THE BEST SOBOLEV TRACE CONSTANT IN A DOMAIN WITH OSCILLATING BOUNDARY

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Abstract. In this paper we study homogenization problems for the best constant for the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ in a bounded smooth domain when the boundary is perturbed by adding an oscillation. We find that there exists a critical size of the amplitude of the oscillations for which the limit problem has a weight on the boundary. For sizes larger than critical the best trace constant goes to zero and for sizes smaller than critical it converges to the best constant in the domain without perturbations.

1. Introduction.

In this paper we consider homogenization problems for the best Sobolev trace constant when a periodic oscillation is added on the boundary.

Sobolev inequalities have been studied by many authors and is by now a classical subject. Relevant for the study of boundary value problems for differential operators is the Sobolev trace inequality that has been intensively studied, see for example, [6, 12, 13] and references therein.

Given a bounded smooth domain $\Omega \subset \mathbb{R}^N$, we deal with the best constant of the Sobolev trace embedding $W^{1,p}(\Omega_\varepsilon) \hookrightarrow L^q(\partial\Omega_\varepsilon)$ where Ω_ε is obtained by adding an oscillating perturbation to the boundary of a fixed domain, Ω.

The interest in problems with oscillating boundary appears in the influence of micro-structures of surfaces (porous medium, composites, micro-materials) over the large scale behavior. The mathematical analysis of problems with oscillating boundary was presented in [19].

Let us describe the involved domains Ω_ε. For simplicity, we consider only perturbations in a region of the boundary $\partial\Omega$ but is clear that the same kind of analysis can be done if the boundary is perturbed everywhere. First, we identify the region of the boundary of $\Omega \subset \mathbb{R}^N$ where the perturbation is localized. We assume that there exits a smooth function $\Phi : U' \subset \mathbb{R}^{N-1} \to \mathbb{R}$, where U' is a connected and open set, such that parameterizes a region Γ_1 of $\partial\Omega$

$$\{(x_1, x') \in \mathbb{R}^N \mid x' \in U',\ x_1 = \Phi(x')\} = \Gamma_1 \subset \partial\Omega.$$

We consider a connected open neighborhood $U = (\delta_1, \delta_2) \times U' \subset \mathbb{R}^N$ such that

$$U \cap \partial\Omega = \Gamma_1,$$

and

$$\Omega \cap U = \{(x_1, x') \in U \mid x' \in U',\ x_1 < \Phi(x')\}.$$

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Now, let $f : \mathbb{R}^{N-1} \to \mathbb{R}$ be a smooth (C^1 is enough) periodic function with period $Y' := [0,1]^{N-1}$ and $f(0) = 0$. We denote the translate cells as $\varepsilon Y'_n = \varepsilon n + \varepsilon Y'$ with $n \in \mathbb{Z}^{N-1}$. We then define the perturbed domain Ω_ε as follows:

$$
\begin{aligned}
&\Omega_\varepsilon \cap U := \{(x_1, x') \in U \mid x' \in U'_\varepsilon,\ x_1 < \Phi(x') + \varepsilon^a f(x'/\varepsilon)\chi_{U'_\varepsilon}(x')\},\\
&\text{where } \ U'_\varepsilon = \bigcup \left\{\varepsilon Y'_n \mid \text{such that } \varepsilon Y'_n \subset U', \quad n \in \mathbb{Z}^N\right\},\\
&\Gamma^1_\varepsilon = \left\{(x_1, x') \in \mathbb{R}^N \mid x' \in U',\ x_1 = \Phi(x') + \varepsilon^a f(x'/\varepsilon)\chi_{U'_\varepsilon}(x')\right\},\\
&\Omega_\varepsilon \cap U^c := \Omega \cap U^c.
\end{aligned}
$$

Here $\chi_{U'_\varepsilon}$ is the characteristic function of U'_ε. Therefore, we are considering oscillations of period ε with size ε^a.

