MULTIPLE POSITIVE SOLUTIONS FOR QUASILINEAR ELLIPTIC PROBLEMS WITH SIGN-CHANGING NONLINEARITIES

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Abstract. Using variational arguments we prove some nonexistence and multiplicity results for positive solutions of a system of $p-$Laplace equations of gradient form. Then we study a $p-$Laplace type problem with nonlinear boundary conditions.

1. Introduction

In a recent paper, [7], the authors studied the existence, multiplicity, and nonexistence of positive classical solutions of the semilinear elliptic boundary-value problem

$$\begin{aligned} -\Delta u = \lambda f(u) \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, $\lambda > 0$ is a parameter, and f is a C^1 sign-changing sub-linear function.

They showed using sub-super solutions arguments and recent results from semipositone problems that there are $\underline{\lambda}$ and $\overline{\lambda}$ such that (1.1) has no positive solution for $\lambda < \underline{\lambda}$ and at least two positive solutions for $\lambda \geq \overline{\lambda}$.

More recently, in [8], the author extends these results to the quasilinear problem

$$\begin{aligned} -\Delta_p u = \lambda f(x,u) \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned}$$

where $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the $p-$Laplacian, $1 < p < \infty$, $\lambda > 0$, and f is a sign-changing Carathéodory function on $\Omega \times [0,\infty)$.

The method in [8] is variational and allowed the author to substantially relax the assumptions on f. More precisely, these assumptions are

(H$_1$) $f(x,0) = 0 \qquad |f(x,t)| \leq C|t|^{p-1}$,
(H$_2$) $\exists \delta > 0$ such that $F(x,t) \leq 0$ for $0 \leq t \leq \delta$
(H$_3$) $\exists t_0 > 0$ such that $F(x,t_0) > 0$,
(H$_4$) $\displaystyle\limsup_{t\to\infty} \frac{F(x,t)}{t^p} \leq 0$ uniformly in x,

where $F(x,t) = \int_0^t f(x,s)\,ds$.

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The purpose of this article is twofold. By applying variational methods, first we extend the results in [8] to quasilinear elliptic systems of the form

$$\begin{aligned} -\Delta_p u &= \lambda F_u(x,u,v) && \text{in } \Omega,\\ -\Delta_q v &= \lambda F_v(x,u,v) && \text{in } \Omega,\\ u = v &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where (F_u, F_v) stands for the gradient of a given potential F and second, we want to see to what extent these variational techniques can be adapted to deal with the nonlinear boundary condition case,

$$\begin{aligned} -\Delta_p u + |u|^{p-2}u &= 0 && \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where $\partial/\partial\nu$ is the outer unit normal derivative.

Systems of the form (1.2) are usually called *gradient systems* and have been widely studied in the past. See, for example, [2] for a comprehensive analysis of such systems. This gradient structure allows us to treat (1.2) variationally. Other kind of elliptic systems that can be treated variationally are the so-called *Hamiltonian systems*, see [3].

However, as far as we know, all the results for (1.2) assume, to begin with, that $F_u, F_v \geq 0$ for $u, v \geq 0$.

For problem (1.3), in a previous paper, [4], the authors studied the problem where the nonlinearity g was assume to be of power-type, i.e. essentially the case $g(x,t) = |t|^{q-2}t$ was consider, so again $g(x,t) \geq 0$ for $t \geq 0$.

The main results of this paper can be formulated as follows:

Under hypotheses similar to (H_1)–(H_4)*, there exists* $0 < \underline{\lambda} < \bar{\lambda}$ *such that if* $0 < \lambda < \underline{\lambda}$ *problem* (1.2) *(or problem* (1.3)*) has no positive solution and if* $\lambda > \bar{\lambda}$ *problem* (1.2) *(or problem* (1.3)*) has, at least, two positive solutions.*

The rest of the paper is organized as follows: In §2 we deal with problem (1.2) and in §3 with (1.3).

