AN OPTIMIZATION PROBLEM FOR NONLINEAR STEKLOV EIGENVALUES WITH A BOUNDARY POTENTIAL

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ABSTRACT. In this paper, we analyze an optimization problem for the first (nonlinear) Steklov eigenvalue plus a boundary potential with respect to the potential function which is assumed to be uniformly bounded and with fixed L^1-norm.

1. INTRODUCTION

In recent years a great deal of attention has been putted in optimal design problems for eigenvalues (both linear and nonlinear) due to many interesting applications. For a comprehensive description of the current developments in the field in the case of linear eigenvalues and very interesting open problems, we refer to [12]. In the nonlinear setting, we refer to the recent research papers [3, 4, 5, 7, 8, 11] and references therein.

To be precise, the eigenvalue problem that we are interested in is the following

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \mathbf{n}} + \sigma\phi|u|^{p-2}u = \lambda|u|^{p-2}u & \text{in } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $\Delta_p u$ is the usual p-Laplace operator defined as $\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$, $\mathbf{n}$ denotes the outer unit normal vector to $\partial\Omega$, $\phi \in L^\infty(\partial\Omega)$ is a nonnegative boundary potential and $\sigma > 0$ is a real parameter.

Under these hypotheses, the functional associated to (1.1) is trivially coercive, that is

$$I(u,\phi) = \int_\Omega |\nabla u|^p + |u|^p dx + \sigma \int_{\partial\Omega} \phi|u|^p d\mathcal{H}^{n-1} \geq \|u\|^p_{W^{1,p}(\Omega)}.$$

This functional is associated to (1.1) in the sense that eigenvalues λ of (1.1) are critical values of I restricted to the manifold $\|u\|_{L^p(\partial\Omega)} = 1$. See [9].

In particular, It is easy to see that the minimum value of I

$$\lambda(\sigma,\phi) = \inf\left\{ I(u,\phi) : u \in W^{1,p}(\Omega), \|u\|_{L^p(\partial\Omega)} = 1 \right\} \tag{1.2}$$

is the first (lowest) eigenvalue of (1.1). Therefore, the existence of the first eigenvalue and the corresponding eigenfunction u follows from the compact embedding $W^{1,p}(\Omega) \subset L^p(\partial\Omega)$.

In this work, we are interested in the minimization problem for $\lambda(\sigma,\phi)$ with respect to different configurations for the boundary potential ϕ. That is, given certain class of admissible potentials $\mathcal{A}$, we look for the minimum possible value of $\lambda(\sigma,\phi)$ when $\phi \in \mathcal{A}$.

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This study complements the ones started in [7]. In that paper, the authors analyzed the Steklov problem but with an interior potential and show the connections of that problem with the one considered in [11].

In this opportunity, we consider the class of uniformly bounded potentials, i.e.

$$\mathcal{A} = \{\phi \in L^\infty(\partial\Omega) \colon 0 \le \phi \le 1\}.$$

Observe that $\mathcal{A}$ is the closure of the characteristic functions in the weak* topology.

Clearly, the minimization problem in the whole class $\mathcal{A}$ has no sense since the infimum is realized with $\phi \equiv 0$. The relevant problem here is to consider the minimization among those potentials in $\mathcal{A}$ that has fixed L^1-norm. That is

$$\Lambda(\sigma, a) = \inf\left\{\lambda(\sigma, \phi) \colon \phi \in \mathcal{A}, \int_{\partial\Omega} \phi \, d\mathcal{H}^{n-1} = a\right\} \tag{1.3}$$

The first result in this paper is the existence of an optimal potential for $\Lambda(\sigma, a)$ and, moreover, it is shown that this optimal potential can be taken as the characteristic function a sub-level set D_σ of the corresponding eigenfunction. See [1, 2] for related results.

