcal{N}_\tau(K)} \phi|\nabla u|^2 \leq \liminf_{\varepsilon\to 0} \iint_{\mathcal{N}_\tau(K)} \phi|\nabla u^\varepsilon|^2,$$

for every nonnegative $\phi \in L^\infty(\mathcal{D})$.

We follow here ideas from [**2**] and [**13**] in order to prove that we have strong convergence.

Since $\Delta u - u_t = 0$ in $\{u > 0\}$, if we take $\delta > 0$ and multiply this equation by $(u-\delta)^+\psi(x)$ with $\psi \in L^\infty(\mathcal{D})$ and nonnegative, we get after integration by parts in $Q_\tau(x_0,t_0)$,

$$\begin{aligned}\iint_{\{u>\delta\}} |\nabla u|^2\psi &= -\iint_{\{u>\delta\}} u\nabla u\nabla\psi + \delta\iint_{\{u>\delta\}} \nabla u\nabla\psi \\ &-\frac{1}{2}\int_{\{u>\delta\}}(u-\delta)^2(x,t_0+\tau^2)\psi(x) + \frac{1}{2}\int_{\{u>\delta\}}(u-\delta)^2(x,t_0-\tau^2)\psi(x).\end{aligned}$$

Now, letting $\delta \to 0$, we get

$$\begin{aligned}\iint_{\{u>0\}} |\nabla u|^2\psi = -\iint_{\{u>0\}} u\nabla u\nabla\psi &- \frac{1}{2}\int_{\{u>0\}} u^2(x,t_0+\tau^2)\psi(x) \\ &+ \frac{1}{2}\int_{\{u>0\}} u^2(x,t_0-\tau^2)\psi(x).\end{aligned}$$

On the other hand, since $\psi \geq 0$, $f_\varepsilon \geq 0$ and $u^\varepsilon \geq -C\varepsilon$, multiplying (1.0.1) by $(u^\varepsilon + C\varepsilon)\psi$ and integrating by parts we get

$$\begin{aligned}\iint_{Q_\tau(x_0,t_0)} |\nabla u^\varepsilon|^2\psi \leq &-\iint_{Q_\tau(x_0,t_0)} u^\varepsilon\nabla u^\varepsilon\nabla\psi - C\varepsilon\iint_{Q_\tau(x_0,t_0)} \nabla u^\varepsilon\nabla\psi \\ &-\frac{1}{2}\int_{B_\tau(x_0)}(u^\varepsilon+C\varepsilon)^2(x,t_0+\tau^2)\psi(x) \\ &+\frac{1}{2}\int_{B_\tau(x_0)}(u^\varepsilon+C\varepsilon)^2(x,t_0-\tau^2)\psi(x).\end{aligned}$$

Thus,

$$\limsup_{\varepsilon\to 0}\iint_{Q_\tau(x_0,t_0)} |\nabla u^\varepsilon|^2\psi \leq \iint_{Q_\tau(x_0,t_0)} |\nabla u|^2\psi,$$

so that

$$\|\psi^{1/2}\nabla u^\varepsilon\|_{L^2(Q_\tau(x_0,t_0))} \to \|\psi^{1/2}\nabla u\|_{L^2(Q_\tau(x_0,t_0))}.$$

Since, in addition,

$$\psi^{1/2}\nabla u^\varepsilon \to \psi^{1/2}\nabla u \ \text{ weakly in } L^2(Q_\tau(x_0,t_0)),$$

it follows that

$$\psi^{1/2}\nabla u^\varepsilon \to \psi^{1/2}\nabla u \ \text{ in } L^2(Q_\tau(x_0,t_0)).$$

Therefore, as $\psi \equiv 1$ in $B_{\tau/2}(x_0)$,

$$\nabla u^\varepsilon \to \nabla u \ \text{ in } L^2(Q_{\tau/2}(x_0,t_0))$$

and since K is compact, this implies that

$$\nabla u^\varepsilon \to \nabla u \ \text{ in } L^2(K).$$

By the same standard diagonal argument used before, the assertion of the Theorem follows. □

Next we show that the limit function u is a solution of the free boundary problem in a very weak sense.

Proposition 1.2.2. *Let $(u^{\varepsilon_j}, Y^{\varepsilon_j})$ be a family of solutions of (1.0.1) in a domain $\mathcal{D} \subseteq \mathbb{R}^{N+1}$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{D}$, $Y^{\varepsilon_j} \geq 0$ and verify (1.0.2). Then, there exists a locally finite measure μ supported on the free boundary $\mathcal{D} \cap \partial\{u > 0\}$ such that $Y^{\varepsilon_j} f_{\varepsilon_j}(u^{\varepsilon_j}) \to \mu$ weakly in $\mathcal{D}$ and therefore*

$$\Delta u - \frac{\partial u}{\partial t} = \mu \quad \text{in } \mathcal{D}.$$

That is $\forall\ \phi \in C_c^\infty(\mathcal{D})$

$$\iint_{\mathcal{D}} (u\phi_t - \nabla u \nabla \phi)\, dxdt = \iint_{\mathcal{D}} \phi \, d\mu. \tag{1.2.3}$$

Proof. Let us multiply (1.0.1) by $\phi \in C_0^\infty(\mathcal{D})$ and integrate by parts. We obtain

$$\iint_{\mathcal{D}} (u^\varepsilon\phi_t - \nabla u^\varepsilon \nabla \phi)\, dxdt = \iint_{\mathcal{D}} Y^\varepsilon f_\varepsilon(u^\varepsilon)\phi \, dxdt. \tag{1.2.4}$$

We want to pass to the limit in (1.2.4). We now that $u^{\varepsilon_j} \to u$ uniformly on compact sets of $\mathcal{D}$ and thus from Proposition 1.2.1, $\nabla u^{\varepsilon_j} \to \nabla u$ in $L^2_{\mathrm{loc}}(\mathcal{D})$, so the convergence of the left hand side follows. Now let $K \subset \mathcal{D}$ be a compact set and choose $\phi = \phi_K \in C_0^\infty(\mathcal{D})$ such that $\phi_K = 1$ in K. Then (1.2.4) yields

$$\iint_K Y^{\varepsilon_j} f_{\varepsilon_j}(u^{\varepsilon_j})\, dxdt \leq C(\phi_K).$$

This L^1_{loc} bound implies that there exists a locally finite measure μ in $\mathcal{D}$, such that (for a subsequence) $Y^{\varepsilon_j} f_{\varepsilon_j}(u^{\varepsilon_j}) \to \mu$ as measures in $\mathcal{D}$. Now, passing to the limit in (1.2.4) we get (1.2.3). In addition, we see that (1.2.3) implies that the whole sequence $Y^{\varepsilon_j} f_{\varepsilon_j}(u^{\varepsilon_j})$ converge to μ and that

$$\Delta u - u_t = \mu \quad \text{in } \mathcal{D}.$$

Finally, since we know that $\Delta u - u_t = 0$ in $\{u > 0\}$, we conclude that

$$\text{support } \mu \subset \mathcal{D} \cap \partial\{u > 0\},$$

and the proof is complete. $\square$

3. A technical lemma

In this section we state an approximation lemma that will be used throughout the rest of the Thesis.

LEMMA 1.3.1. *Let $(u^{\varepsilon_j}, Y^{\varepsilon_j})$ be a family of solutions of (1.0.1) in a domain $\mathcal{D} \subseteq \mathbb{R}^{N+1}$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{D}$, $Y^{\varepsilon_j} \geq 0$ and verify (1.0.2). Let $(x_0, t_0) \in \mathcal{D} \cap \partial\{u > 0\}$ and let $(x_n, t_n) \in \mathcal{D} \cap \partial\{u > 0\}$ be such that $(x_n, t_n) \to (x_0, t_0)$ as $n \to \infty$. Let $\lambda_n \to 0$, $u_{\lambda_n}(x,t) = \frac{1}{\lambda_n} u(x_n + \lambda_n x, t_n + \lambda_n^2 t)$ and $(u^{\varepsilon_j})_{\lambda_n}(x,t) = \frac{1}{\lambda_n} u^{\varepsilon_j}(x_n + \lambda_n x, t_n + \lambda_n^2 t)$. Assume that $u_{\lambda_n} \to U$ as $n \to \infty$ uniformly on compact sets of $\mathbb{R}^{N+1}$. Then, there exists $j(n) \to \infty$ such that for every $j_n \geq j(n)$ there holds that $\frac{\varepsilon_{j_n}}{\lambda_n} \to 0$ and*

1. *$(u^{\varepsilon_{j_n}})_{\lambda_n} \to U$ uniformly on compact sets of $\mathbb{R}^{N+1}$,*
2. *$\nabla (u^{\varepsilon_{j_n}})_{\lambda_n} \to \nabla U$ in $L^2_{loc}(\mathbb{R}^{N+1})$,*
3. *$\frac{\partial}{\partial t}(u^{\varepsilon_{j_n}})_{\lambda_n} \to \frac{\partial}{\partial t} U$ weakly in $L^2(\mathbb{R}^{N+1})$.*
   Also, we deduce that
4. *$\nabla u_{\lambda_n} \to \nabla U$ in $L^2(\mathbb{R}^{N+1})$,*
5. *$\frac{\partial}{\partial t} u_{\lambda_n} \to \frac{\partial}{\partial t} U$ weakly in $L^2(\mathbb{R}^{N+1})$.*

PROOF. The proof is a rather straightforward adaptation of Lemma 3.2 of [**10**] but we include here the proof in order to make the thesis self contained.

Let us find the sequence $j(n)$. In order to verify (1),

$$\begin{aligned}(u^{\varepsilon_j})_{\lambda_n}(x,t) - U(x,t) =& \frac{1}{\lambda_n}\left[u^{\varepsilon_j}(x_n+\lambda_n x, t_n+\lambda_n^2 t)\right.\\ &\left.-u(x_n+\lambda_n x, t_n+\lambda_n^2 t)\right]\\ &+(u_{\lambda_n}(x,t)-U(x,t)) = I + II.\end{aligned}$$

Let us fix $r>0$ such that $Q_{3r}(x_0,t_0) \subset\subset \mathcal{D}$, so for n large

$$Q_r(x_0,t_0) \subset Q_{2r}(x_n,t_n) \subset\subset \mathcal{D}.$$

Let $k>0$ be fixed and $\delta>0$ be arbitrary. We know by hypotheses that $|II|<\delta$ in $Q_k(0,0)$ if $n \geq n(k,\delta)$. Let us bound $|I|$.

For each n there exists $j(n)$ such that, if $j>j(n)$,

$$|u^{\varepsilon_j}(x,t)-u(x,t)| \leq \frac{\lambda_n}{n} \quad \text{for } (x,t)\in Q_r(x_n,t_n).$$

Therefore, if $j>j(n)$ with n large so that $\lambda_n < r/k$ then,

$$|I| \leq \frac{1}{n} \quad \text{for } (x,t)\in Q_k(0,0).$$

So that if $j>j(n)$ and n large,

$$|(u^{\varepsilon_j})_{\lambda_n}(x,t)-U(x,t)| < \delta + \frac{1}{n} \quad \text{for } (x,t)\in Q_k(0,0).$$

Therefore, if $j_n \geq j(n)$, then $(u^{\varepsilon_{j_n}})_{\lambda_n} \to U$ as $n\to+\infty$ uniformly in $Q_k(0,0)$. In particular $(u^{\varepsilon_j})_{\lambda_n}$ are bounded uniformly in n and j in $Q_k(0,0)$ for $j \geq j(n)$ and n large enough.

It is easy to see that $(u^{\varepsilon_j})_{\lambda_n}$ are solutions to

$$\Delta(u^{\varepsilon_j})_{\lambda_n} - \frac{\partial (u^{\varepsilon_j})_{\lambda_n}}{\partial t} = ((u^{\varepsilon_j})_{\lambda_n} + (w^{\varepsilon_j})_{\lambda_n}) f_{\varepsilon_j/\lambda_n}((u^{\varepsilon_j})_{\lambda_n}),$$

where $(w^{\varepsilon_j})_{\lambda_n} = \frac{1}{\lambda_n} w^{\varepsilon_j}(x_n+\lambda_n x, t_n+\lambda_n^2 t)$, in $Q_k(0,0)$ for n large, and we may assume without loss of generality that $\varepsilon_j/\lambda_n < 1/n$ for $j>j(n)$.

By Proposition 1.2.1, for every choice of a sequence (j_n) with $j_n \geq j(n)$ there exists a subsequence $j_{n'}$ such that for the corresponding $\lambda_{n'}$,

$$\nabla(u^{\varepsilon_{j_{n'}}})_{\lambda_{n'}} \to \nabla U \quad \text{in } L^2_{\mathrm{loc}}(Q_k(0,0))$$

and

$$\frac{\partial}{\partial t}(u^{\varepsilon_{j_{n'}}})_{\lambda_{n'}} \to \frac{\partial}{\partial t}U \quad \text{weakly in } L^2_{\mathrm{loc}}(Q_k(0,0)).$$

By the uniqueness of the limit we see that the whole sequence $(u^{\varepsilon_j})_{\lambda_n}$ converges. At this point we want to remark that the sequence $j(n)$ is independent of k. Therefore (1), (2), and (3) are proved.

Let us see that we also have (4). In fact,

$$\begin{aligned}\|\nabla u_{\lambda_n} - \nabla U\|_{L^2(Q_k(0,0))} \leq &\|\nabla u_{\lambda_n} - \nabla (u^{\varepsilon_j})_{\lambda_n}\|_{L^2(Q_k(0,0))} \\ &+ \|\nabla (u^{\varepsilon_j})_{\lambda_n} - \nabla U\|_{L^2(Q_k(0,0))} = I + II\end{aligned}$$

We know that $|II| < \delta$ if $j \geq j(n)$ for n large enough. Let us estimate I.

$$\begin{aligned}&\|\nabla u_{\lambda_n} - \nabla (u^{\varepsilon_j})_{\lambda_n}\|^2_{L^2(Q_k(0,0))} = \\ &\iint_{Q_k(0,0)} |\nabla u - \nabla u^{\varepsilon_j}|^2 (x_n + \lambda_n x, t_n + \lambda^2 t)\, dxdt = \\ &\frac{1}{\lambda_n^{N+2}} \iint_{Q_{\lambda_n k}(x_n,t_n)} |\nabla u - \nabla u^{\varepsilon_j}|^2 (x,t)\, dxdt.\end{aligned}$$

By Proposition 1.2.1 and the fact that the whole sequence u^{ε_j} converges to u, $\nabla u^{\varepsilon_j} \to \nabla u$ in $L^2(Q_r(0,0))$, where $Q_r(x_0,t_0) \subset Q_{2r}(x_n,t_n) \subset\subset \mathcal{D}$. Therefore if j is sufficiently large and n is large enough so that $\lambda_n k \leq r$,

$$\iint_{Q_{\lambda_n k}(x_n,t_n)} |\nabla u - \nabla u^{\varepsilon_j}|^2 (x,t)\, dxdt < \lambda_n^{N+2} \delta^2.$$

Therefore,

$$\|\nabla u_{\lambda_n} - \nabla U\|_{L^2(Q_k(0,0))} \leq 2\delta$$

if n is large and thus (4) follows.

Finally, let us show that (5) holds. Given $k > 0$, we want to bound $\|\frac{\partial}{\partial t} u_{\lambda_n}\|_{L^2(Q_k(0,0))}$.

We first see that the uniform bound for $(u^{\varepsilon_j})_{\lambda_n}$ shown above, together with Proposition 1.2.1, implies that there exists $C > 0$ such that for $j \geq j(n)$ and n large

$$\left\| \frac{\partial}{\partial t} (u^{\varepsilon_j})_{\lambda_n} \right\|_{L^2(Q_k(0,0))} \leq C.$$

Next, it is easy to see that for every function v such that $v_t \in L^2(Q_k(x_0,t_0))$ and for every $\lambda > 0$ such that $\lambda k \leq r$,

$$\left\| \frac{\partial}{\partial t} v_\lambda \right\|_{L^2(Q_k(0,0))} \leq \frac{1}{\lambda^{N+2}} \|v_t\|_{L^2(Q_{\lambda k}(x_0,t_0))}$$

(where $v_\lambda(x,t) = \frac{1}{\lambda} v(x_0 + \lambda x, t_0 + \lambda^2 t)$) and therefore, for n large

$$\begin{aligned}\left\| \frac{\partial}{\partial t} u_{\lambda_n} \right\|_{L^2(Q_k(0,0))} =& \frac{\|u_t\|_{L^2(Q_{\lambda_n k}(x_n,t_n))} - \|u_t^{\varepsilon_j}\|_{L^2(Q_{\lambda_n k}(x_n,t_n))}}{\lambda_n^{N+2}} \\ &+ \left\| \frac{\partial}{\partial t} (u^{\varepsilon_j})_{\lambda_n} \right\|_{L^2(Q_k(0,0))} = I + II.\end{aligned}$$

We already know that for $j \geq j(n)$, $|II| \leq C$. On the other hand since $u_t^{\varepsilon_j} \to u_t$ weakly in weakly in $L^2(Q_{3r}(x_0,t_0))$,

$$\|u_t\|_{L^2(Q_{\lambda_n k}(x_n,t_n))} \leq \liminf_{j \to +\infty} \|u_t^{\varepsilon_j}\|_{L^2(Q_{\lambda_n k}(x_n,t_n))}.$$

Thus for $\delta > 0$ and n large,

$$\|u_t\|_{L^2(Q_{\lambda_n k}(x_n,t_n))} - \|u_t^{\varepsilon_j}\|_{L^2(Q_{\lambda_n k}(x_n,t_n))} \leq \lambda_n^{N+2} \delta$$

if j is large enough. So that

$$\|\tfrac{\partial}{\partial t} u_{\lambda_n}\|_{L^2(Q_k(0,0))} \leq C.$$

Therefore, for a subsequence $\lambda_{n'} \to 0$,

$$\frac{\partial}{\partial t} u_{\lambda_{n'}} \to U_t \quad \text{weakly in } L^2(Q_k(0,0)).$$

By the uniqueness of the limit, the whole sequence $(\frac{\partial}{\partial t} u_{\lambda_n})$ converges to U_t weakly in $L^2(Q_k(0,0))$, and therefore in $L^2_{\text{loc}}(\mathbb{R}^{N+1})$. $\square$

4. Basic examples

In this section, continuing with the local study of the problem, we study the special cases in which the limit function is the difference of two hyperplanes and the limit function $w_0 = \lim \frac{1}{\varepsilon}(Y^\varepsilon - u^\varepsilon)$ in (1.0.4) is constant. First, we show that if $u = \alpha x_1^+$, there holds that $0 \leq \alpha \leq \sqrt{2M_{w_0}}$ where

$$M_{w_0} = \int_{-w_0}^1 (s + w_0) f(s)\, ds.$$

Next we prove that if $u = \alpha x_1^+ + \bar{\alpha} x_1^-$ with $\alpha, \bar{\alpha} > 0$ then $\alpha = \bar{\alpha} \leq \sqrt{2M_{w_0}}$.

These lemmas, will be helpful in the remaining of the thesis where the situations covered by these lemmas appear as a blow-up limit of (1.0.1) (see, for example, Proposition 1.5.1).

Lemma 1.4.1. *Let $(u^{\varepsilon_j}, Y^{\varepsilon_j})$ be a solution to (1.0.1) in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$ such that $Y^{\varepsilon_j} \geq 0$, and verify (1.0.4) in $\mathcal{D}$ with $w_0 =$ constant. Let $(x_0, t_0) \in \mathcal{D}$ and assume that u^{ε_j} converges to $u = \alpha(x - x_0)_1^+$ uniformly on compact subsets of $\mathcal{D}$, with $\alpha \in \mathbb{R}$ and $\varepsilon_j \to 0$. Then,*

$$0 \leq \alpha \leq \sqrt{2M_{w_0}}. \tag{1.4.2}$$

where $M_{w_0} = \int_{-w_0}^{1} (s + w_0) f(s)\, ds$.

Proof. The proof is an adaptation of Proposition 5.2 of [**10**].

Without loss of generality we may assume that $(x_0, t_0) = (0, 0)$.

First we see that necessarily $\alpha \geq 0$ since u is subcaloric in $\mathcal{D}$ and $u(0,0) = 0$. If $\alpha = 0$ there is nothing to prove. So let us assume that $\alpha > 0$.

Let $\psi \in C_c^\infty(\mathcal{D})$. Multiplying (P_ε) by $u^\varepsilon_{x_1}\psi$ and integrating by parts we get

$$\begin{aligned}\iint_{\mathcal{D}} u_t^{\varepsilon_j} u_{x_1}^{\varepsilon_j} \psi =& \frac{1}{2} \iint_{\mathcal{D}} |\nabla u^{\varepsilon_j}|^2 \psi_{x_1} - \iint_{\mathcal{D}} u_{x_1}^{\varepsilon_j} \nabla u^{\varepsilon_j} \nabla \psi \\ &+ \iint_{\mathcal{D}} \mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}, x, t) \psi_{x_1} + \iint_{\mathcal{D}} w_{x_1}^{\varepsilon_j} \Big(\int_{-w_0}^{u^{\varepsilon_j}} f_{\varepsilon_j}(s) ds \Big) \psi,\end{aligned} \tag{1.4.3}$$

where $\mathcal{B}_\varepsilon(u, x, t) = \int_{-w_0\varepsilon}^{u} (s + w^\varepsilon) f_\varepsilon(s) ds$.

In order to pass to the limit in (1.4.3) we observe that, by Proposition 1.2.1

$$(u^{\varepsilon_j})_t \to 0 \qquad \text{weakly in } L^2_{\text{loc}}(\mathcal{D}),$$
$$\nabla u^{\varepsilon_j} \to \alpha \mathcal{X}_{\{x_1 > 0\}} e_1 \qquad \text{in } L^2_{\text{loc}}(\mathcal{D}).$$

On the other hand,

$$\frac{\nabla w^{\varepsilon_j}}{\varepsilon_j} \to 0 \qquad \text{uniformly on compact subsets of } \mathcal{D}.$$

Therefore, in order to pass to the limit in (1.4.3) we only need to analyze the limit of $\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}, x, t)$. On one hand, it is easy to see that

$$\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}(x,t), x, t) \to M_{w_0} \tag{1.4.4}$$

for every (x, t) such that $x_1 > 0$. In fact,

$$\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}, x, t) = \int_{-w_0}^{\frac{u^{\varepsilon_j}}{\varepsilon_j}} \big(s + \frac{w^{\varepsilon_j}}{\varepsilon_j}\big) f(s) ds = \int_{-w_0}^{1} \big(s + \frac{w^{\varepsilon_j}}{\varepsilon_j}\big) f(s) ds$$

if j is large enough. Since $|\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j},x,t)| \le C$ there holds that (1.4.4) holds in $L^1_{\rm loc}(\{x_1 \ge 0\}$.

On the other hand, there exists $\bar M(x,t) \in L^\infty(\mathcal{D})$ such that

$$\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j},x,t) \to \bar M(x,t) \text{ weakly in } L^2_{\rm loc}(\mathcal{D}).$$

Clearly, $\bar M(x,t) = M_{w_0}$ in $\{x_1 > 0\}$. Let us see that $\bar M(x,t) = \bar M(t)$ in $\{x_1 < 0\}$. In fact,

$$\begin{aligned}\nabla(\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}(x,t),x,t)) &= \frac{\partial \mathcal{B}_{\varepsilon_j}}{\partial u}(u^{\varepsilon_j},x,t)\nabla u^{\varepsilon_j} + \nabla \mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j},x,t)\\ &= (u^{\varepsilon_j}+w^{\varepsilon_j})f_{\varepsilon_j}(u^{\varepsilon_j})\nabla u^{\varepsilon_j} + \nabla w^{\varepsilon_j}\int_{-w_0\varepsilon_j}^{u^{\varepsilon_j}} f_{\varepsilon_j}(s)ds\\ &= Y^{\varepsilon_j} f_{\varepsilon_j}(u^{\varepsilon_j})\nabla u^{\varepsilon_j} + \frac{\nabla w^{\varepsilon_j}}{\varepsilon_j}\int_{-w_0}^{\frac{u^{\varepsilon_j}}{\varepsilon_j}} f(s)ds.\end{aligned}$$

Since $Y^{\varepsilon_j} f_{\varepsilon_j}(u^{\varepsilon_j}) \to 0$ in $L^1_{\rm loc}(\{x_1<0\})$, ∇u^{ε_j} is uniformly bounded in $L^\infty(\mathcal{D}')$ if $\mathcal{D}' \subset\subset \mathcal{D}$ and $\frac{\nabla w^{\varepsilon_j}}{\varepsilon_j} \to 0$ uniformly on compact subsets of $\mathcal{D}$, there holds that

$$\nabla(\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j}(x,t),x,t)) \to 0 \qquad \text{in } L^1_{\rm loc}(\{x_1<0\}).$$

So that, passing to the limit in (1.4.3) we get

$$\frac{\alpha^2}{2}\iint_{\{x_1>0\}} \psi_{x_1} = M_{w_0}\iint_{\{x_1>0\}} \psi_{x_1} + \int_{\{x_1<0\}} \bar M(t)\psi_{x_1}.$$

Thus, integrating in the variable x_1 we get

$$\int_{\{x_1=0\}} \big(\frac{\alpha^2}{2} - M_{w_0} + \bar M(t)\big)\psi = 0.$$

Since ψ is arbitrary, we conclude that

$$\frac{\alpha^2}{2} - M_{w_0} + \bar M(t) = 0.$$

Finally, we notice that $\bar M(t) \ge 0$. In fact,

$$\begin{aligned}\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j},x,t) &= \int_{-\frac{w^{\varepsilon_j}}{\varepsilon_j}}^{\frac{u^{\varepsilon_j}}{\varepsilon_j}} \big(s+\frac{w^{\varepsilon_j}}{\varepsilon_j}\big)f(s)ds + \int_{-w_0}^{-\frac{w^{\varepsilon_j}}{\varepsilon_j}} \big(s+\frac{w^{\varepsilon_j}}{\varepsilon_j}\big)f(s)ds\\ &\ge \int_{-w_0}^{-\frac{w^{\varepsilon_j}}{\varepsilon_j}} \big(s+\frac{w^{\varepsilon_j}}{\varepsilon_j}\big)f(s)ds \to 0\end{aligned}$$

since $\frac{w^{\varepsilon_j}}{\varepsilon_j} \to w_0$ uniformly on compact subsets of $\mathcal{D}$.

Thus,

$$\alpha = \sqrt{2(M_{w_0} - \bar{M}(t))} \le \sqrt{2M_{w_0}}$$

and the proof is complete. □

LEMMA 1.4.5. *Let $(u^{\varepsilon_j}, Y^{\varepsilon_j})$ be a solution to (1.0.1) in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$ such that $Y^{\varepsilon_j} \ge 0$ and verify (1.0.4) with $w_0 =$ constant in $\mathcal{D}$. Let $(x_0, t_0) \in \mathcal{D}$ and assume that u^{ε_j} converges to $u = \alpha(x - x_0)_1^+ + \bar{\alpha}(x - x_0)_1^-$ uniformly on compact subsets of $\mathcal{D}$, with $\alpha, \bar{\alpha} > 0$ and $\varepsilon_j \to 0$. Then,*

$$\bar{\alpha} = \alpha \le \sqrt{2M_{w_0}} \tag{1.4.6}$$

where $M_{w_0} = \int_{-w_0}^{1} (s + w_0) f(s)\, ds$.