2. GRADIENT SYSTEMS

In this section, we deal with problem (1.2). First, we prove the nonexistence result. To this end, we assume that $F(x,u,v)$ is a Carathéodory function on $\Omega \times [0,\infty) \times [0,\infty)$, $F(x,\cdot,\cdot)$ is C^1 for a.e. $x \in \Omega$, and F_u, F_v are also Carathéodory functions satisfying

$$\begin{aligned} &F(x,0,0) = F_u(x,0,0) = F_v(x,0,0) = 0,\\ &|uF_u(x,u,v) + vF_v(x,u,v)| \leq C(u^p + v^q),\\ &|F(x,u,v)| \leq C(u^p + v^q), \end{aligned} \tag{2.1}$$

for some constant $C > 0$.

We have the following

Theorem 2.1. *Assume* (2.1)*. Then there is a* $\underline{\lambda}$ *such that* (1.2) *has no positive solution for* $\lambda < \underline{\lambda}$.

For the proof we need the following observation: Let us denote by λ_r the best constant in the Sobolev embedding $W_0^{1,r}(\Omega) \hookrightarrow L^r(\Omega)$. We have

$$\lambda_p \int_\Omega |u|^p\,dx \le \int_\Omega |\nabla u|^p\,dx, \qquad \text{for } u \in W_0^{1,p}(\Omega),$$

$$\lambda_q \int_\Omega |v|^q\,dx \le \int_\Omega |\nabla v|^q\,dx, \qquad \text{for } v \in W_0^{1,q}(\Omega),$$

so if we denote $\lambda_{p,q} = \min\{\lambda_p, \lambda_q\}$ we obtain

$$0 < \lambda_{p,q} \le \frac{\displaystyle\int_\Omega |\nabla u|^p + |\nabla v|^q\,dx}{\displaystyle\int_\Omega |u|^p + |v|^q\,dx}, \qquad \text{for } u \in W_0^{1,p}(\Omega),\ v \in W_0^{1,q}(\Omega) \tag{2.2}$$

and, moreover, one can easily see that $\lambda_{p,q}$ is optimal.

Proof of Theorem 2.1. If (1.2) has a positive solution (u, v), multiplying the first equation of (1.2) by u, the second by v, integrating by parts and adding up, we get

$$\int_\Omega |\nabla u|^p + |\nabla v|^q\,dx = \lambda \int_\Omega F_u(x,u,v)u + F_v(x,u,v)v\,dx.$$

Thus, using (2.1), we obtain

$$\int_\Omega |\nabla u|^p + |\nabla v|^q\,dx \le \lambda C \int_\Omega |u|^p + |v|^q\,dx$$

and hence $\lambda \ge \lambda_{p,q}/C$ by (2.2), proving Theorem 2.1. □

Now, we prove the multiplicity result. To this end, along with (2.1), we also have to assume that

- (F_1) $\exists \delta > 0$ such that $F(x,u,v) \le 0$ for $|u|^p + |v|^q \le \delta$,
- (F_2) $\exists t_0, s_0 > 0$ such that $F(x,t_0,s_0) > 0$,
- (F_3) $\displaystyle\limsup_{|(u,v)|\to\infty} \frac{F(x,u,v)}{u^p + v^q} \le 0$ uniformly in x.

Under these assumptions, we have the following

Theorem 2.2. *Under the assumptions* (2.1), (F_1)–(F_3), *There is a* $\overline{\lambda}$ *such that* (1.2) *has at least two positive solutions* $(u_1, v_1), (u_2, v_2)$ *for* $\lambda \ge \overline{\lambda}$.

For the proof of Theorem 2.2 we use critical point theory. Set $F(x,u,v) = 0$ for $u, v < 0$, and consider the C^1 functional

$$\mathcal{F}_\lambda(u,v) = \int_\Omega \frac{|\nabla u|^p}{p} + \frac{|\nabla v|^q}{q} - \lambda F(x,u,v)\,dx, \quad (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega).$$

Observe that if (u, v) is a critical point of $\mathcal{F}_\lambda$, denoting by u^- and v^- the negative parts of u and v respectively,

$$\begin{aligned} 0 &= (\mathcal{F}'_\lambda(u,v), (u^-, v^-)) \\ &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u^- + |\nabla v|^{q-2}\nabla v \cdot v^- - \lambda(F_u(x,u,v)u^- + F_v(x,u,v)v^-)\,dx \\ &= \|u^-\|^p_{W_0^{1,p}(\Omega)} + \|v^-\|^q_{W_0^{1,q}(\Omega)}, \end{aligned}$$

hence we have that $u, v \geq 0$. Furthermore, by [11], $u, v \in C^{1,\alpha}(\Omega)$ and so, by Harnack inequality (see [10]), it follows that either $u, v > 0$ or $u \equiv v \equiv 0$. Therefore, nontrivial critical points of $\mathcal{F}_\lambda$ are positive solutions of (1.3).

