THE BEST CONSTANT AND EXTREMALS OF THE SOBOLEV EMBEDDINGS IN DOMAINS WITH HOLES: THE L^∞ CASE.

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We study the best constant of the Sobolev trace embedding $W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\partial\Omega)$ for functions that vanish in a subset $A \subset \Omega$, which we call the hole. That is, we deal with the minimization problem $S_A^T = \inf \|u\|_{W^{1,\infty}(\Omega)}/\|u\|_{L^\infty(\partial\Omega)}$ for functions that verify $u\,|_A = 0$. We find that there exists an optimal hole that minimizes the best constant S_A^T among subsets of Ω of prescribed volume and we give a geometrical characterization of this optimal hole. In fact, minimizers associated to these holes are cones centered at some points x_0^* on $\partial\Omega$ with respect to the arc-length metric in Ω and the best holes are of the form $A^* = \Omega \setminus B_d(x_0^*, r^*)$ where the ball is taken again with respect of the arc-length metric.

A similar analysis can be performed for the best constant of the embedding $W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\Omega)$ with holes. In this case we also find that minimizers associated to optimal holes are cones centered at some points x_0^* on $\partial\Omega$ and the best holes are of the form $A^* = \Omega \setminus B_d(x_0^*, r^*)$.

1. Introduction

Sobolev inequalities are relevant for the study of boundary value problems for differential operators. They have been studied by many authors and it is by now a classical subject. It at least goes back to [2], for more references see [6]. In particular, the Sobolev trace inequality has been intensively studied in [4, 7, 8, 9, 12, 18, 20, 21], etc.

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. In this paper we want to study the best constant and extremals for the embeddings $W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\partial\Omega)$ and $W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\Omega)$ restricted among functions that vanish in a subset A of Ω. Note that functions $u \in W^{1,\infty}(\Omega)$ are Lipschitz and therefore they have a Lipschitz extension to $\overline{\Omega}$.

First, we deal with the trace embedding. To this end, for any function $u \in W^{1,\infty}(\Omega)$, we define the associated Rayleigh quotient

$$Q^T(u) = \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\partial\Omega)}}.$$

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For $A \subset \Omega$ we let

$$S_A^T := \inf \left\{ Q^T(u) \colon u \in W^{1,\infty}(\Omega) \text{ s.t. } u \not\equiv 0 \text{ on } \partial\Omega, u = 0 \text{ in } A \right\}.$$

This constant S_A^T is the best Sobolev trace constant for the embedding $W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\partial\Omega)$ restricted to functions that vanish on a subset A of Ω. Since we are dealing with continuous functions we can assume that the set A is closed (otherwise just consider the closure of A).

Variational problems in L^∞ have been recently considered, due to several mathematical difficulties that are involved and where new phenomena have been observed, see for example, [1, 3, 17] and references therein. In particular, L^∞ problems have been obtained as limits as $p \to \infty$ of L^p problems, see [10, 15, 17, 19]. In those papers a PDE approach is used and the notion of viscosity solutions play a key role in most of them. However, in this paper we will not use any PDE nor take the limit as $p \to \infty$, but we use a more direct and geometric approach, taking advantage of the fact that the L^∞ norm gives pointwise information.

Optimization problems for minima of Rayleigh quotients have been extensively studied in the literature due to many applications in several branches of applied mathematics and engineering, especially in optimal design problems, see the survey [16]. Optimal design problems are usually formulated as problems of the minimization of the energy stored in the design under a prescribed loading. For applications to engineering of the optimization for Steklov eigenvalues, see [5].

In view of the above discussion, we consider the following optimization problem:

> *For a fixed $0 < \alpha < |\Omega|$, find a set A^* of measure α that minimizes S_A^T among all measurable subsets $A \subset \Omega$ of measure α. That is,*
>
> $$S^T(\alpha) := \inf_{A \subset \Omega, |A| = \alpha} S_A^T = S_{A^*}^T.$$

In this paper we prove that there exist optimal holes A^* (with their corresponding extremals u^*) for this optimization problem.

This optimization problem in $W^{1,p}(\Omega)$ has been considered recently. In fact, in [13] the existence of an optimal hole for the trace embedding has been established, see also [11] for numerical computations. Then, in [14], the interior regularity of optimal holes was analyzed.

Once existence of an optimal hole is proved, a natural question is what can be said about the extremals u^* and the optimal holes $A^* = \{u^* = 0\}$.