As another application we investigate the connection with the optimization problem considered in [6]. That is, given $E \subset \partial\Omega$, consider the equation

$$\begin{cases} -\Delta_p u + |u|^{p-2}u & = 0 \quad \text{in } \Omega \\ u & = 0 \quad \text{in } E \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} & = \lambda |u|^{p-2}u \quad \text{in } \partial\Omega \setminus E \end{cases} \tag{1.4}$$

whose first eigenvalue is given by

$$\lambda(\infty, E) := \inf\big\{\|u\|^p_{W^{1,p}(\Omega)} \colon \|u\|_{L^p(\partial\Omega)} = 1, u = 0, \mathcal{H}^{n-1} \text{ a.e. in } E\big\}, \tag{1.5}$$

Associated to (1.5) we have the optimal configuration problem

$$\Lambda(\infty, a) = \inf\big\{\lambda(\infty, E) : \mathcal{H}^{n-1}(E) = a\big\}. \tag{1.6}$$

Our second result shows that $\Lambda(\sigma, a) \to \Lambda(\infty, a)$ as $\sigma \to \infty$ and, moreover, the optimal configuration $\phi_\sigma = \chi_{D_\sigma}$ of $\Lambda(\sigma, a)$ converges (in the topology of L^1-convergence of the characteristic functions) to an optimal configuration of the limit problem $\Lambda(\infty, a)$.

The remaining of the paper is devoted to analyze qualitative properties of optimal configurations for $\Lambda(\sigma, a)$.

First, we consider the spherical symmetric case, that is when Ω is a ball, and in this simple case by means of symmetrization arguments we can give a full description of the optimal configurations.

Finally, we address the general problem and study the behavior of $\lambda(\sigma, \chi_D)$ for regular deformations of the set D. We employ the so–called method of Hadamard and prove differentiability of $\lambda(\sigma, \chi_D)$ with respect to regular deformations and provide a simple formula for the derivative of the eigenvalue. The main novelty of this formula is that it involves a $(n-2)-$dimensional integral along the boundary of D relative to $\partial\Omega$. Up to our knowledge, this is the first time that this type of lower-dimensional integrals were observed in this type of computations.

We want to remark that the results in this work are new even in the linear setting, $p = 2$.

2. Preliminary remarks

A simple modification of the arguments in [10] shows that, given $\phi \in \mathcal{A}$ and $\sigma > 0$, the first eigenvalue $\lambda(\sigma, \phi)$ is simple. i.e. any two eigenfunctions are multiple of each other. Therefore, there exists a unique nonnegative, normalized eigenfunction u (normalized means that $\|u\|_{L^p(\partial\Omega)} = 1$).

The purpose of this very short section is to recall some regularity properties of this eigenfunction.

First, we note that by [15], there exists $\alpha > 0$ such that $u \in C^{1,\alpha}_{loc}(\Omega)$. Now, by an usual argument, we have that $|u|$ is an eigenfunction associated to $\lambda(\sigma, \phi)$. Hence, the Harnack inequallity, c.f. [15], implies that any first eigenfunction u has constant sign and, moreover, that $u > 0$ in Ω.

Next, by the results of [14], an eigenfunction of (1.1) is continuous up to the boundary. In fact, $u \in C^\beta(\bar{\Omega})$ for some $\beta > 0$.

Summing up, we have

Proposition 2.1. *Given $\phi \in \mathcal{A}$ and $\sigma > 0$, there exists a unique nonnegative eigenfunction $u \in W^{1,p}(\Omega)$ of* (1.1) *associated to $\lambda(\sigma, \phi)$. Moreover, this eigenfunction u verifies that $u \in C^{1,\alpha}_{loc}(\Omega) \cap C^\beta(\bar{\Omega})$ for some $\alpha, \beta > 0$. Finally, $u > 0$ in Ω.*

3. Existence of optimal configurations

In this section we first establish the existence of optimal configurations for $\Lambda(\sigma, a)$. Then we analyze the limit $\sigma \to \infty$ and show the convergence to the problem $\Lambda(\infty, a)$.

Let us begin with the existence result.

