ESTIMATES FOR THE SOBOLEV TRACE CONSTANT WITH CRITICAL EXPONENT AND APPLICATIONS

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Abstract. In this paper we find estimates for the optimal constant in the critical Sobolev trace inequality $S\|u\|^p_{L^{p_*}(\partial\Omega)} \le \|u\|^p_{W^{1,p}(\Omega)}$ that are independent of Ω. This estimates generalized those of [3] for general p. Here $p_* := p(N-1)/(N-p)$ is the critical exponent for the immersion and N is the space dimension.

Then we apply our results first to prove existence of positive solutions to a nonlinear elliptic problem with a nonlinear boundary condition with critical growth on the boundary, generalizing the results of [16]. Finally, we study an optimal design problem with critical exponent.

1. Introduction

Sobolev inequalities are relevant for the study of boundary value problems for differential operators. They have been studied by many authors and it is by now a classical subject. It at least goes back to [1], for more references see [9]. In particular, the Sobolev trace inequality has been intensively studied in [4, 11, 13, 16, 19], etc.

Let Ω be a bounded smooth domain of $\mathbb{R}^N$. For any $1 < p < N$, the Sobolev trace immersion says that there exists a constant $S > 0$ such that

$$S\Big(\int_{\partial\Omega} |u|^{p_*}\,dS\Big)^{p/p_*} \le \int_\Omega |\nabla u|^p + |u|^p\,dx$$

for any $u \in W^{1,p}(\Omega)$, where $W^{1,p}(\Omega)$ is the usual Sobolev spaces of the functions $u \in L^p(\Omega$ such that $\nabla u \in L^p(\Omega$. Here $p_* := p(N-1)/(N-p)$ is the critical exponent for this inequality.

The optimal constant in the above inequality is the largest possible S, that is

$$S = S_p(\Omega) := \inf \frac{\displaystyle\int_\Omega |\nabla u|^p + |u|^p\,dx}{\Big(\displaystyle\int_{\partial\Omega} |u|^{p_*}\,dS\Big)^{p/p_*}},$$

where the infimum is taken over the set $X := W^{1,p}(\Omega) \setminus W^{1,p}_0(\Omega)$, $W^{1,p}_0(\Omega)$ being the closure for the $W^{1,p}$-norm of the space of smooth functions with compact support in Ω.

The dependance of S with respect to p and Ω has been studied by many authors, specially in the *subcritical case*, i.e. where p_* is replaced by any

Key words and phrases. Sobolev trace embedding, Optimal design problems, Critical exponents.

2000 *Mathematics Subject Classification.* 35J20, 35P30, 49R50.

exponent q such that $1 < q < p_*$. See, for instance [8, 14] and references therein.

The analysis for the critical case is more involved because the immersion $W^{1,p}(\Omega) \hookrightarrow L^{p_*}(\partial\Omega)$ is no longer compact and so the existence of minimizers for S does not follows by standard methods.

To overcome this problem, in [16], the authors use an old idea from T. Aubin [1]. In fact, let K_p^{-1} be the best trace constant for the embedding $W^{1,p}(\mathbb{R}^n_+) \hookrightarrow L^{p_*}(\partial\mathbb{R}^n_+)$, namely

$$K_p^{-1} = \inf_{u\in W^{1,p}(\mathbb{R}^n_+)\setminus W^{1,p}_0(\mathbb{R}^n_+} \frac{\int_{\mathbb{R}^n_+} |\nabla u|^p dx}{\left(\int_{\partial\mathbb{R}^n_+} |u|^{p_*} dS\right)^{p/p_*}}. \tag{1.1}$$

In [16] it is shown, following ideas from [1], that if

$$S_p(\Omega) < K_p^{-1}, \tag{1.2}$$

then there exists an extremal for $S_p(\Omega)$. Taking the function $u \equiv 1$ in the definition of $S_p(\Omega)$ one obtain that if

$$\frac{|\Omega|}{|\partial\Omega|^{\frac{p}{p_*}}} < K_p^{-1},$$

then (1.2) is satisfied. Observe that this is a global condition on Ω.

It follows from Lions [20] that the infimum (1.1) is achieved. The value of K_p is explicitely known when $p = 2$ (see Escobar [11]).