By (F_3) and (2.1), there is a constant $C_\lambda > 0$ such that

$$\lambda F(x, u, v) \leq \frac{\lambda_{p,q}}{2} \left(\frac{|u|^p}{p} + \frac{|v|^q}{q} \right) + C_\lambda$$

and hence

$$\begin{aligned} \mathcal{F}_\lambda(u) &\geq \int_\Omega \frac{|\nabla u|^p}{p} + \frac{|\nabla v|^q}{q} - \frac{\lambda_{p,q}}{2} \left(\frac{|u|^p}{p} + \frac{|v|^q}{q} \right) - C_\lambda \, dx \\ &\geq \frac{1}{2p} \|u\|^p_{W_0^{1,p}(\Omega)} + \frac{1}{2q} \|v\|^q_{W_0^{1,q}(\Omega)} - C_\lambda |\Omega|_N, \end{aligned}$$

where $|\cdot|_d$ denotes the $d-$dimensional Lebesgue measure in $\mathbb{R}^N$, so $\mathcal{F}_\lambda$ is bounded from below and coercive.

Therefore, as $\mathcal{F}_\lambda$ is weakly lower semicontinuous, we obtain a global minimizer (u_1, v_1). Let us show that, if λ is big enough, this minimizer is nontrivial.

Lemma 2.1. *There is a* $\overline{\lambda}$ *such that* $\inf \mathcal{F}_\lambda < 0$, *and hence* $(u_1, v_1) \neq (0, 0)$, *for* $\lambda \geq \overline{\lambda}$.

Proof. Let us consider a sufficiently large compact subset Ω' of Ω and take functions $u_0 \in W_0^{1,p}(\Omega)$, $v_0 \in W_0^{1,q}(\Omega)$ such that $u_0(x) = t_0$ on Ω', $0 \leq u_0(x) \leq t_0$ on $\Omega \setminus \Omega'$, $v_0(x) = s_0$ on Ω', $0 \leq v_0(x) \leq s_0$ on $\Omega \setminus \Omega'$, where t_0, s_0 are as in (F_2).

Then, we obtain

$$\int_\Omega F(x, u_0, v_0) \, dx \geq \int_{\Omega'} F(x, t_0, s_0) \, dx - C(t_0^p + s_0^q) |\Omega \setminus \Omega'|_N > 0,$$

if Ω' is big enough. Hence $\mathcal{F}_\lambda(u_0, v_0) < 0$ for λ large enough. □

We will obtain a critical point (u_2, v_2) with $\mathcal{F}_\lambda(u_2, v_2) > 0$ via the mountain-pass lemma, which would complete the proof since $\mathcal{F}_\lambda(u_2, v_2) > 0 > \mathcal{F}_\lambda(u_1, v_1)$.

Lemma 2.2. *The origin is a strict local minimizer of* $\mathcal{F}_\lambda$.

Proof. Let $U_{(u,v)} = \big\{ x \in \Omega : |u(x)|^p + |v(x)|^q > \delta \big\}$. By ($\mathrm{F}_1$), $F(x, u(x), v(x)) \leq 0$ on $\Omega \setminus U_{(u,v)}$, so

$$\mathcal{F}_\lambda(u, v) \geq \frac{1}{p} \|u\|^p_{W_0^{1,p}(\Omega)} + \frac{1}{q} \|v\|^q_{W_0^{1,q}(\Omega)} - \lambda \int_{U_{(u,v)}} F(x, u, v) \, dx.$$