Theorem 3.1. *For any $\sigma > 0$ and $0 \le a \le \mathcal{H}^{n-1}(\partial\Omega)$ there exist an optimal pair $(u, \phi) \in W^{1,p}(\Omega) \times \mathcal{A}$, which has the following properties*

(1) $u \in C^{1,\alpha}_{loc}(\Omega) \cap C(\bar{\Omega})$
(2) $\phi = \chi_D$ *where, for some s, $\{u < s\} \subset D \subset \{u \le s\}$, $\mathcal{H}^{n-1}(D) = a$*

Proof. We consider a minimizing sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset \mathcal{A}$ of (1.3) and their associated normalized eigenfunctions $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega)$.

From the reflexivity of the Sobolev space $W^{1,p}(\Omega)$, the compactness of the embeddings $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and $L^\infty(\partial\Omega)$ being a dual space, we obtain a subsequence (again denoted $\{u_k, \phi_k\}$) and $u \in W^{1,p}(\Omega)$, $\phi \in L^\infty(\partial\Omega)$ such that

$$u_k \rightharpoonup u \quad \text{in } W^{1,p}(\Omega) \tag{3.1}$$

$$u_k \to u \quad \text{in } L^p(\partial\Omega) \tag{3.2}$$

$$u_k \to u \quad \text{in } L^p(\Omega) \tag{3.3}$$

$$\phi_k \overset{*}{\rightharpoonup} \phi \quad \text{in } L^\infty(\partial\Omega) \tag{3.4}$$

From the admissibility of ϕ_k and (3.4), we get $0 \le \phi \le 1$ and $\int_{\partial\Omega} \phi \, d\mathcal{H}^{n-1} = a$. Using (3.2), we get $\|u\|_{L^p(\partial\Omega)} = 1$. As a consequence of the lower semicontinuity of the norm $\|.\|_{W^{1,p}(\Omega)}$ with respect to weak convergence, we obtain

$$\int_\Omega |\nabla u|^p + |u|^p dx \le \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^p + |u_k|^p dx \tag{3.5}$$

Using (3.2), we can see that $|u_k|^p \to |u|^p \quad \text{in } L^1(\partial\Omega)$. Therefore, taking into account (3.4) we obtain

$$\int_{\partial\Omega} \phi_k |u_k|^p d\mathcal{H}^{n-1} \to \int_{\partial\Omega} \phi |u|^p d\mathcal{H}^{n-1} \tag{3.6}$$

From (3.5) and (3.6), we have (u, ϕ) is an optimal pair for (1.3).

By an elementary variation of the *Bathtub Principle* ([14, Pag. 28]), we can prove that the minimization problem

$$\inf_{\int_{\partial\Omega} \phi d\mathcal{H}^{n-1}=a} \int_{\partial\Omega} \phi |u|^p d\mathcal{H}^{n-1},$$

has a solution of the form $\phi = \chi_D$, where $\{u < s\} \subset D \subset \{u \le s\}$ and $\mathcal{H}^{n-1}(D) = a$ and therefore (χ_D, u) is an optimal pair for $\Lambda(\sigma, a)$. □

Now we prove a Lemma about the continuity of the eigenvalues and eigenfunctions with respect to the potential ϕ in the weak * topology.

Lemma 3.2. *Let $\phi_j, \phi \in L^\infty(\partial\Omega)$ be such that $\phi_j \overset{*}{\rightharpoonup} \phi$ in $L^\infty(\partial\Omega)$. Let $\lambda_j = \lambda(\sigma, \phi_j)$ and $\lambda = \lambda(\sigma, \phi)$ the eigenvalues defined by* (1.2) *and let $u_j, u \in W^{1,p}(\Omega)$ be the positive normalized eigenfunctions associated to λ_j and λ respectively.*

Then $\lambda_j \to \lambda$ and $u_j \to u$ strongly in $W^{1,p}(\Omega)$ as $j \to \infty$.