For any $1 < p < \infty$ and for every subcritical exponent,

$$
1 \le q < p_* := \frac{p(N-1)}{(N-p)_+},
$$

we consider the Sobolev trace inequality, $S\|v\|^p_{L^q(\partial\Omega_\varepsilon)} \le \|v\|^p_{W^{1,p}(\Omega_\varepsilon)}$, valid for all $v \in W^{1,p}(\Omega_\varepsilon)$. The best Sobolev trace constant is the largest S such that the above inequality holds,

$$
S(\varepsilon) := \inf_{v \in W^{1,p}(\Omega_\varepsilon)\setminus W^{1,p}_0(\Omega_\varepsilon)} \frac{\displaystyle\int_{\Omega_\varepsilon} |\nabla v|^p + |v|^p \, dx}{\left(\displaystyle\int_{\partial\Omega_\varepsilon} |v|^q \, dS\right)^{p/q}}. \tag{1.1}
$$

For subcritical exponents, $1 \le q < p_*$, the embedding $W^{1,p}(\Omega_\varepsilon) \hookrightarrow L^q(\partial\Omega_\varepsilon)$ is compact, so we have existence of extremals, i.e., functions where the infimum is attained. These extremals are strictly positive in Ω_ε (see [15]) and $C^{1,\alpha}_{\mathrm{loc}}(\Omega) \cap C^\alpha(\overline{\Omega})$ (see [21, 18]). When one normalize the extremals with

$$
\int_{\partial\Omega_\varepsilon} |u_\varepsilon|^q dS = 1, \tag{1.2}
$$

they are weak solutions to the following problem

$$
\begin{cases}
\Delta_p u_\varepsilon = |u_\varepsilon|^{p-2} u_\varepsilon & \text{in } \Omega_\varepsilon,\\[1ex]
|\nabla u_\varepsilon|^{p-2} \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = S(\varepsilon)|u_\varepsilon|^{q-2} u_\varepsilon & \text{on } \partial\Omega_\varepsilon,
\end{cases} \tag{1.3}
$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the usual $p-$laplacian operator and ν_ε is the unit outward normal vector. In the rest of this article we will assume that the extremals are normalized according to (1.2) and hence solutions to (1.3). Note that when $p = 2$ the equation becomes linear. Even in this case our results are new. Of special importance is the case $q = p$. In this case (1.3) is an eigenvalue problem of Steklov type, see [15, 20], etc.

Problems like (1.3) appears naturally in several branches of applied mathematics, like non-Newtonian fluids, reaction diffusion problems, nonlinear elasticity, glaciology, see [2, 3], etc. Also, the Steklov eigenvalue problem has applications. For instance, problems of minimization of the energy stored in the design under a prescribed loading. Solutions of these loadings are unstable under perturbations of the loading. The stable optimal design problem is formulated as minimization of the stored energy of the project under the most unfavorable load. This most dangerous loading in one that maximizes the stored energy over the class of admissible functions. This problem is reduced to minimization of the Steklov eigenvalues (see [9]). For many others applications of the Steklov eigenvalue problem we refer to [4].

Again, we want to stress that the results in this paper are new even in the linear case. Since our techniques are variational and can be easily extended from the linear setting to more general energies, like the L^p norm of the gradient, we choose to present our results in the Sobolev trace context which also has many theoretical and practical implications (see [6, 10, 12, 15] and references therein).

In view of the above discussion, our concern in this article is the study of the limit of $S(\varepsilon)$ and of the corresponding extremals as ε goes to zero. We find that there is a critical size of the amplitudes for the oscillations such that the extremals converge as the oscillations go to infinity to a solution of an homogenized limit problem and the best trace constant converges to a homogenized best trace constant. For amplitudes larger than the critical one, the size of the boundary becomes too large and the Sobolev trace constant goes to zero. For amplitudes smaller that the critical one, the perturbation is too small, so the Sobolev trace constant converges to the one of the unperturbed domain.

