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ON THE EXISTENCE OF EXTREMALS FOR THE SOBOLEV TRACE EMBEDDING THEOREM WITH CRITICAL EXPONENT

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Abstract

In this paper we study the existence problem for extremals of the Sobolev trace inequality $W^{1,p}(\Omega) \to L^{p_*}(\partial\Omega)$ where Ω is a bounded smooth domain in $\mathbb{R}^N$, $p_* = p(N-1)/(N-p)$ is the critical Sobolev exponent and $1 < p < N$.

1. *Introduction.*

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. Relevant for the study of boundary value problems for differential operators are the two following Sobolev inequalities. For each $1 \le q \le p(N-1)/(N-p) \equiv p_*$, we have a continuous inclusion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$, and for each $1 \le r \le pN/(N-p) \equiv p^*$, $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, hence the following inequalities hold:

$$S_q \|u\|^p_{L^q(\partial\Omega)} \le \|u\|^p_{W^{1,p}(\Omega)}, \qquad \bar{S}_r \|u\|^2_{L^r(\Omega)} \le \|u\|^p_{W_0^{1,p}(\Omega)}.$$

These inequalities are known as the Sobolev trace theorem and the Sobolev embedding theorem respectively. The best constants for these embeddings are the largest S and $\bar{S}$ such that the above inequalities hold, that is,

$$S_q = \inf_{v \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^p + |v|^p \, dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q \, d\sigma \right)^{p/q}} \tag{1.1}$$

and

$$\bar{S}_r = \inf_{v \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla v|^p \, dx}{\left(\displaystyle\int_\Omega |v|^r \, dx \right)^{p/r}}. \tag{1.2}$$

One big difference between these two quantities, is the fact that $\bar{S}_r$ is homogeneous under dilatations of the domain, that is, if we define $\mu\Omega = \{\mu x | \ x \in \Omega\}$, taking $v(x) = u(\mu x)$ in (1.2) and changing variables we get

$$\bar{S}_r(\mu\Omega) = \mu^{(rN-pr-pN)/r} \bar{S}_r(\Omega).$$

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On the other hand, S_q is not homogeneous under dilatations. In fact we have

$$S_q(\mu\Omega) = \mu^\beta \inf_{v\in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega \mu^{-p}|\nabla v|^p + |v|^p\, dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q\, d\sigma\right)^{p/q}}, \tag{1.3}$$

where $\beta = (Nq - pN + p)/q$.

For $1 \le q < p_*$ and $1 \le r < p^*$ the embeddings are compact, so we have existence of extremals, i.e. functions where the infimum is attained. These extremals are weak solutions of the following problems

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial\nu} = \lambda|u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$laplacian, $\frac{\partial}{\partial\nu}$ is the outer unit normal derivative, and

$$\begin{cases} -\Delta_p u = \lambda|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

The asymptotic behavior of $S_q(\mu\Omega)$ in expanding ($\mu \to \infty$) and contracting domains ($\mu \to 0$), was studied in [**7**] and [**11**]. In [**7**] it is proved that for expanding domains and $q > p = 2$, $S_q(\mu\Omega) \to S_q(\mathbb{R}^N_+)$. In [**11**] it is shown that

$$\lim_{\mu\to 0+} \frac{S_q(\mu\Omega)}{\mu^\beta} = \frac{|\Omega|}{|\partial\Omega|^{p/q}}. \tag{1.6}$$

The behavior of the extremals for (1.1) in expanding and contracting domains is also studied in [**7**] and [**11**]. For expanding domains, it is proved in [**7**] (again in the case $q > p = 2$) that the extremals develop a peak near a point where the mean curvature of the boundary maximizes. For contracting domains, we have that the extremals, when rescaled to the original domain as $v(x) = u(\mu x)$, $x \in \Omega$, and normalized with $\|v\|_{L^q(\partial\Omega)} = 1$, are nearly constant in the sense that

$$\lim_{\mu\to 0} v = \frac{1}{|\partial\Omega|^{1/q}} \qquad \text{in } W^{1,p}(\Omega).$$

Another big difference between the Sobolev trace theorem and the Sobolev embedding theorem arises in the behavior of extremals. Namely, if Ω is a ball, $\Omega = B(0,\mu)$, as the extremals do not change sign, from results of [**12**] the extremals for (1.2) are radial while, at least for large μ, extremals for (1.1) are not, since they develop peaking concentration phenomena as is described in [**7**].