Here we prove that minimizers associated to optimal holes are cones centered at some point x_0^* on $\partial\Omega$ with respect to the arc-length metric in Ω and the best holes are of the form $A^* = \Omega \setminus B_d(x_0^*, r^*)$ where the ball is considered with respect to the arc-length metric. Moreover, we find a geometrical characterization of an optimal hole (and its corresponding extremal).

Recall that the arc-length metric in Ω, that we will call $d(x,y)$, is defined by the infimum of the lengths of rectifiable curves in $\overline{\Omega}$ that join x and y. Therefore, the cone centered at y with slope $1/t$ with respect to this metric is given by

$$C_{y,t}(x) := (1 - t^{-1} d(x,y))_+,$$

where $(z)_+$ denotes the positive part of z, i.e., $(z)_+ = z$ if $z > 0$ and $(z)_+ = 0$ otherwise. Observe that for convex domains the arc-length metric coincides with the euclidian metric, $d(x,y) = |x-y|$ and hence the cones are given by

$$C_{y,t}(x) := \Big(1 - \frac{|x-y|}{t}\Big)_+.$$

To give the geometrical characterization of optimal holes, note that for any $x_0 \in \partial\Omega$ there exists a unique radius $r = r(x_0)$ defined by $|\Omega \setminus B_d(x_0, r)| = \alpha$.

Our main result for the trace embedding reads as follows:

Theorem 1.1. *There exists an optimal hole A^* in the sense that it minimizes S_A^T among subsets of Ω with measure α.*

Moreover, every optimal hole is of the form $A^* = \Omega \setminus B_d(x_0^*, r^*)$, with x_0^* such that

$$r^* = r(x_0^*) = \max_{x_0 \in \partial\Omega} r(x_0),$$

and the corresponding extremal is the cone

$$u^*(x) = C_{x_0^*, r^*}(x) = (1 - (r^*)^{-1} d(x, x_0^*))_+.$$

Note that for any $u \in W^{1,\infty}(\Omega)$ it holds that

$$S^T(|\{u = 0\}|) \|u\|_{L^\infty(\partial\Omega)} \le \|u\|_{W^{1,\infty}(\Omega)}$$

Remark that this inequality is sharp. The function $S^T(\alpha)$ can be computed using our result. In fact,

$$S^T(\alpha) = \frac{1}{r^*} + 1, \qquad r^* = r^*(\alpha, \Omega).$$

In some cases this r^* can be computed explicitly. For example, let Ω be the unit cube in $\mathbb{R}^2$, $\Omega = [0,1]^2$. It is clear that the vertex of an optimal cone must be located at one corner of the square. Then we easily obtain,

$$r^* = 2\sqrt{\frac{1-\alpha}{\pi}}, \qquad \text{if } \alpha \ge 1 - \frac{\pi}{4},$$

while r^* is given implicitly by

$$\sqrt{(r^*)^2 - 1} + \int_{\sqrt{(r^*)^2-1}}^{1} \sqrt{(r^*)^2 - x^2}\, dx = 1 - \alpha, \qquad \text{if } \alpha < 1 - \frac{\pi}{4}.$$

In this case it is also clear that there exist exactly four optimal holes for each α.

Now, we can perform a similar analysis for the usual Sobolev embedding $W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\Omega)$ with holes. Let

$$Q(u) = \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\Omega)}}.$$

For $A \subset \Omega$ we let

$$S_A := \inf \left\{ Q(u) \colon u \in W^{1,\infty}(\Omega) \text{ s.t. } u \not\equiv 0 \text{ in } \Omega, u = 0 \text{ in } A \right\}.$$

This constant S_A is the best constant for $W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\Omega)$ restricted to functions that vanish on a subset A of Ω.

Theorem 1.2. *There exists an optimal hole A^* in the sense that it minimizes S_A among subsets with measure α.*

Moreover, the same conclusion as in Theorem 1.1 holds. The best holes are complements of balls centered at x_0^* on the boundary and the best functions are cones.

Organization of the paper: In Section 2 we deal with the Sobolev trace embedding and in Section 3 we briefly explain the main arguments for the Sobolev embedding.

2. The best Sobolev trace constant

As we have mentioned in the introduction, for $u \in W^{1,\infty}(\Omega)$ we define

$$Q^T(u) = \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\partial\Omega)}}$$

and for $A \subset \Omega$,

$$S_A^T := \inf \left\{ Q^T(u) \colon u \in W^{1,\infty}(\Omega) \text{ s.t. } u \not\equiv 0 \text{ on } \partial\Omega, u = 0 \text{ in } A \right\}.$$

Our first lemma shows that S_A^T is attained.