Proof. First, define $v \equiv \mathcal{H}^{n-1}(\partial\Omega)^{-1/p}$ and from (1.2) we get

$$\lambda_j \le I(v, \phi_j) = \frac{|\Omega| + \int_\Omega \phi_j}{\mathcal{H}^{n-1}(\partial\Omega)} \le C$$

for every $j \in \mathbb{N}$. Therefore, since $\|u_j\|_{W^{1,p}(\Omega)} \le \lambda_j$ (recall that the eigenfunctions u_j are normalized) it follows that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$.

From these, we obtain the existence of a function $w \in W^{1,p}(\Omega)$ such that, for a subsequence,

$$\begin{aligned} u_j &\rightharpoonup w && \text{weakly in } W^{1,p}(\Omega) \\ u_j &\to w && \text{strongly in } L^p(\Omega) \\ u_j &\to w && \text{strongly in } L^p(\partial\Omega) \end{aligned}$$

It then follows that $w \ge 0$ and that $\|w\|_{L^p(\partial\Omega)} = 1$.

Now, from the weakly sequentially lower semicontinuity it holds

$$\lambda \le I(w, \phi) \le \liminf I(u_j, \phi) = \liminf I(u_j, \phi_j) + \sigma \int_{\partial\Omega} (\phi - \phi_j)|u_j|^p \, d\mathcal{H}^{n-1}. \tag{3.7}$$

Since $|u_j|^p \to |u|^p$ strongly in $L^1(\partial\Omega)$, it easily follows that

$$\lambda \le \liminf \lambda_j.$$

For the reverse inequality, we proceed in a similar fashion. In fact, from (1.2)

$$\lambda_j \le I(u, \phi_j).$$

Therefore

$$\limsup \lambda_j \le \lim I(u, \phi_j) = I(u, \phi) = \lambda,$$

so $\lambda_j \to \lambda$.

Finally, from (3.7), one obtains that $I(w, \phi) = \lambda$ and since there exists a unique nonnegative normalized eigenfunction associated to λ it follows that $w = u$. Moreover, again from (3.7) it is easily seen that $\|u_j\|_{W^{1,p}(\Omega)} \to \|u\|_{W^{1,p}(\Omega)}$ and so $u_j \to u$

strongly in $W^{1,p}(\Omega)$ and, since the limit is uniquely determined, the whole sequence $\{u_j\}_{j\in\mathbb{N}}$ is convergent. □

The next Lemma, that was proved in [6] gives the strict monotonicity of the quantity $\Lambda(\infty, a)$ with respect to a and will be helpful in showing the behavior of $\Lambda(\sigma, a)$ for $\sigma \to \infty$.

Lemma 3.3 (Corollary 3.7, [6]). *The function $\Lambda(\infty, \cdot)$ is strictly monotonic.*

Now we are ready to prove the convergence of $\Lambda(\sigma, a)$ to $\Lambda(\infty, a)$ as $\sigma \to \infty$.

Theorem 3.4. *If σ_j is a sequence tending to ∞ and (u_j, D_j) associated optimal pairs of* (1.3)*, then there exists a subsequence (that we still call σ_j) and an optimal pair (u, D) of the problem* (1.6) *such that $u_j \rightharpoonup u$ in $W^{1,p}(\Omega)$, $\chi_{D_j} \overset{*}{\rightharpoonup} \chi_D$ in $L^\infty(\partial\Omega)$.*

Proof. We consider $E \subset \partial\Omega$ closed such that $\mathcal{H}^{n-1}(E) = a$ and $v \in W^{1,p}(\Omega)$, $\|v\|_{L^p(\partial\Omega)} = 1$ such that $v = 0$ in E. Therefore

$$\|u_j\|^p_{W^{1,p}(\Omega)} \le I(u_j, \chi_{D_j}) = \Lambda(\sigma_j, a) \le \lambda(\sigma_j, \chi_E) \le I(v, \chi_E) = \|v\|^p_{W^{1,p}(\Omega)}$$