As for the symmetry properties of the extremals of the Sobolev trace constant, it is proved in [**10**] that if Ω is a ball of sufficiently small radius, then the extremals are radial functions. Also in [**10**], the authors use this result to prove that there exists a radial extremal for the immersion $H^1(B(0,\mu)) \to L^{2_*}(\partial B(0,\mu))$ if the radius μ is small enough. See also [**1**] for other geometric conditions that leads to existence of extremals in the case $p = 2$.

So we arrive at the purpose of this article: to find existence of extremals for the Sobolev trace theorem with the critical exponent in a general smooth bounded domain Ω. Our main result is the following:

THEOREM 1. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$ such that*

$$\frac{|\Omega|}{|\partial\Omega|^{p/p_*}} < \frac{1}{K(N,p)}, \tag{1.7}$$

where $K(N,p)$ is given by (2.1). Then there exists an extremal for the immersion $W^{1,p}(\Omega) \to L^{p_*}(\partial\Omega)$.

For the proof of Theorem 1 we use the same approach as in [**2**] (see also [**16**]), properly adapted to our new context.

Existence result for elliptic problems with critical Sobolev exponents have deserved a great deal of attention since the pioneer work [**5**] and is a intensive area of research nowadays. Best Sobolev inequalities have been studied by many authors and is by now a classical subject. It at least goes back to [**2**], [**3**].

The other key ingredient in the proof is the result of [**4**] where the author compute the optimal constant in the Sobolev trace inequality (see Theorem 2 below). See also [**13**] for a similar result in the case $p = 2$. For more references on Sobolev inequalities see [**6**].

REMARK 1. Let Ω be any smooth bounded domain in $\mathbb{R}^N$ and let

$$\Omega_\mu = \mu\Omega = \{\mu x \mid x \in \Omega\},$$

where $\mu > 0$. We observe that when μ is small enough, precisely

$$\mu < \frac{1}{K(N,p)^{1/p}} \frac{|\partial\Omega|^{1/p_*}}{|\Omega|^{1/p}},$$

then Ω_μ verifies the hypotheses of Theorem 1 and hence there is an extremal for the immersion $W^{1,p}(\Omega_\mu) \to L^{p_*}(\partial\Omega_\mu)$.

REMARK 2. We observe that with the same ideas and computations, we can consider a problem of the form

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = a(x)u^{p_*-1} & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

with $a \in L^\infty(\partial\Omega)$ bounded away from zero. This corresponds to a Sobolev trace immersion with a weight, a, on the boundary. In this case the condition on Ω and the weight function a is

$$|\Omega| \left(\sup_{\partial\Omega} a\right)^{p/p_*} < \frac{1}{K(N,p)} \left(\int_{\partial\Omega} a\, d\sigma\right)^{p/p_*}.$$

REMARK 3. Observe that from the proof of Theorem 1, we obtain the existence of extremals for every domain Ω that satisfies

$$S_{p_*} = S_{p_*}(\Omega) < \frac{1}{K(N,p)}. \tag{1.9}$$

Condition (1.7) is the simplest geometric condition that ensures (1.9).

2. *Proof of Theorem 1*

For the proof of Theorem 1 we use the following Theorem due to [**4**]

THEOREM 2. *For every $\varepsilon > 0$, there exists a constant $B = B(\varepsilon) > 0$ such that,*

$$\left(\int_{\partial\Omega} v^{p_*}\, d\sigma \right)^{p/p_*} \le (K(N,p) + \varepsilon) \int_\Omega |\nabla v|^p\, dx + B \int_\Omega v^p\, dx$$

for every $v \in W^{1,p}(\Omega)$, where

$$\frac{1}{K(N,p)^p} = \inf_{\nabla w \in L^p(\mathbb{R}^N_+),\ w \in L^{p_*}(\partial\mathbb{R}^N_+)} \frac{\displaystyle\int_{\mathbb{R}^N_+} |\nabla w|^p\, dx}{\left(\displaystyle\int_{\partial\mathbb{R}^N_+} |w|^{p_*}\, dx' \right)^{p/p_*}}. \tag{2.1}$$

REMARK 4. The constant $K(N,p)$ in Theorem 2 is sharp.