Lemma 2.1. *Consider a fixed hole $A \subset \Omega$, with $|A| = \alpha$. Then there exists $u \in W^{1,\infty}(\Omega)$ that minimizes S_A^T.*

Proof. Consider a minimizing sequence $u_n \in W^{1,\infty}(\Omega)$. We can assume that $\|u_n\|_{L^\infty(\partial\Omega)} = 1$, if not, just consider the normalized sequence $v_n = \frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}}$.

Then our sequence u_n is bounded in $W^{1,\infty}(\Omega)$, as $\|u_n\|_{W^{1,\infty}(\Omega)} \leq S_A^T + 1$ for n large. Therefore, using that the embedding $W^{1,\infty}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact, we can extract a subsequence (that we still call u_n) such that

$$u_n \to u$$

weakly-* in $W^{1,\infty}(\Omega)$ and uniformly in $\overline{\Omega}$.

By the weak-* convergence we have

$$\|\nabla u\|_{L^\infty(\Omega)} \leq \liminf \|\nabla u_n\|_{L^\infty(\Omega)},$$

and by the uniform convergence up to the boundary

$$\|u_n\|_{L^\infty(\Omega)} \to \|u\|_{L^\infty(\Omega)} \quad \text{and} \quad \|u_n\|_{L^\infty(\partial\Omega)} \to \|u\|_{L^\infty(\partial\Omega)} .$$

Therefore $\|u\|_{L^\infty(\partial\Omega)} = 1$, $u = 0$ in A and

$$Q^T(u) \le \liminf Q^T(u_n).$$

It follows that u is a minimizer of S^T_A. □

Next we want to show the existence of an optimal hole A^* for S^T_A. For this we define

$$S^T(\alpha) = \inf_{A\subset\Omega, |A|=\alpha} S^T_A. \tag{2.1}$$

Note that $S^T(\alpha)$ also has the following variational characterization

$$S^T(\alpha) = \inf\{Q^T(u)\colon u \in W^{1,\infty}(\Omega),\ |\{u = 0\}| \ge \alpha, u \not\equiv 0 \text{ on } \partial\Omega\}.$$

The next result shows that there exists an optimal hole.

Theorem 2.1. *There exists a hole A^* with $|A^*| \ge \alpha$ such that $S^T_{A^*} = S^T(\alpha)$.*

Proof. Our problem is to find extremals for (2.1).

If we consider sets A with $|A| \ge \alpha$ we only extend our number of test functions and therefore

$$S^T(\alpha) = \inf\left\{S^T_A \text{ with } A \subset \Omega, |A| \ge \alpha\right\}.$$

Further note that we can always restrict ourselves to nonnegative test functions by replacing u by $|u|$.

So let A_n be a minimizing sequence for $S^T(\alpha)$ with extremals u_n normalized with $\|u_n\|_{L^\infty(\partial\Omega)} = 1$. Like in the proof of the previous lemma we can assume that u_n converges weakly-* in $W^{1,\infty}(\Omega)$ and uniformly in $\overline{\Omega}$ to a function $u \in W^{1,\infty}(\Omega)$ with $\|u\|_{L^\infty(\partial\Omega)} = 1$.

Now we have to consider the limiting set of the sequence of holes A_n. Since the characteristic functions of A_n are bounded in $L^\infty(\Omega)$ we can extract a subsequence such that $\chi_{A_n} \rightharpoonup^\star \phi$ with $0 \le \phi \le 1$. So that in particular, for $A = \{\phi > 0\}$ we have

$$|A| \ge \int_\Omega \phi = \lim \int_\Omega \chi_{A_n} = \lim |A_n| \ge \alpha.$$

Since $u \ge 0$, $\phi \ge 0$ and

$$\int_\Omega u\phi = \lim \int_\Omega u_n \chi_{A_n} = 0$$

we get that u vanishes in A, where A has measure $|A| \ge \alpha$. Hence, u vanishes on $A^* = \overline{A}$ with $|A^*| \ge |A| \ge \alpha$. Since $u \not\equiv 0$, $\{u > 0\}$ is a nonempty open set and therefore A^* is a proper subset of $\overline{\Omega}$.

As before, the convergence of u_n to u (in different topologies) implies that

$$\liminf Q^T(u_n) \ge Q^T(u).$$

As u is an admissible function we conclude that A^* is an optimal set and that u is an extremal for $S^T(\alpha)$. $\square$

Now we want to specify properties of extremals of S_A^T. We begin with the proof of the following lemma.