Hence, the sequence u_j is bounded in $W^{1,p}(\Omega)$. Therefore we can assume that there exists $u_\infty \in W^{1,p}(\Omega)$ and $\phi_\infty \in L^\infty(\partial\Omega)$ such that

$$u_j \rightharpoonup u_\infty \text{ in } W^{1,p}(\Omega) \tag{3.8}$$

$$u_j \to u_\infty \text{ in } L^p(\Omega) \tag{3.9}$$

$$u_j \to u_\infty \text{ in } L^p(\partial\Omega) \tag{3.10}$$

$$\chi_{D_j} \overset{*}{\rightharpoonup} \phi_\infty \text{ in } L^\infty(\partial\Omega) \tag{3.11}$$

From (3.10) and (3.11) we have that $\|u_\infty\|_{L^p(\partial\Omega)} = 1$, $\int_{\partial\Omega} \phi_\infty d\mathcal{H}^{n-1} = a$ and $0 \le \phi_\infty \le 1$. The rest of the proof follows in a completely analogous way, using Lemma 3.3, to [7, Theorem 1.2] □

4. Symmetry

Throughout this section we assume that Ω is the unit ball $B(0, 1)$. The goal of the section is to show that there exists an optimal pair (u, χ_D) of the problem (1.1) with D a spherical cup in $S^{n-1} = \partial\Omega$. A key tool is played by the *spherical symmetrization.*

The spherical symmetrization of a set $A \subset \mathbb{R}^n$ with respect to an axis given by a unit vector e is defined as follows: Given $r > 0$ we consider $s_r > 0$ such that $\mathcal{H}^{n-1}(A \cap \partial B(0, r)) = \mathcal{H}^{n-1}(B(re, s_r) \cap \partial B(0, r))$. We note that the sets $A \cap \partial B(0, r)$ are $\mathcal{H}^{n-1}$-measurable for almost every $r \ge 0$. Now we put:

$$A^* = \bigcup_{0\le r\le 1} B(re, s_r) \cap \partial B(0, r)$$

The set A^* is well defined and measurable whence A is a measurable set. If $u \ge 0$ is a measurable function, we define its symmetrized function u^* so that satisfies the relation $\{u^* \ge t\} = \{u \ge t\}^*$. We refer to [13] for an exhaustive study of this symmetrization. In particular, we need the following known results:

Theorem 4.1. *Let $0 \le u \in W^{1,p}(\Omega)$ and let $u*$ be its symmetrized function. Then*

(1) $u^* \in W^{1,p}(\Omega)$

(2) *u^* and u are equi-measurable, i.e. they have the same distribution function, Hence for every continuos increasing function Φ: $\int_\Omega \Phi(u^*)dx = \int_\Omega \Phi(u)dx$*
(3) *$\int_\Omega uvdx \le \int_\Omega u^* v^* dx$, for every measurable positive function v.*
(4) *In a similar way u and u^* are equimeasurable respect to the Hausdorff measure on boundary of balls. Therefore, the two previous items holds with $\partial\Omega$ and $d\mathcal{H}^{n-1}$ instead of Ω and dx, respectively.*
(5) *$\int_\Omega |\nabla u^*|^p dx \le \int_\Omega |\nabla u|^p dx$.*

With these preliminaries, we can now prove the main result of the section.

Theorem 4.2. *Let $\Omega = B(0,1)$. Then there exists an optimal pair (u, χ_E) of the problem (1.1) with E a spherical cup in $\partial\Omega$.*

Proof. Let (u, χ_D) be an optimal pair. We define $E := ((D^c)^*)^c$. Since $(D^c)^*$ is a spherical cup it follows that E is also a spherical cup.

We note that $\chi_E = 1 - (\chi_{D^c})^*$, therefore it is easy to show, from (c) in Theorem 4.1 that

$$\int_{\partial\Omega} \chi_E |u^*|^p d\mathcal{H}^{n-1} \le \int_{\partial\Omega} \chi_D |u|^p d\mathcal{H}^{n-1}.$$

