A non-existence result for the reinforced membrane problem

Zhiwei Cheng[a], Julian Fernandez Bonder[b], Hayk Mikayelyan[c,*]

[a] *School of Statistics and Mathemtices, Zhejiang Gongshang University, 18 Xuezheng Street, Qiantang District, Hangzhou 310018, Zhejiang, PR China*

[b] *Departamento de Matemática, FCEyN - Universidad de Buenos Aires and IC - CONICET, Ciudad Universitaria, $0+\infty$ Building (1428) Av. Cantilo s/n. Buenos Aires, Argentina*

[c] *Mathematical Sciences, University of Nottingham Ningbo China, 199 Taikang East Road, Ningbo 315100, Zhejiang, PR China*

Abstract

The paper deals with the minimization problem considered by Henrot and Maillot in [1], known as the *reinforced membrane problem*:

$$F(E) := \int_\Omega f u_E \, dx \mapsto \min \qquad \text{over} \qquad E \subset \Omega,\ |E| = \beta,$$

where $u_E \in W_0^{1,2}(\Omega)$ models the displacement of the membrane reinforced on a subset E

$$-\Delta u_E(x) + \chi_E u_E(x) = f(x)$$

and $F(E)$ is the work done by the given loading f. The non-existence of an optimal reinforcement shape $E \subset \Omega$ is proven under rather general conditions.

In addition, a counterexample for the radially symmetric decreasing right-hand side is presented, which rules out the possibility of a simple existence/non-existence alternative depending on the value of $|E| = \beta$ even in the radially symmetric situation.

Keywords: reinforced membrane problem, optimal shape
2020 MSC: 49Q10, 74P05, 35J15

1. Introduction

We consider the following shape optimization problem: let u_E model the displacement of a composite membrane over a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, subject to the loading f, which is reinforced on a subset $E \subset \Omega$ of higher stiffness, where the reinforcement is modeled by the following Dirichlet boundary value problem

$$\begin{cases} -\Delta\ u_E(x) + \chi_E u_E(x) = f(x) & \text{in } \Omega, \\ u_E(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

*Corresponding author.
Email addresses: zhiweicheng@ustc.edu.cn (Zhiwei Cheng), jfbonder@dm.uba.ar (Julian Fernandez Bonder), hayk.mikayelyan@nottingham.edu.cn (Hayk Mikayelyan)

Further, let

$$F(E) = \int_\Omega |\nabla u_E|^2 + \chi_E u_E^2 \, dx = \int_\Omega f u_E \, dx$$

be the work done by the loading f.

The minimization problem

$$F(E) \mapsto \min \qquad \text{over} \qquad E \subset \Omega, \ |E| = \beta, \tag{2}$$

models the optimal shape of the membrane reinforcement problem, where the unknown set E has a given volume.

This problem introduced in [1] is a version of a broader class of optimal shape problems and their relaxed versions considered by various authors (see [2, 3, 4, 5, 6, 7]). This problem is particularly interesting due to the failure of the lower semi continuity of F for the γ-convergence, which makes the application of the classical existence result by Buttazzo and Dal Maso possible (see [8]).

The relaxed version of the problem is the following: let

$$F(l) = \int_\Omega |\nabla u_l|^2 + l u_l^2 \, dx = \int_\Omega f u_l \, dx \tag{3}$$

where u_l is the unique solution of the Dirichlet boundary value problem with nonnegative function $f(x)$

$$\begin{cases} -\Delta \ u_l(x) + l u_l(x) = f(x) & \text{in } \Omega, \\ u_l(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

and l belongs to the set

$$\bar{\mathcal{R}}_\beta = \left\{ l \in L^\infty(\Omega) \colon 0 \le l \le 1, \int_\Omega l \, dx = \beta \right\},$$

where $\bar{\mathcal{R}}_\beta$ is the weak*-closure of the rearrangement class

$$\mathcal{R}_\beta = \{ l = \chi_E \colon E \subset \Omega, \ |E| = \beta \}.$$

For the minimization problem

$$F(l) \mapsto \min \qquad \text{over} \qquad \bar{\mathcal{R}}_\beta, \tag{5}$$

Henrot and Maillot in [1] have proven the following two theorems for $N = 2$ and 3. However, one can easily check that same proofs work in higher dimensions, too. One just needs to use different argument to obtain the continuity of function u^* in the Remark 2.4 in [1] (see [9]).