PROOF. We argue in a similar way as in Proposition 5.3 of [**10**].

We will denote $Q_r = Q_r(0,0)$. Without loss of generality we will assume that $(x_0, t_0) = (0,0)$ and that $Q_2 \subset\subset \mathcal{D}$.

As before, u^ε satisfies

$$\begin{aligned}\iint_{\mathcal{D}} u_t^\varepsilon u_{x_1}^\varepsilon \psi =& \frac{1}{2} \iint_{\mathcal{D}} |\nabla u^\varepsilon|^2 \psi_{x_1} - \iint_{\mathcal{D}} u_{x_1}^\varepsilon \nabla u^\varepsilon \nabla \psi \\ &+ \iint_{\mathcal{D}} \mathcal{B}_\varepsilon(u^\varepsilon, x, t) \psi_{x_1} + \iint_{\mathcal{D}} w_{x_1}^\varepsilon \left(\int_0^{u^\varepsilon} f_\varepsilon(s) ds \right) \psi.\end{aligned}$$

We want to pass to the limit. By Proposition 1.2.1 and the fact that u^{ε_j} converge to $\alpha x_1^+ + \bar{\alpha} x_1^-$ we have that

$$u_t^{\varepsilon_j} \to 0 \qquad \text{weakly in } L^2_{\text{loc}}(\mathcal{D}),$$

$$\nabla u^{\varepsilon_j} \to \alpha \mathcal{X}_{\{x_1 > 0\}} e_1 - \bar{\alpha} \mathcal{X}_{\{x_1 < 0\}} e_1 \quad \text{in } L^2_{\text{loc}}(\mathcal{D}).$$

Clearly, as $\alpha, \bar{\alpha} > 0$, $\mathcal{B}(u^{\varepsilon_j}, x, t) \to M$ in $L^1_{\text{loc}}(\mathcal{D})$.

So, passing to the limit in the latter equation for the subsequence ε_j, we get

$$-\frac{\alpha^2}{2} \iint_{\{x_1 > 0\}} \psi_{x_1} - \frac{\overline{\alpha}^2}{2} \iint_{\{x_1 < 0\}} \psi_{x_1} + M_{w_0} \iint \psi_{x_1} = 0.$$

Integrating in the x_1 variable, we conclude that

$$\alpha = \bar{\alpha}.$$

Next, we will assume that $\alpha > \sqrt{2M_{w_0}}$ and arrive at a contradiction.

First, let us consider z^{ε_j}, defined in Q_2, the solution to

$$\Delta z^{\varepsilon_j} - z_t^{\varepsilon_j} = \left(\beta_{\varepsilon_j}(z^{\varepsilon_j}) + W_{\varepsilon_j} f_{\varepsilon_j}(z^{\varepsilon_j})\right) \rho_{\varepsilon_j}(z^{\varepsilon_j}/\varepsilon_j) \quad \text{in } Q_2 \tag{1.4.7}$$

with boundary conditions

$$z^{\varepsilon_j} = u - b^{\varepsilon_j} \quad \text{on } \partial_p Q_2$$

where $\beta_\varepsilon(s) = s f_\varepsilon(s)$, $W_\varepsilon = \sup_{Q_2} w^\varepsilon$, $b^{\varepsilon_j} = \sup_{Q_2} |u^{\varepsilon_j} - u|$ and ρ_{ε_j} is a smooth cutoff function with support in $[-(w_0 + 2C_{\varepsilon_j}), 3]$ and $\rho_{\varepsilon_j} \equiv 1$ in $[-(w_0 + C_{\varepsilon_j}), 2]$ (Here $C_{\varepsilon_j} \to 0^+$ is such that $\left|w^{\varepsilon_j}/\varepsilon_j - w_0\right| \le C_{\varepsilon_j}$ in Q_2 so that $u^{\varepsilon_j}/\varepsilon_j \ge -(w_0 + C_{\varepsilon_j})$ in Q_2).

Observe that $z^{\varepsilon_j}(x_1, x', t) = z^{\varepsilon_j}(-x_1, x', t)$ in Q_2.

It is easy to see that the proofs of Propositions 1.1.3 and 1.1.5 can be adapted to z^{ε_j} so that, for a subsequence, that we still call ε_j, there holds that $z^{\varepsilon_j} \to z$ uniformly on compact sets of Q_2. We will show that $z = u$.

First,

$$\begin{aligned}\Delta u^{\varepsilon_j} - u_t^{\varepsilon_j} &= (u^{\varepsilon_j} + w^{\varepsilon_j}) f_{\varepsilon_j}(u^{\varepsilon_j}) \le \beta_{\varepsilon_j}(u^{\varepsilon_j}) + W_{\varepsilon_j} f_{\varepsilon_j}(u^{\varepsilon_j}) \\ &= \left(\beta_{\varepsilon_j}(u^{\varepsilon_j}) + W_{\varepsilon_j} f_{\varepsilon_j}(u^{\varepsilon_j})\right)\rho_{\varepsilon_j}(u^{\varepsilon_j}/\varepsilon_j) \qquad \text{in } Q_2.\end{aligned}$$

From the fact that $z^{\varepsilon_j} \le u^{\varepsilon_j}$ on $\partial_p Q_2$, we deduce that $z^{\varepsilon_j} \le u^{\varepsilon_j}$ in Q_2 and therefore $z \le u$.

In order to see that $u \le z$, we consider $a^{\varepsilon_j} \in C^2(\mathbb{R})$ such that

$$\begin{aligned}&a_{ss}^{\varepsilon_j} = \left(\beta(a^{\varepsilon_j}) + \tfrac{W_{\varepsilon_j}}{\varepsilon_j} f(a^{\varepsilon_j})\right) \rho_{\varepsilon_j}(a^{\varepsilon_j}),\ s \in \mathbb{R} \\ &a^{\varepsilon_j}(0) = 1,\ a_s^{\varepsilon_j}(0) = \alpha\end{aligned}$$

Integrating the equation we get, for every $s \in \mathbb{R}$, that

$$0 < \gamma - \kappa_{\varepsilon_j} \le a_s^{\varepsilon_j}(s) \le \alpha$$

where $\frac{1}{2}\gamma^2 \equiv \frac{1}{2}\alpha^2 - M_{w_0} > 0$ and $\kappa_{\varepsilon_j} \to 0$ when $j \to \infty$.

It follows that there exists $\overline{s}_{\varepsilon_j} < 0$ such that

$$a^{\varepsilon_j}(s) = \begin{cases} 1 + \alpha s & s \ge 0 \\ (\gamma - \kappa_{\varepsilon_j})(s - \overline{s}_{\varepsilon_j}) & s \le \overline{s}_{\varepsilon_j} \end{cases}$$

and it is easy to see that $\overline{s}_{\varepsilon_j}$ are uniformly bounded by below and moreover, there exists $\overline{s} < 0$ such that $\overline{s}_{\varepsilon_j} \to \overline{s}$.

Now let

$$\tilde{a}^{\varepsilon_j}(x) = \varepsilon_j a^{\varepsilon_j}\big(\frac{x_1}{\varepsilon_j} - \frac{b^{\varepsilon_j}}{(\gamma - \kappa_{\varepsilon_j})\varepsilon_j} + \overline{s}_{\varepsilon_j}\big).$$

Using that $\tilde{a}^{\varepsilon_j}(0, x', t) = -b^{\varepsilon_j}$ and the bounds on $a_s^{\varepsilon_j}$, we deduce that

$$\tilde{a}^{\varepsilon_j} \leq u - b^{\varepsilon_j} \text{ in } Q_2.$$

Now, since $\tilde{a}^{\varepsilon_j} \leq z^{\varepsilon_j}$ on $\partial_p Q_2$, and $\tilde{a}^{\varepsilon_j}$ is a one dimensional stationary solution to (1.4.7), we have that $\tilde{a}^{\varepsilon_j} \leq z^{\varepsilon_j}$ in Q_2. Since $\tilde{a}^{\varepsilon_j} \to u$ uniformly on compact subsets of $\{x_1 > 0\}$, we deduce that $u \leq z$ in $Q_2 \cap \{x_1 > 0\}$.

Finally, we notice that $z^{\varepsilon_j}(x_1, x', t) = z^{\varepsilon_j}(-x_1, x', t)$, so we conclude that $u \leq z$ in Q_2.

Now, let

$$\mathcal{R} = \{(x,t) | 0 < x_1 < 1, |x'| < 1, |t| < 1\}.$$

Let us multiply (1.4.7) by $z_{x_1}^{\varepsilon_j}$ and integrate in $\mathcal{R}$. Then, we have

$$\begin{aligned} E_j &= \iint_{\mathcal{R}} \frac{\partial}{\partial x_1} \left(\frac{1}{2} (z_{x_1}^{\varepsilon_j})^2 - \mathcal{F}_{\varepsilon_j}(z^{\varepsilon_j}) \right) \\ &= \iint_{\mathcal{R}} z_t^{\varepsilon_j} z_{x_1}^{\varepsilon_j} - \iint_{\mathcal{R}} \Delta_{x'} z^{\varepsilon_j} z_{x_1}^{\varepsilon_j} = F_j - G_j, \end{aligned}$$

where $\mathcal{F}_{\varepsilon_j}(z) = \int_{-W_{\varepsilon_j}}^{z} (s + W_{\varepsilon_j}) f_{\varepsilon_j}(s) \rho_{\varepsilon_j}(s/\varepsilon_j)\, ds$.

Since every z^{ε_j} is symmetric in the x_1 variable, we deduce that $z_{x_1}^{\varepsilon_j}(0, x', t) = 0$ and therefore,

$$E_j \geq \int_{\partial\mathcal{R} \cap \{x_1 = 1\}} \left(\frac{1}{2} (z_{x_1}^{\varepsilon_j})^2 - \mathcal{F}_{\varepsilon_j}(z^{\varepsilon_j}) \right) dx' dt.$$

Since $z^{\varepsilon_j} \to u = \alpha x_1^+ + \alpha x_1^-$ uniformly on compact subsets of Q_2, we deduce that, in $Q_{3/2} \cap \{x_1 > \frac{1}{2}\}$, $z^{\varepsilon_j} \geq \varepsilon_j$ is j is large and

$$\begin{aligned} &z_{x_1}^{\varepsilon_j} \to \alpha \quad \text{uniformly}, \\ &\mathcal{F}_{\varepsilon_j}(z^{\varepsilon_j}) = \int_{-W_{\varepsilon_j}/\varepsilon_j}^{1} \left(s + \frac{W_{\varepsilon_j}}{\varepsilon_j} \right) f(s) \rho_{\varepsilon_j}(s)\, ds \to M_{w_0}. \end{aligned}$$

Then we get

$$\liminf_{j \to +\infty} E_j \geq \int_{\partial\mathcal{R} \cap \{x_1 = 1\}} \left(\frac{1}{2} \alpha^2 - M_{w_0} \right) dx' dt.$$

On the other hand, we know from Lemma 1.2.1 that

$$\begin{aligned} &z_t^{\varepsilon_j} \to u_t = 0 \quad \text{weakly in } L^2(Q_{3/2}), \\ &z_{x_1}^{\varepsilon_j} \to u_{x_1} \quad \text{in } L^2(Q_{3/2}), \end{aligned}$$

which implies that $F_j \to 0$.

Finally, integrating by parts, we get

$$-G_j \le \int_{\partial\mathcal{R}\cap\{|x'|=1\}} |z^{\varepsilon_j}_{x_1}||\nabla_{x'} z^{\varepsilon_j}|\, dSdt + \int_{\partial\mathcal{R}\cap\{x_1=1\}} \frac{1}{2}|\nabla_{x'} z^{\varepsilon_j}|^2\, dx'dt.$$

From the convergence of $z^{\varepsilon_j} \to u$ it follows that

$$|\nabla_{x'} z^{\varepsilon_j}| \to 0 \quad \text{pointwise in } Q_2 \cap \{x_1 > 0\}.$$

If we now use that z^{ε_j} are locally uniformly Lipschitz in space, we deduce that

$$\limsup_{j\to+\infty}(-G_j) \le 0,$$

which gives that $\frac{1}{2}\alpha^2 - M_{w_0} \le 0$, a contradiction. This finishes the proof. □

5. Behavior of limit functions near the free boundary

In this section we analyze the behavior of a limit function $u = \lim u^{\varepsilon_j}$ near an arbitrary free boundary point.

First we show that every limit function u is a supersolution of problem (P) under the assumption (1.0.4), and then we show that if U is a global limit and w_0 in (1.0.4) is constant, there holds that $|\nabla U^+| \le \sqrt{2M_{w_0}}$, where $M_{w_0} = \int_{-w_0}^1 (s+w_0) f(s)\, ds$.

Proposition 1.5.1. *Let $(u^{\varepsilon_j}, Y^{\varepsilon_j})$ be a solution to (1.0.1) in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$ such that $Y^{\varepsilon_j} \ge 0$ and verify (1.0.4) with $w_0 > -1$. Assume that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{D}$. Then u is a supersolution of (P) in the sense that*

i. $\Delta u - u_t = 0$ *in* $\{u > 0\} \cap \mathcal{D}$

ii. $\limsup_{(x,t)\to(x_0,t_0)} |\nabla u(x,t)| \le \sqrt{2M(x_0,t_0)}$ *for* $(x_0,t_0) \in \partial\{u > 0\} \cap \mathcal{D}$, $u(x,t) > 0$.

Proof. We only have to show ii. The proof is a rather simple modification of Theorem 6.1 of [**10**].

Let $\alpha = \limsup_{(x,t)\to(x_0,t_0)} |\nabla u(x,t)|$ with $u(x,t) > 0$.

Since $u \in Lip_{\mathrm{loc}}(1, \frac{1}{2})$ in $\mathcal{D}$, we know that $\alpha < +\infty$. If $\alpha = 0$ there is nothing to prove. So let us assume that $\alpha > 0$ and let $(x_n, t_n) \to (x_0, t_0)$

be such that $u(x_n,t_n)>0$ and $|\nabla u(x_n,t_n)|\to\alpha$. Let $(z_n,s_n)\in\mathcal{D}\cap\partial\{u>0\}$ be such that

$$\begin{aligned} d_n &\equiv \max\{|x_n-z_n|,|t_n-s_n|^{1/2}\}\\ &= \inf_{(z,s)\in\partial\{u>0\}}\left\{\max\{|x_n-z|,|t_n-s|^{1/2}\}\right\}. \end{aligned}$$

Let us consider the sequence

$$u_{d_n}(x,t)=\frac{1}{d_n}u(z_n+d_nx,s_n+d_n^2t).$$

Since $u\in Lip_{\mathrm{loc}}(1,\frac12)$ in $\mathcal{D}$ and $d_n\to0$, given a compact set $K\subset\mathbb{R}^{N+1}$ the functions u_{d_n} are uniformly bounded in $Lip(1,\frac12)$ seminorm in K, if n is large enough. On the other hand, $u_{d_n}(0,0)=0$ for every n. So that u_{d_n} are uniformly bounded on compact sets of $\mathbb{R}^{N+1}$. Therefore, for a subsequence (that we still call u_{d_n}), $u_{d_n}\to u_0$ uniformly on compact sets of $\mathbb{R}^{N+1}$, where $u_0\in Lip(1,\frac12)$ in $\mathbb{R}^{N+1}$.

Let $\bar x_n=(x_n-z_n)/d_n$, $\bar t_n=(t_n-s_n)/d_n^2$. Then $(\bar x_n,\bar t_n)\in\partial Q_1(0,0)$ so that (for a subsequence) $(\bar x_n,\bar t_n)\to(\bar x,\bar t)\in\partial Q_1(0,0)$.

On the other hand, since $u_{d_n}>0$ in $Q_1(\bar x_n,\bar t_n)$ we deduce that in $Q_1(\bar x,\bar t)$, $u_0(x,t)\ge0$ and u_0 is a solution of the heat equation.

Let us consider the sequence

$$\nu_n=\frac{\nabla u_{d_n}(\bar x_n,\bar t_n)}{|\nabla u_{d_n}(\bar x_n,\bar t_n)|}=\frac{\nabla u(x_n,t_n)}{|\nabla u(x_n,t_n)|}.$$

We may assume that $\nu_n\to\nu$. Let us see that

$$|\nabla u(x_n,t_n)|\to\frac{\partial u_0}{\partial\nu}(\bar x,\bar t).$$

To this end, we will show that $\nabla u_{d_n}\to\nabla u_0$ uniformly on compact subsets of $Q_1(\bar x,\bar t)$. But this is a consequence of the fact that any such compact set is at a fixed positive distance from the boundary of $Q_1(\bar x_n,\bar t_n)$, in n is large enough. In fact, let K be any such compact set and let $\tau>0$ be such that $\mathcal{N}_{2\tau}(K)\subset Q_1(\bar x_n,\bar t_n)$ for n large. We have, for n,m large,

$$\begin{aligned} |\nabla u_{d_n}(x,t)-\nabla u_{d_m}(x,t)| &= |\nabla(u_{d_n}-u_{d_m})(x,t)|\\ &\le C\max_{\mathcal{N}_{2\tau}(K)}|u_{d_n}-u_{d_m}| \quad \text{for any } (x,t)\in K, \end{aligned}$$

since every u_{d_n} is a solution of the heat equation in $Q_1(\bar x_n,\bar t_n)$. Therefore we have the uniform convergence of the gradients, so that

$$\frac{\partial u_0}{\partial\nu}(\bar x,\bar t)=\alpha.$$

On the other hand, it is easy to see that $|\nabla u_0^+| \leq \alpha$ in $\mathbb{R}^{N+1}$. In fact, let $R > 0$ and $\delta > 0$ be fixed. There exists λ_0 such that

$$|\nabla u^+(x,t)| \leq \alpha + \delta \quad \text{for } (x,t) \in Q_{\lambda R}(x_0, t_0)$$

if $\lambda < \lambda_0$.

Since $Q_{d_n R}(z_n, s_n) \subset Q_{3\lambda_n R}(x_0, t_0)$ if $\lambda_n = \max\{|x_n - x_0|, |t_n - t_0|^{1/2}\}(\geq d_n)$ and $R > 1$ and since $\lambda_n \to 0$ as $n \to +\infty$, we deduce that

$$|\nabla u_{d_n}^+(x,t)| \leq \alpha + \delta \quad \text{for } (x,t) \in Q_{\lambda R}(0,0)$$

if n is large enough. Thus $\nabla u_{d_n}^+ \to \nabla u_0^+$ *-weakly in $L^\infty(Q_R(0,0))$ and therefore $|\nabla u_0^+| \leq \alpha + \delta$ in $Q_R(0,0)$. Since δ and R where arbitrary we deduce that

$$|\nabla u_0^+| \leq \alpha \quad \text{in } \mathbb{R}^{N+1}.$$

Let $V = \partial u_0 / \partial \nu$. We know that V is a solution of the heat equation in $\{u_0 > 0\}$ since u_0 is a solution of the heat equation in this set. On the other hand, we know that $V \leq \alpha$ in $\{u_0 > 0\}$ and $V(\bar{x}, \bar{t}) = \alpha$. Since $\alpha > 0$ we must have $u_0(\bar{x}, \bar{t}) > 0$ (otherwise $u_0 \equiv 0$ in $Q_1^-(\bar{x}, \bar{t})$) and thus $u_0 > 0$ in $Q_\rho(\bar{x}, \bar{t})$, for some $\rho > 0$. It follows that $V \equiv \alpha$ in $Q_\rho^-(\bar{x}, \bar{t})$. Moreover, if we call $\mathcal{R}$ the set of points in $\{u_0 > 0\} \cap \{t < \bar{t}\}$ which can be connected to $(\bar{x}, \bar{t})$ by a continuous curve in $\{u_0 > 0\}$ along which the $t-$coordinate is nondecreasing, we see that $V \equiv \alpha$ in $\mathcal{R}$.

Since $|\nabla u_0| \leq \alpha$ in $\{u_0 > 0\}$ we deduce that $\nabla u_0 = V\nu$ in $\mathcal{R}$. Let us assume, for the sake of simplicity, that $\nu = e_1$. Then by the considerations above

$$u_0(x,t) = \alpha x_1 + b(t) \quad \text{in } \mathcal{R}.$$

Since u_0 is caloric in $\mathcal{R}$, $b(t)$ must be constant. Thus there exists $\tilde{x} \in \mathbb{R}^N$ such that

$$u_0(x,t) = \alpha(x - \tilde{x})_1 \quad \text{in } \mathcal{R}.$$

It is not hard to see that $\mathcal{R} = \{(x - \tilde{x})_1 > 0, \ t < \bar{t}\}$. Hence,

$$u_0(x,t) = \alpha(x - \tilde{x})_1 \quad \text{in } \{(x - \tilde{x})_1 > 0, \ t < \bar{t}\}.$$

By Corollary A.1 of [**10**], we get for some $\bar{\alpha} \geq 0$

$$u_0(x,t) = \bar{\alpha}(x - \tilde{x})_1^- + o(|x - \tilde{x}| + |t - \bar{t}|^{1/2}) \quad \text{in } \{(x - \tilde{x})_1 < 0, \ t < \bar{t}\}.$$

Let us consider for $\lambda > 0$ the function $(u_0)_\lambda(x,t) = (1/\lambda) u_0(\lambda x + \tilde{x}, \lambda^2 t + \bar{t})$. Now, one can check that $(u_0)_\lambda$ converges uniformly on compact sets of $\mathbb{R}^{N+1}$ to u_{00} where

$$u_{00}(x,t) = \alpha x_1^+ + \bar{\alpha} x_1^- \quad \text{in } \{t \leq 0\}.$$

Let

$$(u^{\varepsilon_j})_{d_n}(x,t) = \frac{1}{d_n} u^{\varepsilon_j}(z_n + d_n x, s_n + d_n^2 t).$$

By Lemma 1.3.1, there exists a sequence $j_n \to +\infty$ such that $(u^{\varepsilon_{j_n}})_{d_n} \to u_0$ uniformly on compact sets of $\mathbb{R}^{N+1}$ and $\varepsilon_{j_n}/d_n \to 0$. It is easy to see that $(u^{\varepsilon_{j_n}})_{d_n}$ is a solution to

$$\Delta(u^{\varepsilon_{j_n}})_{d_n} - \frac{\partial}{\partial t}(u^{\varepsilon_{j_n}})_{d_n} = ((u^{\varepsilon_{j_n}})_{d_n} + (w^{\varepsilon_{j_n}})_{d_n}) f_{\varepsilon_{j_n}/d_n}((u^{\varepsilon_{j_n}})_{d_n})$$

in $Q_1(\tilde{x}, \bar{t})$ for n large, where $(w^{\varepsilon_{j_n}})_{d_n} = (1/d_n) w^{\varepsilon_{j_n}}(z_n + d_n x, s_n + d_n^2 t)$.

By calling $\varepsilon_n^0 = \varepsilon_{j_n}/d_n$, $u^{\varepsilon_n^0} = (u^{\varepsilon_{j_n}})_{d_n}$ and $w^{\varepsilon_n^0} = (w^{\varepsilon_{j_n}})_{d_n}$, then $u^{\varepsilon_n^0}$ are solutions to $(P_{\varepsilon_n^0})$ with

$$\frac{w^{\varepsilon_n^0}}{\varepsilon_n^0} \to w_0(x_0, t_0) \quad \text{uniformly on compact sets of } Q_1(\tilde{x}, \bar{t}),$$
$$u^{\varepsilon_n^0} \to u_0 \quad \text{uniformly on compact sets of } Q_1(\tilde{x}, \bar{t}),$$
$$(u_0)_{\lambda_k} \to u_{00} \quad \text{uniformly on compact sets of } \mathbb{R}^{N+1},$$

$\varepsilon_n^0 \to 0$ and $\lambda_k \to 0$. Therefore we can apply Lemma 1.3.1 again and find a sequence $\varepsilon_n^{00} \to 0$ and solutions $u^{\varepsilon_n^{00}}$ to $(P_{\varepsilon_n^{00}})$ in $Q_1(0,0)$ such that

$$\frac{w^{\varepsilon_n^{00}}}{\varepsilon_n^{00}} \to w_0(x_0, t_0) \quad \text{uniformly on compact sets of } Q_1(0,0),$$
$$u^{\varepsilon_n^{00}} \to u_{00} = \alpha x_1^+ + \bar{\alpha} x_1^- \quad \text{uniformly on compact sets of } Q_1(0,0).$$

Finally, if $\bar{\alpha} = 0$ we apply Lemma 1.4.1 and if $\bar{\alpha} > 0$ we apply Lemma 1.4.5 to deduce that

$$\alpha \leq \sqrt{2M(x_0, t_0)}.$$

So the Proposition is proved. $\square$

LEMMA 1.5.2. *Let $(u^{\varepsilon_j}, Y^{\varepsilon_j})$ be a solution to (1.0.1) in a domain $\mathcal{D}_j$ such that $Y^{\varepsilon_j} \geq 0$ and satisfies (1.0.4) in $\mathcal{D}_j$ with $w_0 =$ constant. Here $\mathcal{D}_j$ is such that $\mathcal{D}_j \subset \mathcal{D}_{j+1}$ and $\cup_j \mathcal{D}_j = \mathbb{R}^{N+1}$. Let us assume that $u^{\varepsilon_j} \to U$ uniformly on compact subsets of $\mathbb{R}^{N+1}$ as $j \to \infty$ and $\varepsilon_j \to 0$, with $U \geq 0$, $U \in Lip(1, 1/2)$ and $\partial\{U > 0\} \neq \emptyset$. Then,*

$$|\nabla U| \leq \sqrt{2M_{w_0}} \qquad \text{in } \mathbb{R}^{N+1} \tag{1.5.3}$$

with $M_{w_0} = \int_{-w_0}^1 (s + w_0) f(s) ds$.

PROOF. The proof is similar to that of Theorem 6.2 in [**10**]. Here we use Lemmas 1.4.1 and 1.4.5 instead of Propositions 5.2 and 5.3 in [**10**].

Let $\alpha = \sup|\nabla U^+|$. By assumption, $\alpha < +\infty$. If $\alpha = 0$ there is nothing to prove. Let us assume that $\alpha > 0$ and let (x_n, t_n) be such that $U(x_n, t_n) > 0$ and $|\nabla U(x_n, t_n)| \to \alpha$ as $n \to +\infty$. Let $(z_n, s_n) \in \partial\{U > 0\}$ be such that

$$\begin{aligned} d_n &:= \max\{|x_n - z_n|, |t_n - s_n|^{1/2}\} \\ &= \inf_{(z,s)\in\partial\{u>0\}} \left\{\max\{|x_n - z|, |t_n - s|^{1/2}\}\right\}. \end{aligned}$$

Let

$$U_{d_n}(x, t) = \frac{1}{d_n} U(z_n + d_n x, s_n + d_n^2 t).$$

Then the family U_{d_n} is uniformly bounded in $Lip(1, \frac{1}{2})$ seminorm in $\mathbb{R}^{N+1}$ and since $U_{d_n}(0, 0) = 0$, the family is uniformly bounded on compact sets of $\mathbb{R}^{N+1}$. So that we may assume (by taking a subsequence that we still call U_{d_n}) that $U_{d_n} \to U_0$ uniformly on compact sets of $\mathbb{R}^{N+1}$, where $U_0 \in Lip(1, \frac{1}{2})$ in $\mathbb{R}^{N+1}$.