The precise statement of our result is as follows:

Theorem 1. *Let $S(\varepsilon)$ be the best Sobolev trace constant given by* (1.1)*.*

(1) *If $a < 1$, then $S(\varepsilon)$ goes to zero as $\varepsilon \to 0$. Moreover, it holds*

$$S(\varepsilon) \le C\varepsilon^{(1-a)p/q} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{1.4}$$

(2) *If $a > 1$, then $S(\varepsilon)$ converges as $\varepsilon \to 0$ to $S(0)$ defined by*

$$S(0) := \inf_{v \in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^p + |v|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q\,dS\right)^{p/q}}, \tag{1.5}$$

i.e, $S(0)$ is the best Sobolev trace constant of the unperturbed domain Ω. The corresponding normalized extremals, rescaled to Ω, converge (along subsequences) strongly in $W^{1,p}(\Omega)$ to an extremal of (1.5)*.*

(3) *If $a = 1$, then $S(\varepsilon)$ converges as $\varepsilon \to 0$ to S^* defined by*

$$S^* = \inf_{v \in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^p + v^p\,dx}{\left(\displaystyle\int_{\partial\Omega} m(x)|v|^q\,dS\right)^{p/q}}, \tag{1.6}$$

with

$$m(x) := \begin{cases} \dfrac{\displaystyle\int_Y \sqrt{1+|\nabla\Phi(x')+\nabla f(y)|^2}\,dy}{\sqrt{1+|\nabla\Phi(x')|^2}} & \text{for } x \in \partial\Omega\cap U, \\ 1 & \text{elsewhere} \end{cases} \tag{1.7}$$

Moreover, the normalized extremals (rescaled to Ω in a suitable way) converge (along subsequences) weakly in $W^{1,p}$ to an extremal of (1.6)*.*

To end the introduction, we briefly describe related results for problems with oscillating boundary for second order elliptic equations. In [7] and [17], the asymptotic behavior of solutions to the Neumann boundary value problem with respect to the oscillating boundary shows a limiting macrostucture. In [1] the authors study the behavior of the Laplace equation in as oscillating domain imposing nonhomogeneous Dirichlet boundary conditions on the oscillating part of the boundary. In [8] and [16] a rapidly oscillating boundary with unlimited growth and inhomogeneous Fourier boundary condition is studied. The limiting problem can involve Dirichlet, Fourier or Neumann boundary conditions depending on the structure. There exists references that deal with quasilinear operators and oscillating boundaries, see [5] and [11]. On the other hand, the best Sobolev trace constant in domains with holes was recently studied in [14] using homogenization techniques.

2. Proofs of the results

2.1. **Subcritical case.** $a < 1$**.** This is the easiest case. The result follows just by taking $v \equiv 1$ as a test in the variational characterization of $S(\varepsilon)$, (1.1). Doing so we get the following inequality

$$S(\varepsilon) \leq \frac{|\Omega_\varepsilon|}{|\partial\Omega_\varepsilon|^{p/q}}.$$

It is clear that

$$\lim_{\varepsilon\to 0} |\Omega_\varepsilon| = |\Omega|.$$

Let us estimate $|\partial\Omega_\varepsilon|$. We have

$$\begin{aligned} |\partial\Omega_\varepsilon| \geq |\partial\Omega_\varepsilon \cap U| &\geq \int_{U'_\varepsilon} \sqrt{1 + |\nabla\Phi(x') + \varepsilon^{a-1}\nabla f(x'/\varepsilon)|^2}\, dx' \\ &= \varepsilon^{a-1} \int_{U'_\varepsilon} \sqrt{\varepsilon^{2(1-a)} + |\varepsilon^{1-a}\nabla\Phi(x') + \nabla f(x'/\varepsilon)|^2}\, dx'. \end{aligned}$$

As $a < 1$, it is easy to see that

$$\int_{U'_\varepsilon} \sqrt{\varepsilon^{2(1-a)} + |\varepsilon^{1-a}\nabla\Phi(x') + \nabla f(x'/\varepsilon)|^2}\, dx' \to m(|\nabla f|) := \int_{Y'} |\nabla f(y')|\, dy' > 0.$$

Hence, $|\partial\Omega_\varepsilon| \geq c\varepsilon^{a-1}$. From where it follows that

$$S(\varepsilon) \leq C\varepsilon^{(1-a)p/q} \to 0 \qquad \text{as } \varepsilon \to 0,$$

as we wanted to show.

Remark 2.1. *From the previous proof we have that the constant* C *in* (1.4) *can be any constant larger than* $|\Omega|/(m(|\nabla f|))^{p/q}$.