We note that $\int_{\partial\Omega} |u^*|^p d\mathcal{H}^{n-1} = 1$, so u^* is an admissible function in (1.2). Moreover,

$$\int_\Omega |\nabla u^*|^p + |u^*|^p dx + \sigma \int_{\partial\Omega} \chi_E |u^*|^p d\mathcal{H}^{n-1} \le \int_\Omega |\nabla u|^p + |u|^p dx + \sigma \int_{\partial\Omega} \chi_D |u|^p d\mathcal{H}^{n-1}.$$

Consequently, (u^*, χ_E) is an optimal pair. □

5. Derivative of Eigenvalues

Henceforth we put $\Gamma := \partial\Omega$. In this section we compute derivatives of the eigenvalues $\lambda(\sigma, \chi_D)$ with respect to perturbations of the set D. We also assume that the set $D \subset \Gamma$ is the closure of a regular relatively open set.

For this purpose, we introduce the vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ supported on a narrow neighborhood of Γ with $V \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the outer normal vector. We consider the flow

$$\begin{cases} \frac{d}{dt}\Psi_t(x) & = V(\Psi_t(x)) \\ \Psi_0(x) & = x \end{cases} \tag{5.1}$$

We note that the condition $V \cdot \mathbf{n} = 0$ implies that $\Psi_t(\Gamma) = \Gamma$. From (5.1), it follows the asymptotic expansions

$$D\Psi_t = I + tDV + o(t), \tag{5.2}$$

$$(D\Psi_t)^{-1} = I - tDV + o(t), \tag{5.3}$$

$$J\Psi_t = 1 + t\mathrm{div}V + o(t). \tag{5.4}$$

Here $D\Psi_t$ and $J\Psi_t$ denote the differential matrix of Ψ_t and its jacobian, respectively. See [12].

In order to try with surface integrals, we need the following formulas whose proofs can be founded in [12]. The tangential Jacobian of Ψ_t is given by

$$J_\Gamma \Psi_t(x) = |(D\Psi(x))^{-1}\mathbf{n}| J\Psi(x) = 1 + t\mathrm{div}_\Gamma V + o(t) \quad x \in \Gamma$$

where $\mathrm{div}_\Gamma V$ is the tangential divergence operator defined by

$$\mathrm{div}_\Gamma V = \mathrm{div} V - \mathbf{n}^T DV\mathbf{n}.$$

The main result here is the following

Theorem 5.1. *Let $\sigma > 0$ be fixed and $D \subset \Gamma$ be the closure of a smooth relatively open set. Let $u \in W^{1,p}(\Omega)$ be the nonnegative normalized eigenfunction for $\lambda(\sigma, \chi_D)$.*

Then, the function $\lambda(t) := \lambda(\sigma, \chi_{D_t})$ where $D_t = \Psi_t(D)$ is differentiable at $t = 0$ and

$$\lambda'(0) = -\sigma \int_{\partial_\Gamma D} |u_0|^p (\mathbf{n}_\Gamma \cdot V) d\mathcal{H}^{n-2}$$

where $\mathbf{n}_\Gamma$ denotes the unit normal vector exterior to $\partial_\Gamma D$ relative to the tangent space of Γ.

Remark 5.2. *Observe that the results of Lemma 3.2 immediately imply the continuity of $\lambda(t)$ at $t = 0$ and also that the associated eigenfunctions u_t strongly converge to the associated eigenfunction u of $\lambda(0)$ in $W^{1,p}(\Omega)$.*

Proof of Theorem 5.1. We will follow the same line that [8, Theorem 1.1]. Let $u \in W^{1,p}(\Omega)$. We call $\overline{u} = u \circ \Psi_t$, then the following asymptotic expansions hold