Let $\bar{x}_n = (x_n - z_n)/d_n$, $\bar{t}_n = (t_n - s_n)/d_n^2$. It is easy to see (by taking a subsequence) that $(\bar{x}_n, \bar{t}_n) \to (\bar{x}, \bar{t}) \in \partial Q_1(0, 0)$. Also,

$$\frac{\nabla U_{d_n}(\bar{x}_n, \bar{t}_n)}{|\nabla U_{d_n}(\bar{x}_n, \bar{t}_n)|} \to \nu.$$

We will assume without loss of generality that $\nu = e_1$.

Proceeding in a similar way as in the proof of Proposition 1.5.1 we see that necessarily

$$U_0(x, t) = \begin{cases} \alpha(x - \tilde{x})_1^+ & \text{in } (x - \tilde{x})_1 > 0,\ t < \bar{t} \\ \bar{\alpha}(x - \tilde{x})_1^- + o(|x - \tilde{x}| + |t - \bar{t}|^{1/2}) & \text{in } (x - \tilde{x})_1 < 0,\ t < \bar{t} \end{cases}$$

for some point $\tilde{x} \in \mathbb{R}^N$ and some $\bar{\alpha} \geq 0$.

Let $U_{00} = \lim_{\lambda\to 0}(U_0)_\lambda$ where $(U_0)_\lambda(x, t) = (1/\lambda)U_0(\tilde{x} + \lambda x, \bar{t} + \lambda^2 t)$. Then, $U_{00}(x, t) = \alpha x_1^+ + \bar{\alpha} x_1^-$ and the proof follows as in Proposition 1.5.1 $\square$

CHAPTER 2

The Free Boundary Problem

In this chapter, we find the free boundary condition for the limit problem and we show that the limit function u is a solution to the free boundary problem (P) in a pointwise sense, under the assumption that the free boundary admits an inward spatial normal in a parabolic measure theoretic sense (Definition 2.1.3). Then we show that, under suitable assumptions, the limit function u is a viscosity solution of the free boundary problem (P).

Finally, we end this Chapter with some applications of the results and construct a family $(u^\varepsilon, Y^\varepsilon)$ of solutions to (1.0.1) with $w^\varepsilon/\varepsilon \to w_0 \neq 0$ such that $u = \lim u^\varepsilon$ is a viscosity solution to (P), by showing that the local assumptions in Theorem 2.2.9 can be fulfilled by imposing conditions on the initial data $(u_0^\varepsilon, Y_0^\varepsilon)$.

1. The free boundary condition

Throughout this section we will assume that (1.0.4) holds and that for every $K \subset \{u \equiv 0\}^\circ$ compact there exists $0 < \eta < 1$ and $\varepsilon_0 > 0$ such that, for $\varepsilon < \varepsilon_0$

$$\frac{u^\varepsilon}{\varepsilon} \leq \eta \qquad \text{in } K. \tag{2.1.1}$$

This assumption is a natural one in applications, roughly speaking it means that the mixture temperature reaches the flame temperature only if some combustion is taking place.

As a consequence there holds that

$$w_0 = \lim_{\varepsilon \to 0} \frac{v^\varepsilon - u^\varepsilon}{\varepsilon} \geq - \limsup_{\varepsilon \to 0} \frac{u^\varepsilon}{\varepsilon} \geq -\eta > -1 \quad \text{in } K.$$

So that, for the sake of simplicity we will assume from now on that $w_0 > -1$ in $\mathcal{D}$.

We start this section with a lemma that is the essential ingredient in the subsequent proofs.

LEMMA 2.1.2. *Let* $(u^{\varepsilon_k}, Y^{\varepsilon_k})$ *be a solution to (1.0.1) in a domain* $\mathcal{D} \subset \mathbb{R}^{N+1}$ *such that* $Y^{\varepsilon_k} \geq 0$ *and (1.0.4) and (2.1.1) are satisfied with* $w_0 > -1$. *Let* $u = \lim u^{\varepsilon_k}$, *with* $\varepsilon_k \to 0$, *and* $\mathcal{B}_{\varepsilon_k}(u, x, t) = \int_{-w_0\varepsilon_k}^{u} \big(s + w^{\varepsilon_k}\big) f_{\varepsilon_k}(s) ds$. *Then,*

$$\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t) \to M(x,t)\mathcal{X}_{\{u>0\}}, \quad in\ L^1_{loc}(\mathcal{D}).$$

where $M(x,t) = \int_{-w_0(x,t)}^{1} (s + w_0(x,t)) f(s) ds$.

PROOF. First, let us observe that

$$\begin{aligned}\int_{-w_0\varepsilon}^{\varepsilon} (w^\varepsilon + s) f_\varepsilon(s) ds &= \int_{-w_0\varepsilon}^{\varepsilon} (w^\varepsilon + s) \frac{1}{\varepsilon^2} f\left(\frac{s}{\varepsilon}\right) ds \\ &= \int_{-w_0}^{1} \big(\frac{w^\varepsilon}{\varepsilon} + s\big) f(s) ds.\end{aligned}$$

Therefore,

$$\lim_{\varepsilon_k \to 0} \int_{-w_0\varepsilon_k}^{\varepsilon_k} (w^{\varepsilon_k} + s) f_{\varepsilon_k}(s) ds = M(x,t).$$

uniformly on compact sets of $\mathcal{D}$.

Let us now see that $\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t) \to M(x,t)$ uniformly on compact subsets of $\{u > 0\}$.

Let $K \subset\subset \{u > 0\}$, then there exists $\lambda > 0$ and ε_0 such that $u^{\varepsilon_k}(x,t) > \lambda \ \ \forall \varepsilon_k < \varepsilon_0$, $(x,t) \in K$. Thus, we have

$$\begin{aligned}\lim_{k\to\infty} \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}(x,t), x, t) &= \lim_{k\to\infty} \int_{-w_0\varepsilon_k}^{u^{\varepsilon_k}(x,t)} (w^\varepsilon + s) f_{\varepsilon_k}(s) ds \\ &= \lim_{k\to\infty} \int_{-w_0\varepsilon_k}^{\varepsilon_k} (w^\varepsilon + s) f_{\varepsilon_k}(s) ds = M(x,t).\end{aligned}$$

Since $|\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t)| \leq C$ on every compact subset of $\mathcal{D}$, there holds, for a subsequence that we still call ε_k that

$$\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t) \to \bar{M}(x,t) \qquad \text{weakly in } L^2_{\text{loc}}(\mathcal{D}).$$

Clearly, $\bar{M}(x,t) = M(x,t)$ in $\{u > 0\}$. Let us see that $\bar{M}(x,t) = 0$ in $\{u \equiv 0\}^\circ$. In fact, let K be a compact subset of $\{u \equiv 0\}^\circ$. For every

$\varepsilon_1, \varepsilon_2 > 0$ there holds that, for k large enough

$$|\{(x,t) \in K \,/\, \varepsilon_1 < \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t) < M(x,t) - \varepsilon_2\}| \leq$$
$$\Big|\{(x,t) \in K \,/\, \frac{u^{\varepsilon_k}}{\varepsilon_k}(x,t) > -w_0(x,t),$$
$$\frac{\varepsilon_1}{2} < \int_{-w_0}^{\frac{u^{\varepsilon_k}}{\varepsilon_k}} (s + w_0) f(s) ds < M - \frac{\varepsilon_2}{2}\}\Big|.$$

In fact, let $\delta > 0$ be such that

$$\int_{-w_0}^{-w_0 - \delta} (s + w_0) f(s) ds < \frac{\varepsilon_1}{2}.$$

Since $u^{\varepsilon_k}/\varepsilon_k$ is bounded in K,

$$\left|\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t) - \int_{-w_0}^{u^{\varepsilon_k}/\varepsilon_k} (s + w_0) f(s) ds\right| < \min(\frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2})$$

if $k \geq k_0$. On the other hand

$$\frac{u^{\varepsilon_k}}{\varepsilon_k} \geq -\frac{w^{\varepsilon_k}}{\varepsilon_k} \geq -w_0 - \delta$$

if $k \geq k_1$. Thus, if $k \geq \max(k_0, k_1)$ and $u^{\varepsilon_k}/\varepsilon_k \leq -w_0$ there holds that

$$\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t) < \varepsilon_1.$$

Therefore,

$$\Big|\{(x,t) \in K \,/\, \frac{u^{\varepsilon_k}}{\varepsilon_k}(x,t) > -w_0(x,t),$$
$$\frac{\varepsilon_1}{2} < \int_{-w_0}^{\frac{u^{\varepsilon_k}}{\varepsilon_k}} (s + w_0) f(s) ds < M - \frac{\varepsilon_2}{2}\}\Big|$$
$$\leq |\{(x,t) \in K \,/\, -w_0(x,t) + \mu \leq \frac{u^{\varepsilon_k}}{\varepsilon_k} \leq 1 - \mu\}|$$
$$\leq |\{(x,t) \in K \,/\, \frac{Y^{\varepsilon_k}}{\varepsilon_k} \geq \frac{\mu}{2}\,,\ \frac{u^{\varepsilon_k}}{\varepsilon_k} \leq 1 - \mu\}|$$
$$\leq |\{(x,t) \in K \,/\, Y^{\varepsilon_k} f_{\varepsilon_k}(u^{\varepsilon_k}) \geq \frac{C_\mu}{2\varepsilon_k}\}|.$$

Since $Y^{\varepsilon_k} f_{\varepsilon_k}(u^{\varepsilon_k}) \to 0$ as measures in K and $Y^{\varepsilon_k} \geq 0$, $f_{\varepsilon_k} \geq 0$ there holds that

$$Y^{\varepsilon_k} f_{\varepsilon_k}(u^{\varepsilon_k}) \to 0 \qquad \text{in } L^1(K).$$

Therefore,

$$|\{(x,t) \in K \,/\, \varepsilon_1 < \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t) < M(x,t) - \varepsilon_2\}| \to 0.$$

On the other hand, let $1 > \eta > \sup_K(-w_0)$ be the constant in (2.1.1) in K, there holds that

$$\begin{aligned}\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t) &= \int_{-\frac{w^{\varepsilon_k}}{\varepsilon_k}}^{\frac{u^{\varepsilon_k}}{\varepsilon_k}} \big(s+\frac{w^{\varepsilon_k}}{\varepsilon_k}\big)f(s)ds + \int_{-w_0}^{-\frac{w^{\varepsilon_k}}{\varepsilon_k}} \big(s+\frac{w^{\varepsilon_k}}{\varepsilon_k}\big)f(s)ds \\ &\le \int_{-\frac{w^{\varepsilon_k}}{\varepsilon_k}}^{\eta} \big(s+\frac{w^{\varepsilon_k}}{\varepsilon_k}\big)f(s)ds + \int_{-w_0}^{-\frac{w^{\varepsilon_k}}{\varepsilon_k}} \big(s+\frac{w^{\varepsilon_k}}{\varepsilon_k}\big)f(s)ds \\ &= \int_{-w_0}^{\eta} \big(s+\frac{w^{\varepsilon_k}}{\varepsilon_k}\big)f(s)ds \to \int_{-w_0}^{\eta} (s+w_0)f(s)ds < M(x,t),\end{aligned}$$

since $-\frac{w^{\varepsilon_k}}{\varepsilon_k} \le \frac{u^{\varepsilon_k}}{\varepsilon_k} \le \eta$ in K. Therefore,

$$\limsup \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t) \le \int_{-w_0}^{\eta} (s+w_0)f(s)ds < M(x,t).$$

So that, for $\varepsilon_2 > 0$ small we get

$$\begin{aligned}&|\{(x,t)\in K \,/\, \varepsilon_1 < \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t)\}| = \\ &|\{(x,t)\in K \,/\, \varepsilon_1 < \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t) < M-\varepsilon_2\}| \to 0.\end{aligned}$$

Let us now see that $\bar{M}(x,t) = 0$ in K. As in Lemma 1.4.1 we see that $\bar{M}(x,t) \ge 0$. Now assume that for some $\varepsilon_1 > 0$ we have $|\{\bar{M}(x,t) > \varepsilon_1\}| > 0$. Then, there exists m such that $|\{\bar{M}(x,t) > \varepsilon_1 + \frac{1}{m}\}| := |A_m| > 0$.

Now,

$$\int_{A_m} \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t) \to \int_{A_m} \bar{M}(x,t) > \big(\varepsilon_1 + \frac{1}{m}\big)|A_m|$$

but,

$$\begin{aligned}\int_{A_m} \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t) = &\int_{A_m\cap\{\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t)>\varepsilon_1\}} \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t) \\ &+ \int_{A_m\cap\{\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t)\le\varepsilon_1\}} \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k},x,t).\end{aligned}$$

Since the first term in the right hand side goes to zero and the second is bounded by $\varepsilon_1|A_m|$, we get a contradiction.

The proof is finished. $\square$

Let us give the definition of a regular point.

DEFINITION 2.1.3. *We say that ν is the interior unit spatial normal to the free boundary $\partial\{u>0\}$ at a point $(x_0,t_0)\in\partial\{u>0\}$ in the parabolic measure theoretic sense, if $\nu\in\mathbb{R}^N$, $|\nu|=1$ and*

$$\lim_{r\to 0}\frac{1}{r^{N+2}}\iint_{Q_r(x_0,t_0)}|\mathcal{X}_{\{u>0\}}-\mathcal{X}_{\{(x,t)/\ \langle x-x_0,\nu\rangle>0\}}|dxdt=0.$$

DEFINITION 2.1.4. *We say that (x_0,t_0) is a regular point of $\partial\{u>0\}$ if there exists an interior unit spatial normal to $\partial\{u>0\}$ at (x_0,t_0) in the parabolic measure theoretic sense.*

We can now prove the main result of this section.

THEOREM 2.1.5. *Let $(u^{\varepsilon_j},Y^{\varepsilon_j})$ be a family of uniformly bounded solutions of (1.0.1) in a domain $\mathcal{D}\subset\mathbb{R}^{N+1}$ such that $u^{\varepsilon_j}\to u$ uniformly on compact subsets of $\mathcal{D}$, $Y^{\varepsilon_j}\geq 0$ and verify (1.0.4) and (2.1.1), with $w_0>-1$. If (x_0,t_0) is a regular point of $\mathcal{D}\cap\partial\{u>0\}$, then u has the asymptotic development*

$$u(x,t)=\alpha\langle x-x_0,\nu\rangle^+ + o(|x-x_0|+|t-t_0|^{1/2}),$$

with $\alpha=\sqrt{2M(x_0,t_0)}$, where $M(x,t)=\int_{-w_0(x,t)}^1(s+w_0(x,t))f(s)\,ds$. Here ν is the interior unit spatial normal to the free boundary at (x_0,t_0) in the parabolic measure theoretic sense.

PROOF. We assume, without loss of generality, that $(x_0,t_0)=(0,0)$ and $\nu=e_1=(1,0,...,0)$.

Let $\psi\in C_c^\infty(\mathcal{D})$. We proceed as in Lemma 1.4.1. Let us multiply the equation for u^ε by $u^\varepsilon_{x_1}\psi$ and integrate by parts. We have

$$\iint_{\mathcal{D}}u^\varepsilon_t u^\varepsilon_{x_1}\psi=\frac{1}{2}\iint_{\mathcal{D}}|\nabla u^\varepsilon|^2\psi_{x_1}-\iint_{\mathcal{D}}u^\varepsilon_{x_1}\nabla u^\varepsilon\nabla\psi+\iint_{\mathcal{D}}\mathcal{B}_\varepsilon(u^\varepsilon,x,t)\psi_{x_1}$$
$$+\iint_{\mathcal{D}}\left(\frac{w^\varepsilon}{\varepsilon}-w_0\right)f(-w_0)(w_0)_{x_1}\psi+\iint_{\mathcal{D}}\frac{w^\varepsilon_{x_1}}{\varepsilon}\left(\int_{-w_0}^{\frac{u^\varepsilon}{\varepsilon}}f(s)ds\right)\psi.$$

Since

$$\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j},x,t)=\int_{-w_0}^{\frac{u^{\varepsilon_j}}{\varepsilon_j}}(s+w_0)f(s)ds+\int_{-w_0}^{\frac{u^{\varepsilon_j}}{\varepsilon_j}}\big(\frac{w^{\varepsilon_j}}{\varepsilon_j}-w_0\big)f(s)ds,$$

and $\mathcal{B}_{\varepsilon_j}(u^{\varepsilon_j},x,t)\to 0$ weakly in $L^1(K)$ for every $K\subset\{u\equiv 0\}^\circ$ compact, there holds that

$$F_{\varepsilon_j}(x,t):=\int_{-w_0}^{\frac{u^{\varepsilon_j}}{\varepsilon_j}}(s+w_0)f(s)ds\to 0\qquad\text{weakly in } L^1(K).$$

Since F_{ε_j} is nonnegative there holds that

$$F_{\varepsilon_j} \to 0 \qquad \text{in } L^1(K).$$

So that, for a subsequence that we still call ε_j there holds that

$$F_{\varepsilon_j} \to 0 \qquad a.e.\ K.$$

Thus,

$$\frac{u^{\varepsilon_j}}{\varepsilon_j} \to -w_0 \qquad a.e.\ K.$$

Therefore,

$$\int_{-w_0}^{\frac{u^{\varepsilon_j}}{\varepsilon_j}} f(s)ds \to \left(\int_{-w_0}^{1} f(s)ds\right) \mathcal{X}_{\{u>0\}} \qquad a.e.\ \mathcal{D}. \tag{2.1.6}$$

By using Proposition 1.2.1, Lemma 2.1.2 and (2.1.6) we can pass to the limit (for the sequence $\varepsilon_j \to 0$) in the latter equation and get

(2.1.7)
$$\begin{aligned}\iint_{\mathcal{D}} u_t u_{x_1}\psi =& \frac{1}{2}\iint_{\mathcal{D}} |\nabla u|^2 \psi_{x_1} - \iint_{\mathcal{D}} u_{x_1}\nabla u \nabla\psi \\ &+ \iint_{\{u>0\}} M(x,t)\psi_{x_1} + \iint_{\{u>0\}} (w_0)_{x_1}\left(\int_{-w_0}^{1} f(s)ds\right)\psi\end{aligned}$$

for every $\psi \in C_c^\infty(\mathcal{D})$.

Now, let $\psi^\lambda(x,t) = \lambda\psi(\frac{x-x_0}{\lambda}, \frac{t-t_0}{\lambda^2})$. Replacing ψ by ψ^λ in (2.1.7) and changing variables, we get for $u_\lambda(x,t) = \frac{1}{\lambda}u(x_0+\lambda x, t_0+\lambda^2 t)$,

(2.1.8)
$$\begin{aligned}\iint (u_\lambda)_t (u_\lambda)_{x_1}\psi &= \frac{1}{2}\iint |\nabla u_\lambda|^2 \psi_{x_1} - \iint (u_\lambda)_{x_1}\nabla u_\lambda \nabla\psi \\ &+ \iint_{\{u_\lambda>0\}} M(\lambda x,\lambda^2 t)\psi_{x_1} + \iint_{\{u>0\}} (w_0)_{x_1}\left(\int_{-w_0}^{1} f(s)ds\right)\psi^\lambda.\end{aligned}$$

Let $r > 0$ be such that $Q_r(x_0,t_0) \subset\subset \mathcal{D}$. We have that $u_\lambda \in Lip(1,1/2)$ in $Q_{r/\lambda}(0,0)$ uniformly in λ, and $u_\lambda(0,0) = 0$. Therefore, for every $\lambda_n \to 0$, there exists a subsequence $\lambda_{n'} \to 0$ and a function $U \in Lip(1,1/2)$ in $\mathbb{R}^{N+1}$ such that $u_{\lambda_{n'}} \to U$ uniformly on compact sets of $\mathbb{R}^{N+1}$.

By our assumption on (x_0,t_0), we can easily see that for every $k > 0$

$$|\{u_\lambda > 0\} \cap \{x_1 < 0\} \cap Q_k(0,0)| \to 0 \text{ as } \lambda \to 0, \tag{2.1.9}$$

and

$$|\{u_\lambda = 0\} \cap \{x_1 > 0\} \cap Q_k(0,0)| \to 0 \text{ as } \lambda \to 0. \tag{2.1.10}$$

Now, using lemma 1.3.1 and the fact that $\psi^\lambda \to 0$ uniformly in $\mathcal{D}$ and $\operatorname{supp}\psi^\lambda \subset \operatorname{supp}\psi$, we can pass to the limit in (2.1.8) and get

(2.1.11)
$$\iint_{\{x_1>0\}} U_t U_{x_1}\psi = \frac{1}{2}\iint_{\{x_1>0\}} |\nabla U|^2 \psi_{x_1} - \iint_{\{x_1>0\}} U_{x_1}\nabla U \nabla\psi + M(0,0)\iint_{\{x_1>0\}} \psi_{x_1}.$$

Our aim is to prove that $U = \alpha x_1^+$. First, by (2.1.9) and (2.1.10), we deduce that $U = 0$ in $\{x_1 < 0\}$. On the other hand, U is a solution to the heat equation in $\{U > 0\} \subset \{x_1 > 0\}$. By Corollary A.1 in [**10**], for every $\bar{x}' \in \mathbb{R}^{N-1}$, $\bar{t} \in \mathbb{R}$ there exists $\alpha \geq 0$ such that

$$U(x,t) = \alpha x_1^+ + o(|(x_1,x') - (0,\bar{x}')| + |t-\bar{t}|^{1/2}) \quad \text{in } \{x_1 > 0\} \cap \{t < \bar{t}\}.$$

Replacing the test function ψ by $\Phi^\lambda(x,t) = \lambda\Phi(\frac{x_1}{\lambda}, \frac{x'-\bar{x}'}{\lambda}, \frac{t-\bar{t}}{\lambda^2})$ with $\Phi \in C_c^\infty(\{t<0\})$ and proceeding as above we get

$$-\frac{\alpha^2}{2}\iint_{\{x_1>0\}} \Phi_{x_1} + M(0,0)\iint_{\{x_1>0\}} \Phi_{x_1} = 0. \tag{2.1.12}$$

In order to pass to the limit for a sequence $\lambda_n \to 0$ we have used Lemma 1.3.1.

Thus, $\alpha = \sqrt{2M(0,0)}$.

In order to see that $U = \alpha x_1^+$ we use Lemma 1.5.2. In fact, by Lemma 1.3.1 there exists a sequence $j_n \to \infty$ such that

$$u^{\delta_n} := \frac{1}{\lambda_n} u^{\varepsilon_{j_n}}(\lambda_n x, \lambda_n^2 t) \to U(x,t)$$

uniformly on compact subsets of $\mathbb{R}^{N+1}$. We recall that $(u^{\delta_n}, Y^{\delta_n})$ is a solution to 1.0.1 with ε replaced by δ_n. Moreover,

$$\frac{w^{\delta_n}}{\delta_n} = \frac{w^{\varepsilon_{j_n}}(\lambda_n x, \lambda_n^2 t)}{\varepsilon_{j_n}} \to w_0(0,0)$$

uniformly on compact sets of $\mathbb{R}^{N+1}$.

On the other hand, $U \geq 0$ and $\partial\{U>0\} \neq \emptyset$. By Lemma 1.5.2 we have that $|\nabla U| \leq \alpha = \sqrt{2M(0,0)}$. Since $U \equiv 0$ in $\{x_1 = 0\}$ we deduce that

$$U \leq \alpha x_1 \qquad \text{in } \{x_1 > 0\}.$$

By Hopf's Principle, we deduce that

$$U = \alpha x_1 \qquad \text{in } \{x_1 > 0\}.$$

Since the limit of $u_{\lambda_{n'}}$ is αx_1^+ with α independent of the sequence $\lambda_{n'}$, we deduce that $u_\lambda \to \alpha x_1^+$ so that

$$u(x,t) = \alpha x_1^+ + o(|x| + |t|^{1/2}).$$

The Theorem is proved. □

REMARK 2.1.13. *It is clear from the proof that the result is still true if we replace condition* (2.1.1) *by the following property:* $\frac{u^{\varepsilon_j}}{\varepsilon_j} \to -w_0$ *a.e.* $\{u \equiv 0\}^\circ$.

2. Viscosity solutions

In this section we prove that, under suitable assumptions, the limit function u is a viscosity solution of the free boundary problem (P).

For the sake of completeness, we state here the definition of viscosity solution that was introduced in [**11**] for the two phase case of this problem when $w_0 = 0$.

DEFINITION 2.2.1. *Let Q be a cylinder in $\mathbb{R}^N \times (0,T)$ and let $v \in C(\overline{Q})$. Then v is called a classical subsolution (supersolution) of (P) in Q if $v \geq 0$ and*

(1) $\Delta v - v_t \geq 0$ (≤ 0) *in* $\Omega^+ \equiv Q \cap \{v > 0\}$.
(2) $v \in C^1(\overline{\Omega^+})$.
(3) *For any* $(x,t) \in \partial\Omega^+ \cap Q$, $\nabla v(x,t) \neq 0$, *and*

$$|\nabla v(x,t)| \geq \sqrt{2M(x,t)} \quad (\leq \sqrt{2M(x,t)}).$$

We say that v is a classical solution in Q if it is both a classical subsolution and a classical supersolution.

DEFINITION 2.2.2. *Let u be a continuous nonnegative function in Q; u is called a viscosity subsolution (supersolution) of (P) in Q if, for every subcylinder $Q' \subset\subset Q$ and for every classical supersolution (subsolution) v in Q',*

$$\begin{array}{ll} u \leq v \text{ on } \partial_p Q' & (u \geq v \quad \text{on } \partial_p Q') \quad \text{and} \\ v > 0 \text{ on } \overline{\{u > 0\}} \cap \partial_p Q' & (u > 0 \text{ on } \overline{\{v > 0\}} \cap \partial_p Q') \end{array}$$

implies that $u \leq v$ $(u \geq v)$ in Q'.

The function u is called a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

DEFINITION 2.2.3. *Let u be a continuous nonnegative function in $\mathcal{D}$ and let $(x_0, t_0) \in \partial\{u > 0\} \cap \mathcal{D}$. We say that (x_0, t_0) is a regular point from the nonpositive side, if there exists a regular nonnegative function v in $\mathcal{D}$ such that $v > u$ in $\{u > 0\}$ for $t < t_0$ and $v(x_0, t_0) = u(x_0, t_0)$.*

Finally we need the following definition of nondegeneracy.

DEFINITION 2.2.4. *Let u be a continuous nonnegative function in $\mathcal{D}$. Let $(x_0, t_0) \in \mathcal{D}$ be such that $u(x_0, t_0) = 0$. We say that u does not degenerate at (x_0, t_0) if there exist $r_0 > 0$ and $C > 0$ such that*

$$\sup_{\partial_p Q_r^-(x_0,t_0)} u \geq C\, r \qquad \textit{for } 0 < r \leq r_0.$$

We now prove that, under suitable assumptions on the limit function u, there holds that u is a viscosity solution to the free boundary problem.