Lemma 2.2. *Let $A \subset \Omega$, $|A| = \alpha < |\Omega|$ and u an extremal of S_A^T. Then u attains its maximum on the boundary of Ω.*

Proof. Let u be an optimal function of S_A^T for a hole $A \subset \Omega$ with $|A| = \alpha < |\Omega|$. Because $u \in W^{1,\infty}(\Omega)$, u is Lipschitz continuous and therefore attains a maximum in $\overline{\Omega}$. Let $x_0 \in \overline{\Omega}$ be a point where the maximum is attained

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x).$$

As before we assume that u is normalized with $u(x_0) = 1$.

We want to prove that the maximum is attained at the boundary. Assume not, that is $x_0 \in \Omega$ and

$$\|u\|_{L^\infty(\Omega)} = 1, \quad \|u\|_{L^\infty(\partial\Omega)} = k < 1.$$

Define a new function

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } u(x) \leq k, \\ k & \text{if } u(x) > k. \end{cases}$$

So $\bar{u}$ still vanishes on A, $\bar{u}(x) = u(x)$ for $x \in \partial\Omega$, $\|\nabla\bar{u}\|_{L^\infty(\Omega)} \leq \|\nabla u\|_{L^\infty(\Omega)}$ and $\|\bar{u}\|_{L^\infty(\Omega)} = k < 1 = \|u\|_{L^\infty(\Omega)}$. But then it follows that

$$Q^T(\bar{u}) = \frac{\|\nabla\bar{u}\|_{L^\infty(\Omega)} + \|\bar{u}\|_{L^\infty(\Omega)}}{\|\bar{u}\|_{L^\infty(\partial\Omega)}} < \frac{\|\nabla u\|_{L^\infty(\Omega)} + 1}{\|u\|_{L^\infty(\partial\Omega)}} = Q^T(u),$$

which is a contradiction to our assumption that u is an extremal of $Q^T(v)$. It follows that u attains its maximum on the boundary of Ω. $\square$

As the problem is posed in $W^{1,\infty}(\Omega)$ test functions are Lipschitz continuous in Ω. Therefore cones are natural candidates to evaluate the quotient $Q^T(u)$ and then to estimate the infimum S_A^T. Moreover, in the next theorem, we find that cones are extremals for S_A^T.

With the knowledge that an extremal of S_A^T attains its maximum value in a point x_0 on the boundary of Ω we can further prove that the cone with center in x_0 and radius

$$dist(x_0, A) := \min_{y \in A} d(x_0, y)$$

is an extremal for S_A^T. Recall from the introduction that the cone with vertex at y and slope $1/t$ is given by,

$$C_{y,t}(x) = \left(1 - \frac{d(x,y)}{t}\right)_+ .$$

Theorem 2.2. *Let $A \subset \Omega$, $|A| = \alpha < |\Omega|$ and u be an extremal for S_A^T. Then the cone $C_{x_0,r}$ with $x_0 \in \partial\Omega$ where $u(x_0) = \max_{x\in\overline{\Omega}} u(x)$ and $r = dist(x_0, A)$ is an extremal for S_A^T.*

Proof. Let $u \in W^{1,\infty}(\Omega)$ be an extremal of S_A^T. From Lemma 2.2 we know that u attains its maximum in a point $x_0 \in \partial\Omega$. Without loss of generality we assume $u(x_0) = 1$. Then it follows that

$$Q^T(u) = \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\partial\Omega)}} = \|\nabla u\|_{L^\infty(\Omega)} + 1.$$

Now consider an arbitrary point $y \in \partial A$. By the mean value theorem, considering paths γ that joins x_0 with y with $|\dot{\gamma}| = 1$ in the definition of $d(x_0, y)$, we get that

$$\frac{|u(x_0) - u(y)|}{d(x_0, y)} \leq |\nabla u(\xi)| \leq \|\nabla u\|_{L^\infty(\Omega)},$$

for a point $\xi = \gamma(\tau)$ between x_0 and y on the curve γ. As $u(x_0) = 1$ and $u(y) = 0$ we get

$$\frac{1}{d(x_0, y)} \leq \|\nabla u\|_{L^\infty(\Omega)},$$

for every $y \in \partial A$, and hence

$$\frac{1}{r} = \frac{1}{dist(x_0, A)} \leq \|\nabla u\|_{L^\infty(\Omega)}.$$

It follows that

$$\frac{1}{r} + 1 \leq Q^T(u). \tag{2.2}$$

On the other hand, choose as a test function $v = C_{x_0,r}$. Note that $v(x_0) = \max v(x) = 1$. We obtain

$$Q^T(u) \leq Q^T(v) = \frac{\|\nabla v\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)}}{\|v\|_{L^\infty(\partial\Omega)}}.$$