By (2.1), Hölder's inequality, and Sobolev embedding,

$$\begin{aligned} \int_{U_{(u,v)}} F(x, u, v) \, dx &\leq C \int_{U_{(u,v)}} u^p + v^q \, dx \\ &\leq C \left(|U_{(u,v)}|_N^{1-\frac{p}{r}} \|u\|^p_{W_0^{1,p}(\Omega)} + |U_{(u,v)}|_N^{1-\frac{q}{s}} \|v\|^q_{W_0^{1,q}(\Omega)} \right), \end{aligned}$$

where $r = Np/(N-p)$ if $p < N$ and $r > p$ if $p \geq N$, and $s = Nq/(N-q)$ if $q < N$ and $s > q$ if $q \geq N$ so in order to finish the proof we need to show that $|U_{(u,v)}|_N \to 0$ as $\|u\|_{W_0^{1,p}(\Omega)} + \|v\|_{W_0^{1,q}(\Omega)} \to 0$.

Now

$$\begin{aligned}\|u\|^p_{W_0^{1,p}(\Omega)} + \|v\|^q_{W_0^{1,q}(\Omega)} &\geq \lambda_{p,q} \int_\Omega u^p + v^q \, dx \geq \lambda_{p,q} \int_{U_{(u,v)}} u^p + v^q \, dx \\ &\geq \lambda_{p,q} \delta |U_{(u,v)}|_N,\end{aligned}$$

as we wanted to show. ☐

Now, we are in position to finish the proof of Theorem 2.2.

Proof of Theorem 2.2. As $\mathcal{F}_\lambda$ is coercive, every Palais-Smale sequence is bounded and hence contains a convergent subsequence as usual. Now the mountain-pass lemma gives a critical point (u_2, v_2) of $\mathcal{F}_\lambda$ at the level

$$c := \inf_{\gamma \in \Gamma} \max_{(u,v) \in \gamma([0,1])} \mathcal{F}_\lambda(u, v) > 0,$$

where $\Gamma = \big\{\gamma \in C([0,1], W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)) : \gamma(0) = 0, \gamma(1) = (u_1, v_1)\big\}$ is the class of paths joining the origin to (u_1, v_1) (see [9]). ☐

3. The nonlinear boundary condition case

In this section, we deal with the nonlinear boundary condition case, problem (1.3). The main ideas and structures of the proofs are the same as in the previous section, so we only sketch them and stress the differences between the two cases.

We begin with the nonexistence result. To this end, we assume that g is a Carathéodory function on $\partial\Omega \times [0, \infty)$ satisfying

$$g(x, 0) = 0, \quad -ct^{r-1} \leq g(x, t) \leq Ct^{p-1}, \tag{3.1}$$

for some $1 \leq r \leq p$ and some constants $C, c > 0$.

We have,

Theorem 3.1. *There is a* $\underline{\lambda}$ *such that* (1.3) *has no positive solution for* $\lambda < \underline{\lambda}$.

For the proof we need some knowledge on the following eigenvalue problem

$$\begin{aligned} -\Delta_p u + |u|^{p-2} u &= 0 \quad \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2} u \quad \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

This problem was studied in [4, 6] (see also [5]). It was proved there that problem (3.2) has a first positive eigenvalue λ_1 given by

$$\lambda_1 = \min_{u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p + |u|^p \, dx}{\displaystyle\int_{\partial\Omega} |u|^p \, d\sigma}, \tag{3.3}$$

where $d\sigma$ is the boundary measure. In the linear case, $p = 2$, problem (3.2) is known as the *Steklov problem* (see [1]).

Proof of Theorem 3.1. If (1.3) has a positive solution u, multiplying (1.3) by u, integrating by parts, and using (3.1) gives

$$\int_\Omega |\nabla u|^p + |u|^p \, dx = \lambda \int_{\partial\Omega} f(x, u) u \, d\sigma \leq C\lambda \int_{\partial\Omega} |u|^p \, d\sigma,$$

and hence $\lambda \geq \lambda_1 / C$ by (3.3), proving Theorem 3.1. ☐

Now we prove the multiplicity result.

The assumptions in this case are: Let $G(x,t) = \int_0^t g(x,s)ds$, and assume

- (G$_1$) $\exists\delta > 0$ such that $G(x,t) \le 0$ for $0 \le t \le \delta$,
- (G$_2$) $\exists t_0 > 0$ such that $G(x,t_0) > 0$,
- (G$_3$) $\displaystyle\limsup_{t\to\infty} \frac{G(x,t)}{t^p} \le 0$ uniformly in x.