Now, to prove our result, we will use the same approach used by [**2**] (see also [**16**]). Let us consider, for each $1 < q \le p_*$ the quotient

$$Q_q(u) = \frac{\displaystyle\int_\Omega |\nabla u|^p + u^p\, dx}{\left(\displaystyle\int_{\partial\Omega} u^q\, d\sigma \right)^{p/q}} \tag{2.2}$$

and

$$S_q = \inf_{u \in W^{1,p}(\Omega) \setminus W^{1,p}_0(\Omega)} Q_q(u).$$

To simplify the notation, we denote

$$Q(u) = Q_{p_*}(u) \qquad \text{and} \qquad S = S_{p_*}. \tag{2.3}$$

We have the following

LEMMA 1. *If $q < p_*$, the constant S_q is attained.*

Proof. It follows from the compactness of the immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$. □

From now on, we will denote by u_q an extremal related to S_q, normalized such that $\|u_q\|_{L^q(\partial\Omega)} = 1$.

For the proof of the Theorem, we need the following proposition, which is a straightforward modification of [**16**].

PROPOSITION 1. *Let $q_i \to r \le p_*$ and assume that*

$$\sup_{x \in \overline{\Omega}} u_{q_i}(x) \le A.$$

Then there exists a subsequence $u_{q_{i_j}}$ *such that* $u_{q_{i_j}} \to u$ *in the sense of distributions and* u *is a solution of*

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega \\ \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{r-2}u & \text{on } \partial\Omega, \end{cases}$$

where $\lambda = \lim S_{q_{i_j}}$. *Moreover* $\|u\|_{L^r(\partial\Omega)} = 1$.

The next proposition we believe that has independent interest.

Proposition 2. *The constant* S_q *is continuous with respect to* q *in* $1 \le q \le p_*$.

Proof. Let $v \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)$ be fixed. As $v \in L^{p_*}(\partial\Omega)$, it follows from the Lebesgue's dominated convergence Theorem that $Q_q(v)$ is a continuous function with respect to q.

From this fact it follows that S_q is an upper semicontinuous function for $q \in [1, p_*]$. In fact, by definition, for any $\varepsilon > 0$ there exists $v \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)$ such that $Q_q(v) < S_q + \varepsilon$. On the other hand, $S_r \le Q_r(v)$ and, as $Q_r(v) \to Q_q(v)$ as $r \to q$, it follows that

$$\limsup_{r \to q} S_r < S_q + \varepsilon, \tag{2.4}$$

as we wanted to show.

Let us now show the left continuity on $[1, p_*]$. For $1 \le r < q \le p_*$ we have

$$\left(\int_{\partial\Omega} v^r \, d\sigma\right)^{1/p} \le \left(\int_{\partial\Omega} v^q \, d\sigma\right)^{1/q} |\partial\Omega|^{\frac{1}{r}-\frac{1}{q}},$$

hence

$$Q_r(v) \ge Q_q(v)|\partial\Omega|^{\frac{1}{r}-\frac{1}{q}}$$

and then

$$S_r \ge S_q|\partial\Omega|^{\frac{1}{r}-\frac{1}{q}}. \tag{2.5}$$

Now the claim follows by taking limit in (2.5) together with (2.4).

It remains to prove the continuity of S_q in $[1, p_*)$. To this end observe that the functions u_q are uniformly bounded for $1 \le q \le q_0 < p_*$, so we can apply Proposition 1.

Assume that there exists $r \in [1, p_*)$ such that S_q is not continuous in r. Then there exists a sequence $q_i \to r$ such that $\lim_{i\to\infty} S_{q_i} = \lambda \ne S_r$. By (2.4) we must have $\lambda < S_r$, but, by Proposition 1, there exists a function u satisfying (2.6) and $\|u\|_{L^r(\partial\Omega)} = 1$ and hence $Q_r(u) = \lambda$ from where it follows that $\lambda \ge S_r$, a contradiction.

This finishes the proof. □

Now we are ready to prove the main theorem.