THEOREM 2.2.5. *Let $u = \lim u^{\varepsilon_k}$, where $(u^{\varepsilon_k}, Y^{\varepsilon_k})$ are uniformly bounded solutions to (1.0.1) with $Y^{\varepsilon_k} \geq 0$, satisfying (1.0.4) in $\mathcal{D}$, with $w_0 > -1$, and such that u^{ε_k} either satisfies (2.1.1) or $u_t^{\varepsilon_k} \leq 0$ in $\mathcal{D}$.*

If u^+ does not degenerate at every point of the free boundary which is regular from the nonpositive side, then u is a viscosity solution of (P).

In order to prove Theorem 2.2.5, we need to show that u is both a viscosity super- and subsolution. We perform this in Theorems 2.2.6 and 2.2.9 respectively.

We want to remark that every limit function u is a viscosity supersolution to problem (P) (i.e. we do not need the nondegeneracy, monotonicity nor condition (2.1.1)).

Let us first show that every limit function u is a viscosity supersolution.

THEOREM 2.2.6. *Let $u = \lim u^{\varepsilon_k}$, where $(u^{\varepsilon_k}, Y^{\varepsilon_k})$ are uniformly bounded solutions to (1.0.1) with $Y^{\varepsilon_k} \geq 0$, satisfying (1.0.4) in $\mathcal{D}$, with $w_0 > -1$.*

Then u is a viscosity supersolution of (P).

PROOF. The proof of this Theorem is analogous to that of Theorem 4.1 in [**11**]. We include the details here for the sake of completeness.

Let $Q \subset\subset \mathcal{D}$ be a cylinder which will be assumed to be $B_1(0)\times(0,T)$ and let v be a classical subsolution in Q satisfying

$$u \geq v \text{ on } \partial_p Q$$

and

$$u > 0 \text{ on } \overline{\{v > 0\}} \cap \partial_p Q \text{ if } \overline{\{v > 0\}} \cap \partial_p Q \neq \emptyset.$$

We will show that $u \geq v$ in Q.

If $\overline{\{v > 0\}} \cap \partial_p Q = \emptyset$, then $v \leq 0$ in $\partial_p Q$ and therefore $v \leq 0$ in Q. As $u \geq 0$ everywhere, we see that $u \geq v$ in Q.

If $\overline{\{v > 0\}} \cap \partial_p Q \neq \emptyset$, it follows from the continuity of u and v that $u > 0$ in $\overline{\{v > 0\}} \cap Q$ for $0 \leq t < \tau$, for some small $\tau > 0$. It is not hard to see that if $u > 0$ in $\overline{\{v > 0\}} \cap Q$ for $0 \leq t < s$, then $u \geq v$ in $Q \cap \{0 \leq t \leq s\}$. We set

$$t_0 = \sup\{0 < s < T : \; u > 0 \text{ in } \overline{\{v > 0\}} \cap Q \cap \{0 \leq t < s\}\},$$

and we will arrive at a contradiction assuming that $t_0 < T$.

We have that $t_0 > 0$ and $u \geq v$ in $Q \cap \{0 \leq t \leq t_0\}$. In addition, there exists a sequence $(x_n, t_n) \to (x_0, t_0) \in \bar{Q}$ such that $u(x_n, t_n) = 0$, $(x_n, t_n) \in \overline{\{v > 0\}} \cap Q$. Then, $u(x_0, t_0) = v(x_0, t_0) = 0$ and $(x_0, t_0) \in \partial\{v > 0\} \cap Q$. Since v is a classical subsolution, there exists a sequence $y_n \to x_0$ such that $0 < v(y_n, t_0) \leq u(y_n, t_0)$, so we have proved that

$$\begin{aligned} &u \geq v \text{ in } Q \cap \{0 \leq t \leq t_0\}, \\ &(x_0, t_0) \in \partial\{u > 0\} \cap \partial\{v > 0\} \cap Q. \end{aligned}$$

Now consider for $\lambda > 0$

$$\begin{aligned} u_\lambda(x, t) &= \frac{1}{\lambda} u(x_0 + \lambda x, t_0 + \lambda^2 t), \\ v_\lambda(x, t) &= \frac{1}{\lambda} v(x_0 + \lambda x, t_0 + \lambda^2 t). \end{aligned}$$

Since $u, v \in Lip(1, \frac{1}{2})$ in Q, and $u_\lambda(0,0) = v_\lambda(0,0) = 0$, there exists a sequence $\lambda_n \to 0$ and $u_0, v_0 \in Lip(1, \frac{1}{2})$ in $\mathbb{R}^{N+1}$ such that $v_{\lambda_n} \to v_0$ and $u_{\lambda_n} \to u_0$ uniformly on compact sets of $\mathbb{R}^{N+1}$. Since v is a classical subsolution, if we assume that $\nabla v^+(x_0, t_0)/|\nabla v^+(x_0, t_0)| = e_1$ and we set $\bar{\alpha} = |\nabla v^+(x_0, t_0)|$, we see that (as $v \geq 0$)

$$v_0(x, t) = \bar{\alpha} x_1^+, \qquad \bar{\alpha} \geq \sqrt{2M(x_0, t_0)}.$$

Moreover, $u_0 \geq v_0$ when $t < 0$, so that u_0 is caloric in $\{x_1 > 0, \; t < 0\}$. In addition $u_0(0,0) = 0$.

There are two possibilities depending whether the following assertion holds or not:

(2.2.7)

There exists $\delta < 0$ such that $u_0 - v_0 > 0$ when $x_1 > 0$, $\delta < t < 0$.

CASE I. Suppose that (2.2.7) does not hold. Then there exists a sequence (x_n, t_n) in $\{x_1 > 0,\ t < 0\}$ such that $t_n \to 0$ and $u_0(x_n, t_n) - v_0(x_n, t_n) = 0$. From the strong maximum principle, it follows that

$$u_0 \equiv v_0 = \bar{\alpha} x_1^+ \quad \text{in } \{x_1 \geq 0,\ t \leq 0\},$$

implying that

$$\frac{1}{\lambda_n}(u - v)(x_0 + \lambda_n e_1, t_0) \to 0 \quad \text{as } n \to +\infty. \tag{2.2.8}$$

We denote $(x, t) = (x_1, x', t)$ and for small $\rho, r > 0$ we define

$$E = \{g(x', t) < x_1 < g(x', t) + \rho,\ |x' - x_0'| < r,\ |t - t_0| < r^2\},$$

where g is a C^1 function in a neighborhood of (x_0', t_0) such that for a small $r_0 > 0$

$$B_{r_0}(x_0, t_0) \cap \partial\{v > 0\} = B_{r_0}(x_0, t_0) \cap \{(x, t) \mid x_1 = g(x', t)\}$$

and

$$B_{r_0}(x_0, t_0) \cap \{v > 0\} = B_{r_0}(x_0, t_0) \cap \{(x, t) \mid x_1 > g(x', t)\}.$$

If r and ρ are small enough, then $E \subset \{v > 0\}$ and therefore, $u - v$ is positive and supercaloric in $E \cap \{t < t_0\}$ and in addition, $u - v \geq \mu > 0$ in $\tilde{B}$, for some small μ and some ball $\tilde{B}$ with center in $\partial_p E \cap \{t < t_0\}$ and not touching $\partial\{v > 0\}$.

Let w_1 be a caloric function in E with smooth boundary data satisfying

$$w_1 = 0 \text{ on } \partial_p E \setminus \tilde{B}, \qquad 0 < w_1 \leq \mu \text{ on } \partial_p E \cap \tilde{B},$$

and let w_2 be the caloric function in E such that $w_2 = v$ on $\partial_p E$.

We have

$$u - v \geq w_1 \geq C w_2 \qquad \text{in } E \cap B_{r_2}(x_0, t_0) \cap \{t \leq t_0\}$$

for some constants $C > 0$ and $r_2 > 0$ small, the last inequality following from Theorem 3 in [**14**]. Hence,

$$u - v \geq C v \qquad \text{in } E \cap B_{r_2}(x_0, t_0) \cap \{t \leq t_0\}$$

and therefore,

$$\liminf_{\lambda \to 0^+} \frac{1}{\lambda}(u - v)(x_0 + \lambda e_1, t_0) \geq C\bar{\alpha} > 0,$$

which contradicts (2.2.8).

Case II. The argument above implies that necessarily (2.2.7) holds. Then, from Lemma A.1 in [**10**], it follows that

$$(u_0 - v_0)(x,t) = \sigma x_1^+ + o(|x| + |t|^{1/2})$$

when $x_1 > 0$, $t < 0$, for some $\sigma > 0$. That is,

$$u_0(x,t) = \alpha x_1^+ + o(|x| + |t|^{1/2}) \quad \text{in } \{x_1 > 0,\ t < 0\} \text{ with } \alpha > \bar{\alpha}.$$

Now consider for $\lambda > 0$,

$$(u_0)_\lambda(x,t) = \frac{1}{\lambda} u_0(\lambda x, \lambda^2 t), \qquad (v_0)_\lambda(x,t) = \frac{1}{\lambda} v_0(\lambda x, \lambda^2 t).$$

As before, there exists a sequence $\lambda_n \to 0$ and $u_{00}, v_{00} \in Lip(1, \frac{1}{2})$ in $\mathbb{R}^{N+1}$ such that

$$(u_0)_{\lambda_n} \to u_{00}, \qquad (v_0)_{\lambda_n} \to v_{00}$$

uniformly on compact sets of $\mathbb{R}^{N+1}$. Clearly $u_{00} \geq v_{00}$ when $t < 0$ and moreover,

$$\begin{aligned} &v_{00}(x,t) = \bar{\alpha} x_1^+ \text{ and} \\ &u_{00}(x,t) = \alpha x_1^+ \text{ in } \{x_1 > 0,\ t < 0\}. \end{aligned}$$

Since u_{00} is caloric in $\{u_{00} > 0\}$, we can apply Corollary A.1 in [**10**] to u_{00} in $\{x_1 \leq 0,\ t < 0\}$ and hence,

$$u_{00}(x,t) = \begin{cases} \alpha x_1^+ & \text{in } \{x_1 > 0,\ t < 0\}, \\ \gamma x_1^- + o(|x| + |t|^{1/2}) & \text{in } \{x_1 < 0,\ t < 0\}, \end{cases}$$

for some $\gamma \geq 0$. We consider

$$(u_{00})_\lambda(x,t) = \frac{1}{\lambda} u_{00}(\lambda x, \lambda^2 t), \qquad (v_{00})_\lambda(x,t) = \frac{1}{\lambda} v_{00}(\lambda x, \lambda^2 t).$$

There is a sequence $\lambda_n \to 0$ and $u_{000}, v_{000} \in Lip(1, \frac{1}{2})$ in $\mathbb{R}^{N+1}$ such that

$$(u_{00})_{\lambda_n} \to u_{000}, \qquad (v_{00})_{\lambda_n} \to v_{000}$$

uniformly on compact sets of $\mathbb{R}^{N+1}$ and moreover,

$$\begin{aligned} &v_{000}(x,t) = \bar{\alpha} x_1^+ \text{ and} \\ &u_{000}(x,t) = \alpha x_1^+ + \gamma x_1^- \end{aligned}$$

for $t \leq 0$.

Applying Lemma 1.3.1 three times, we find a sequence ${\varepsilon_j}^{000} \to 0$ and solutions $u^{{\varepsilon_j}^{000}}, Y^{{\varepsilon_j}^{000}}$ of (1.0.1) in $Q_1(0,0)$ such that $u^{{\varepsilon_j}^{000}} \to u_{000}$ and $w^{{\varepsilon_j}^{000}}/{\varepsilon_j}^{000} \to w(x_0,t_0)$ uniformly on compact subsets of $Q_1(0,0)$.

We recall that

$$u_{000}(x,t) = \alpha x_1^+ + \gamma x_1^- \quad \text{with } \alpha > 0, \gamma \geq 0$$

for $t \leq 0$.

If $\gamma = 0$, we apply Lemma 1.4.1 to u_{000} in a neighborhood of some point $(0, \bar{t})$ with $\bar{t} < 0$ and deduce that

$$\alpha \leq \sqrt{2M(x_0, t_0)}.$$

If $\gamma > 0$, we apply Lemma 1.4.5 and conclude that $\gamma = \alpha$ and

$$\alpha \leq \sqrt{2M(x_0, t_0)}.$$

In any case, as $\alpha > \bar{\alpha} \geq \sqrt{2M(x_0, t_0)}$ we get a contradiction and this finishes the proof. □

Finally, we end this section (and the proof of Theorem 2.2.5) by showing that, under the nondegeneracy assumption, a limit function u is a viscosity subsolution.

THEOREM 2.2.9. *Let* $u = \lim u^{\varepsilon_k}$*, where* $(u^{\varepsilon_k}, Y^{\varepsilon_k})$ *are uniformly bounded solutions to (1.0.1) with* $Y^{\varepsilon_k} \geq 0$*, satisfying (1.0.4) in* $\mathcal{D}$*, with* $w_0 > -1$*, and such that* u^{ε_k} *either satisfies (2.1.1) or* $u_t^{\varepsilon_k} \leq 0$ *in* $\mathcal{D}$*.*

If u^+ *does not degenerate at every point of the free boundary which is regular from the nonpositive side, then* u *is a viscosity subsolution of (P).*

PROOF. In order to see that u is a viscosity subsolution, let v be a classical supersolution such that

$$u \leq v \text{ in } \partial_p Q \text{ and } v > 0 \text{ in } \overline{\{u > 0\}} \cap \partial_p Q$$

we want to see that $u \leq v$ in Q.

If not, we define

$$t_0 = \sup\{0 < s < T : \ v > 0 \text{ in } \overline{\{u > 0\}} \cap Q \cap \{0 \leq t < s\}\}.$$

From the definition of t_0, it follows that $t_0 > 0$ and, from our hypotheses we deduce that $v \geq u$ in $Q \cap \{0 \leq t \leq t_0\}$. In addition, there exists a sequence $(x(s), t(s)) \to (x_0, t_0) \in \overline{Q}$ such that $v(x(s), t(s)) = 0$, $(x(s), t(s)) \in \overline{\{u > 0\}} \cap Q$. Clearly, $u(x_0, t_0) = v(x_0, t_0) = 0$ and $(x_0, t_0) \in \partial\{u > 0\} \cap Q$. If $(x_0, t_0) \in \{v = 0\}^\circ$ then, for τ small we have $u \leq v = 0$ in $B_\tau(x_0, t_0) \cap \{t < t_0\}$ and therefore, $u \equiv 0$ there, which contradicts our hypothesis. Thus

$$v \geq u \text{ in } Q \cap \{0 \leq t \leq t_0\},$$

$$(x_0, t_0) \in \partial\{u > 0\} \cap \partial\{v > 0\} \cap Q.$$

We may assume, without loss of generality, that $(x_0, t_0) = (0, 0)$ and $Q_1(0, 0) = Q_1 \subset Q$ (consider instead of u the function $\frac{1}{\lambda_0} u(x_0 + \lambda_0 x, t_0 + \lambda_0^2 t)$ for certain $\lambda_0 > 0$ small, and analogously with v). Let us take

$$v_\lambda(x, t) = \frac{1}{\lambda} v(\lambda x, \lambda^2 t), \quad u_\lambda(x, t) = \frac{1}{\lambda} u(\lambda x, \lambda^2 t).$$

It is easy to see that there exists a sequence $\lambda_n \to 0$ and functions $u_0,\ v_0$ such that $v_\lambda \to v_0,\ u_\lambda \to u_0$.

Since v is regular, we have that $v_0(x, t) = \beta x_1^+$ with $0 \leq \beta \leq \sqrt{2M(0, 0)}$ (for some system of coordinates).

Let us see that also $u_0(x, t) = \alpha x_1^+$ for some $\alpha \geq 0$,

We may think that in Q_1, $\partial\{v > 0\}$ is the graph of some function $\psi(x', t) = x_1,\ x = (x_1, x')$ with $\psi \in Lip(1, 1/2)$, where $\psi(0, 0) = 0$ and $\{v > 0\} = \{x_1 > \psi(x', t)\}$.

Hence, we have that

$$|\psi(x', t)| \leq C \left(|x'| + |t|^{1/2}\right).$$

Let $\mathcal{R} = \left\{(x, t) \in Q_1 :\ x_1 < -C \left(|x'| + |t|^{1/2}\right)\right\}$. Then $\mathcal{R} \cap \{v > 0\} = \emptyset$ and let w be the caloric function in $\mathcal{O} = Q_1^- \setminus \mathcal{R}$ with $w = 0$ in $\partial_p \mathcal{R}$ and $w = L \geq \|u\|_\infty$ in the rest of $\partial_p \mathcal{O}$.

Since u is globally subcaloric and $u \leq w$ on $\partial_p \mathcal{O}$, then $u < w$ in $\mathcal{O}$.

Now, $w - u$ is supercaloric in $\mathcal{O}$, $w - u > 0$ in the interior and $w - u = 0$ at $(0, 0)$, then, by lemma A.1 of [**10**], we have that $w - u = \delta x_1^+ + o(|x| + |t|^{1/2})$ and, since by the same lemma, w has an asymptotic development at $(0, 0)$,

$$u(x, t) = \alpha x_1^+ + o(|x| + |t|^{1/2}), \quad \text{with } \alpha \geq 0.$$

Since by hypothesis u^+ does not degenerate, there follows that $\alpha > 0$.

On the other hand, since v is regular, v admits an asymptotic development at the origin in the form $v(x, t) = \beta x_1^+ + o(|x| + |t|^{1/2})$. Clearly, $\beta \geq \alpha$.

Now, let h be the caloric function in $\tilde{\mathcal{O}} := Q_1^- \cap \{v > 0\} \cap \{-\mu < t < 0\}$ for some small $\mu > 0$, with $h = v - u$ on $\partial_p \tilde{\mathcal{O}}$. And, let g be the caloric function in $\tilde{\mathcal{O}}$ with $g = v$ on $\partial_p \tilde{\mathcal{O}}$. Then, $h = g = 0$ in $Q_1^- \cap \partial\{v > 0\} \cap \{-\mu < t < 0\}$ and $h > 0$, $g > 0$ in $\tilde{\mathcal{O}}$.

Therefore, by the results in [**1**], there exists $\sigma > 0$ such that $h \geq \sigma g$ in $Q^-_{1/2} \cap \{v > 0\} \cap \{-\frac{\mu}{2} < t < 0\}$.

Since u is subcaloric in Q^-_1 and $u \leq v$ in Q^-_1 we deduce that $v - u \geq \sigma u$ in $Q^-_{1/2} \cap \{v > 0\} \cap \{-\frac{\mu}{2} < t < 0\}$. In particular $\beta - \alpha \geq \sigma\alpha > 0$.

The theorem will be finished if we show that $\alpha = \sqrt{2M(0,0)}$.

<u>Case 1:</u> u^{ε_k} verifies (2.1.1).

As in Theorem 2.1.5, we obtain

$$\iint_{\mathcal{D}} u_t u_{x_1} \psi = \frac{1}{2} \iint_{\mathcal{D}} |\nabla u|^2 \psi_{x_1} - \iint_{\mathcal{D}} u_{x_1} \nabla u \nabla \psi$$
$$+ \iint_{\mathcal{D} \cap \{u>0\}} M(x,t)\psi_{x_1} + \iint_{\mathcal{D} \cap \{u>0\}} (w_0)_{x_1} \left(\int_{-w_0}^{1} f(s)ds \right) \psi$$

for every test function ψ. Then, taking $\psi^\lambda(x,t) = \lambda\psi(\frac{x}{\lambda}, \frac{t}{\lambda^2})$ and changing variables, we get

$$\iint_{\mathcal{D}} (u_\lambda)_t (u_\lambda)_{x_1} \psi = \frac{1}{2} \iint_{\mathcal{D}} |\nabla u_\lambda|^2 \psi_{x_1} - \iint_{\mathcal{D}} (u_\lambda)_{x_1} \nabla u_\lambda \nabla \psi$$
$$+ \iint_{\mathcal{D} \cap \{u_\lambda>0\}} M(\lambda x, \lambda^2 t)\psi_{x_1} + \iint_{\mathcal{D} \cap \{u>0\}} (w_0)_{x_1} \left(\int_{-w_0}^{1} f(s)ds \right) \psi^\lambda$$

By Lemma 1.3.1, we get (for some sequence $\lambda_k \to 0$)

$$0 = -\frac{1}{2}\alpha^2 \iint_{\mathcal{D} \cap \{x_1>0\}} \psi_{x_1} + \lim_{k\to\infty} \iint_{\mathcal{D} \cap \{u_{\lambda_k}>0\}} M(\lambda_k x, \lambda_k^2 t)\psi_{x_1}.$$

We want to check that $\mathcal{X}_{\{u_{\lambda_k}>0\}} \to \mathcal{X}_{\{x_1>0\}}$ a.e. or, equivalently,

(1) $\{x_1 > 0\} \subset \cup_{n=1}^\infty \cap_{k\geq n} \{u_{\lambda_k} > 0\} = \liminf\{u_{\lambda_k} > 0\}$ a.e.
(2) $\cap_{n=1}^\infty \cup_{k\geq n} \{u_{\lambda_k} > 0\} = \limsup\{u_{\lambda_k} > 0\} \subset \{x_1 > 0\}$ a.e.

Let us see (1). If $x_1 > 0$, we get that $\alpha x_1 > 0$ and since $u_{\lambda_k}(x,t) \to \alpha x_1$ it follows that $u_{\lambda_k}(x,t) > 0 \quad \forall k \geq k_0$.

Let us see (2). If exists $k_j \to \infty$ with $u_{\lambda_{k_j}}(x,t) > 0$ then it must be $x_1 \geq 0$, because if $x_1 < 0$, we have that $v_{\lambda_{k_j}}(x,t) = 0$ for $j \geq j_0$ (because as v is regular, $\{v_{\lambda_k} > 0\} \to \{x_1 > 0\}$). Since $u_{\lambda_{k_j}} \leq v_{\lambda_{k_j}}$ we get a contradiction.

Therefore,

$$0 = -\frac{1}{2}\alpha^2 \iint_{\mathcal{D} \cap \{x_1>0\}} \psi_{x_1} + M(0,0) \iint_{\mathcal{D} \cap \{x_1>0\}} \psi_{x_1}.$$

So that,

$$0 = \int_{\mathcal{D}\cap\{x_1=0\}} \left(\frac{1}{2}\alpha^2 - M(0,0)\right) \psi dx' dt.$$

Since ψ is arbitrary, $\frac{1}{2}\alpha^2 = M(0,0)$, so that,

$$\alpha = \sqrt{2M(0,0)}$$

and the proof is finished

Case 2: $u_t^{\varepsilon_k} \leq 0$

We already know that, if we consider $u_\lambda(x,t) = \frac{1}{\lambda}u(\lambda x, \lambda^2 t)$, then it follows that

$$u_\lambda(x,t) \to u_0(x,t) \equiv \alpha x_1^+,$$

uniformly on compact subsets of $\mathbb{R}^{N+1}$.

As before

$$\begin{aligned}\iint_{\mathcal{D}} u_t^{\varepsilon_k} u_{x_1}^{\varepsilon_k} \psi =& \frac{1}{2}\iint_{\mathcal{D}} |\nabla u^{\varepsilon_k}|^2 \psi_{x_1} - \iint_{\mathcal{D}} u_{x_1}^{\varepsilon_k} \nabla u^{\varepsilon_k} \nabla \psi \\ &+ \iint_{\mathcal{D}} \mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t)\psi_{x_1} + \iint_{\mathcal{D}} w_{x_1}^{\varepsilon_k} \left(\int_{-w_0\varepsilon_k}^{u^{\varepsilon_k}} f_{\varepsilon_k}(s)ds\right)\psi \\ &+ \iint_{\mathcal{D}} (w_0)_{x_1}\big(\frac{w^{\varepsilon_k}}{\varepsilon_k} - w_0\big) f(-w_0)\psi\end{aligned}$$

Now, as in the previous case, if we consider first $\psi^\lambda(x,t) = \lambda\psi(\frac{x}{\lambda}, \frac{t}{\lambda^2})$ and change variables, we obtain

(2.2.10)

$$\begin{aligned}\iint (u_\lambda^{\varepsilon_k})_t (u_\lambda^{\varepsilon_k})_{x_1} \psi =& \frac{1}{2}\iint |\nabla u_\lambda^{\varepsilon_k}|^2 \psi_{x_1} - \iint (u_\lambda^{\varepsilon_k})_{x_1} \nabla u_\lambda^{\varepsilon_k} \nabla \psi \\ &+ \iint \mathcal{B}_{\varepsilon_k/\lambda}^{\lambda}(u_\lambda^{\varepsilon_k}, x, t)\psi_{x_1} + \iint_{\mathcal{D}} \frac{(w^{\varepsilon_k})_{x_1}}{\varepsilon_k} \left(\int_{-w_0}^{\frac{u_k^\varepsilon}{\varepsilon_k}} f(s)ds\right)\psi^\lambda \\ &+ \iint_{\mathcal{D}} (w_0)_{x_1}\big(\frac{w^{\varepsilon_k}}{\varepsilon_k} - w_0\big) f(-w_0)\psi^\lambda\end{aligned}$$

where $\mathcal{B}_\varepsilon^\lambda(u,x,t) = \int_{-w_0(\lambda x, \lambda^2 t)\varepsilon}^{u} (s + w^\varepsilon(x,t)) f_\varepsilon(s)ds$. We want to pass to the limit as both ε_k and λ go to zero.

Using Lemma 1.3.1, we see that for every sequence $\lambda_n \to 0$ there exists a sequence $k_n \to \infty$ such that $\delta_n := \varepsilon_{k_n}/\lambda_n \to 0$ and $u^{\delta_n} := (u^{\varepsilon_{k_n}})_{\lambda_n} \to u_0$ uniformly on compact sets of $\mathbb{R}^{N+1}$. By Proposition

1.2.1 we see that we can pass to the limit in the first three terms of (2.2.10) (with $\varepsilon = \varepsilon_{k_n}$ and $\lambda = \lambda_n$).

Let us study the limit of $\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t)$.

It is easy to see that in $\{x_1 > 0\}$, $\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t) \to M(0,0)$ uniformly on compact sets. Now, let $K \subset \{x_1 < 0\}$ be compact. We will show that

$$\nabla(\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t)) \to 0 \quad \text{in } L^1(K)$$

In fact,

$$\begin{aligned}\nabla(\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t)) &= Y^{\delta_n} f_{\delta_n}(u^{\delta_n})\nabla u^{\delta_n}\\ &+ \lambda_n \nabla w_0(\lambda_n x, \lambda_n^2 t)\big(\frac{w^{\delta_n}}{\delta_n}(x,t) - w_0(\lambda_n x,\lambda_n^2 t)\big) f(-w_0(\lambda_n x, \lambda_n^2 t))\\ &+ \frac{\nabla w^{\delta_n}}{\delta_n}\int_{-w_0(\lambda_n x,\lambda_n^2 t)}^{\frac{u^{\delta_n}}{\delta_n}} f(s)ds\end{aligned}$$

Since $Y^{\delta_n} f_{\delta_n}(u^{\delta_n}) \to 0$ as measures in K and is nonnegative, we deduce that the convergence takes place in $L^1(K)$. On the other hand, ∇u^{δ_n} is uniformly bounded. Therefore, the first term goes to zero in $L^1(K)$.