The first theorem establishes a necessary condition on the function f to guarantee the existence of the optimal shape $\hat{E}$ given by $\hat{l} = \chi_{\hat{E}} \in \mathcal{R}_\beta$.

Theorem A. Let $f \ge 0$ be bounded and $u_\infty \in W_0^{1,2}(\Omega)$ solve the problem

$$-\Delta u_\infty = f \text{ in } \Omega.$$

If one of the conditions below holds

(i) $u_\infty \le f$ in Ω,
(ii) $f \le -\Delta f$ in Ω,
(iii) $\beta > |\{x \in \Omega \colon u_\infty(x) > \alpha\}|$, where $\alpha = \inf\{f(x) \colon x \text{ such that } u_\infty(x) > f(x)\}$,

then the minimization problem (5) has a unique minimizer, which is a characteristic function $\chi_E \in \mathcal{R}_\beta$.

The second theorem in [1] gives a necessary and sufficient condition for a minimizer.

Theorem B. Let $\hat{u}$ solve (4) with a design function $\hat{l} \in \bar{\mathcal{R}}_\beta$, and

$$\Omega_0 = \left\{x \in \Omega \colon \hat{l}(x) = 0\right\}, \quad \Omega_* = \left\{x \in \Omega \colon 0 < \hat{l}(x) < 1\right\}, \quad \Omega_1 = \left\{x \in \Omega \colon \hat{l}(x) = 1\right\}.$$

Then $\hat{l}$ minimizes J if and only if the following two conditions hold

(i) $\gamma_{\hat{l}} = \sup_{x\in\Omega_0} \hat{u}(x) = \inf_{x\in\Omega_1} \hat{u}(x)$.
(ii) If $|\Omega_*| > 0$, then $\hat{u}(x) = \gamma_{\hat{l}}$ a.e. in Ω_*.

Ideally, we would like to find a necessary and sufficient condition on the function f for the existence of an optimal shape (i.e., the minimizer is a characteristic function), and it would be a stronger version of Theorem A above.

In this paper we obtain a negative result in this direction. More precisely, in Section 2 we provide a general non-existence result: under natural conditions the minimizer cannot be characteristic. Then, in Section 3, we construct an explicit counterexample in the radially symmetric case, which shows that even when f is radial and radially decreasing the minimizer may fail to be characteristic for values of β below certain threshold. This indicates that finding a full necessary and sufficient condition on f for the existence of optimal shapes is a highly delicate problem.

2. A non-existence result for the optimal shape

This section is devoted to our main general non-existence result. We show that, under suitable assumptions on f and α, the minimizer of problem (4) cannot be characteristic. The proof relies on a careful comparison between the minimizer of a convex functional and the solution of a normalized obstacle problem.

Let $f \in L^\infty(\Omega)$, $f \ge 0$ in Ω and

$$-\Delta u_\infty = f, \quad u_\infty \in W_0^{1,2}(\Omega).$$

Further, for $0 < \alpha < M = \max u_\infty$ let v_α solve the normalized obstacle problem

$$\Delta v = \chi_{\{v>0\}} f, \tag{6}$$

with constant boundary condition $v = \alpha$ on $\partial\Omega$.

Theorem 2.1. *If $0 < f < \alpha$ in $\{v_\alpha = 0\}$, then the solution to the minimization problem (4) with $\beta = \alpha^{-1} \int_{\{v_\alpha=0\}} f(x)\,dx$ is non-characteristic.*