Recently, Biezuner [4] proved that K_p is also the best first constant in the inequality,

$$\left(\int_{\partial\Omega} |u|^{p_*} dS\right)^{\frac{p}{p_*}} \le A \int_\Omega |\nabla u|^p dx + B \int_\Omega |u|^p dx,$$

in the sense that, for any $\epsilon > 0$, there exists a constant C_ϵ such that

$$\left(\int_{\partial\Omega} |u|^{p_*} dS\right)^{\frac{p}{p_*}} \le (K_p + \epsilon) \int_\Omega |\nabla u|^p dx + C_\epsilon \int_\Omega |u|^p dx, \tag{1.3}$$

for every $u \in W^{1,p}(\Omega)$, and K_p is the lowest possible constant. This fact will be used in a crucial way in the course of the paper.

On the other hand a local condition ensuring (1.2), depending only on local geometric properties of Ω, is known to hold in the case $p = 2$. Indeed Adimurthi-Yadava [3] obtained (1.2) assuming the existence of a "good point" $x \in \partial\Omega$, i.e. a point x at which the mean curvature of $\partial\Omega$ is positive and such that, in a neighborhood of x, Ω lies on one side of the tangent plane at x. The method in their proof is the use as test-functions of a suitable rescaling of the extremals of (1.1).

These extremals are explicitly known for $p = 2$ since Escobar's work [11] who conjectured the result for any $p \in (1, N)$. This conjecture has recently been proved by Nazaret [21] using a mass-transportation method. It turns out that all the extremals of (1.1) are of the form

$$
\begin{aligned}
U_{\epsilon,y_0}(y,t) =& \frac{\epsilon^{\frac{N-p}{p(p-1)}}}{[(t+\epsilon)^2+|y-y_0|^2]^{\frac{N-p}{2(p-1)}}} \\
=&\epsilon^{-\frac{N-p}{p}} U\left(\frac{y-y_0}{\epsilon}, \frac{t}{\epsilon}\right)
\end{aligned}
\tag{1.4}
$$

where $\epsilon > 0$ and $y, y_0 \in \mathbb{R}^{N-1} = \partial\mathbb{R}^N_+$, $t > 0$, with

$$
U(y,t) = \frac{1}{[(t+1)^2+|y|^2]^{\frac{N-p}{2(p-1)}}}. \tag{1.5}
$$

The knowledge of this extremals allows us first to compute the explicit value of K_p:

Proposition 1.1. *The value of K_p is*

$$
K_p^{-1} = \left(\frac{N-p}{p-1}\right)^{p-1} \pi^{\frac{p-1}{2}} \left(\frac{\Gamma\left(\frac{N-1}{2(p-1)}\right)}{\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)}\right)^{\frac{p-1}{N-1}}.
$$

Applying a similar technique as in [3], we can use the rescaled extremals for K_p and obtain a local (geometrical) condition on Ω such that (1.2) is satisfied.

In fact, we can deal with a slightly more general problem. Namely

$$
\lambda = \lambda(p,\Omega) := \inf \frac{\displaystyle\int_\Omega |\nabla u|^p + h(x)|u|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |u|^{p_*}\,dS\right)^{p/p_*}} \tag{1.6}
$$

where the infimum is taken over X and the function $h \in C^1(\overline{\Omega})$ is such that there exists $c > 0$ satisfying

$$
\int_\Omega |\nabla u|^p + h(x)|u|^p\,dx \geq c\|u\|^p_{W^{1,p}(\Omega)} \tag{1.7}
$$

for any $u \in X$.

We are lead to the following generalization of the notion of “good point” to our case: we say that a point $x \in \partial\Omega$ is a “good point” if there exists $r > 0$ such that $\Omega \cap B_r(x)$ lies on one side of the tangent plane at x and either $H(x) > 0$ or, if $H(x) = 0$, either

$$
h(x) < 0 \text{ if } N = 2,3,4 \text{ and } p < \sqrt{N}
$$

or, if $N \geq 5$,

$$
\begin{aligned}
&h(x) < 0 \text{ if } p < 2, \\
&\frac{N}{N-1}\sum \lambda_i^2 - 2\sum_{i<j}\lambda_i\lambda_j < \frac{-8(N-1)h(x)}{(N-2)(N-4)} \text{ if } p = 2, \\
&\frac{p+N-2}{N-1}\sum \lambda_i^2 - 2\sum_{i<j}\lambda_i\lambda_j < 0 \text{ if } 2 < p < (N+2)/3.
\end{aligned}
$$

where the λ_i's are the principal curvatures at x and $H(x)$ is the mean curvature at x.

Remark that our method gives the restriction $1 < p < (N+1)/2$ and also that a "good point" in the sense of Adimurthi-Yadava is also a "good point" in our sense.