In order to see that the second and third terms go to zero uniformly in K we only need to observe that

$$\frac{u^{\delta_n}}{\delta_n}(x,t) = \frac{u^{\varepsilon_{k_n}}}{\varepsilon_{k_n}}(\lambda_n x, \lambda_n^2 t)$$

and a similar formula holds for $\frac{w^{\delta_n}}{\delta_n}$. So that

$$\begin{aligned}&\Big|\frac{w^{\delta_n}}{\delta_n}(x,t) - w_0(\lambda_n x,\lambda_n^2 t)\Big| \to 0 \quad \text{uniformly on compact sets of } \mathbb{R}^{N+1},\\ &\frac{u^{\delta_n}}{\delta_n} \ge -\frac{w^{\delta_n}}{\delta_n} \ge -C,\\ &\frac{|\nabla w^{\delta_n}|}{\delta_n}(x,t) = \lambda_n \frac{|\nabla w^{\varepsilon_{k_n}}|}{\varepsilon_{k_n}}(\lambda_n x, \lambda_n^2 t) \to 0 \qquad \text{uniformly on compact sets of } \mathbb{R}^{N+1}.\end{aligned}$$

On the other hand, $|\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t)| \le C_K$, so that we have,

$$\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t) \to \bar{M}(t) \qquad \text{weakly in } L^2(K).$$

Let us now see that, actually, the convergence takes place in $L^1(K)$.

There holds that

$$\frac{\partial}{\partial t}\left(\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t)\right) = Y^{\delta_n} f_{\delta_n}(u^{\delta_n})(u^{\delta_n})_t + \frac{\partial}{\partial t}\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n},x,t)$$
$$\leq \frac{\partial}{\partial t}\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n},x,t) \leq C_K \quad \text{in } K.$$

On the other hand, for every $(x_0,t_0)\in K,$ and $Q_\tau(x_0,t_0)\subset\{x_1<0\}$

$$\iint_{Q_\tau(x_0,t_0)} \frac{\partial}{\partial t}\left(\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t)\right) =$$
$$\int_{B_\tau(x_0)} \mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t_0+\tau^2),x,(t_0+\tau^2))\,dx$$
$$-\int_{B_\tau(x_0)} \mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t_0-\tau^2),x,(t_0-\tau^2))\,dx \geq\ -C_\tau$$

since $|\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t)|\leq C_K$ for every compact set K.

Therefore there exists $C_K>0$ such that

$$\|\mathcal{B}_{\delta_n}^{\lambda_n}(u^{\delta_n}(x,t),x,t)\|_{W^{1,1}(K)}\leq C_K.$$

Hence the convergence takes place in $L^1(K)$ (for a subsequence).

Now arguing as in Lemma 2.1.2, we get that $\bar{M}(t)=0$ or $\bar{M}(t)=M(0,0)$.

We can now take the limit in (2.2.10) for the sequences ε_{k_n} and λ_n and we obtain

$$0=-\frac{1}{2}\alpha^2\iint_{\mathcal{D}\cap\{x_1>0\}}\psi_{x_1}+M(0,0)\iint_{\mathcal{D}\cap\{x_1>0\}}\psi_{x_1}+\iint_{\mathcal{D}\cap\{x_1<0\}}\bar{M}(t)\psi_{x_1}.$$

So that,

$$0=\int_{\mathcal{D}\cap\{x_1=0\}}\left(\frac{1}{2}\alpha^2-M(0,0)-\bar{M}(t)\right)\psi\,dx'dt.$$

Since ψ is arbitrary we get $\frac{1}{2}\alpha^2 = M(0,0)-\bar{M}(t)$. So that, in particular, $\bar{M}(t)$ is constant and then we have that $\bar{M}(t)\equiv 0$ or $\bar{M}(t)\equiv M(0,0)$. Since $\alpha>0$ we deduce that $\bar{M}(t)\equiv 0$ and

$$\alpha=\sqrt{2M(0,0)}.$$

The proof is finished. □

3. Consequences and applications

In this section, we study some consequences of the results collected in Sections 1 and 2.

First, we prove a result that guarantees the nondegeneracy of a limit function u and then we combine this result with Theorem 2.2.5 and construct a family u^ε of solutions to (P_ε) such that the limit function $u = \lim u^\varepsilon$ is a viscosity solution to (P) for rather general initial data.

Now we prove a proposition that says that, under suitable assumptions, u^+ does not degenerate at the free boundary. The proof is similar to Theorem 6.3 in [**10**], where the nondegeneracy of u^+ was proved in the strictly two phase case. Here we assume, instead of (2.1.1) the somewhat stronger condition that for every $K \subset \mathcal{D}$ compact, there exist $0 < \eta < 1$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$

$$\frac{u^\varepsilon}{\varepsilon} \le \eta \qquad \text{in } K \cap \{u \equiv 0\}^\circ. \tag{2.3.1}$$

Proposition 2.3.2. *Let $u = \lim u^{\varepsilon_k}$, where $(u^{\varepsilon_k}, Y^{\varepsilon_k})$ are uniformly bounded solutions to (1.0.1) satisfying (1.0.4) with $w_0 > -1$, such that $Y^{\varepsilon_k} \ge 0$ and the functions u^{ε_k} satisfy (2.3.1). Let $(x_0, t_0) \in \partial\{u > 0\}$.*

Let us assume that there exists $\nu \in \mathbb{R}^N$, with $|\nu| = 1$ such that

$$\liminf_{r\to 0^+} \frac{|\{u > 0\} \cap \{\langle x - x_0, \nu\rangle > 0\} \cap Q_r^-(x_0,t_0)|}{|Q_r^-(x_0,t_0)|} > \alpha_1$$

and

$$\liminf_{r\to 0^+} \frac{|\{u = 0\}^\circ \cap \{\langle x - x_0, \nu\rangle < 0\} \cap Q_r^-(x_0,t_0)|}{|Q_r^-(x_0,t_0)|} > \alpha_2$$

with $\alpha_1 + \alpha_2 > \frac{1}{2}$, then there exists a constant $C > 0$ depending only on N, f, $\alpha_1 + \alpha_2$ and $r_0 > 0$ such that, if $0 < r \le r_0$,

$$\sup_{\partial_p Q_r^-(x_0,t_0)} u \ge Cr.$$

Proof. Without loss of generality, we may assume that $(x_0, t_0) = (0, 0)$ and that $\nu = e_1 = (1, 0, ..., 0)$.

We will note $Q_r^- = Q_r^-(0,0)$ and

$$(u^{\varepsilon_k})_r(x,t) = \frac{1}{r} u^{\varepsilon_k}(rx, r^2 t), \quad (Y^{\varepsilon_k})_r(x,t) = \frac{1}{r} Y^{\varepsilon_k}(rx, r^2 t),$$

$$u_r(x,t) = \frac{1}{r} u(rx, r^2 t).$$

STEP 1. Let us see that there exists $r_0 > 0$ and a constant c such that if $r < r_0$ and $\varepsilon_k < \varepsilon_0 = \varepsilon_0(r)$, then

$$\iint_{Q_1^-} (Y^{\varepsilon_k})_r f_{\varepsilon_k/r}((u^{\varepsilon_k})_r) dx \geq c.$$

Without loss of generality we may assume that r_0 is small so that $|u^\varepsilon| \leq 1$ in $Q_{r_0}^-$.

Let $0 < 4\lambda < \alpha_1 + \alpha_2 - \frac{1}{2}$. From our assumptions, it follows that there exists $r_0 > 0$ such that, for $r \leq r_0$,

$$\frac{|\{u_r > 0\} \cap \{x_1 > 0\} \cap Q_1^-|}{|Q_1^-|} + \frac{|\{u_r = 0\}^\circ \cap \{x_1 < 0\} \cap Q_1^-|}{|Q_1^-|} > \frac{1}{2} + 2\lambda.$$

We now fix r with this property. Then, there exists $\gamma > 0$ small such that

$$\frac{|\{u_r > \gamma\} \cap \{x_1 > 0\} \cap Q_1^-|}{|Q_1^-|} + \frac{|\{u_r = 0\}^\circ \cap \{x_1 < 0\} \cap Q_1^-|}{|Q_1^-|} \geq \frac{1}{2} + \lambda.$$

Let us now define

$$A_r = \{u_r > \gamma\} \cap \{x_1 > 0\} \cap Q_1^-, \quad B_r = \{u_r = 0\}^\circ \cap \{x_1 < 0\} \cap Q_1^-$$
$$\text{and } -B_r = \{(x_1, x', t)/\ (-x_1, x', t) \in B_r\}.$$

Then, we have

$$|A_r \cap (-B_r)| \geq \lambda |Q_1^-| = \tilde{\lambda}.$$

For $0 \leq x_1 < 1$, let

$$g(x_1) = |\{(x', t)\ :\ (x_1, x', t) \in A_r \cap (-B_r)\}|.$$

Let $0 < \rho < 1$ be fixed, then

$$|\{x_1\ :\ g(x_1) > \rho\tilde{\lambda}\}| > 0.$$

In fact, if not

$$\tilde{\lambda} \leq |A_r \cap (-B_r)| = \int_0^1 g(x_1)\, dx_1 = \int_{\{g \leq \rho\tilde{\lambda}\}} g(x_1)\, dx_1 \leq \rho\tilde{\lambda}$$

which is a contradiction. Therefore, we have in particular that there exists $0 < x_1^r < 1$ such that

$$g(x_1) = |\Lambda_r| = |\{(x', t)/\ (x_1^r, x', t) \in A_r \cap (-B_r)\}| > \rho\tilde{\lambda}$$

Let $\eta > 0$ be the constant in (2.3.1) in $Q_1(0,0)$, let $0 < \delta' < \delta$, $0 < b < b' < 1$ be such that

$$\eta < -w_0(0,0) + \delta < b.$$

Let $\kappa > 0$ be such that

$$f(s) > \kappa > 0 \quad \text{for } s \leq b'.$$

Then, for $(x', t) \in \Lambda_r$ we have

$$\begin{aligned}
&\frac{1}{\varepsilon_k/r}(u^{\varepsilon_k})_r(x_1^r, x', t) > \frac{\gamma}{2(\varepsilon_k/r)} > b,\\
&\frac{1}{\varepsilon_k/r}(u^{\varepsilon_k})_r(-x_1^r, x', t) < -w_0(0,0) + \delta
\end{aligned}$$

if $\varepsilon_k < \varepsilon_1 = \varepsilon_1(r)$ is small. So that, for every $(x', t) \in \Lambda_r$ there exists $\tilde{x}_1^r \in (-1, 1)$ such that $-w_0(0,0) + \delta \leq \frac{1}{\varepsilon_k/r}(u^{\varepsilon_k})_r(\tilde{x}_1^r, x', t) \leq b$.

Now, by the uniform Lipschitz regularity of $(u^{\varepsilon_k})_r$ and $(Y^{\varepsilon_k})_r$, and (1.0.4), we have that for $\varepsilon_k \leq \varepsilon_0 (\leq \varepsilon_1)$ and $r \leq r_0$,

$$\frac{(u^{\varepsilon_k})_r}{\varepsilon_k/r}(x_1, x', t) \leq b' \quad \text{and} \quad \frac{(Y^{\varepsilon_k})_r}{\varepsilon_k/r}(x_1, x', t) \geq \delta' \quad \text{if } |x_1 - \tilde{x}_1^r| < C\frac{\varepsilon_k}{r}$$

where C depends on δ, δ', b, b', on the Lipschitz constant of u^{ε_k} and Y^{ε_k} in Q_1^- and r_0 depends only on w_0.

Finally we have

$$\begin{aligned}
\iint_{Q_1^-} (Y^{\varepsilon_k})_r f_{\varepsilon_k/r}((u^{\varepsilon_k})_r) &= \iint_{Q_1^-} \frac{(Y^{\varepsilon_k})_r}{\varepsilon_k/r} \frac{1}{\varepsilon_k/r} f\Big(\frac{(u^{\varepsilon_k})_r}{\varepsilon_k/r}\Big)\\
&\geq \delta' \frac{\kappa}{\varepsilon_k/r} \left| \left\{ (x,t) \in Q_1^- /\ \frac{(Y^{\varepsilon_k})_r}{\varepsilon_k/r} \geq \delta' \text{ and } f\Big(\frac{(u^{\varepsilon_k})_r}{\varepsilon_k/r}\Big) \geq \kappa \right\} \right|\\
&\geq \delta' \frac{\kappa}{\varepsilon_k/r} |\Lambda_r| 2C \frac{\varepsilon_k}{r} \geq 2C\delta'\kappa\rho\tilde{\delta} \equiv c.
\end{aligned}$$

Step 2. Now we will prove that there exists a constant $C > 0$ such that, for every $r > 0$ small,

$$\sup_{\partial_p Q_r^-} u \geq Cr.$$

We will proceed by contradiction. If the result were false, there would exist a sequence $r_n \to 0$ such that

$$\sup_{\partial_p Q_{2r_n}^-} u \leq \frac{1}{n} r_n.$$

Since $u^{\varepsilon_k} \to u$ uniformly in $\overline{Q_{r_0}^-}$, there holds that

$$\sup_{\partial_p Q_{2r_n}^-} u^{\varepsilon_k} \leq \frac{2}{n} r_n,$$

for $k \geq k(n)$. Therefore we have

$$\sup_{\partial_p Q_2^-} (u^{\varepsilon_k})_{r_n} \leq \frac{2}{n}.$$

In addition, by STEP 1,

$$\iint_{Q_1^-} (Y^{\varepsilon_k})_{r_n} f_{\varepsilon_k/r_n}((u^{\varepsilon_k})_{r_n}) \geq c,$$

as long as $r_n \leq r_0$ and $\varepsilon_k \leq \varepsilon_0$.

Since $(u^{\varepsilon_k})_{r_n}$ are solutions to

$$\Delta(u^{\varepsilon_k})_{r_n} - \frac{\partial}{\partial t}(u^{\varepsilon_k})_{r_n} = (Y^{\varepsilon_k})_{r_n} f_{\varepsilon_k/r_n}((u^{\varepsilon_k})_{r_n})$$

in Q_2^-, there holds the representation formula

$$(u^{\varepsilon_k})_{r_n}(0,0) = \int_{\partial_p Q_2^-} (u^{\varepsilon_k})_{r_n} P - \iint_{Q_2^-} (Y^{\varepsilon_k})_{r_n} f_{\varepsilon_k/r_n}((u^{\varepsilon_k})_{r_n}) G,$$

where

$$P \geq 0 \text{ on } \partial_p Q_2^-, \qquad \int_{\partial_p Q_2^-} P = 1,$$
$$G \geq 0 \text{ in } Q_2^-, \qquad G \geq \mu > 0 \text{ in } Q_1^-.$$

It follows that

$$(u^{\varepsilon_k})_{r_n}(0,0) \leq \frac{2}{n} - \mu c < -\frac{\mu}{2} c,$$

if n is big enough and $\varepsilon_k \leq \min\{\varepsilon_0, \varepsilon_{k(n)}\}$. But this gives a contradiction since $(u^{\varepsilon_k})_{r_n}(0,0) \to 0$ as $k \to \infty$. Thus the proof is complete. □

REMARK 2.3.3. *Proposition 2.3.2 remains true if we change the hypothesis that u^{ε_k} satisfies (2.3.1) by*

$$\frac{u^{\varepsilon_k}}{\varepsilon_k} \to -w_0 \qquad a.e. \;\; \{u \equiv 0\}^\circ. \tag{2.3.4}$$

In fact, as in the proof of Proposition 2.3.2 we consider for each $0 < r < 1$ the sets A_r and B_r. So that, for some $0 < \lambda < 1$

$$|A_r \cap (-B_r)| \geq \lambda |Q_1^-|.$$

Since $B_r \subset \{u_r \equiv 0\}^\circ$, there holds that

$$\frac{(u^\varepsilon)_r}{\varepsilon/r}(-x_1, x', t) \to -w_0(-rx_1, rx', r^2 t) \qquad a.e. \;\; A_r \cap (-B_r).$$

Let $0 < \mu < 1$. There exists $C_r \subset \big(A_r \cap (-B_r)\big)$ such that $|C_r| = \mu|A_r \cap (-B_r)|$ and

$$\frac{(u^\varepsilon)_r}{\varepsilon/r}(-x_1, x', t) \to -w_0(-rx_1, rx', r^2 t) \quad \text{uniformly in } C_r.$$

Let $\delta > 0$. There exists $\varepsilon_1 = \varepsilon_1(r)$ such that

$$\frac{(u^\varepsilon)_r}{\varepsilon/r}(-x_1, x', t) \le -w_0(-rx_1, rx', r^2 t) + \frac{\delta}{2} \le -w_0(0,0) + \delta \quad \text{in } C_r$$

if $\varepsilon < \varepsilon_1$ and $r < r_0 = r_0(\delta)$. Now, the proof follows as in Proposition 2.3.2 by taking $\tilde\lambda = \mu\lambda|Q_1^-|$ and

$$\Lambda_r := \{(x', t) \,/\, (x_1^r, x', t) \in C_r\}.$$

Remark 2.3.5. *Proposition 2.3.2 remains true if we change condition (2.3.1) by condition (2.1.1). In fact, as in the proof of Theorem 2.1.5 we see that condition (2.1.1) implies that*

$$\mathcal{B}_{\varepsilon_k}(u^{\varepsilon_k}, x, t) \to 0 \qquad L^1_{loc}(\{u \equiv 0\}^\circ).$$

As in Theorem 2.1.5 we deduce that u^{ε_k} satisfies (2.3.4).

Using Remark 2.3.3, Remark 2.3.5 and Theorem 2.2.5 we get the following Corollaries.

Corollary 2.3.6. *Let $u = \lim u^{\varepsilon_k}$ where $(u^{\varepsilon_k}, Y^{\varepsilon_k})$ are uniformly bounded solutions to (1.0.1) in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$ with $Y^{\varepsilon_k} \ge 0$, which verify (1.0.4) with $w_0 > -1$ and such that u^{ε_k} satisfies (2.1.1). If the free boundary $\mathcal{D} \cap \partial\{u > 0\}$ is given by $x_1 = g(x', t)$ with $g \in Lip(1, 1/2)$, then, u is a viscosity solution of the free boundary problem (P).*

Corollary 2.3.7. *Let $u = \lim u^{\varepsilon_k}$ where $(u^{\varepsilon_k}, Y^{\varepsilon_k})$ are uniformly bounded solutions to (1.0.1) in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$ with $Y^{\varepsilon_k} \ge 0$, which verify (1.0.4) with $w_0 > -1$ and such that u^{ε_k} satisfies (2.3.4) and $u_t^{\varepsilon_k} \le 0$. If, for every $(x_0, t_0) \in \mathcal{D} \cap \partial\{u > 0\}$, $\{x \in \mathbb{R}^N \,/\, (x, t_0) \in \mathcal{D} \cap \{u > 0\}\}$ is given by $x_1 > \Phi(x')$ with Φ, Lipschitz continuous then, u is a viscosity solution of the free boundary problem (P).*

Proof. We only need to see that u does not degenerate at points of the free boundary which are regular from the zero side. Let (x_0, t_0) be any such point. We see that we can apply Remark 2.3.3 at that point. In fact, since $u_t^{\varepsilon_k} \le 0$, u is decreasing in time. Therefore,

$$\{(x, t) \,/\, x_1 > \Phi(x)\,,\, t \le t_0\} \subset \{u > 0\}$$

and the parabolic density of this set is positive. □

In particular, Corollary 2.3.7 can be applied to solutions of (1.0.1) with u_0^ε constructed as in [**13**] and Y_0^ε a small perturbation of u_0^ε. In fact we have the following result.

COROLLARY 2.3.8. *Let $u_0 \in C(\mathbb{R}^N) \cap C^2(\overline{\{u_0 > 0\}})$ be such that $\|u_0\|_{C^2(\overline{\{u_0>0\}})} < \infty$, $\Delta u_0 \leq 0$ and $(u_0)_{x_1} - \lambda|\nabla u_0| \geq 0$ in $\{u_0 > 0\}$ with $\lambda > 0$. Assume, moreover that $0 < a_2 \leq |\nabla u_0| \leq a_1 < \sqrt{2M_0}$ in a neighborhood of the free boundary: $\{x \in \{u_0 > 0\} \,/\, dist(x, \{u_0 = 0\}) \leq \gamma\}$, and $M_0 = \int_0^1 sf(s)$. Then, there exists a sequence $(u_0^\varepsilon, Y_0^\varepsilon) \in \big(C^1(\mathbb{R}^N)\big)^2$ with $u_0^\varepsilon \to u_0$ uniformly in $\mathbb{R}^N$ (so that u_0^ε are uniformly bounded) and, moreover, they satisfy*

(2.3.9)

(1) $\Delta u_0^\varepsilon - Y_0^\varepsilon f_\varepsilon(u_0^\varepsilon) \leq 0$

(2) $(u_0^\varepsilon)_{x_1} - \lambda|\nabla u_0^\varepsilon| \geq 0$

(3) $\dfrac{Y_0^\varepsilon - u_0^\varepsilon}{\varepsilon} \to w_0$ *uniformly on compact sets, with* $w_0 > -1$.

$w_0 \in \mathbb{R}$ is any constant such that $w_0 \geq -\eta$ with $\eta > 0$ small enough.

Let $(u^\varepsilon, Y^\varepsilon)$ be the solution to (1.0.1) with initial datum $(u_0^\varepsilon, Y_0^\varepsilon)$ (so that, in particular, u^ε and Y^ε are uniformly bounded). For every sequence $\varepsilon_j \to 0$ there exists a subsequence ε_{j_k} such that there exists

$$u = \lim_{k\to\infty} u^{\varepsilon_{j_k}}$$

and u is a viscosity solution to the free boundary problem (P).

PROOF. Let u_0^ε be the approximations constructed in [**13**]. The approximations are constructed in the following way. First we extend u_0 to a neighborhood of $\{u_0 > 0\}$: $S := \{x \in \mathbb{R}^N \,/\, \mathrm{dist}\,(x, \{u_0 > 0\}) \leq \gamma\}$ in such a way that $\|u_0\|_{C^2(S)} < \infty$. For ε small enough we define

$$u_0^\varepsilon(x) = \varepsilon F\left(\frac{1}{\sqrt{2M_0}}\big(1 - \frac{u_0(x)}{\varepsilon}\big)\right) \qquad \text{in } \{-C\varepsilon \leq u_0 \leq \varepsilon\}.$$

where $F \in C^2(\mathbb{R})$ is such that

$$F'' \leq (1+\delta)Ff(F) + \alpha F', \quad F(0) = 1, \quad F'(0) = -\sqrt{2M_0}.$$

Here $\delta > 0$, $\alpha > 0$ are such that F has a strict minimum at a finite point $\bar{s}$ such that $\bar{s}\sqrt{2M_0} > 1$. ($\bar{s} \to +\infty$ as $\delta \to 0$), and F is decreasing for $s < \bar{s}$.

The constant C is taken as $C = \bar{s}\sqrt{2M_0} - 1$.

We define

$$\begin{aligned} u_0^\varepsilon &= u_0 && \text{in } \{u_0 > \varepsilon\} \\ u_0^\varepsilon &= \varepsilon F(\bar s) && \text{in } \mathbb{R}^N \setminus \{u_0 > -C\varepsilon\}. \end{aligned}$$

As in [**13**], we see that $u_0^\varepsilon \in C^1(\mathbb{R}^N)$.

Let $w_0 \in \mathbb{R}$ be such that $w_0 \geq -\eta > -F(\bar s)$ with $\eta > 0$ to be fixed later and let

$$Y_0^\varepsilon = u_0^\varepsilon + \varepsilon w_0.$$

Then, $Y_0^\varepsilon \geq 0$. It is immediate to verify that (2.3.9) 1) is satisfied in $\{u_0 > \varepsilon\}$ and $\mathbb{R}^N \setminus \{u_0 > -C\varepsilon\}$. Let us see that it is satisfied in $\{-C\varepsilon \leq u_0 \leq \varepsilon\}$. In fact,

$$\begin{aligned} \Delta u_0^\varepsilon - Y_0^\varepsilon f_\varepsilon(u_0^\varepsilon) =& \frac{1}{2M_0\varepsilon} F'' |\nabla u_0|^2 - \frac{1}{\sqrt{2M_0}} F' \Delta u_0 - \frac{1}{\varepsilon} F f(F) - \frac{w_0}{\varepsilon} f(F) \\ \leq& \frac{1+\delta}{2M_0\varepsilon} F f(F) |\nabla u_0|^2 + \frac{\alpha}{2M_0\varepsilon} F' |\nabla u_0|^2 - \frac{a}{\sqrt{2M_0}} F' \\ & - \frac{1}{\varepsilon} F f(F) - \frac{w_0}{\varepsilon} f(F) \end{aligned}$$

where $a > 0$ is such that $|\Delta u_0| \leq a$.

Let $0 < \mu < 1$ be such that $a_1 \leq (1-\mu)^{1/2} A \sqrt{2M_0}$ with $0 < A < 1$, and let δ in the definition of F be such that $(1+\delta)A^2 \leq 1$. Then, if ε is small enough so that $\alpha a_2^2/\sqrt{2M_0} > a\varepsilon$ there holds that

$$\begin{aligned} \Delta u_0^\varepsilon - Y_0^\varepsilon f_\varepsilon(u_0^\varepsilon) \leq& \frac{1}{\varepsilon} \Big[[(1+\delta)(1-\mu)A^2 - 1] F f(F) \\ & + \big(\frac{\alpha a_2^2}{2M_0} - \frac{a\varepsilon}{\sqrt{2M_0}} \big) F' - w_0 f(F) \Big] \\ \leq& \frac{1}{\varepsilon} [-\mu F - w_0] f(F) \leq \frac{1}{\varepsilon} [-\mu F(\bar s) - w_0] f(F) \leq 0 \end{aligned}$$

if $\eta = \mu F(\bar s)$.

Clearly, (2.3.9) 3) holds. Let us see that (2.3.9) 2) also holds. We only need to verify this property in the set $\{-C\varepsilon < u_0 < \varepsilon\}$ and this is clear from the fact that

$$\nabla u_0^\varepsilon = -\frac{1}{\sqrt{2M_0}} F' \left(\frac{1}{\sqrt{2M_0}} \big(1 - \frac{u_0}{\varepsilon}\big) \right) \nabla u_0.$$

Now, by the results of Chapter 1, for every sequence $\varepsilon_j \to 0$ there exists a subsequence and a continuous function u such that $u^{\varepsilon_{j_k}} \to u$ uniformly on compact subsets of $\mathbb{R}^N \times (0, \infty)$.