Proof. Inspired by Theorem B let us define the auxiliary function $u_\alpha \in W_0^{1,2}(\Omega)$ as the unique minimizer of the convex functional

$$J_\alpha(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + H_\alpha(u) - fudx \tag{7}$$

where $H_\alpha(t) = \frac{1}{2}\chi_{\{t>\alpha\}}(t^2 - \alpha^2)$. Then u_α satisfies the inequalities

$$\chi_{\{u>\alpha\}}u \leq \Delta u + f \leq \chi_{\{u\geq\alpha\}}u. \tag{8}$$

Let us figure out under which conditions $\alpha - v_\alpha = u_\alpha$. To this end, let $\tilde{u} = \alpha - v_\alpha$, then

$$\Delta\tilde{u} + f = \chi_{\{\tilde{u}\geq\alpha\}}f,$$

and we will have $\tilde{u} = u_\alpha$ if and only if

$$0 = \chi_{\{\tilde{u}>\alpha\}}\tilde{u} \leq \chi_{\{\tilde{u}\geq\alpha\}}f \leq \chi_{\{\tilde{u}\geq\alpha\}}\tilde{u} = \chi_{\{\tilde{u}=\alpha\}}\alpha. \tag{9}$$

This leads to the condition

$$0 \leq \chi_{\{v_\alpha=0\}}f \leq \chi_{\{v_\alpha=0\}}\alpha. \tag{10}$$

For the function u_α, we have

$$-\Delta u_\alpha + \chi_{\{v_\alpha=0\}}f = f$$

or

$$-\Delta u_\alpha + \chi_{\{u_\alpha=\alpha\}}\frac{f(x)}{\alpha}u_\alpha = f.$$

Since the function

$$l(x) = \chi_{\{u_\alpha=\alpha\}}\frac{f(x)}{\alpha}$$

satisfies the conditions of Theorem B above, i.e., $0 < l(x) < 1$ in $\{u_\alpha = \alpha\}$, $l(x) = 0$ in $\{u_\alpha < \alpha\}$, and $\{u_\alpha > \alpha\} = \emptyset$. Thus, the non-characteristic function $l(x) \in \bar{\mathcal{R}}_\beta \setminus \mathcal{R}_\beta$ is the solution of the minimization problem (3)-(4) with $\beta = \alpha^{-1}\int_{\{v_\alpha=0\}} f(x)dx$. $\square$

Corollary 2.1. *If $0 \leq f \leq m < M$, in Ω, then the solution of the minimization problem* (4) *is non-characteristic for all*

$$0 < \beta < m^{-1}\int_{\{v_m=0\}} f(x)dx,$$

where v_m is defined in (6).

3. Radially symmetric case

The result of the previous section shows that non-existence of optimal shapes can occur for small β under pretty general conditions. A natural question is whether such pathologies may be avoided under additional structure, for instance, in the radially symmetric case. In [1], it was proven that in B_1, if f is radially symmetric and radially decreasing, then the minimizer is characteristic for arbitrary β. One may then wonder whether this result extends to larger radii and specific values of β, and thus the following question becomes natural:

Question. Let u_α be the minimizer of (7) in $B_R \subset \mathbb{R}^2$, where $f \geq 0$ is radially decreasing. Is it true that for $0 < \alpha < M$:

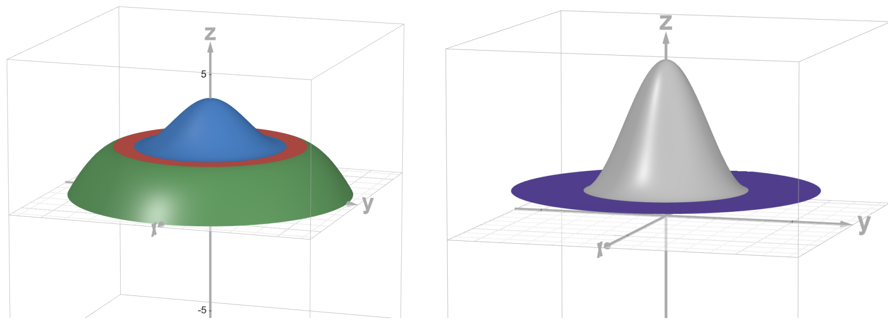

Figure 1: Functions u on the left and f on the right in the construction of the counterexample

- either $\operatorname{int}\{u = \alpha\} \neq \emptyset$ and $\{u > \alpha\} = \emptyset$
- or $\operatorname{int}\{u = \alpha\} = \emptyset$ and $\{u > \alpha\} \neq \emptyset$?