Theorem 3.2. *Assume* (3.1) *and* (G$_1$) – (G$_3$). *Then there is a* $\overline{\lambda}$ *such that* (1.3) *has at least two positive solutions* $u_1 > u_2$ *for* $\lambda \ge \overline{\lambda}$.

Observe that for problem (1.3) we can prove that the two solutions are ordered. We believe that this should hold also for (1.2) but the truncation argument used in the proof does not work because it destruct the variational structure of (1.2).

Again, set $g(x,t) = 0$ for $t < 0$, and consider the C^1 functional

$$\mathcal{G}_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p\,dx - \lambda\int_{\partial\Omega} G(x,u)\,d\sigma, \qquad u \in W^{1,p}(\Omega).$$

Arguing as before, if u is a critical point of $\mathcal{G}_\lambda$, denoting by u^- the negative part of u,

$$\begin{aligned} 0 &= (\mathcal{G}_\lambda'(u), u^-) \\ &= \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla u^- + |u|^{p-2}uu^-\,dx - \lambda\int_{\partial\Omega} g(x,u)u^-\,d\sigma \\ &= \|u^-\|^p_{W^{1,p}(\Omega)}, \end{aligned}$$

hence we have that $u \ge 0$. Furthermore, by [11], $u \in C^{1,\alpha}(\Omega)$ and so, by the strong maximum principle and Hopf's Lemma (see [12]), it follows that either $u > 0$ in $\overline{\Omega}$ or $u \equiv 0$. Therefore, nontrivial critical points of $\mathcal{F}_\lambda$ are positive solutions of (1.3). Observe that in this case, the solution u is positive up to the boundary.

By (G$_3$) and (3.1), there is a constant $C_\lambda > 0$ such that

$$\lambda G(x,t) \le \frac{\lambda_1}{2p}|t|^p + C_\lambda$$

and hence

$$\mathcal{G}_\lambda(u) \ge \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p\,dx - \int_{\partial\Omega}\frac{\lambda_1}{2p}|u|^p + C_\lambda\,d\sigma \ge \frac{1}{2p}\|u\|^p_{W^{1,p}(\Omega)} - C_\lambda|\partial\Omega|_{N-1},$$

so $\mathcal{G}_\lambda$ is bounded from below and coercive.

Therefore, as $\mathcal{G}_\lambda$ is weakly lower semicontinuous, we obtain a global minimizer u_1. Once again, if λ is big enough, this minimizer is nontrivial.

Lemma 3.1. *There is a* $\overline{\lambda}$ *such that* $\inf\mathcal{G}_\lambda < 0$, *and hence* $u_1 \ne 0$, *for* $\lambda \ge \overline{\lambda}$.

Proof. Take the constant function $u_0 \equiv t_0$, where t_0 is as in (G$_2$).

Then, we obtain

$$\int_{\partial\Omega} G(x,u_0)\,d\sigma = \int_{\partial\Omega} G(x,t_0)\,d\sigma > 0.$$

Hence $\mathcal{G}_\lambda(u_0) < 0$ for λ large enough. □

The main difference in the arguments arrive at this point. As we mentioned before, by a truncation argument we can prove that the two solutions are ordered. In fact, fix $\lambda \geq \overline{\lambda}$. Let

$$\widetilde{g}(x,t) = \begin{cases} g(x,t), & t \leq u_1(x), \\ g(x,u_1(x)), & t > u_1(x), \end{cases} \quad \text{and} \quad \widetilde{G}(x,t) = \int_0^t \widetilde{g}(x,s)ds. \tag{3.4}$$

Then consider

$$\widetilde{\mathcal{G}}_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p \, dx - \lambda \int_{\partial\Omega} \widetilde{G}(x,u)\, d\sigma.$$