Since v is a cone it follows that

$$\|\nabla v\|_{L^\infty(\Omega)} = \frac{1}{r}.$$

Therefore,

$$Q^T(u) \leq \frac{\frac{1}{r} + 1}{1} = \frac{1}{r} + 1. \tag{2.3}$$

Combining (2.2) and (2.3) we get that

$$Q^T(u) = \frac{1}{r} + 1 = Q^T(C_{x_0,r}).$$

It follows that the cone $C_{x_0,r}$ is an extremal for S_A^T. $\square$

Further we want to prove that the cone defined in Theorem 1.1 is an extremal for $S^T(\alpha)$ and gives an optimal hole A^* as the complement of a ball in Ω. As we have mentioned in the introduction for any $x_0 \in \partial\Omega$ there exists a unique radius $r = r(x_0)$ defined by $|\Omega \setminus B_d(x_0, r)| = \alpha$. Observe that r is a continuous function on $\partial\Omega$.

Now we can proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. It remains to show that every optimal hole is as described in Theorem 1.1.

Let A^* be an optimal hole with measure $|A^*| \geq \alpha$. Then there exists an extremal for $S^T_{A^*}$ that is the cone $C_{x_0,r}(x)$ with $x_0 \in \partial\Omega$ and $r = dist(x_0, A^*)$.

Let $x_0^* \in \partial\Omega$ be a point such that

$$dist(x_0^*, A^*) = \sup\{dist(x, A^*) \colon x \in \partial\Omega\} =: r^*.$$

Observe that for $C_{y,t}$ we have

$$Q^T(C_{y,t}) = \frac{1}{t} + 1.$$

So, among cones, $Q^T(C_{y,t})$ is minimized when the radius t is the largest possible, that is when $y = x_0^*$ and $t = r^*$.

Now we remark that the measure of A^* is exactly α, $|A^*| = \alpha$. In fact, assume that $|A^*| > \alpha$. Then we show that the cone is not optimal, since there exists $r_0 > 0$ with $r^* < r_0 < diam(\Omega)$ such that $|\Omega \setminus B_d(x_0^*, r_0)| = \alpha$. Then

$$Q^T(C_{x_0^*,r_0}) = \frac{1}{r_0} + 1 < Q^T(C_{x_0^*,r^*}),$$

violating the minimality of $Q^T(C_{x_0^*,r^*})$. Hence $|A^*| = \alpha$.

Therefore, as A^* is an optimal hole it must be of the form

$$A^* = \Omega \setminus B_d(x_0^*, r^*).$$

Now, to end the proof, consider a normalized extremal u^* associated to an optimal hole $A^* = \Omega \setminus B_d(x_0^*, r^*)$. As u^* vanishes on A^*, attains its maximum at x_0^* and $\|\nabla u^*\|_{L^\infty(\Omega)} = 1/r^*$, u^* restricted to every line that joins x_0^* and $y \in \partial A^* \cap \Omega = \partial B_d(x_0^*, r^*) \cap \Omega$ is a linear function with slope $1/r^*$. Therefore, we conclude that

$$u^*(x) = C_{x_0^*,r^*}(x),$$

as we wanted to prove. □

3. The best Sobolev constant

Now we consider the best constant for the usual Sobolev embedding $W^{1,\infty}(\Omega) \hookrightarrow L^\infty(\Omega)$.

Like for the best Sobolev trace constant in the previous section we want to show that the cone with vertex at a point x_0^* on the boundary that maximizes the radius such that $|\Omega \setminus B_d(x_0^*, r)| = \alpha$ is an extremal for the optimization problem of minimizing

$$S(\alpha) = \inf_{A \subset \Omega,\, |A| \geq \alpha} S_A.$$

We just sketch the arguments since they are completely analogous to the previous ones. Details are left to the reader.

The existence of extremals for S_A and the existence of an optimal hole A^* can be shown in a completely analogous way as in the previous section, see Lemma 2.1 and Theorem 2.1.

Next we have that if we consider a fixed hole $A \subset \Omega$ with $|A| = \alpha < |\Omega|$ and a corresponding extremal u, then there exists an extremal for S_A of the form $C_{x_0,r}$, with $r = dist(x_0, A)$, $u(x_0) = \max_{x \in \bar{\Omega}} u(x)$. This plays a key role in the proof of Theorem 1.2, and is the analogous to Theorem 2.2 with a similar proof.