(5.5)
$$\begin{aligned}
&\int_\Omega |\nabla \bar{u}|^p + |\bar{u}|^p dx = \int_\Omega \big(|D\Psi_t \nabla u|^p + |u|^p\big) J\Psi_t^{-1} dx \\
&= \int_\Omega \big(|(I + tDV + o(t))\nabla u|^p + |u|^p\big)(1 - t\mathrm{div}V + o(t))dx \\
&= \int_\Omega |\nabla u|^p + |u|^p dx - t\big(\mathrm{div}V(|\nabla u|^p + |u|^p) - p|\nabla u|^{p-2}(\nabla u)^t DV \nabla u\big)dx + o(t)
\end{aligned}$$

$$\begin{aligned}
\int_\Gamma \chi_{D_t} |\bar{u}|^p d\mathcal{H}^{n-1} &= \int_\Gamma \chi_D |u|^p J_\Gamma \Psi_t^{-1}\, d\mathcal{H}^{n-1} \\
&= \int_\Gamma \chi_D |u|^p (1 - t\mathrm{div}_\Gamma V) d\mathcal{H}^{n-1} + o(t)
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
\int_\Gamma |\bar{u}|^p d\mathcal{H}^{n-1} &= \int_\Gamma |u|^p J_\Gamma \Psi_t^{-1} d\mathcal{H}^{n-1} \\
&= \int_\Gamma |u|^p (1 - t\mathrm{div}_\Gamma V) d\mathcal{H}^{n-1} + o(t)
\end{aligned} \tag{5.7}$$

From (5.5) and (5.6), we obtain

$$I(\bar{u}, \chi_{D_t}) = F(u) - tG(u) + o(t) \tag{5.8}$$

where

$$F(u) = \int_\Omega |\nabla u|^p + |u|^p dx + \sigma \int_\Gamma \chi_D |u|^p \, d\mathcal{H}^{n-1} \tag{5.9}$$

and

(5.10)
$$G(u) = \int_\Omega \mathrm{div}V(|\nabla u|^p + |u|^p) - p|\nabla u|^{p-2}(\nabla u)^t DV \nabla u \, dx + \sigma \int_\Gamma \chi_D |u|^p \mathrm{div}_\Gamma V \, d\mathcal{H}^{n-1}$$

Now, take u to be a normalized eigenfunction associated to $\lambda(0)$. Then we have

$$\begin{aligned}
\lambda(t) &\le \frac{I(\bar u, \chi_{D_t})}{\int_\Gamma |\bar u|^p \, d\mathcal{H}^{n-1}} = \frac{F(u) - tG(u) + o(t)}{\int_\Gamma |u|^p \, d\mathcal{H}^{n-1} - t\int_\Gamma |u|^p \mathrm{div} V d\mathcal{H}^{n-1} + o(t)} \\
&= \frac{F(u)}{\int_\Gamma |u|^p \, d\mathcal{H}^{n-1}} + t\left(F(u)\frac{\int_\Gamma |u|^p \mathrm{div} V \, d\mathcal{H}^{n-1}}{\left(\int_\Gamma |u|^p \, d\mathcal{H}^{n-1}\right)^2} - \frac{G(u)}{\int_\Gamma |u|^p \, d\mathcal{H}^{n-1}}\right) + o(t) \\
&= \lambda(0) + t\left(\lambda(0)\int_\Gamma |u|^p \mathrm{div} V \, d\mathcal{H}^{n-1} - G(u)\right) + o(t)
\end{aligned}$$

Therefore

$$\lambda(t) - \lambda(0) \le t\left(\lambda(0)\int_\Gamma |u|^p \mathrm{div} V \, d\mathcal{H}^{n-1} - G(u)\right) + o(t) \tag{5.11}$$

Now, take $u_t \in W^{1,p}(\Omega)$ a normalized eigenfunction associated to $\lambda(t)$ and denote by $\bar u_t = u_t \circ \Psi_{-t}$. So

$$\begin{aligned}
\lambda(0) &\le \frac{I(\bar u_t, \chi_D)}{\int_\Gamma |\bar u_t|^p \, d\mathcal{H}^{n-1}} = \frac{F(u_t) + tG(u_t) + o(t)}{\int_\Gamma |u_t|^p \, d\mathcal{H}^{n-1} + t\int_\Gamma |u_t|^p \mathrm{div}_\Gamma V \, d\mathcal{H}^{n-1} + o(t)} \\
&= \frac{F(u_t)}{\int_\Gamma |u_t|^p \, d\mathcal{H}^{n-1}} - t\left(F(u_t)\frac{\int_\Gamma |u_t|^p \mathrm{div} V \, d\mathcal{H}^{n-1}}{\left(\int_\Gamma |u_t|^p \, d\mathcal{H}^{n-1}\right)^2} - \frac{G(u_t)}{\int_\Gamma |u_t|^p \, d\mathcal{H}^{n-1}}\right) + o(t) \\
&= \lambda(t) - t\left(\lambda(t)\int_\Gamma |u_t|^p \mathrm{div} V \, d\mathcal{H}^{n-1} - G(u_t)\right) + o(t)
\end{aligned}$$