On the other hand, u_t^ε is a solution to the following equation

$$\Delta U - U_t = \beta_\varepsilon'(u)U.$$

Here $\beta_\varepsilon(s) = sf_\varepsilon(s)$. Since, for ε small enough $u_t^\varepsilon(x,0) \le 0$ we conclude that

$$u_t^\varepsilon \le 0 \qquad \text{in } \mathbb{R}^N \times (0,\infty). \tag{2.3.10}$$

In a similar way we see that $u_{x_1} - \lambda u_{x_i} \ge 0$ for every i. So that

$$u_{x_1}^\varepsilon - \frac{\lambda}{N}|\nabla u^\varepsilon| \ge 0 \qquad \text{in } \mathbb{R}^N \times (0,\infty). \tag{2.3.11}$$

Clearly (2.3.10) and (2.3.11) imply that

$$u_t \le 0 \qquad \text{and} \qquad u_{x_1} - \frac{\lambda}{N}|\nabla u| \ge 0 \qquad \text{in } \{u > 0\}.$$

In particular, the free boundary is Lipschitz in space.

So that, in order to apply Corollary 2.3.7 we only need to verify that u^{ε_k} satisfies (2.3.4). On one hand, given $K \subset \{u_0 \equiv 0\}^\circ$ compact, there exists ε_0 such that for $\varepsilon < \varepsilon_0$

$$\mathcal{B}_\varepsilon(u_0^\varepsilon, x, 0) = \int_{-w_0}^{\frac{u_0^\varepsilon}{\varepsilon}(x)} (s + w_0) f(s) = \int_{-w_0}^{F(\bar s)} (s + w_0) f(s).$$

On the other hand,

$$\frac{\partial}{\partial t}\big(\mathcal{B}_\varepsilon(u^\varepsilon, x, t)\big) = Y^\varepsilon f_\varepsilon(u^\varepsilon) u_t^\varepsilon \le 0.$$

Therefore,

$$\mathcal{B}_\varepsilon(u^\varepsilon, x, t) \le \int_{-w_0}^{F(\bar s)} (s + w_0) f(s) \qquad \text{for } x \text{ in } K\,,\ t > 0.$$

As in the proof of Theorem 2.2.5 we see that, since $u_t^\varepsilon \le 0$, there holds that $\mathcal{B}_\varepsilon(u^\varepsilon, x, t) \to \bar M(x,t)$ in $L^1_{\text{loc}}(\{u \equiv 0\}^\circ)$ and, for almost every (x,t) we either have $\bar M(x,t) = 0$ or $\bar M(x,t) = M = \int_{-w_0}^1 (s + w_0) f(s)$. Since

$$\nabla\big(B_\varepsilon(u^\varepsilon, x, t)\big) = Y^\varepsilon f_\varepsilon(u^\varepsilon)\nabla u^\varepsilon \to 0 \quad \text{in } L^1_{\text{loc}}(\{u \equiv 0\}^\circ)$$

there holds that $\bar M(x,t) = \bar M(t)$ in $\{u \equiv 0\}^\circ$. Therefore,

$$\bar M(t) \le \int_{-w_0}^{F(\bar s)} (s + w_0) f(s) \quad a.e.\ \ \{u \equiv 0\}^\circ.$$

Since $F(\bar s) < 1$, there holds that $\bar M(t) \equiv 0$.

Thus, for every sequence $\varepsilon_k \to 0$

$$\int_{-w_0}^{\frac{u^{\varepsilon_k}}{\varepsilon_k}} (s + w_0) f(s) \to 0 \qquad a.e. \;\; \{u \equiv 0\}^\circ$$

and we deduce that u^{ε_k} satisfies (2.3.4). $\square$

Combining the regularity results for viscosity solutions of [**17**], Corollary 2.3.6 and Corollary 2.3.7 we have the following regularity result for limit functions.

COROLLARY 2.3.12. *Let u as in Corollary 2.3.6 or Corollary 2.3.7. If, moreover, the free boundary $\mathcal{D} \cap \partial\{u > 0\}$ is given by $x_1 = g(x', t)$ with g Lipschitz continuous, then, u is a classical solution of the free boundary problem (P).*

CHAPTER 3

Uniqueness of limit solutions

The main point in this Chapter is to give a positive answer to the question of whether the limit of a sequence of solutions to (P_ε) is determined once the limit of $w^{\varepsilon_k}/\varepsilon_k$ and of u^{ε_k} are fixed. So we study the uniqueness character of the limit functions (or limit solutions) studied in the previous Chapters.

Some geometric assumptions are needed. In fact, uniqueness of the limit fails, in a general setting, even in the case $w^\varepsilon \equiv 0$. These geometric assumptions are similar to that used to prove uniqueness of the limit for the case $u^\varepsilon = Y^\varepsilon$ in [**30**]. We state these assumptions in Section 1.

In fact, we follow here some of the ideas in [**30**] which are based on the fact that any limit function is a supersolution to (P). This is still true in our case. Unfortunately the simple construction in [**30**] of supersolutions of (P_ε) that approximate a strict supersolution of (P), when $w^\varepsilon \equiv 0$, does not work in the general case unless one asks for a lot of complementary conditions on the reaction function f.

Therefore, we follow here the construction done in [**22**]. The proof that this construction works in based on blow up of the constructed functions.

In Section 2 we prove some technical lemmas needed in the proof of the uniqueness result.

In Section 3 we prove that, under the geometric assumptions in consideration, a semi-classical supersolution of (P) is the uniform limit of supersolutions of (P_ε), and restate an analogous result for subsolutions.

In Section 4 we arrive at the main point of the Chapter, we prove that, under suitable assumptions, there exists a unique limit.

We end this Chapter with a discussion of different geometries where these results hold.

1. Preliminaries

Following [**30**] we give the definition of supersolution of problem (P) that will be needed in this Chapter. Note that this definition differs from the one given in Definition 2.2.1 since we are not assuming that the function be C^1 up to the free boundary or that the free boundary be C^1.

DEFINITION 3.1.1. *A continuous nonnegative function u in $\overline{Q}_T = \mathbb{R}^N \times [0,T]$, $T > 0$, is called a semi-classical supersolution of (P) if $u \in C^1(\{u > 0\})$ and*

(i) $\Delta u - u_t \leq 0$ *in* $\Omega = \{u > 0\}$*;*
(ii) $\limsup_{\Omega \ni (y,s) \to (x,t)} |\nabla u(y,s)| \leq \sqrt{2M(x,t)}$ *for every* $(x,t) \in \partial\Omega \cap Q_T$*;*
(iii) $u(\cdot, 0) \geq u_0$.

Respectively, u is a semi-classical subsolution of (P) if conditions (i), (ii) and (iii) are satisfied with reversed inequalities and lim inf *instead of* lim sup *in (ii).*

A function u is a classical solution of (P) if it is both a semi-classical subsolution and a semi-classical supersolution of (P), $u \in C^1(\overline{\{u > 0\}})$ and the free boundary $\partial\{u > 0\} \cap Q_T$ is a C^1 surface.

Next, a semi-classical supersolution u of (P) is a strict semi-classical supersolution of (P) if there is a $\delta > 0$ such that the stronger inequalities

(ii') $\limsup_{\Omega \ni (y,s) \to (x,t)} |\nabla u(y,s)| \leq \sqrt{2M(x,t) - \delta}$ *for every* $(x,t) \in \partial\Omega \cap Q_T$*;*
(iii') $u(\cdot, 0) \geq u_0 + \delta$ *on* $\Omega_0 = \{u_0 > 0\}$

hold. Analogously a strict semi-classical subsolution is defined.

As a consequence of the results in Chapter 1, one can check that every limit solution $u = \lim_{j\to\infty} u^{\varepsilon_j}$ of (P) is a semi-classical supersolution in the sense of Definition 3.1.1. In fact,

PROPOSITION 3.1.2. *Let u^{ε_j} be solutions to (P_{ε_j}) – with w^{ε_j} satisfying (0.2.4) and $w_0 > -1$ – such that $u^{\varepsilon_j} \to u$ uniformly on compact sets and $\varepsilon_j \to 0$. Assume that the initial datum u_0 is Lipschitz continuous and that the approximations of the initial datum verify $|u_0^\varepsilon(x)|, |\nabla u_0^\varepsilon(x)| \leq C$ and $u_0^\varepsilon \in C^1(\overline{\{u_0^\varepsilon > 0\}})$. Then u is a semi-classical supersolution of (P).*

Proof. We have to verify conditions (i)-(iii) of Definition 3.1.1.

By Proposition 1.5.1, (i) and (ii) hold.

Now, from our assumptions on the initial datum u_0, by Proposition 5.2.1 of [**27**], we have that $u^\varepsilon \to u$ uniformly on compact sets of $\overline{Q_T}$ so that u is continuous up to $t = 0$ and (iii) also holds. $\square$

Let us suppose that the initial datum u_0 of problem (P) is star-shaped with respect to a point x_0, that we always assume to be 0, in the following sense: For every $\lambda \in (0,1)$ and $x \in \mathbb{R}^N$,

$$u_0(\lambda x) \geq u_0(x), \qquad \lambda\Omega_0 \subset\subset \Omega_0, \tag{3.1.3}$$

where $\Omega_0 = \{u_0 > 0\}$.

Also, assume that

$$w_0(\lambda x, 0) \leq w_0(x, 0) \quad \text{if } x \in \mathbb{R}^N\ ,\ 0 < \lambda < 1 \tag{3.1.4}$$

and

$$w_0 > -1 + \delta_1 \quad \text{for some } \delta_1 > 0. \tag{3.1.5}$$

Let u be a semi-classical supersolution of (P). Let λ and λ' be two real numbers with $0 < \lambda < \lambda' < 1$. Define

$$u_\lambda(x,t) = \frac{1}{\lambda'} u(\lambda x, \lambda^2 t) \tag{3.1.6}$$

in Q_{T/λ^2}. The rescaling is taken so that u_λ satisfies the heat equation in

$$\Omega_\lambda = \{(x,t) :\ (\lambda x, \lambda^2 t) \in \Omega\}. \tag{3.1.7}$$

Moreover, the fact that $0 < \lambda < \lambda' < 1$ makes u_λ a strict semi-classical supersolution of (P).

In fact, let us first see that

$$M(\lambda x, \lambda^2 t) \leq M(x,t) \qquad \text{if } 0 < \lambda < 1.$$

This is a consequence of the fact that the function

$$a \longrightarrow \int_{-a}^{1} (s+a) f(s)\, ds$$

is nondecreasing and

$$w_0(\lambda x, \lambda^2 t) \leq w_0(x,t) \qquad \text{if } 0 < \lambda < 1. \tag{3.1.8}$$

In fact, the function $w_\lambda(x,t) = w_0(\lambda x, \lambda^2 t)$ is caloric and $w_\lambda(x,0) \leq w_0(x,0)$ if $0 < \lambda < 1$ by hypothesis. Thus, by the comparison principle, $w_\lambda(x,t) \leq w_0(x,t)$ in Q_T.

Now, let $(x_0,t_0)\in\partial\{u_\lambda>0\}$. Then,

$$\begin{aligned}\limsup_{\Omega_\lambda\ni(x,t)\to(x_0,t_0)}|\nabla u_\lambda(x,t)| &= \limsup_{\Omega\ni(\lambda x,\lambda^2 t)\to(\lambda x_0,\lambda^2 t_0)}|\frac{\lambda}{\lambda'}\nabla u(\lambda x,\lambda^2 t)|\\ &\le \frac{\lambda}{\lambda'}\sqrt{2M(\lambda x_0,\lambda^2 t_0)}\\ &\le \sqrt{2M(x_0,t_0)}-\Big(1-\frac{\lambda}{\lambda'}\Big)\sqrt{2M_0},\end{aligned}$$

where $0<M_0<M(x,t)$ in Q_T, by (3.1.5).

On the other hand, since $\lambda\Omega_0\subset\subset\Omega_0$, there holds that

$$u_0(\lambda x)\ge\gamma>0 \qquad \text{if } x\in\Omega_0.$$

Thus, for $x\in\Omega_0$,

$$\begin{aligned}u_\lambda(x,0)=\frac{1}{\lambda'}u_0(\lambda x)&=u_0(\lambda x)+\Big(\frac{1}{\lambda'}-1\Big)u_0(\lambda x)\\ &\ge u_0(x)+\Big(\frac{1}{\lambda'}-1\Big)\gamma.\end{aligned}$$

The following comparison lemma for problem (P) can be proved as Lemma 2.4 in [**30**].

Lemma 3.1.9. *Let u_0 satisfy* (3.1.3) *and w_0 satisfy (3.1.4)-(3.1.5). Then every semi-classical subsolution of (P) with bounded support, is smaller than every semi-classical supersolution of (P). i.e. if u' is a semi-classical subsolution such that Ω' is bounded and u is a semi-classical supersolution then*

$$\Omega'\subset\Omega \quad \textit{and} \quad u'\le u,$$

where $\Omega'=\{u'>0\}$ and $\Omega=\{u>0\}$.

Proof. Let u' be a subsolution and u be a supersolution of (P) in Q_T. We only need to show that $\Omega'\subset\Omega$ since the comparison between u' and u will follow from this inclusion by the maximum principle.

Suppose first that $u'\in C^1(\overline{\Omega'})$ and $u\in C^1(\overline{\Omega})$. Let

$$\lambda_0=\sup\{\lambda\in(0,1):\ \Omega'\subset\Omega_\lambda\},$$

where Ω_λ is defined in (3.1.7). We have to show that $\lambda_0=1$. Suppose not, then $\lambda_0<1$ and $\Omega'\subset\Omega_{\lambda_0}$, and there is a common point $(x_0,t_0)\in\partial\Omega'\cap\partial\Omega_{\lambda_0}\cap Q_T$. Let $\lambda_0<\lambda_0'<1$ and u_{λ_0} be as in (3.1.6). Then

$u' \leq u_{\lambda_0}$ in Ω'. At (x_0, t_0), (as u' and u are regular) by Hopf's Lemma we have

$$-\frac{\partial u'}{\partial \nu}(x_0, t_0) < -\frac{\partial u}{\partial \nu}(x_0, t_0),$$

where ν is the outward spatial normal for Ω' at (x_0, t_0). Now since

$$-\frac{\partial u'}{\partial \nu}(x_0, t_0) = |\nabla u'(x_0, t_0)| \geq \sqrt{2M(x_0, t_0)}$$

and

$$-\frac{\partial u}{\partial \nu}(x_0, t_0) = |\nabla u(x_0, t_0)| < \sqrt{2M(x_0, t_0)},$$

we arrive at a contradiction. Observe that here, we do not need the strong inequality (ii'), so we only need the weaker assumption $w_0 > -1$ in Q_T instead of (3.1.5) in this case.

The general case, can be reduced to the previous one as in [**30**]. In fact, let $\widetilde{u}$ be a supersolution. Choose $0 < \lambda < \lambda' < 1$ close to 1 and regularize $\widetilde{u}$ by

$$u(x, t) = (\widetilde{u}_\lambda(x, t + h) - \eta)^+,$$

for small $h, \eta > 0$. Analogously regularize a subsolution $\widetilde{u'}$. Then we will fall into the hypotheses of the previous case and then we can finish the proof by letting first $h, \eta \to 0+$ and then $\lambda \to 1-$. □

2. Auxiliary results

This section contains results on the following problem:

$$(P_0) \qquad \Delta u - u_t = (u + \omega_0) f(u),$$

where the function f is as in Section 1 and ω_0 is a constant, $\omega_0 > -1$. The results will be used in the next sections where (P_0) appears as a blow-up limit.

These results and their proofs are analogous to those in Section 4 in [**22**] where the case $\omega_0 = 0$ was analyzed. We prove them here for the sake of completeness.

Lemma 3.2.1. *Let $a, b \geq 0$ and let $\psi = \psi_{a,b}$ be the classical solution to*

$$\begin{aligned} &\psi_{ss} = (\psi + \omega_0) f(\psi) \quad \text{for } s > 0, \\ &\psi(0) = a, \quad \psi_s(0) = -\sqrt{2b}. \end{aligned} \tag{3.2.2}$$

Let $B(\tau) = \int_{-\omega_0}^{\tau} (\rho+\omega_0) f(\rho)\, d\rho$.

(3.2.3) *If* $b = 0$ *and* $a \in \{-\omega_0\} \cup [1, +\infty)$, *then* $\psi \equiv a$.

(3.2.4) *If* $b = 0$ *and* $a \in (-\omega_0, 1)$, *then* $\lim_{s\to+\infty} \psi(s) = +\infty$.

(3.2.5) *If* $b \in \big(0,\, B(a)\big)$, *then* $\lim_{s\to+\infty} \psi(s) = +\infty$.

(3.2.6) *If* $0 < b = B(a)$, *then* $\psi_s < 0$ *and* $\lim_{s\to+\infty} \psi(s) = -\omega_0$.

(3.2.7) *If* $b \in \big(B(a),\, +\infty\big)$, *then* $\psi_s < 0$ *and* $\lim_{s\to+\infty} \psi(s) = -\infty$.

PROOF. We first recall that the function f is Lipschitz continuous and therefore, there is a unique classical solution to (3.2.2).

Let us multiply equation (3.2.2) by ψ_s. We get

$$\psi_{ss}\psi_s = (\psi+\omega_0) f(\psi)\psi_s = \frac{d}{d\,s}\big(B(\psi)\big), \qquad \text{for } s > 0.$$

Then, if we integrate the expression above, we deduce that

$$\frac{1}{2}\psi_s^2(s) - B(\psi(s)) = \frac{1}{2}\psi_s^2(0) - B(\psi(0)) = b - B(a), \tag{3.2.8}$$

for every $s \geq 0$.

I. Assume $b = 0$ and $a \in \{-\omega_0\} \cup [1, +\infty)$. Then, (3.2.3) follows easily if we recall that $(s+\omega_0) f(s) = 0$ for $s \in \{-\omega_0\} \cup [1, +\infty)$.

II. Assume $b = 0$ and $a \in (-\omega_0, 1)$. Since $\psi_{ss} \geq 0$, then $\psi_s(s) \geq 0$. Moreover, $\psi_s(s) > 0$ if $s > 0$ (otherwise $\psi \equiv a$ in some interval, which is not possible). In particular, given $s_0 > 0$, we must have, for $s > s_0$,

$$\psi(s) \geq \psi(s_0) + \psi_s(s_0)(s - s_0)$$

and hence, (3.2.4) follows.

III. Assume $b \in \big(0,\, B(a)\big)$. From (3.2.8) we deduce

$$B(\psi(s)) \geq B(a) - b > 0,$$

which implies, for some constant μ,

$$\psi(s) \geq \mu > -\omega_0. \tag{3.2.9}$$

Let us suppose $a > 1$. Then, $\psi_{ss} = (\psi+\omega_0) f(\psi) = 0$ near the origin. Hence

$$\psi(s) = a - \sqrt{2b}\, s,$$

as long as $\psi(s) > 1$. In any case ($a > 1$ or $a \leq 1$), there exists $s_0 \geq 0$ such that $\psi(s_0) \leq 1$ and $\psi_s(s_0) = -\sqrt{2b}$, and therefore, there exists $s_1 \geq 0$ such that

$$\psi(s_1) < 1, \quad \psi_s(s_1) < 0.$$

If we had $\psi_s \leq 0$ for $s \geq s_1$, then, from (3.2.9) and from equation (3.2.2), we would get, for $s \geq s_1$,

$$-\omega_0 < \mu \leq \psi(s) \leq \psi(s_1) < 1 \quad \text{and} \quad \psi_{ss}(s) > \delta > 0,$$

for some constant δ. Thus,

$$0 \geq \psi_s(s) \geq \psi_s(s_1) + \delta(s - s_1),$$

for $s \geq s_1$, which is not possible.

That is, we have shown that there exists $s_2 > 0$ such that $\psi_s(s_2) > 0$. Then, $\psi_{ss} \geq 0$ now gives, for $s \geq s_2$,

$$\psi(s) \geq \psi(s_2) + \psi_s(s_2)(s - s_2),$$

that is, (3.2.5) holds.

IV. Assume $0 < b = B(a)$. Now, (3.2.8) gives

$$\frac{1}{2}\psi_s^2(s) = B(\psi(s)), \quad \text{for } s \geq 0. \tag{3.2.10}$$

If there existed $s_0 \geq 0$ such that $\psi_s(s_0) = 0$, then $B(\psi(s_0)) = 0$, implying $\psi(s_0) = -\omega_0$. The uniqueness of (3.2.2) would give $\psi(s) \equiv \psi(s_0)$, a contradiction.

Hence, $\psi_s(s) < 0$ and thus $B(\psi(s)) > 0$. This implies that $\psi(s) > -\omega_0$ and that there exists

$$\lim_{s \to +\infty} \psi(s) = \gamma, \quad -\omega_0 \leq \gamma < +\infty.$$

If $\gamma > -\omega_0$, it follows from (3.2.10) that

$$\lim_{s \to +\infty} \psi_s(s) = -\sqrt{2B(\gamma)} < 0,$$

and then $\psi(s) < -\omega_0$ for s large. This gives a contradiction and thus, (3.2.6) holds.

V. Finally, assume $b \in \big(B(a), +\infty\big)$. Then, (3.2.8) gives

$$\frac{1}{2}\psi_s^2(s) \geq b - B(a) > 0.$$

In particular, ψ_s never vanishes and we have, $\psi_s(s) \leq -\sqrt{2\big(b - B(a)\big)}$. It follows that

$$\psi(s) \leq \psi(0) - \sqrt{2\big(b - B(a)\big)}\, s,$$

for $s > 0$, then (3.2.7) holds and the proof is complete. $\square$

LEMMA 3.2.11. *Let* $B(\tau) = \int_{-\omega_0}^{\tau} (\rho + \omega_0) f(\rho)\, d\rho$.

a) Let $\psi^n \geq -\omega_0$, *symmetric with respect to* $s = \frac{n}{2}$, *be a solution to*

$$\begin{aligned} &\psi_{ss} = (\psi + \omega_0) f(\psi) \quad \textit{in } (0, n), \\ &\psi(0) = \psi(n) = a \in (-\omega_0, 1). \end{aligned} \tag{3.2.12}$$

Then, $\psi_s^n(0) = -\sqrt{2b_n}$ *with* $b_n \nearrow B(a)$ *as* $n \to \infty$.

b) Let $\psi^n \geq -\omega_0$ *be a solution to*

$$\begin{aligned} &\psi_{ss} = (\psi + \omega_0) f(\psi) \quad \textit{in } (0, n), \\ &\psi(0) = a \in (-\omega_0, 1], \\ &\psi(n) = -\omega_0. \end{aligned} \tag{3.2.13}$$

Then, $\psi_s^n(0) = -\sqrt{2b_n}$ *with* $b_n \searrow B(a)$ *as* $n \to \infty$.

PROOF. Part a). Since ψ^n is symmetric, $\psi_s^n(\frac{n}{2}) = 0$.

On the other hand, since

$$\frac{1}{2}(\psi_s^n)^2 - B(\psi^n) = b_n - B(a),$$

there holds that

$$-B\big(\psi^n(n/2)\big) = b_n - B(a).$$

In particular, there holds that $b_n \leq B(a)$.

We claim that $\psi^n(\frac{n}{2}) \to -\omega_0$ as $n \to \infty$. In fact, if not there would exist $\alpha > -\omega_0$ such that, for a subsequence that we still call ψ^n,

$$\psi^n(s) \geq \psi^n(n/2) \geq \alpha, \quad \text{for } 0 \leq s \leq n.$$

On the other hand, there holds that $\psi^n(s) \leq a$ for $0 \leq s \leq n$. Thus, $(\psi^n + \omega_0) f(\psi^n(s)) \geq \beta_0 > 0$ for $0 \leq s \leq n$. Therefore, $\psi_{ss}^n \geq \beta_0$ for $0 \leq s \leq n$ and thus

$$\psi^n(s) \geq \alpha + \frac{\beta_0}{2}\big(s - n/2\big)^2, \quad \text{for } s \in [n/2, n].$$

In particular,

$$a = \psi^n(n) \geq \alpha + (\beta_0/8) n^2 \to \infty, \quad \text{as } n \to \infty$$

which is a contradiction. Thus,

$$b_n - B(a) = -B\big(\psi^n(n/2)\big) \to 0, \quad \text{as } n \to \infty.$$

Part b). Since

$$\frac{1}{2}(\psi_s^n)^2 - B(\psi^n) = b_n - B(a),$$

there holds that

$$\frac{1}{2}(\psi_s^n(n))^2 = b_n - B(a) \geq 0.$$

We claim that $\psi_s^n(n) \to 0$ as $n \to \infty$. In fact, if not, there would exist $\alpha > 0$ such that, for a subsequence that we still call ψ^n, $\psi_s^n(n) \leq -\alpha$. Since $\psi_{ss}^n \geq 0$, there holds that

$$\psi_s^n(n) \geq \psi_s^n(s),$$

for $0 \leq s \leq n$. Thus,

$$\psi_s^n(s) \leq -\alpha \quad \text{for } 0 \leq s \leq n.$$

Therefore,

$$a + \omega_0 = \psi^n(0) - \psi^n(n) = -\psi_s^n(\theta)n \geq \alpha n \to \infty, \quad \text{as } n \to \infty$$

which is a contradiction. Therefore, $\psi_s^n(n) \to 0$ as $n \to \infty$ and there holds that

$$b_n \to B(a),$$

as $n \to \infty$. □

LEMMA 3.2.14. *Let $B(\tau)$ be as in the previous Lemma and let $\mathcal{R}_\gamma = \{(x,t) \in \mathbb{R}^{N+1} / x_1 > 0\,, -\infty < t \leq \gamma\}$, $0 \leq \theta < 1 + \omega_0$ and let $U \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\mathcal{R}_\gamma})$ be such that*

$$\begin{aligned} \Delta U - U_t = (U + \omega_0) f(U) \quad & \textit{in } \mathcal{R}_\gamma, \\ U = 1 - \theta \quad & \textit{on } \{x_1 = 0\}, \\ -\omega_0 \leq U \leq 1 - \theta \quad & \textit{in } \overline{\mathcal{R}_\gamma}. \end{aligned}$$

1) If $\theta = 0$, then $|\nabla U| \leq \sqrt{2B(1)}$ on $\{x_1 = 0\}$.

2) If $0 < \theta < 1 + \omega_0$ and $0 < \sigma < B(1)$ is such that

$$\int_{-\omega_0}^{1-\theta} (\rho + \omega_0) f(\rho)\, d\rho = B(1) - \sigma,$$

then $|\nabla U| = \sqrt{2(B(1) - \sigma)}$ on $\{x_1 = 0\}$.

PROOF. For $\theta \in [0, 1 + \omega_0)$, let V_n be the bounded solution to

$$\begin{aligned} \Delta V - V_t &= (V + \omega_0) f(V) \qquad \text{in } \{0 < x_1 < n,\, x' \in \mathbb{R}^{N-1}\,,\, t > 0\}, \\ V(0, x', t) &= 1 - \theta, \\ V(n, x', t) &= -\omega_0, \\ V(x, 0) &= 0, \end{aligned}$$

and let W_n be the bounded solution to

$$\begin{aligned} \Delta W - W_t &= (W+\omega_0)f(W) \quad \text{ in } \{0<x_1<n,\, x'\in\mathbb{R}^{N-1}\,,\, t>0\},\\ W(0,x',t) &= 1-\theta,\\ W(n,x',t) &= 1-\theta,\\ W(x,0) &= 1-\theta. \end{aligned}$$

Let us point out that V_n and W_n are actually functions of (x_1,t).