We get the following theorem:

Theorem 1.1. *Let $1 < p < (N+1)/2$. If there exist a "good point" $x \in \partial\Omega$, then*

$$\lambda < K_p^{-1}. \tag{1.8}$$

As a consequence of Theorem 1.1 we have

Corollary 1.1. *Under the hypotheses of Theorem 1.1, the infimum* (1.6) *is achieved.*

Observe that any extremal u can be taken to be nonnegative (just replace u by $|u|$), and if we take it *normalized* as $\|u\|_{L^{p_*}(\partial\Omega)} = 1$, it is an eigenfunction associated to the eigenvalue λ in the sense that it is a weak solution of the following Steklov-like eigenvalue problem

$$\begin{cases} -\Delta_p u + h(x)u^{p-1} = 0 & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda u^{p_*-1} & \text{on } \partial\Omega \end{cases} \tag{1.9}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplacian and ν is the unit outward normal of Ω.

Then it follows by the results of Cherrier [5] that u is smooth on Ω and continuous up to the boundary. Moreover, it is strictly positive in $\overline{\Omega}$ (see, for instance, [15]) so any extremal has constant sign.

As an application of Theorem 1.1, we study a shape optimization problem related to λ. Given $\alpha \in (0, |\Omega|)$, where $|\Omega|$ denotes the volume of Ω, and a measurable subset $A \subset \Omega$ of volume α, we first consider the minimization problem

$$\lambda_A = \inf \frac{\displaystyle\int_\Omega |\nabla u|^p + h(x)|u|^p\,dx}{\Big(\displaystyle\int_{\partial\Omega} |u|^{p_*}\,dS\Big)^{p/p_*}} \tag{1.10}$$

where the infimum is taken over $X_A := \{u \in X \mid u|_A = 0 \text{ a.e.}\}$ and the function $h \in C^1(\overline{\Omega})$ is such that the coercivity assumption (1.7) holds

As a consequence of Theorem 1.1, we have

Theorem 1.2. *Let $1 < p < (N+1)/2$ and let $A \subset \Omega$ be such that $|A| = \alpha$. Assume that there exists a "good point" $x \in \partial\Omega$ such that $B_r(x) \cap A = \emptyset$ for some $r > 0$. Then λ_A is attained by some nonnegative nontrivial u_A.*

These extremals u_A are eigenfunctions associated to the eigenvalue λ_A in the sense that, if A is closed, they are weak solutions of the following

Steklov-like eigenvalue problem

$$\begin{cases} -\Delta_p u + h(x)u^{p-1} = 0 & \text{in } \Omega \setminus A \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda_A u^{p_*-1} & \text{on } \partial\Omega \setminus A \\ u = 0 & \text{in } A \end{cases} \tag{1.11}$$

We consider the following shape optimization problem:

For a fixed $0 < \alpha < |\Omega|$, find a set A_* of measure α that minimizes λ_A among all measurable subsets $A \subset \Omega$ of measure α. That is,

$$\lambda(\alpha) := \inf_{A\subset\Omega, |A|=\alpha} \lambda_A = \lambda_{A_*}.$$

In this paper we prove that there exist an optimal set A_* (with their corresponding extremals u_*) for this optimization problem.

This optimization problem in the subcritical case (that is, when p_* is replaced by an exponent q with $1 < q < p_*$) has been considered recently. In fact, in [17] the existence of an optimal set has been established, see also [12] for numerical computations. Then, in [18], the interior regularity of optimal sets was analyzed in the case $p = 2$. We remark that in the result of [18] the subcriticality plays no role, so this local regularity result holds true also for this critical case.

We prove,

Theorem 1.3. *Let $1 < p < (N+1)/2$. If there exists a "good point" $x \in \partial\Omega$, then $\lambda(\alpha)$ is achieved.*

Problems of optimal design related to eigenvalue problems like (1.11) appear in several branches of applied mathematics, specially in the case $p = 2$. For example in problems of minimization of the energy stored in the design under a prescribed loading. We refer to [6] for more details.

We want to stress that Theorem 1.3 is new, even in the case $p = 2$.

Organization of the paper. In the next section we deal with the proof of the applications of the estimate $\lambda < K_p^{-1}$, that is, we deal with the proof of Corollary 1.1 and Theorems 1.2 and 1.3. We leave for the final section the computation of K_p and the proof of Theorem 1.1.

2. Applications of Theorem 1.1

In this section we use Theorem 1.1, that is proved in the Section 3, and prove Corollary 1.1, Theorem 1.2 and Theorem 1.3.