Proof of Theorem 1. First, we claim that there exists a sequence $q_j \nearrow p_*$ such that $u_{q_j} \to u$ weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\partial\Omega)$. Moreover, the limit u

satisfies

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = S u^{p_*-1} & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

In order to see this, we observe that

$$\|u_q\|_{W^{1,p}(\Omega)} \le S_q \le C$$

by Proposition 2 (see also [**9**]). Hence, as $W^{1,p}(\Omega) \to L^{p_*}(\partial\Omega)$ continuously,

$$\|u_q\|_{L^{p_*}(\partial\Omega)} \le C.$$

Moreover, as $p_*/(p_*-1) < p_*/(q-1)$, we have u_q^{q-1} is uniformly bounded in $L^{p_*/(p_*-1)}(\partial\Omega)$. As a consequence of these uniform bounds, there exists a sequence $q_j \nearrow p_*$ and a function $u \in H^1(\Omega)$ such that

$$\begin{array}{ll} u_{q_j} \rightharpoonup u & \text{weakly in } W^{1,p}(\Omega), \\ \\ u_{q_j} \to u & \text{strongly in } L^p(\Omega) \text{ and in } L^p(\partial\Omega), \\ \\ u_{q_j} \to u & \text{a.e. } \Omega \text{ and a.e. } \partial\Omega, \\ \\ u_{q_j}^{q_j-1} \rightharpoonup u^{p_*-1} & \text{weakly in } L^{p_*/(p_*-1)}(\partial\Omega). \end{array} \tag{2.7}$$

Now, let $\psi \in W^{1,p}(\Omega)$. We have, by (2.7),

$$\int_{\partial\Omega} u_{q_j}^{q_j-1}\psi\, d\sigma \to \int_{\partial\Omega} u^{p_*-1}\psi\, d\sigma,$$

$$\int_\Omega \nabla u_{q_j}\nabla\psi + u_{q_j}\psi\, dx \to \int_\Omega \nabla u \nabla\psi + u\psi\, dx.$$

By the continuity of S_q (Proposition 2) we have

$$\lim_{q_j \nearrow p_*} S_{q_j} = S = S_{p_*},$$

therefore $u \in H^1(\Omega)$ is a weak solution of (2.6) and the claim follows.

Now, as $u_q \ge 0$, it follows that $u \ge 0$ and, by classical regularity theory (see [**14**]), u is smooth ($C^{1,\alpha}$) up to the boundary. By the strong maximum principle and Hopf's lemma (see [**15**]), it follows that either $u > 0$ or $u \equiv 0$. So, in order to complete the proof of the theorem, we have to rule out the possibility of $u \equiv 0$. To do this, we adapt the argument given in [**2**] to show that $\|u\|_{L^p(\Omega)} \ne 0$. In fact, by Theorem 2, given $\varepsilon > 0$, there exists a constant $B = B(\varepsilon)$ such that

$$\left(\int_{\partial\Omega} v^{p_*}\, d\sigma\right)^{p/p_*} \le (K(N,p)+\varepsilon)\int_\Omega |\nabla v|^p\, dx + B\int_\Omega |v|^p\, dx$$

for every $v \in W^{1,p}(\Omega)$. Recall that u_q are normalized such that $\|u_q\|_{L^q(\partial\Omega)} = 1$, so, by Hölder's inequality,

$$\begin{aligned} |\partial\Omega|^{\frac{p}{p_*}-\frac{p}{q}}\left(\int_{\partial\Omega} u_q^q\, d\sigma\right)^{p/q} &\le \left(\int_{\partial\Omega} u_q^{p_*}\, d\sigma\right)^{p/p_*} \\ &\le (K(N,p)+\varepsilon)\int_\Omega |\nabla u_q|^p\, dx + B\int_\Omega u_q^p\, dx \end{aligned}$$

and hence

$$|\partial\Omega|^{\frac{p}{p_*}-\frac{p}{q}} \le (K(N,p)+\varepsilon)S_q + (B - K(N,p) - \varepsilon)\int_\Omega u_q^p\,dx. \tag{2.8}$$

Passing to the limit $q_j \nearrow p_*$ in (2.8) we arrive at

$$1 \le (K(N,p)+\varepsilon)S + (B - K(N,p) - \varepsilon)\int_\Omega u^p\,dx$$

therefore, if $S < K(N,p)^{-1}$ we are done choosing ε small enough. By taking $v \equiv 1$ in (1.1), we get

$$S \le \frac{|\Omega|}{|\partial\Omega|^{p/p_*}}.$$

Hence, if

$$\frac{|\Omega|}{|\partial\Omega|^{p/p_*}} < K(N,p)^{-1},$$

we have the existence of an extremal. □

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