2.2. **Supercritical case.** $a > 1$**.** To transform integrals in Ω_ε into integrals in Ω, let us perform the following change of variables

$$\bar{x}' = x', \qquad \bar{x}_1 = x_1 - \varepsilon^a f(x'/\varepsilon)\varphi(x_1, x'),$$

where φ is a smooth cut-off function with bounded derivatives that vanish outside U. For every $u \in C^1(\overline{\Omega_\varepsilon})$ consider

$$u(x_1, x') = v(\bar{x}_1, \bar{x}').$$

We obtain that $v \in C^1(\overline{\Omega})$. For any $u \in W^{1,p}(\Omega_\varepsilon)$ and $v \in W^{1,p}(\Omega)$, we denote

$$Q_\varepsilon(u) := \frac{\displaystyle\int_{\Omega_\varepsilon} |\nabla u|^p + u^p \, dx}{\left(\displaystyle\int_{\partial\Omega_\varepsilon} |u|^q \, dS\right)^{p/q}} \qquad \text{and} \qquad Q_0(v) := \frac{\displaystyle\int_{\Omega} |\nabla v|^p + v^p \, dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q \, dS\right)^{p/q}}.$$

To change variables in $Q_\varepsilon(u)$, let us compute the jacobian of the change of variables

$$J^{-1} = 1 - \varepsilon^a f(x'/\varepsilon)\varphi_{x_1}.$$

The derivatives of v and u are related by

$$u_{x_1} = v_{\bar{x}_1}(1 - \varepsilon^a f(x'/\varepsilon)\varphi_{x_1})$$

and

$$\nabla_{x'} u = -v_{\bar{x}_1}(\varepsilon^{a-1}\nabla_{x'} f(x'/\varepsilon)\varphi + \varepsilon^a f(x'/\varepsilon)\nabla_{x'}\varphi) + \nabla_{\bar{x}'} v.$$

Therefore, as φ and f have bounded derivatives,

$$\int_{\Omega_\varepsilon} |\nabla u|^p \, dx = (1 + O(\varepsilon^{(a-1)})) \int_{\Omega} |\nabla v|^p \, d\bar{x},$$

$$\int_{\Omega_\varepsilon} |u|^p \, dx = (1 + O(\varepsilon^a)) \int_{\Omega} |v|^p \, d\bar{x}$$

and

$$\int_{\partial\Omega_\varepsilon} |u|^q \, dS = (1 + O(\varepsilon^{(a-1)})) \int_{\partial\Omega} |v|^q \, dS.$$

Thus we obtain, as $a > 1$,

$$Q_\varepsilon(u) = Q_0(v) + \delta_\varepsilon, \qquad \text{with } \delta_\varepsilon \to 0, \quad \text{as } \varepsilon \to 0. \tag{2.1}$$

Since φ and f have bounded derivatives, it can be checked that δ_ε can be taken uniformly on bounded sets of $W^{1,p}(\Omega)$. Then,

$$Q_\varepsilon(u) \geq S(0) + \delta_\varepsilon.$$

Now let u_ε be an extremal for (1.1) normalized by (1.2). Taking $u \equiv 1$ in (1.1) we get

$$\|u_\varepsilon\|^p_{W^{1,p}(\Omega)} = \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^p + u_\varepsilon^p \, dx \leq \frac{|\Omega_\varepsilon|}{|\partial\Omega_\varepsilon|^{p/q}} \leq C.$$

When we change variables we get that v_ε is bounded in $W^{1,p}(\Omega)$ independently of ε, from where it follows that

$$\liminf_{\varepsilon \to 0} S(\varepsilon) \geq S(0). \tag{2.2}$$

To obtain the upper bound, given $\rho > 0$ we take a $C^1(\overline{\Omega})$ function v such that

$$Q_0(v) \leq S(0) + \rho.$$

From (2.1) we obtain

$$S(\varepsilon) \leq Q_\varepsilon(u) = Q_0(v) + \delta_\varepsilon \leq S(0) + \rho + \delta_\varepsilon.$$