This last inequality together with (5.11) give us

$$\begin{aligned}
t\Big(\lambda(t)\int_\Gamma |u_t|^p \mathrm{div} V \, d\mathcal{H}^{n-1} - G(u_t)\Big) + o(t) &\le \lambda(t) - \lambda(0) \\
&\le t\left(\lambda(0)\int_\Gamma |u|^p \mathrm{div} V \, d\mathcal{H}^{n-1} - G(u)\right) + o(t)
\end{aligned}$$

So, by Remark 5.2 one gets

$$\lambda'(0) = \left(\lambda(0)\int_\Gamma |u|^p \mathrm{div} V \, d\mathcal{H}^{n-1} - G(u)\right).$$

It remains to further simplify the expression for $\lambda'(0)$. Let

$$\begin{aligned}
G(u) &= \int_\Omega \mathrm{div} V(|\nabla u|^p + |u|^p) - p|\nabla u|^{p-2}(\nabla u)^t DV \nabla u \, dx \\
&\quad + \sigma\int_\Gamma \chi_D |u|^p \mathrm{div}_\Gamma V \, d\mathcal{H}^{n-1} \\
&= I_1 + I_2
\end{aligned}$$

Now using $V \cdot \nabla u$ as test function in the equation $-\Delta_p u + |u|^{p-2}u = 0$ and the boundary condition in (1.1) we obtain:

$$
\begin{aligned}
I_1 &= -p \int_\Gamma |\nabla u|^{p-2} \frac{\partial u}{\partial \mathbf{n}} V \cdot \nabla u \, d\mathcal{H}^{n-1} \\
&= -p \int_\Gamma \lambda(0)|u|^{p-2}u(V \cdot \nabla u) - \sigma \chi_D |u|^{p-2}u(V \cdot \nabla u) d\mathcal{H}^{n-1} \\
&= -\lambda(0) \int_\Gamma \nabla(|u|^p) \cdot V \, d\mathcal{H}^{n-1} + \sigma \int_\Gamma \chi_D \nabla(|u|^p) \cdot V \, d\mathcal{H}^{n-1} \\
&= \lambda(0) \int_\Gamma |u|^p \mathrm{div}_\Gamma V \, d\mathcal{H}^{n-1} - \sigma \int_D |u|^p \mathrm{div}_\Gamma V \, d\mathcal{H}^{n-1} + \sigma \int_{\partial_\Gamma D} |u|^p V \cdot \mathbf{n}_\Gamma \, d\mathcal{H}^{n-2} \\
&= \lambda(0) \int_\Gamma |u|^p \mathrm{div}_\Gamma V \, d\mathcal{H}^{n-1} + \sigma \int_{\partial_\Gamma D} |u|^p V \cdot \mathbf{n}_\Gamma \, d\mathcal{H}^{n-2} - I_2
\end{aligned}
$$

So,

$$
G(u) = \lambda(0) \int_\Gamma |u|^p \mathrm{div}_\Gamma V \, d\mathcal{H}^{n-1} + \sigma \int_{\partial_\Gamma D} |u|^p V \cdot \mathbf{n}_\Gamma \, d\mathcal{H}^{n-2}
$$

and therefore

$$
\lambda'(0) = -\sigma \int_{\partial_\Gamma D} |u|^p V \cdot \mathbf{n}_\Gamma \, d\mathcal{H}^{n-2}
$$

This completes the proof of the Theorem. □

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