Once this result is proved the proof of Theorem 1.2 follows by the same arguments as used in the proof of Theorem 1.1.

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References

[1] G. Aronsson, M.G. Crandall and P. Juutinen, *A tour of the theory of absolutely minimizing functions.* Bull. Amer. Math. Soc., 41 (2004), 439–505.

[2] T. Aubin. *Équations différentielles non linéaires et le problème de Yamabe concernant la courbure scalaire.* J. Math. Pures et Appl., 55 (1976), 269–296.

[3] E. N. Barron and R. Jensen. *Minimizing the L^∞ norm of the gradient with an energy constraint.* Comm. Partial Differential Equations, 30(10-12), (2005), 1741–1772.

[4] R.J. Biezuner. *Best constants in Sobolev trace inequalities.* Nonlinear Analysis, 54 (2003), 575–589.

[5] A. Cherkaev and E. Cherkaeva. *Optimal design for uncertain loading condition.* Homogenization, 193–213, Ser. Adv. Math. Appl. Sci., 50, World Sci. Publishing, River Edge, NJ, 1999.

[6] O. Druet and E. Hebey. *The AB program in geometric analysis: sharp Sobolev inequalities and related problems.* Mem. Amer. Math. Soc. 160 (761) (2002).

[7] M. del Pino and C. Flores. *Asymptotic behavior of best constants and extremals for trace embeddings in expanding domains.* Comm. Partial Differential Equations, 26 (11-12) (2001), 2189–2210.

[8] J. F. Escobar. *Sharp constant in a Sobolev trace inequality.* Indiana Math. J., 37 (3) (1988), 687–698.

[9] J. Fernández Bonder, E. Lami Dozo and J.D. Rossi. *Symmetry properties for the extremals of the Sobolev trace embedding.* Ann. Inst. H. Poincaré. Anal. Non Linéaire, 21(6), (2004), 795–805.

[10] J. Fernandez Bonder, R. Ferreira and J. D. Rossi. *Uniform bounds for the best Sobolev trace constant.* Adv. Nonlinear Studies, 3(2), (2003), 181–192.

[11] J. Fernández Bonder, P. Groisman and J.D. Rossi. *Optimization of the first Steklov eigenvalue in domains with holes: a shape derivative approach.* Ann. Mat. Pura Appl., 186(2), (2007), 341–358.

[12] J. Fernández Bonder and J.D. Rossi. *Asymptotic behavior of the best Sobolev trace constant in expanding and contracting domains.* Comm. Pure Appl. Anal., 1 (3), (2002), 359–378.

[13] J. Fernández Bonder, J.D. Rossi and N. Wolanski. *Behavior of the best Sobolev trace constant and extremals in domains with holes.* Bull. Sci. Math., 130 (2006), 565-579.

[14] J. Fernández Bonder, J.D. Rossi and N. Wolanski. *Regularity of the free boundary in an optimization problem related to the best Sobolev trace constant.* SIAM J. Control Optimz., 44(5), (2005), 1614–1635.

[15] J. Garcia-Azorero, J. J. Manfredi, I. Peral and J. D. Rossi. *Steklov eigenvalues for the ∞-Laplacian.* Rend. Lincei Mat. Appl., 17(3), (2006), 199–210.

[16] A. Henrot. *Minimization problems for eigenvalues of the Laplacian.* J. Evol. Equ. 3(3), (2003), 443–461.

[17] P. Juutinen, P. Lindqvist and J. J. Manfredi, *The $\infty-$eigenvalue problem.* Arch. Rational Mech. Anal., 148 (1999), 89–105.

[18] E. Lami Dozo and O. Torne, *Symmetry and symmetry breaking for minimizers in the trace inequality.* Commun. Contemp. Math. 7(6), (2005), 727–756.

[19] A. Le, *On the first eigenvalue of the Steklov eigenvalue problem for the infinity Laplacian.* Electron. J. Diff. Eqns., 2006(111), (2006), 1–9.

[20] Y. Li and M. Zhu. *Sharp Sobolev trace inequalities on Riemannian manifolds with boundaries.* Comm. Pure Appl. Math., 50 (1997), 449–487.

[21] S. Martinez and J. D. Rossi, *Isolation and simplicity for the first eigenvalue of the p-laplacian with a nonlinear boundary condition.* Abst. Appl. Anal., 7 (5), (2002), 287–293.

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