Therefore

$$\limsup_{\varepsilon \to 0} S(\varepsilon) \leq S(0) + \rho.$$

Since this inequality holds for every $\rho > 0$, we have

$$\limsup_{\varepsilon \to 0} S(\varepsilon) \leq S(0). \tag{2.3}$$

Combining (2.2) and (2.3) we conclude

$$\lim_{\varepsilon\to 0} S(\varepsilon) = S(0).$$

Now we deal with the convergence of the extremals. Let u_ε be an extremal for Q_ε. From our previous arguments we have that the rescaled functions v_ε are bounded in $W^{1,p}(\Omega)$. Therefore we can extract a subsequence (that we still call v_ε) such that $v_\varepsilon \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$. We have

$$1 = \int_{\partial\Omega_\varepsilon} |u_\varepsilon|^q \, dS = (1 + O(\varepsilon^{(a-1)})) \int_{\partial\Omega} |v_\varepsilon|^q \, dS.$$

Hence, by the compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$,

$$\int_{\partial\Omega} |v|^q \, dS = 1.$$

Moreover,

$$\|v\|^p_{W^{1,p}(\Omega)} \le \liminf_{\varepsilon\to 0} \|v_\varepsilon\|^p_{W^{1,p}(\Omega)} = S(0) \le \|v\|^p_{W^{1,p}(\Omega)}.$$

Therefore,

$$\lim_{\varepsilon\to 0} \|v_\varepsilon\|^p_{W^{1,p}(\Omega)} = \|v\|^p_{W^{1,p}(\Omega)},$$

and we conclude that the sequence v_ε converges strongly to an extremal of $S(0)$.

2.3. **Critical case.** $a = 1$. Since the size of the oscillations is small the perturbations of Ω are contained in a small neighborhood of the perturbed portion of the boundary. In fact, the perturbations lie in the set

$$A_\varepsilon = \{x \in U \cap \Omega \ : \ \mathrm{dist}(x, \partial\Omega) < \varepsilon\}.$$

Observe that $|A_\varepsilon| \sim \varepsilon$.

As before, taking $u \equiv 1$ in (1.1) we get that $S(\varepsilon)$ is bounded independently of ε. Thus, the $W^{1,p}(\Omega_\varepsilon)$ norm of the normalized extremals u_ε is bounded independently of ε.

The key point to handle this case is to perform a change of variables like the one that is used in the supercritical case, but now with a cut-off function φ depending on ε. Let

$$\bar{x}' = x', \qquad \bar{x}_1 = x_1 - \varepsilon f(x'/\varepsilon)\varphi_\varepsilon(x_1, x'),$$

where φ_ε is a smooth cut-off function supported in A_ε such that $\varphi_\varepsilon \equiv 1$ on $\partial\Omega_\varepsilon \cap A_\varepsilon$. For $u \in C^1(\overline{\Omega_\varepsilon})$ consider

$$u(x_1, x') = v(\bar{x}_1, \bar{x}').$$

The derivatives of v and u are related by

$$u_{x_1} = v_{\bar{x}_1}(1 - \varepsilon f(x'/\varepsilon)(\varphi_\varepsilon)_{x_1})$$

and

$$\nabla_{x'} u = -v_{\bar{x}_1}(\nabla_{x'} f(x'/\varepsilon)\varphi_\varepsilon + \varepsilon f(x'/\varepsilon)\nabla_{x'}\varphi_\varepsilon) + \nabla_{\bar{x}'} v.$$

We obtain that $v \in C^1(\overline{\Omega})$. Moreover, the $W^{1,p}(\Omega)$ norm of the rescaled extremals v_ε is bounded independently of ε. Hence we may assume, taking a subsequence if necessary, that $v_\varepsilon \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$.

Since the derivatives of φ_ε are bounded by C/ε, the jacobian of the change of variables verifies $J^{-1} = 1$ in $\Omega \setminus A_\varepsilon$ and $J^{-1} \le C$ in A_ε. Since f has bounded derivatives, the derivatives of φ_ε are bounded by C/ε and the measure of A_ε is of order ε, we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla\theta\,dx = \int_\Omega |\nabla v|^{p-2}\nabla v\nabla\theta\,d\bar{x} \tag{2.4}$$

and

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |u_\varepsilon|^{p-2}u_\varepsilon\theta\,dx = \int_\Omega |v|^{p-2}v\theta\,d\bar{x}. \tag{2.5}$$