In this section, we answer this question negatively by constructing an explicit counterexample. Consider a radially symmetric function in $B_\pi(0) \subset \mathbb{R}^2$

$$u(x, y) = u(r) = 3 + \cos r.$$

Then

$$-\Delta u + u = -(u'' + \frac{1}{r}u') + u = 3 + 2\cos r + \frac{\sin r}{r} =: f(r)$$

is a decreasing function in $[0, \pi]$.

Let us now choose $R_4 > 4$ such that the solution U_4 of

$$\Delta U = \chi_{\{U>0\}}$$

with boundary values $U_4 = 2$ on ∂B_{R_4}, has the coincidence set $\{U_4 = 0\} = B_4$.

We can now define

$$u(r) = \begin{cases} 3 + \cos r & \text{for } 0 \leq r \leq \pi \\ 2 & \text{for } \pi \leq r \leq 4 \\ 2 - U_4(r) & \text{for } 4 \leq r \leq R_4, \end{cases}$$

and

$$f(r) = \begin{cases} 3 + 2\cos r + \frac{\sin r}{r} & \text{for } 0 \leq r \leq \pi \\ 1 & \text{for } \pi \leq r \leq R_4. \end{cases}$$

See Figure 1. One can observe that for f above, the function u is the minimizer of (7), by checking the condition (8).

On the other hand, for the same choice of f, the function

$$l(r) = \begin{cases} 1 & \text{for } 0 \leq r \leq \pi \\ 1/2 & \text{for } \pi \leq r \leq 4 \\ 0 & \text{for } 4 \leq r \leq R_4, \end{cases}$$

and $u(r)$ above satisfy (4), as well as the necessary and sufficient conditions of Theorem B. Thus, l is the non-characteristic minimizer of the functional (3).

Acknowledgments

Julian Fernandez Bonder is partially supported by UBACYT Prog. 2018 20020170100445BA, CONICET PIP 11220210100238CO and ANPCyT PICT 2019-03837 and PICT 2019-00985.

References

[1] A. Henrot, H. Maillot, Optimization of the shape and the location of the actuators in an internal control problem, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 4 (3) (2001) 737–757.

[2] J.-L. Lions, Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles, Dunod, Paris; Gauthier-Villars, Paris, 1968, avant propos de P. Lelong.

[3] G. Buttazzo, G. Dal Maso, Shape optimization for Dirichlet problems: relaxed formulation and optimality conditions, Appl. Math. Optim. 23 (1) (1991) 17–49. doi:10.1007/BF01442391. URL `https://doi.org/10.1007/BF01442391`

[4] R. V. Kohn, G. Strang, Optimal design and relaxation of variational problems. I, Comm. Pure Appl. Math. 39 (1) (1986) 113–137. doi:10.1002/cpa.3160390107. URL `https://doi.org/10.1002/cpa.3160390107`

[5] Z. Cheng, H. Mikayelyan, Optimization of the shape for a non-local control problem, Fract. Calc. Appl. Anal. 27 (5) (2024) 2482–2501. doi:10.1007/s13540-024-00318-9. URL `https://doi.org/10.1007/s13540-024-00318-9`

[6] M. Chipot, G. Dal Maso, Relaxed shape optimization: the case of nonnegative data for the Dirichlet problem, Adv. Math. Sci. Appl. 1 (1) (1992) 47–81.

[7] A. Henrot, M. Pierre, Variation et optimisation de formes, Vol. 48 of Mathématiques & Applications (Berlin) [Mathematics & Applications], Springer, Berlin, 2005, une analyse géométrique. [A geometric analysis]. doi:10.1007/3-540-37689-5. URL `https://doi.org/10.1007/3-540-37689-5`

[8] G. Buttazzo, G. Dal Maso, An existence result for a class of shape optimization problems, Arch. Rational Mech. Anal. 122 (2) (1993) 183–195. doi:10.1007/BF00378167. URL `https://doi.org/10.1007/BF00378167`

[9] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Classics in Mathematics, Springer-Verlag, Berlin, 2001, reprint of the 1998 edition.