For $k\in\mathbb{N}$, let $V_n^k(x,t)=V_n(x,t+k)$ and $W_n^k(x,t)=W_n(x,t+k)$. Since V_n^k, U and W_n^k are bounded solutions to equation P_0 in the domain $\{0<x_1<n,\, x'\in\mathbb{R}^{N-1}\,,\, -k<t\le\gamma\}$, and on the parabolic boundary of this domain, we have $V_n^k\le U\le W_n^k$. It follows that

$$V_n^k(x,t)\le U(x,t)\le W_n^k(x,t)$$

in $\{0<x_1<n,\, x'\in\mathbb{R}^{N-1}\,,\, -k<t\le\gamma\}$. On the other hand (see [**19**]), $V_n(x,t)\to\psi_-^n(x_1)$ uniformly as $t\to\infty$, where $\psi_-^n\ge 0$ is a solution to (3.2.13) with $a=1-\theta$.

Analogously, $W_n(x,t)\to\psi_+^n(x_1)$ uniformly as $t\to\infty$, where $\psi_+^n\ge 0$, symmetric with respect to $x_1=\frac{n}{2}$, is a solution to (3.2.12) with $a=1-\theta$.

Therefore, letting $k\to\infty$ we get

$$\psi_-^n(x_1)\le U(x,t)\le\psi_+^n(x_1)\quad\text{for } 0\le x_1\le n\,,\; t\le\gamma.$$

In particular,

$$(\psi_-^n)_s(0)\le U_{x_1}(0,x',t)\le(\psi_+^n)_s(0),\quad\text{for } t\le\gamma.$$

Let $\theta=0$. We deduce from Lemma 3.2.11, b) that

$$-|\nabla U(0,x',t)|=U_{x_1}(0,x',t)\ge\lim_{n\to\infty}(\psi_-^n)_s(0)=-\sqrt{2B(1)}.$$

Let $\theta>0$. We deduce from Lemma 3.2.11, a) and b) that

$$\begin{aligned} -\sqrt{2(B(1)-\sigma)} &= \lim_{n\to\infty}(\psi_-^n)_s(0)\le U_{x_1}(0,x',t)\\ &\le\lim_{n\to\infty}(\psi_+^n)_s(0)=-\sqrt{2(B(1)-\sigma)}. \end{aligned}$$

Therefore,

$$-|\nabla U(0,x',t)|=U_{x_1}(0,x',t)=-\sqrt{2(B(1)-\sigma)}.$$

$\square$

LEMMA 3.2.15. *Let ε_j, γ_{ε_j} and τ_{ε_j} be sequences such that $\varepsilon_j > 0$, $\varepsilon_j \to 0$, $\gamma_{\varepsilon_j} > 0$, $\gamma_{\varepsilon_j} \to \gamma$, with $0 \le \gamma \le +\infty$, $\tau_{\varepsilon_j} > 0$, $\tau_{\varepsilon_j} \to \tau$ with $0 \le \tau \le +\infty$, and such that $\tau < +\infty$ implies that $\gamma = +\infty$. Let $\rho > 0$ and*

$$\mathcal{A}_{\varepsilon_j} = \left\{ (x,t) \ / \ |x| < \frac{\rho}{\varepsilon_j}\,,\ -\min(\tau_{\varepsilon_j}, \frac{\rho^2}{{\varepsilon_j}^2}) < t < \min(\gamma_{\varepsilon_j}, \frac{\rho^2}{{\varepsilon_j}^2}) \right\}.$$

Assume that $0 \le \theta < 1 + w_0(x_0,t_0)$ and let $\bar u^{\varepsilon_j}$ be weak solutions to

$$\begin{aligned}
&\Delta \bar u^{\varepsilon_j} - \bar u^{\varepsilon_j}_t = \left(\bar u^{\varepsilon_j} + \frac{w^{\varepsilon_j}(\varepsilon_j x + x_{\varepsilon_j}, {\varepsilon_j}^2 t + t_{\varepsilon_j})}{\varepsilon_j} \right) f(\bar u^{\varepsilon_j}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \{x_1 > \bar h_{\varepsilon_j}(x',t)\} \cap \mathcal{A}_{\varepsilon_j}, \\
&\bar u^{\varepsilon_j} = 1 - \theta \qquad\qquad\qquad\qquad\qquad \text{on } \{x_1 = \bar h_{\varepsilon_j}(x',t)\} \cap \mathcal{A}_{\varepsilon_j}, \\
&-\frac{w^{\varepsilon_j}(\varepsilon_j x + x_{\varepsilon_j}, {\varepsilon_j}^2 t + t_{\varepsilon_j})}{\varepsilon_j} \le \bar u^{\varepsilon_j} \le 1 - \theta \qquad \text{in } \{x_1 \ge \bar h_{\varepsilon_j}(x',t)\} \cap \overline{\mathcal{A}_{\varepsilon_j}},
\end{aligned}$$

where $(x_{\varepsilon_j}, t_{\varepsilon_j}) \to (x_0,t_0)$, with $\bar u^{\varepsilon_j} \in C(\{x_1 \ge \bar h_{\varepsilon_j}(x',t)\} \cap \overline{\mathcal{A}_{\varepsilon_j}})$, and $\nabla \bar u^{\varepsilon_j} \in L^2$. Here $\bar h_{\varepsilon_j}$ are continuous functions such that $\bar h_{\varepsilon_j}(0,0) = 0$ with $\bar h_{\varepsilon_j} \to 0$ uniformly on compact subsets of $\mathbb{R}^{N-1} \times (-\tau,\gamma)$. Moreover, we assume that $\|\bar h_{\varepsilon_j}\|_{C^1(K)} + \|\nabla_{x'} \bar h_{\varepsilon_j}\|_{C^{\alpha,\frac{\alpha}{2}}(K)}$ are uniformly bounded, for every compact set $K \subset \mathbb{R}^{N-1} \times (-\tau,\gamma)$.

Then, there exists a function $\bar u$ such that, for a subsequence,

$$\begin{aligned}
\bar u &\in C^{2+\alpha,1+\frac{\alpha}{2}}\big(\{x_1 \ge 0,\ \gamma > t > -\tau\}\big), \\
\bar u^{\varepsilon_j} &\to \bar u \quad \text{uniformly on compact subsets of } \{x_1 > 0,\ \gamma > t > -\tau\}, \\
\Delta \bar u - \bar u_t &= (\bar u + w_0(x_0,t_0)) f(\bar u) \quad \text{in } \{x_1 > 0,\ \gamma > t > -\tau\}, \\
\bar u &= 1 - \theta \quad \text{on } \{x_1 = 0,\ \gamma > t > -\tau\}, \\
-w_0(x_0,t_0) &\le \bar u \le 1 - \theta \quad \text{in } \{x_1 \ge 0,\ \gamma > t > -\tau\}.
\end{aligned}$$

If $\gamma < +\infty$, we require, in addition, that

$$\|\bar h_{\varepsilon_j}(x', t + \gamma_{\varepsilon_j} - \gamma)\|_{C^1(K)} + \|\nabla_{x'} \bar h_{\varepsilon_j}(x', t + \gamma_{\varepsilon_j} - \gamma)\|_{C^{\alpha,\frac{\alpha}{2}}(K)}$$

be uniformly bounded for every compact set $K \subset \mathbb{R}^{N-1} \times (-\infty,\gamma]$. And we deduce that

$$\bar u \in C^{2+\alpha,1+\frac{\alpha}{2}}\big(\{x_1 \ge 0,\ t \le \gamma\}\big).$$

If $\tau < +\infty$, we let

$$\mathcal{B}_{\varepsilon_j} = \left\{ x \ / \ |x| < \frac{\rho}{\varepsilon_j}\,,\ x_1 > \bar h_{\varepsilon_j}(x', -\tau_{\varepsilon_j}) \right\},$$

and we require, in addition, that for every $R>0$,

$$\|\bar u^{\varepsilon_j}(x,-\tau_{\varepsilon_j})\|_{C^\alpha\left(\overline{\mathcal{B}}_{\varepsilon_j}\cap \overline{B}_R(0)\right)}\le C_R,$$

and that there exists $r>0$ such that

$$\|\bar u^{\varepsilon_j}(x,-\tau_{\varepsilon_j})\|_{C^{1+\alpha}\left(\overline{\mathcal{B}}_{\varepsilon_j}\cap \overline{B}_r(0)\right)}\le C_r.$$

Moreover, we assume that $\|\bar h_{\varepsilon_j}(x',t-\tau_{\varepsilon_j}+\tau)\|_{C^1(K)}+\|\nabla_{x'}\bar h_{\varepsilon_j}(x',t-\tau_{\varepsilon_j}+\tau)\|_{C^{\alpha,\frac{\alpha}{2}}(K)}$ are uniformly bounded for every compact set $K\subset\mathbb{R}^{N-1}\times[-\tau,+\infty)$.

Then, there holds that

$$\bar u\in C^{\alpha,\frac{\alpha}{2}}\big(\{x_1\ge 0,\, t\ge-\tau\}\big),\ \nabla\bar u\in C\big(\{0\le x_1<r,\, t\ge-\tau\}\big),$$
$$\bar u^{\varepsilon_j}(x,-\tau_{\varepsilon_j})\to\bar u(x,-\tau)\quad \textit{uniformly on compact subsets of } \{x_1>0\}.$$

In any case (τ,γ be infinite or finite)

$$|\nabla\bar u^{\varepsilon_j}(0,0)|\to|\nabla\bar u(0,0)|.$$

PROOF. We will drop the subscript j when referring to the sequences defined in the statement and $\varepsilon\to0$ will mean $j\to\infty$.

Case I. $\tau=+\infty$, $\gamma=+\infty$.

In order to prove the result, we first apply suitable changes of variables to straighten up the boundaries $x_1=\bar h_\varepsilon(x',t)$. Namely, for every ε, we let

$$y=H^\varepsilon(x,t)$$

where

$$H_1^\varepsilon=x_1-\bar h_\varepsilon(x',t),\quad H_i^\varepsilon=x_i,\quad i>1,$$

and we define

$$\bar v^\varepsilon(y,t)=\bar u^\varepsilon(x,t).$$

Let $R>0$ be fixed and let

$$B_R^+=\{(y,t)\ /\ y_1>0\}\cap B_R(0,0)$$

and let

$$\mathcal{L}\bar v^\varepsilon=\sum_{i,\,j}\frac{\partial}{\partial y_i}\left(a_{ij}^\varepsilon(y,t)\frac{\partial\bar v^\varepsilon}{\partial y_j}\right)+\sum_i b_i^\varepsilon(y,t)\frac{\partial\bar v^\varepsilon}{\partial y_i}-\frac{\partial\bar v^\varepsilon}{\partial t},$$

where

$$a_{ij}^\varepsilon(y,t)=\sum_k\frac{\partial H_i^\varepsilon}{\partial x_k}\frac{\partial H_j^\varepsilon}{\partial x_k},\quad b_i^\varepsilon(y,t)=-\frac{\partial H_i^\varepsilon}{\partial t}.\tag{3.2.16}$$

Note that there exists $C_R > 0$ such that

$$\|a_{ij}^\varepsilon\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{B_R^+})} \le C_R, \quad \|b_i^\varepsilon\|_{L^\infty(B_R^+)} \le C_R. \tag{3.2.17}$$

Moreover, there exists $\lambda > 0$ such that, if ε is small enough,

$$\sum_{i,\,j} a_{ij}^\varepsilon(y,t)\xi_i\,\xi_j \ge \lambda|\xi|^2 \quad \text{ for } (y,t) \in \overline{B_R^+}. \tag{3.2.18}$$

Here we have used the fact that $|(DH^\varepsilon)^{-1}|$ are uniformly bounded on any compact set, if ε is small enough.

Then the function $\bar v^\varepsilon \in C(\overline{B_R^+})$, with $\nabla \bar v^\varepsilon \in L^2(B_R^+)$ is a weak solution to

$$\begin{aligned}
&\mathcal{L}\bar v^\varepsilon = \left(\bar v^\varepsilon + \frac{w^\varepsilon(\varepsilon x + x_\varepsilon, \varepsilon^2 t + t_\varepsilon)}{\varepsilon}\right) f(\bar v^\varepsilon) && \text{in } B_R^+,\\
&\bar v^\varepsilon = 1-\theta && \text{on } \overline{B_R^+} \cap \{y_1 = 0\},\\
&-\frac{w^{\varepsilon_j}(\varepsilon_j x + x_\varepsilon, {\varepsilon_j}^2 t + t_\varepsilon)}{\varepsilon_j} \le \bar v^\varepsilon \le 1-\theta && \text{in } \overline{B_R^+},
\end{aligned}$$

if ε is small enough.

By Theorem 10.1, Chapter III in [**21**], there exists $C_R > 0$ such that

$$||\bar v^\varepsilon||_{C^{\alpha,\frac{\alpha}{2}}(\overline{B^+_{\frac{R}{2}}})} \le C_R.$$

On the other hand, by Theorem 1.4.3 in [**12**] we also have that

$$||\nabla \bar v^\varepsilon||_{L^\infty(\overline{B^+_{\frac{R}{2}}})} \le C_R.$$

Moreover, by Theorem 1.4.10 in [**12**], the functions $\nabla \bar v^\varepsilon$ are continuous in $\overline{B^+_{\frac{R}{2}}}$ with a modulus of continuity independent of ε.

Therefore, there exists a function $\overline{u} \in C^{\alpha,\frac{\alpha}{2}}(\overline{B^+_{\frac{R}{2}}})$ and a subsequence that we still call $\bar v^\varepsilon$ such that $\bar v^\varepsilon \to \overline{u}$ and $\nabla \bar v^\varepsilon \to \nabla \overline{u}$ uniformly in $\overline{B^+_{\frac{R}{2}}}$.

Clearly,

$$\begin{aligned}
\overline{u} &= 1-\theta && \text{in } \{y_1 = 0\} \cap \overline{B^+_{\frac{R}{2}}},\\
-w_0(x_0,t_0) \le \overline{u} &\le 1-\theta && \text{in } \overline{B^+_{\frac{R}{2}}}.
\end{aligned}$$

Since $\bar h_\varepsilon \to 0$ and $\nabla_{x'}\bar h_\varepsilon \to 0$ uniformly on compact sets, it is easy to see that we actually have that

$$\bar u^\varepsilon \to \overline{u}, \quad \text{and} \quad \nabla \bar u^\varepsilon \to \nabla \overline{u} \qquad \text{uniformly on compact sets of } B^+_{\frac{R}{2}}.$$

Clearly $\overline{u}$ is a solution of $\Delta \overline{u} - \overline{u}_t = (\overline{u} + w_0(x_0, t_0)) f(\overline{u})$ in $B^+_{\frac{R}{2}}$. Standard Schauder estimates imply that $\bar{u} \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{B^+_{\frac{R}{4}}})$.

Since $\bar{h}_\varepsilon(0,0) = 0$, $\nabla_{x'} \bar{f}_\varepsilon(0,0) \to 0$ and $\nabla \bar{v}^\varepsilon(0,0) \to \nabla \bar{u}(0,0)$, it is easy to see that $\nabla \bar{u}^\varepsilon(0,0) \to \nabla \bar{u}(0,0)$.

Since R is arbitrary, a standard procedure gives the result in

$$\{y_1 > 0\,,\, -\tau < t < \gamma\} \quad \text{for } \tau = +\infty \text{ and } \gamma = +\infty.$$

Case II. $\tau < +\infty$.

As in the previous case, we apply suitable changes of variables to straighten up the boundaries $x_1 = \bar{h}_\varepsilon(x', t)$. Namely, for every ε, we let

$$y = H^\varepsilon(x,t), \quad s = t + \tau_\varepsilon - \tau,$$

where

$$H_1^\varepsilon(x,t) = x_1 - \bar{h}_\varepsilon(x', t), \quad H_i^\varepsilon(x,t) = x_i, \quad i > 1,$$

and we define

$$\bar{v}^\varepsilon(y,s) = \bar{u}^\varepsilon(x,t).$$

Let $R > 0$ be fixed and let

$$B^+_{R,\tau} = \{(y,s) \;/\; y_1 > 0\,,\, s > -\tau\} \cap B_R(0,0)$$

and let, as before,

$$\mathcal{L}\bar{v}^\varepsilon = \sum_{i,\,j} \frac{\partial}{\partial y_i}\left(a^\varepsilon_{ij}(y,s)\frac{\partial \bar{v}^\varepsilon}{\partial y_j}\right) + \sum_i b^\varepsilon_i(y,s)\frac{\partial \bar{v}^\varepsilon}{\partial y_i} - \frac{\partial \bar{v}^\varepsilon}{\partial s},$$

where $a^\varepsilon_{ij}(y,s)$ and $b^\varepsilon_i(y,s)$ are defined in $B^+_{R,\tau}$ in a way analogous to (3.2.16) and moreover, they satisfy estimates similar to those in (3.2.17) and (3.2.18) in $B^+_{R,\tau}$.

Then the function $\bar{v}^\varepsilon \in C(\overline{B^+_{R,\tau}})$, with $\nabla \bar{v}^\varepsilon \in L^2(B^+_{R,\tau})$ is a weak solution to

$$\begin{aligned}
&\mathcal{L}\bar{v}^\varepsilon = \left(\bar{v}^\varepsilon + \frac{w^\varepsilon(\varepsilon x + x_\varepsilon, \varepsilon^2 t + t_\varepsilon)}{\varepsilon}\right) f(\bar{v}^\varepsilon) && \text{in } B^+_{R,\tau},\\
&\bar{v}^\varepsilon = 1 - \theta && \text{on } \overline{B^+_{R,\tau}} \cap \{y_1 = 0\},\\
&-\frac{w^{\varepsilon_j}(\varepsilon_j x + x_\varepsilon, {\varepsilon_j}^2 t + t_\varepsilon)}{\varepsilon_j} \le \bar{v}^\varepsilon \le 1 - \theta && \text{in } \overline{B^+_{R,\tau}},\\
&\bar{v}^\varepsilon = g^\varepsilon(y) && \text{in } \overline{B^+_{R,\tau}} \cap \{s = -\tau\},
\end{aligned}$$

if ε is small enough, where we have called $g^\varepsilon(y) = \bar{v}^\varepsilon(y, -\tau)$. In addition,

$$\|g^\varepsilon\|_{C^\alpha(\overline{B_R(0)}\cap\{y_1\geq 0\})} \leq C_R \qquad \text{and} \qquad \|g^\varepsilon\|_{C^{1+\alpha}(\overline{B_r(0)}\cap\{y_1\geq 0\})} \leq C_r.$$

Moreover, $g^\varepsilon = 1 - \theta \quad \text{on } \{y_1 = 0\}$.

By Theorem 10.1, Chapter III in [**21**], there exists $C_R > 0$ such that

$$||\bar{v}^\varepsilon||_{C^{\alpha,\frac{\alpha}{2}}(\overline{B^+_{\frac{R}{2},\tau}})} \leq C_R.$$

On the other hand, by Remark 1.4.11 in [**12**], applied to the functions $\hat{v}^\varepsilon = \bar{v}^\varepsilon - g^\varepsilon$, we also have that

$$||\nabla\bar{v}^\varepsilon||_{L^\infty\left((\overline{B_{\frac{r}{2}}}(0)\times[-\tau,\frac{R}{2}])\cap\{y_1\geq 0\}\right)} \leq C_R$$

and that the functions $\nabla\bar{v}^\varepsilon$ are continuous in $(\overline{B_{\frac{r}{2}}}(0)\times[-\tau,\frac{R}{2}])\cap\{y_1 \geq 0\}$ with a modulus of continuity independent of ε.

Proceeding as in the case $\tau = +\infty$ and using that $\tau_\varepsilon \to \tau$ we see that there exists a function $\overline{u} \in C^{\alpha,\frac{\alpha}{2}}(\overline{B^+_{\frac{R}{2},\tau}})$ such that for a subsequence

$$\begin{aligned}
&\bar{v}^\varepsilon \to \overline{u} \quad \text{uniformly in } \overline{B^+_{\frac{R}{2},\tau}},\\
&\nabla\bar{v}^\varepsilon \to \nabla\overline{u} \quad \text{uniformly on compact sets of } B^+_{\frac{R}{2},\tau},\\
&\bar{u}^\varepsilon \to \overline{u}, \quad \nabla\bar{u}^\varepsilon \to \nabla\overline{u} \quad \text{uniformly on compact sets of } B^+_{\frac{R}{2},\tau},\\
&\bar{u}^\varepsilon(y,-\tau_\varepsilon) \to \overline{u}(y,-\tau) \quad \text{uniformly on compact sets of}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \{y_1 > 0\}\cap B_{\frac{R}{2}}(0),\\
&\nabla\bar{v}^\varepsilon \to \nabla\overline{u} \quad \text{uniformly in } \left(\overline{B_{\frac{r}{2}}}(0)\times[-\tau,\frac{R}{2}]\right)\cap\{y_1\geq 0\}.
\end{aligned}$$

This function $\overline{u}$ satisfies

$$\begin{aligned}
\bar{u} &\in C^{2+\alpha,1+\frac{\alpha}{2}}\left(\{y_1\geq 0,\, t > -\tau\}\cap B_{\frac{R}{2}}(0,0)\right),\\
\Delta\bar{u} - \bar{u}_t = (\bar{u} + w_0(x_0,t_0))f(\bar{u}) &\quad \text{in } \{y_1 > 0,\, t > -\tau\}\cap B_{\frac{R}{2}}(0,0),\\
\bar{u} = 1-\theta &\quad \text{on } \{y_1 = 0,\, t \geq -\tau\}\cap B_{\frac{R}{2}}(0,0),\\
-w_0(x_0,t_0) \leq \bar{u} \leq 1-\theta &\quad \text{in } \{y_1 \geq 0,\, t \geq -\tau\}\cap B_{\frac{R}{2}}(0,0).
\end{aligned}$$

Moreover, there holds that $\nabla\bar{u}^\varepsilon(0,0) \to \nabla\bar{u}(0,0)$.

Since R is arbitrary, Case II is proved.

Case III. $\gamma < +\infty$.

We proceed as in the previous cases. For every ε, we let

$$y = H^\varepsilon(x,t), \quad s = t - \gamma_\varepsilon + \gamma,$$

where

$$H_1^\varepsilon(x,t) = x_1 - \bar{h}_\varepsilon(x',t), \quad H_i^\varepsilon(x,t) = x_i, \quad i > 1,$$

and we define

$$\bar{v}^\varepsilon(y,s) = \bar{u}^\varepsilon(x,t).$$

Let $R > 0$ be fixed and let

$$B_{R,\gamma}^+ = \{(y,s) \ / \ y_1 > 0 \, , \ s < \gamma\} \cap B_R(0,0).$$

As in the previous cases, by using Theorem 10.1, Chapter III in [**21**], and Theorems 1.4.3 and 1.4.10 in [**12**] we deduce that there exists a function $\overline{u} \in C^{\alpha,\frac{\alpha}{2}}\left(\overline{B_{\frac{R}{2},\gamma}^+}\right)$ such that for a subsequence

$$\begin{aligned}
&\bar{v}^\varepsilon \to \overline{u}, \quad \nabla \bar{v}^\varepsilon \to \nabla \overline{u} \quad \text{uniformly in } \overline{B_{\frac{R}{2},\gamma}^+},\\
&\bar{u}^\varepsilon \to \overline{u}, \quad \nabla \bar{u}^\varepsilon \to \nabla \overline{u} \quad \text{uniformly on compact sets of } B_{\frac{R}{2},\gamma}^+.
\end{aligned}$$

This function $\overline{u}$ satisfies

$$\begin{aligned}
\bar{u} \in C^{2+\alpha,1+\frac{\alpha}{2}}\big(\{y_1 \geq 0, \, t \leq \gamma\} \cap B_{\frac{R}{2}}(0,0)\big),&\\
\Delta \bar{u} - \bar{u}_t = (\bar{u} + w_0(x_0,t_0))f(\bar{u}) \quad &\text{in } \{y_1 > 0, \, t < \gamma\} \cap B_{\frac{R}{2}}(0,0),\\
\bar{u} = 1 - \theta \quad &\text{on } \{y_1 = 0, \, t \leq \gamma\} \cap B_{\frac{R}{2}}(0,0),\\
-w_0(x_0,t_0) \leq \bar{u} \leq 1 - \theta \quad &\text{in } \{y_1 \geq 0, \, t \leq \gamma\} \cap B_{\frac{R}{2}}(0,0).
\end{aligned}$$

Moreover, there holds that $\nabla \bar{u}^\varepsilon(0,0) \to \nabla \bar{u}(0,0)$.

Since R is arbitrary, the lemma is proved. $\square$

3. Approximation results

In this section we prove that, under certain assumptions, a strict semi-classical supersolution to problem (P) is the uniform limit of a family of supersolutions to problem (P_ε) (Theorem 3.3.1), and we state an analogous result for subsolutions (Theorem 3.3.7). Also, we prove that for compactly supported initial data, limit solutions have bounded support (Proposition 3.3.8).

The following construction follows the lines of Theorem 5.2 in [**22**]. In our case we have to be more careful with the construction of the initial data.

THEOREM 3.3.1. *Let $\widetilde{u}$ be a semi-classical supersolution to (P) in Q_T with $\widetilde{u} \in C^1(\overline{\{\widetilde{u} > 0\}})$ and such that $\{\widetilde{u} > 0\}$ is bounded. Assume,* *in addition, that there exist $\delta_0, s_0 > 0$ such that*

$$|\nabla\widetilde{u}^+| \le \sqrt{2M(x,t)-\delta_0} \quad \text{on } Q\cap\partial\{\widetilde{u}>0\},$$
$$|\nabla\widetilde{u}| > \delta_0 \quad \text{in } Q\cap\{0<\widetilde{u}<s_0\}.$$

Let w^ε be a solution of the heat equation in Q_T such that $\frac{w^\varepsilon(x,t)}{\varepsilon} \to w_0(x,t)$ uniformly in $\overline{Q_T}$ with $w_0 \in C(\overline{Q_T}) \cap L^\infty(Q_T)$ and verifies (3.1.5).

Then, there exists a family $u^\varepsilon \in C(\overline{Q_T})$, with $\nabla u^\varepsilon \in L^2_{\mathrm{loc}}(\overline{Q_T})$, of weak supersolutions to (P_ε) in Q_T, such that, as $\varepsilon \to 0$, $u^\varepsilon \to \widetilde{u}$ uniformly in $\overline{Q_T}$.

PROOF. *Step* I. Construction of the family u^ε. Let $0<\theta<\delta_1$ be such that

$$\int_{1-\theta}^1 (s+W)f(s)\,ds = \frac{\delta_0}{8},$$

where W is a suitable uniform bound of $\|w^\varepsilon/\varepsilon\|_{L^\infty(\{\widetilde{u}>0\})}$. For every $\varepsilon>0$ small, we define the domain $D^\varepsilon = \{\widetilde{u} < (1-\theta)\varepsilon\} \subset Q_T$.