Concerning the boundary term we have

$$\begin{aligned}\int_{\partial\Omega_\varepsilon\cap U} |u_\varepsilon|^{q-2}u_\varepsilon\theta\,dS = &\int_{U'\setminus U'_\varepsilon} |v_\varepsilon|^{q-2}v_\varepsilon\theta\sqrt{1+|\nabla\Phi(\bar{x}')|^2}\,dx' \\ &+\int_{U'_\varepsilon} |v_\varepsilon|^{q-2}v_\varepsilon\theta\sqrt{1+|\nabla\Phi(\bar{x}')+\nabla f(\frac{\bar{x}'}{\varepsilon})|^2}\,dx'.\end{aligned}$$

When $\varepsilon\to 0$, we get that $U'_\varepsilon\to U'$ and

$$\sqrt{1+|\nabla\Phi(\bar{x}')+\nabla f(\frac{\bar{x}'}{\varepsilon})|^2} \overset{*}{\rightharpoonup} \int_Y \sqrt{1+|\nabla\Phi(x')+\nabla f(y)|^2}\,dy,$$

$*$-weakly in $L^\infty(U')$. Therefore,

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon\cap U} |u_\varepsilon|^{q-2}u_\varepsilon\theta\,dS = \int_{U'} |v|^{q-2}\,v\,\theta\,m(\bar{x}')\sqrt{1+|\nabla\Phi(\bar{x}')|^2}\,dx',$$

where m is given by (1.7). Hence, we get

$$\int_{\partial\Omega_\varepsilon} |u_\varepsilon|^{q-2}u_\varepsilon\theta\,dS \to \int_{\partial\Omega} |v|^{q-2}v\,\theta\,m(x)\,dS. \tag{2.6}$$

Since the extremals u_ε are solutions to (1.3), they satisfy for every $\theta\in C^\infty(\mathbb{R}^N)$

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla\theta\,dx + \int_{\Omega_\varepsilon} |u_\varepsilon|^{p-2}u_\varepsilon\theta\,dx = S(\varepsilon)\int_{\partial\Omega_\varepsilon} |u_\varepsilon|^{q-2}u_\varepsilon\theta\,dS. \tag{2.7}$$

Using (2.4), (2.5) and (2.6) we obtain that a weak limit of the sequence v_ε in $W^{1,p}(\Omega)$ satisfies

$$\int_\Omega |\nabla v|^{p-2}\nabla v\nabla\theta\,dx + \int_\Omega |v|^{p-2}v\theta\,dx = \bar{S}\int_{\partial\Omega} m|v|^{q-2}v\theta\,dS.$$

That is to say that v is a weak solution to

$$\begin{cases} \Delta_p v = |v|^{p-2}v & \text{in } \Omega,\\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial\nu} = \bar{S}m(x)|v|^{q-2}v & \text{on } \partial\Omega.\end{cases}$$

Therefore

$$\int_\Omega |\nabla v|^p + v^p\,dx = \bar{S}\int_{\partial\Omega} m(x)|v|^q\,dS.$$

Moreover, from our previous calculations, we have

$$1 = \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} |u_\varepsilon|^q\,dS = \int_{\partial\Omega} m(x)\,|v|^q dS.$$

Now, for every $w \in W^{1,p}(\Omega)$ we define $w_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$ thanks to the change of variables $w_\varepsilon(x) = w(\bar{x})$. These w_ε verify

$$S(\varepsilon) \int_{\partial\Omega_\varepsilon} |w_\varepsilon|^q \, dS \le \int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^p + |w_\varepsilon|^p \, dx.$$

Taking limits in the above inequality, we arrive to

$$\bar{S} \int_{\partial\Omega} m(x)|w|^q \, dS \le \int_{\Omega} |\nabla w|^p + |w|^p \, dx.$$

But, v is an extremal for this inequality. We conclude that $\bar{S} = S^*$ (given by (1.6)) and that v is an extremal. This proves that

$$\lim_{\varepsilon \to 0} S(\varepsilon) = S^*.$$

Moreover, we have proved that the rescaled extremals v_ε converge weakly to v in $W^{1,p}(\Omega)$. □

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