If u is a critical point of $\widetilde{\mathcal{G}}_\lambda$, then $u \geq 0$ as before. Now,

$$\begin{aligned} 0 =& \big(\widetilde{\mathcal{G}}'_\lambda(u) - \mathcal{G}'_\lambda(u_1), (u-u_1)^+\big) \\ =& \int_\Omega \big[\big(|\nabla u|^{p-2}\nabla u - |\nabla u_1|^{p-2}\nabla u_1\big) \cdot \nabla(u-u_1)^+ \\ &+ \big(|u|^{p-2}u - |u_1|^{p-2}u_1\big)\,(u-u_1)^+\big]\, dx \\ &- \lambda \int_{\partial\Omega} \big(\widetilde{g}(x,u) - g(x,u_1)\big)(u-u_1)^+ \, d\sigma \\ =& \int_{\{u>u_1\}} \big[\big(|\nabla u|^{p-2}\nabla u - |\nabla u_1|^{p-2}\nabla u_1\big) \cdot (\nabla u - \nabla u_1) \\ &+ \big(|u|^{p-2}u - |u_1|^{p-2}u_1\big)\,(u-u_1)^+\big]\, dx \\ \geq& \int_{\{u>u_1\}} \big[\big(|\nabla u|^{p-1} - |\nabla u_1|^{p-1}\big)\,(|\nabla u| - |\nabla u_1|) \\ &+ \big(|u|^{p-1} - |u_1|^{p-1}\big)\,(|u| - |u_1|)\big]\, dx \geq 0, \end{aligned}$$

so $u \leq u_1$. Therefore u is a solution of (1.3).

Now, as in the previous case, we will obtain the second solution as a critical point of $\widetilde{\mathcal{G}}$, u_2, with $\widetilde{\mathcal{G}}_\lambda(u_2) > 0$ via the mountain-pass lemma, which would complete the proof since $\widetilde{\mathcal{G}}_\lambda(0) = 0 > \widetilde{\mathcal{G}}_\lambda(u_1)$.

Lemma 3.2. *The origin is a strict local minimizer of* $\widetilde{\mathcal{G}}_\lambda$.

Proof. Let $\Gamma_u = \big\{x \in \partial\Omega : u(x) > \min\big\{u_1(x), \delta\big\}\big\}$.

By (3.4) and (G$_1$), $\widetilde{G}(x,u(x)) \leq 0$ on $\partial\Omega \setminus \Gamma_u$, so

$$\widetilde{\mathcal{G}}_\lambda(u) \geq \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \lambda \int_{\Gamma_u} \widetilde{G}(x,u)\, d\sigma.$$

By (3.1), Hölder's inequality, and Sobolev trace embedding,

$$\int_{\Gamma_u} \widetilde{G}(x,u)\, d\sigma \leq C \int_{\Gamma_u} u^p \, d\sigma \leq C|\Gamma_u|_{N-1}^{1-\frac{p}{q}} \|u\|^p_{W^{1,p}(\Omega)},$$

where $q = (N-1)p/(N-p)$ if $p < N$ and $q > p$ if $p \geq N$, so in order to finish the proof we need to show that $|\Gamma_u|_{N-1} \to 0$ as $\|u\|_{W^{1,p}(\Omega)} \to 0$.

Let $k = \min\{\min_{\partial\Omega} u_1; \delta\}$ where δ is given in (G$_1$). Then

$$\|u\|^p_{W^{1,p}(\Omega)} \geq C \int_{\partial\Omega} u^p \, d\sigma \geq C \int_{\Gamma_u} u^p \, d\sigma \geq Ck^p |\Gamma_u|_{N-1},$$

as we wanted to show. $\square$

Now, we are in position to finish the proof of Theorem 3.2.

Proof of Theorem 3.2. The same argument used for $\mathcal{G}_\lambda$ shows that $\widetilde{\mathcal{G}}_\lambda$ is also coercive, so every Palais-Smale sequence of $\widetilde{\mathcal{G}}_\lambda$ is bounded and hence contains a convergent subsequence as usual. Now the mountain-pass lemma gives a critical point u_2 of $\widetilde{\mathcal{G}}_\lambda$ at the level

$$c := \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} \widetilde{\mathcal{G}}_\lambda(u) > 0,$$

where $\Gamma = \big\{\gamma \in C([0,1], W^{1,p}(\Omega)) : \gamma(0) = 0, \gamma(1) = u_1\big\}$ is the class of paths joining the origin to u_1. ☐

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