Let z^ε be the bounded solution to

$$\Delta z^\varepsilon - z^\varepsilon_t = (z^\varepsilon + w^\varepsilon) f_\varepsilon(z^\varepsilon) \quad \text{in } D^\varepsilon,$$

with boundary data

$$z^\varepsilon(x,t) = \begin{cases} (1-\theta)\varepsilon & \text{on } \partial D^\varepsilon \cap t>0, \\ z_0^\varepsilon(x) & \text{in } D^\varepsilon \cap \{t=0\}. \end{cases}$$

In order to give the initial data z_0^ε, we let $\psi^\varepsilon(s,x)$ be the solution to (3.2.2) with

$$a = 1-\theta, \quad b = \int_{-w^\varepsilon(x,0)/\varepsilon}^{1-\theta} (s + \frac{w^\varepsilon(x,0)}{\varepsilon}) f(s)\,ds, \quad \omega_0 = \frac{w^\varepsilon(x,0)}{\varepsilon}.$$

Assume first that $|\nabla\widetilde{u}|$ is smooth. Then we let

$$\varphi^\varepsilon(\xi,x) = \psi^\varepsilon\Big(\frac{1-\theta-\xi}{|\nabla\widetilde{u}(x,0)|}, x\Big),$$

and we define

$$z_0^\varepsilon(x) = \varepsilon\varphi^\varepsilon\Big(\frac{1}{\varepsilon}\widetilde{u}(x,0), x\Big).$$

If $\widetilde{u}$ is not regular enough, we can replace $|\nabla\widetilde{u}(x,0)|$ by a smooth approximation $F_\varepsilon(x)$ so that the initial datum z_0^ε is $C^{1+\alpha}$. We leave the details to the reader.

Finally, we define the family u^ε as follows:

$$u^\varepsilon = \begin{cases} \widetilde{u} & \text{in } \{\widetilde{u} \geq (1-\theta)\varepsilon\}, \\ z^\varepsilon & \text{in } \overline{D^\varepsilon}. \end{cases}$$

Step II. Passage to the limit. If $(x,0) \in \overline{D^\varepsilon}$, we have $0 \leq \frac{1}{\varepsilon}\widetilde{u}(x,0) \leq 1-\theta$. Since, from Lemma 3.1, we know that $-w^\varepsilon(x,0)/\varepsilon \leq \psi(s,x) \leq 1-\theta$ for $s \geq 0$, it follows that $-w^\varepsilon(x,0) \leq z^\varepsilon(x,0) \leq (1-\theta)\varepsilon$. Since $f_\varepsilon(s) \geq 0$, constant functions larger than $-w^\varepsilon(x,t)$ are supersolutions to (P_ε). Therefore, $(1-\theta)\varepsilon$ is a supersolution if $\varepsilon < \varepsilon_1$ and we may apply the comparison principle for bounded super and subsolutions of (P_ε) to conclude that $-w^\varepsilon \leq z^\varepsilon \leq (1-\theta)\varepsilon$.

Hence,

$$\sup_{\overline{Q_T}} |u^\varepsilon - \widetilde{u}| = \sup_{D^\varepsilon} |z^\varepsilon - \widetilde{u}| \leq C\varepsilon$$

and therefore, the convergence of the family u^ε follows.

Step III. Let us show that there exists $\varepsilon_0 > 0$ such that the functions u^ε are supersolutions to (P_ε) for $\varepsilon < \varepsilon_0$.

If $u^\varepsilon > (1-\theta)\varepsilon$, then $u^\varepsilon = \widetilde{u}$, which by hypothesis is supercaloric. Since $f_\varepsilon(s) \geq 0$ and $(1-\theta)\varepsilon \geq -w^\varepsilon$ if $\varepsilon < \varepsilon_1$, it follows that u^ε are supersolutions to (P_ε) here.

If $u^\varepsilon < (1-\theta)\varepsilon$, then we are in D^ε and therefore, by construction, u^ε are solutions to (P_ε).

That is, the u^ε's are continuous functions, and they are piecewise supersolutions to (P_ε). In order to see that u^ε are globally supersolutions to (P_ε), it suffices to see that the jumps of the gradients (which occur at smooth surfaces), have the right sign.

To this effect, we will show that there exists $\varepsilon_0 > 0$ such that

$$(3.3.2) \quad |\nabla u^\varepsilon| \geq \sqrt{2M(x,t) - \delta_0/2} \quad \text{on } \{\widetilde{u} = (1-\theta)\varepsilon\}, \text{ for } \varepsilon < \varepsilon_0.$$

Assume that (3.3.2) does not hold. Then, for every $j \in \mathbb{N}$, there exist $\varepsilon_j > 0$ and $(x_{\varepsilon_j}, t_{\varepsilon_j}) \in Q$, with

$$\varepsilon_j \to 0 \quad \text{and} \quad (x_{\varepsilon_j}, t_{\varepsilon_j}) \to (x_0, t_0) \in \partial\{\widetilde{u} > 0\} \cap \{\widetilde{u} = 0\},$$

such that

(3.3.3)

$$u^{\varepsilon_j}(x_{\varepsilon_j}, t_{\varepsilon_j}) = (1-\theta)\varepsilon_j \quad \text{and} \quad |\nabla u^{\varepsilon_j}(x_{\varepsilon_j}, t_{\varepsilon_j})| < \sqrt{2M(x_{\varepsilon_j}, t_{\varepsilon_j}) - \delta_0/2}.$$

From now on we will drop the subscript j when referring to the sequences defined above and $\varepsilon \to 0$ will mean $j \to \infty$.

We can assume (performing a rotation in the space variables if necessary) that there exists a family g_ε of smooth functions such that, in a neighborhood of $(x_\varepsilon, t_\varepsilon)$,

$$\begin{aligned} &\{u^\varepsilon = (1-\theta)\varepsilon\} = \{(x,t) \,/\, x_1 - x_{\varepsilon 1} = g_\varepsilon(x' - {x_\varepsilon}', t - t_\varepsilon)\}, \\ &\{u^\varepsilon < (1-\theta)\varepsilon\} = \{(x,t) \,/\, x_1 - x_{\varepsilon 1} > g_\varepsilon(x' - {x_\varepsilon}', t - t_\varepsilon)\}, \end{aligned} \tag{3.3.4}$$

where there holds that

$$g_\varepsilon(0,0) = 0, \quad |\nabla_{x'} g_\varepsilon(0,0)| \to 0, \quad \varepsilon \to 0.$$

We can assume that (3.3.4) holds in $\big(B_\rho(x_\varepsilon) \times (t_\varepsilon - \rho^2, t_\varepsilon + \rho^2)\big) \cap \{0 \le t \le T\}$ for some $\rho > 0$.

Let us now define

$$\bar u^\varepsilon(x,t) = \frac{1}{\varepsilon} u^\varepsilon(x_\varepsilon + \varepsilon x, t_\varepsilon + \varepsilon^2 t), \quad \bar g_\varepsilon(x', t) = \frac{1}{\varepsilon} g_\varepsilon(\varepsilon x', \varepsilon^2 t),$$

and let

$$\tau_\varepsilon = \frac{t_\varepsilon}{\varepsilon^2}\ ,\ \gamma_\varepsilon = \frac{T - t_\varepsilon}{\varepsilon^2}.$$

We have, for a subsequence,

$$\tau_\varepsilon \to \tau\ ,\ \gamma_\varepsilon \to \gamma$$

where $0 \le \tau, \gamma \le +\infty$ and τ and γ cannot be both finite.

We now let

$$\mathcal{A}_\varepsilon = \left\{ (x,t)\ /\ |x| < \frac{\rho}{\varepsilon},\ -\min(\tau_\varepsilon, \frac{\rho^2}{\varepsilon^2}) < t < \min(\gamma_\varepsilon, \frac{\rho^2}{\varepsilon^2}) \right\}.$$

Then, the functions $\bar u^\varepsilon$ are weak solutions to

$$\begin{aligned} &\Delta \bar u^\varepsilon - \bar u^\varepsilon_t = \left(\bar u^\varepsilon + \frac{w^\varepsilon(x_\varepsilon + \varepsilon x, t_\varepsilon + \varepsilon^2 t)}{\varepsilon} \right) f(\bar u^\varepsilon) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \{x_1 > \bar g_\varepsilon(x', t)\} \cap \mathcal{A}_\varepsilon, \\ &\bar u^\varepsilon = 1 - \theta \qquad\qquad\qquad\qquad\qquad\quad \text{on } \{x_1 = \bar g_\varepsilon(x', t)\} \cap \mathcal{A}_\varepsilon, \\ &-\frac{w^\varepsilon(x_\varepsilon + \varepsilon x, t_\varepsilon + \varepsilon^2 t)}{\varepsilon} \le \bar u^\varepsilon \le 1 - \theta \qquad \text{in } \{x_1 \ge \bar g_\varepsilon(x', t)\} \cap \overline{\mathcal{A}_\varepsilon}. \end{aligned}$$

Note that we are under the hypotheses of Lemma 3.2.15. Then, there exists a function $\bar u$ such that, for a subsequence,

$$\bar u \in C^{2+\alpha, 1+\frac{\alpha}{2}}\big(\{x_1 \ge 0,\, -\tau < t < \gamma\}\big),$$

$$\bar u^\varepsilon \to \bar u \quad \text{uniformly on compact subsets of } \{x_1 > 0,\, -\tau < t < \gamma\},$$

$$\begin{aligned} &\Delta \bar u - \bar u_t = (\bar u + w_0(x_0, t_0)) f(\bar u) && \text{in } \{x_1 > 0,\, -\tau < t < \gamma\}, \\ &\bar u = 1 - \theta && \text{on } \{x_1 = 0,\, -\tau < t < \gamma\}, \\ &-w_0(x_0, t_0) \le \bar u \le 1 - \theta && \text{in } \{x_1 \ge 0,\, -\tau < t < \gamma\}. \end{aligned}$$

We will divide the remainder of the proof into two cases, depending on whether $\tau = +\infty$ or $\tau < +\infty$.

Case I. Assume $\tau = +\infty$.

In this case, Lemma 3.2.15 also gives

$$|\nabla \bar{u}^\varepsilon(0,0)| \to |\nabla \bar{u}(0,0)|.$$

On the other hand, $\bar{u}$ satisfies the hypotheses of Lemma 3.2.14 and therefore,

$$|\nabla \bar{u}| \geq \sqrt{2M(x_0,t_0) - \delta_0/4} \quad \text{on } \{x_1 = 0\},$$

which yields

$$|\nabla \bar{u}^\varepsilon(0,0)| \geq \sqrt{2M(x_0,t_0) - 3\delta_0/8},$$

for ε small. But this gives

$$|\nabla u^\varepsilon(x_\varepsilon,t_\varepsilon)| \geq \sqrt{2M(x_\varepsilon,t_\varepsilon) - \delta_0/2},$$

for ε small. This contradicts (3.3.3) and completes the proof in case $\tau = +\infty$.

Case II. Assume $\tau < +\infty$. (In this case $\gamma = +\infty$.)

There holds that $\bar{u}^\varepsilon(x, -\tau_\varepsilon) = \frac{1}{\varepsilon} u^\varepsilon(x_\varepsilon + \varepsilon x, 0)$, then

$$\bar{u}^\varepsilon(x, -\tau_\varepsilon) = \varphi^\varepsilon\Big(\frac{1}{\varepsilon}\widetilde{u}(x_\varepsilon + \varepsilon x, 0), x_\varepsilon + \varepsilon x\Big). \tag{3.3.5}$$

Here we want to apply the result of Lemma 3.2.15 corresponding to $\tau < +\infty$. In fact, we can see that there exist $C, r > 0$ such that $\|\bar{u}^\varepsilon(\cdot, -\tau_\varepsilon)\|_{C^{1+\alpha}\left(\overline{B}_r(0)\right)} \leq C$.

Now Lemma 3.2.15 gives, for a subsequence,

$$\bar{u} \in C^{\alpha,\frac{\alpha}{2}}\big(\{x_1 \geq 0,\, t \geq -\tau\}\big),$$
$$\bar{u}^\varepsilon(x, -\tau_\varepsilon) \to \bar{u}(x, -\tau) \quad \text{uniformly on compact subsets of } \{x_1 > 0\}.$$

Therefore, we get that (recall that in the case we are considering $t_0 = 0$),

$$\bar{u}(x, -\tau) = \bar{\varphi}\Big(1 - \theta - |\nabla \widetilde{u}^+(x_0,t_0)|\, x_1, x_0\Big).$$

where $\bar{\varphi}(s,x) = \psi\Big(\dfrac{1-\theta-s}{|\nabla \widetilde{u}(x,0)|}, x\Big)$ and $\psi(s,x)$ is the solution of (3.2.2) with

$$a = 1-\theta, \quad b = \int_{-w_0(x,0)}^{1-\theta} (s + w_0(x,0)) f(s)\, ds, \quad \omega_0 = w_0(x,0).$$

Thus,

$$\bar{u}(x, -\tau) = \psi(x_1, x_0).$$

Since the function $\psi(x_1, x_0)$ is a stationary solution to equation (P_0), bounded for $x_1 \geq 0$, and $\bar{u} = \psi$ on the parabolic boundary of the domain $\big\{x_1 > 0,\, t > -\tau\big\}$, we conclude that

$$\bar{u}(x,t) = \psi(x_1, x_0) \quad \text{ in } \big\{x_1 \geq 0,\, t \geq -\tau\big\}.$$

It follows from Lemma 3.2.1 and the choice of θ that, on $\{x_1 = 0,\ t \geq -\tau\}$,

$$\begin{aligned} \frac{1}{2}|\nabla \bar{u}|^2 = \frac{1}{2}\big(\psi_s(0, x_0)\big)^2 &= \int_{-w_0(x_0,t_0)}^{1-\theta} (s + w_0(x_0, t_0)) f(s)\, ds \\ &\geq M(x_0, t_0) - \frac{\delta_0}{8}. \end{aligned}$$

This is,

$$|\nabla \bar{u}| \geq \sqrt{2M(x_0, t_0) - \delta_0/4} \quad \text{ on } \{x_1 = 0, t \geq -\tau\}.$$

But Lemma 3.2.15 gives

$$|\nabla \bar{u}^\varepsilon(0,0)| \to |\nabla \bar{u}(0,0)|,$$

which yields

$$|\nabla \bar{u}^\varepsilon(0,0)| \geq \sqrt{2M(x_0,t_0) - 3\delta_0/8},$$

for ε small. Then,

$$|\nabla u^\varepsilon(x_\varepsilon, t_\varepsilon)| \geq \sqrt{2M(x_\varepsilon, t_\varepsilon) - \delta_0/2},$$

for ε small. This contradicts (3.3.3) and completes the proof in case $\tau < +\infty$. $\square$

REMARK 3.3.6. *Observe that from the construction of u^ε done in the previous proof, it follows that*

$$u^\varepsilon \equiv \widetilde{u} \qquad \text{in } \{\widetilde{u} > (1-\theta)\varepsilon\}.$$

We state without proof the following Theorem.

THEOREM 3.3.7. *Let $\widetilde{u}$ be a semi-classical subsolution to (P) in Q_T with $\widetilde{u} \in C^1(\overline{\{\widetilde{u} > 0\}})$ such that $\{\widetilde{u} > 0\}$ is bounded. Assume, in addition, that there exist $\delta_0 > 0$ such that*

$$|\nabla \widetilde{u}^+| \geq \sqrt{2M(x,t) + \delta_0} \qquad \text{on } Q \cap \partial\{\widetilde{u} > 0\}.$$

Let w^ε be a solution of the heat equation in Q_T such that $\frac{w^\varepsilon(x,t)}{\varepsilon} \to w_0(x,t)$ uniformly in $\overline{Q_T}$. And assume, moreover that $w_0 \in C(\overline{Q_T}) \cap L^\infty(Q_T)$ and verifies (3.1.5).

Then, there exists a family $u^\varepsilon \in C(\overline{Q_T})$, with $\nabla u^\varepsilon \in L^2_{\mathrm{loc}}(\overline{Q_T})$, of weak subsolutions to (P_ε) in Q_T, such that, as $\varepsilon \to 0$, $u^\varepsilon \to \widetilde{u}$ uniformly in $\overline{Q_T}$.

PROOF. The proof is analogous to Theorem 3.3.1. See [**22**] for a similar result in the case $w^\varepsilon = 0$. $\square$

Finally, we end this Section by showing that, for compactly supported initial data, the support of a limit solution of problem (P) is bounded.

PROPOSITION 3.3.8. *Let $u_0 \in C(\mathbb{R}^N)$ with compact support. Let u_0^ε converge uniformly to u_0 with supports converging to the support of u_0 and let w^ε be a solution of the heat equation in Q_T such that $\frac{w^\varepsilon(x,t)}{\varepsilon} \to w_0(x,t)$ uniformly in $\overline{Q_T}$. And assume, moreover that $w_0 \in C(\overline{Q_T}) \cap L^\infty(Q_T)$ and verifies (3.1.5). Finally, let u^ε be the solution to (P_ε) with function w^ε and initial condition u_0^ε.*

Let $u = \lim u^{\varepsilon_j}$. Then $\{u > 0\}$ is bounded. Moreover, u vanishes in finite time.

PROOF. Let $-1 < \omega_0 < w^\varepsilon(x,t)/\varepsilon$. Then it is easy to check that

$$M_{\omega_0} = \int_{-\omega_0}^{1} (s + \omega_0) f(s)\, ds < M(x,t) = \int_{-w_0(x,t)}^{1} (s + w_0(x,t)) f(s)\, ds. \tag{3.3.9}$$

Let us now consider the following self-similar function

$$V(x,t;T) = (T-t)^{1/2} h(|x|(T-t)^{-1/2}),$$

where $h = h(r)$ is a solution of

$$\begin{aligned} & h'' + \left(\frac{N-1}{r} + \frac{1}{2} r\right) h' + \frac{1}{2} h = 0, \quad 0 < r < R, \\ & h'(0) = 0, \quad h(r) > 0, \quad 0 \le r < R, \\ & h(R) = 0, \quad h'(R) = -\sqrt{2 M_{\omega_0}}. \end{aligned} \tag{3.3.10}$$

It is proved in [**13**], Proposition 1.1, that there exists a unique $R > 0$ and a unique h solution of (3.3.10).

Moreover, it can be checked that if one picks T sufficiently large, then

$$V(x,0;T) \ge u_0 + 1 \quad \text{in } \{u_0 > 0\},$$

and so $V(x,t;T)$ is a strict semi-classical supersolution of (P) with bounded support and positive gradient near its free boundary.

Now, let u^{ε_j} be solutions to (P_{ε_j}) – with initial data $u_0^{\varepsilon_j}$ converging unifomly to u_0 such that support $u_0^{\varepsilon_j} \to$ support u_0 – such that $u = \lim u^{\varepsilon_j}$.

By Theorem 3.3.1, there exists a family v^{ε_j} of supersolutions of (P_{ε_j}) such that $v^{\varepsilon_j} \to V$ uniformly on compact sets, and $v^{\varepsilon_j}(x,0) \geq u^{\varepsilon_j}(x,0)$. Therefore, by the comparison principle, we obtain $u^{\varepsilon_j} \leq v^{\varepsilon_j}$ and passing to the limit $u(x,t) \leq V(x,t;T)$, and the result follows. □

4. Uniqueness of the limit solution

In this section we arrive at the main point of the Chapter: we prove that, under certain assumptions, there exists a unique limit solution to the initial and boundary value problem associated to (P) as long as condition (0.2.3) is satisfied.

Let us begin with the following Proposition that is the key ingredient in the proof of our main result.

PROPOSITION 3.4.1. *Let $\widetilde{u}$ be a strict semi-classical supersolution to (P) with bounded support in Q_T such that there exists $s_0 > 0$ so that $|\nabla \widetilde{u}| > 0$ in $\{0 < \widetilde{u} < s_0\}$ and let $w^\varepsilon/\varepsilon$ be solutions to the heat equation in Q_T converging to w_0 uniformly with $w_0 \in C(\overline{Q_T}) \cap L^\infty(Q_T)$ and verifies (3.1.5).*

Let u^ε be solutions to (P_ε) with function w^ε and initial condition u_0^ε, where u_0^ε are uniform approximations of u_0 with support $u_0^\varepsilon \to$ support u_0. *Then*

$$\limsup_{\varepsilon \to 0+} u^\varepsilon(x,t) \leq \widetilde{u}(x,t)$$

for every $(x,t) \in Q_T$.

PROOF. Let $\widetilde{u}$ be a strict semi-classical supersolution of (P). Let us first, define the following regularization

$$u(x,t) = (\widetilde{u}(x,t+h) - \eta)^+,$$

for $h, \eta > 0$ small. So that u is a strict semi-classical supersolution of (P) with C^1 free boundary, $C^1(\overline{\{u > 0\}})$ and $|\nabla u| > \delta_0 > 0$ in a neighborhood of its free boundary. So, by Theorem 3.3.1, there exists v^ε supersolution of (P_ε) such that $v^\varepsilon \to u$ uniformly in Q_{T-h}.

Now, using the comparison principle, we conclude that $u^\varepsilon \leq v^\varepsilon$ in Q_{T-h}, and the Proposition now follows letting first $\varepsilon \to 0+$ and then $h, \eta \to 0+$. □

Finally, we arrive at the main point of the paper: The uniqueness of limit solutions of (P).

THEOREM 3.4.2. *Let the initial datum u_0 be Lipschitz, with compact support and satisfy the condition* (3.1.3). *Then there exists at most one limit solution such that its gradient does not vanish near its free boundary as long as the function w^ε in problem (P_ε) satisfies condition (0.2.4).*

More precisely, let $u_0^{\varepsilon_j}, \tilde{u}_0^{\varepsilon_k}$ be uniformly Lipschitz continuous in $\mathbb{R}^N$ with uniformly bounded Lipschitz norms and ε_j, $\varepsilon_k \to 0$. Assume that $u_0^{\varepsilon_j} \in C^1(\overline{\{u_0^{\varepsilon_j} > 0\}})$, $\tilde{u}_0^{\varepsilon_k} \in C^1(\overline{\{\tilde{u}_0^{\varepsilon_k} > 0\}})$, $u_0^{\varepsilon_j}, \tilde{u}_0^{\varepsilon_k} \to u_0$ uniformly and support $u_0^{\varepsilon_j}$, support $\tilde{u}_0^{\varepsilon_k} \to$ support u_0. *Let $w^{\varepsilon_j}/\varepsilon_j$ and $\tilde{w}^{\varepsilon_k}/\varepsilon_k$ be solutions of the heat equation converging uniformly to the same function $w_0 \in C(\overline{Q_T}) \cap L^\infty(Q_T)$, that verifies (3.1.5). Also, assume that w_0 satisfies the monotonicity condition (3.1.4).*

Let u^{ε_j} (resp. $\tilde{u}^{\varepsilon_k}$) be the solution to (P_{ε_j}) with function w^{ε_j} and initial datum $u_0^{\varepsilon_j}$ (resp. solution to (P_{ε_k}) with function $\tilde{w}^{\varepsilon_k}$ and initial datum $\tilde{u}_0^{\varepsilon_k}$). Let $u = \lim u^{\varepsilon_j}$ and $\tilde{u} = \lim \tilde{u}^{\varepsilon_k}$. If there exists $s_0 > 0$ such that $|\nabla \tilde{u}| > 0$ in $\{0 < \tilde{u} < s_0\}$.

Then, $u \le \tilde{u}$.

PROOF. Since $\tilde{u}$ is a semi-classical supersolution of (P), $\tilde{u} \in C^1(\{\tilde{u} > 0\})$ and, by Propositon 3.3.8, its support is bounded, the function $\tilde{u}_\lambda$ as defined in (3.1.6) satisfies the hypotheses of Proposition 3.4.1 in $Q_{T/\lambda^2} \supset Q_T$. So by letting $\lambda \to 1-$ we arrive at

$$u(x,t) \le \tilde{u}(x,t). \tag{3.4.3}$$

This finishes the proof. ∎

THEOREM 3.4.4. *Let the initial datum u_0 be as in Theorem 3.4.2. Assume that there exists a semi-classical solution v to (P) with initial datum u_0 and let $u_0^{\varepsilon_j}$ be uniformly Lipschitz continuous in $\mathbb{R}^N$ with $\varepsilon_j \to 0$, such that $u_0^{\varepsilon_j} \in C^1(\overline{\{u_0^{\varepsilon_j} > 0\}})$, $u_0^{\varepsilon_j} \to u_0$ uniformly and* support $u_0^{\varepsilon_j} \to$ support u_0. *Assume $w^{\varepsilon_j}/\varepsilon_j$ is a solution of the heat equation converging to w_0 uniformly with $w_0 \in C(\overline{Q_T}) \cap L^\infty(Q_T)$ and verifying (3.1.5). Also, assume that w_0 satisfies the monotonicity condition (3.1.4).*

Let u^{ε_j} be the solution to (P_{ε_j}) with function w^{ε_j} and initial datum $u_0^{\varepsilon_j}$ and let $u = \lim u^{\varepsilon_j}$. Then, $u = v$.

In particular, there exists at most one classical solution to (P).

PROOF. Since u is a semi-classical supersolution to (P) and v is a semi-classical subsolution, Lemma 2.1 applies and we get that $v \leq u$.

On the other hand, if we define v_λ as in (3.1.6), with $0 < \lambda < \lambda' < 1$, we have that v_λ satisfies the hypotheses of Proposition 3.4.1. Thus, there exists a family v^{ε_j} of supersolutions to (P_{ε_j}) with function w^{ε_j} such that, for a subsequence, $v^{\varepsilon_j} \to v$ with initial data converging uniformly to u_0. So by the comparison principle

$$u = \lim u^{\varepsilon_j} \leq \lim v^{\varepsilon_j} = v.$$

This finishes the proof. □

5. Conclusions

We have proved that the limits of sequences of solutions to (P_ε) with different constitutive functions w^ε and initial data u_0^ε coincide – as long as certain monotonicity assumptions are made – if the limit of $w^\varepsilon/\varepsilon$ and of u_0^ε are prescribed.

The monotonicity assumptions are necessary to provide strict semi-classical supersolutions as close as we want to any semi-classical supersolution. This kind of condition was also used with the same purpose – in the case in which $w^\varepsilon = 0$ – in [**30, 22**]. In the latter, a different geometry was considered namely, the domain was a cylinder, Neumann boundary conditions were given on the boundary of the cylinder and monotonicity in the direction of the cylinder axis was assumed. In [**22**] it was proved that, if a classical solution exists and $w^\varepsilon = 0$, then it is equal to any limit of solutions to (P_ε).

In our case, this is with $w^\varepsilon \neq 0$ satisfying (0.2.4) and nondecreasing in the direction of the cylinder axis, the uniqueness result in the presence of a classical solution still holds.

The cylindrical geometry has the advantage of giving the condition of nonvanishing gradient in the positivity set of any limit solution. Since in dimension 2 one can prove that limit solutions are semi-classical supersolutions up to the fixed boundary, the uniqueness of limit solutions follows in this case without further assumptions.

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