2.1. **Proof of Corollary 1.1.** We first prove that λ is attained as soon as (1.8) is satisfied. Since this kind of criterion is classical (see e.g. [7] or [16]), we only sketch the proof for the reader's convenience.

Let $\{u_n\}_{n\in\mathbb{N}} \subset X$ be a minimizing sequence for (1.6) normalized such that $\|u_n\|_{L^{p_*}(\partial\Omega)} = 1$. According to (1.7), this sequence is bounded in X and thus it converges up to a subsequence to some $u \in X$ weakly in X, strongly in $L^p(\Omega)$ and a.e.

Using Ekeland's variational principle (see [23] Theorems 8.5 and 8.14), we can assume that $\{u_n\}_{n\in\mathbb{N}}$ is a Palais-Smale sequence for the functional $J : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + h(x)|u|^p\,dx - \frac{\lambda}{p_*}\int_{\partial\Omega} |u|^{p_*}\,dS,$$

in the sense that the sequence $\{J(u_n)\}_{n\in\mathbb{N}}$ is bounded and $DJ(u_n) \to 0$ strongly in $(W^{1,p}(\Omega))^*$. Letting $v_n := u_n - u$, we can also assume that, up to a subsequence,

$$|v_n|^{p_*}\,dS \rightharpoonup d\nu, \qquad |\nabla v_n|^p\,dx \rightharpoonup d\mu,$$

weakly in the sense of measures, where μ and ν are nonnegative measures such that $\mathrm{supp}(\nu) \subset \partial\Omega$.

According to (1.3), we have for any $\phi \in C^1(\overline{\Omega})$ that

$$\left(\int_{\partial\Omega} |\phi v_n|^{p_*}\,dS\right)^{p/p_*} \le (K_p + \epsilon)\int_\Omega |\nabla(\phi v_n)|^p\,dx + C_\epsilon \int_\Omega |\phi v_n|^p\,dx.$$

Passing to the limit in this expression, first in $n \to \infty$ and then in $\epsilon \to 0$, we get that

$$\left(\int_{\partial\Omega} |\phi|^{p_*}\,d\nu\right)^{p/p_*} \le K_p \int_\Omega |\phi|^p\,d\mu$$

for any $\phi \in C^1(\overline{\Omega})$. From this inequality, we can deduce as in [20] Lemma 2.3, the existence of a sequence of points $\{x_i\}_{i\in I} \subset \partial\Omega$, $I \subset \mathbb{N}$, and two sequences of positive real numbers $\{\nu_i\}_{i\in I}$, $\{\mu_i\}_{i\in I}$ such that

$$\nu = \sum_{i\in I} \nu_i \delta_{x_i}, \quad \mu \ge \sum_{i\in I} \mu_i \delta_{x_i} \quad \text{and} \quad \mu_i \ge K_p^{-1}\nu_i^{p/p_*} \quad \forall\, i \in I.$$

Therefore,

$$\begin{cases} |u_n|^{p_*}dS & \rightharpoonup |u|^{p_*}dS + \sum_{i\in I}\nu_i\delta_{x_i} \\ |\nabla u_n|^p dx & \rightharpoonup |\nabla u_n|^p dx + \mu \ge |\nabla u_n|^p dx + \sum_{i\in I}\mu_i\delta_{x_i} \\ \mu_i & \ge K_p^{-1}\nu_i^{p/p_*}\ \forall\, i\in I. \end{cases} \tag{2.1}$$

It can also be shown that $\{v_n\}_{n\in\mathbb{N}}$ is a Palais-Smale sequence for the functional $I : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I(u) := J(u) - \int_\Omega h(x)|u|^p\,dx$$

(see e.g. [22]). In particular, for any $\phi \in C^1(\overline{\Omega})$,

$$\begin{aligned} o(1) &= DI(v_n)(v_n\phi) \\ &= \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla(v_n\phi)\,dx - \lambda\int_{\partial\Omega} |v_n|^{p_*}\phi\,dS. \end{aligned}$$

Passing to the limit, we get that $\int_\Omega \phi\,d\mu = \lambda\int_{\partial\Omega}\phi\,d\nu$ for any $\phi \in C^1(\overline{\Omega})$. Hence $\mu = \lambda\nu$. Using (2.1), we then obtain the estimates

$$\nu_i \ge (\lambda K_p)^{-\frac{n-1}{p-1}}, \quad \mu_i \ge K_p^{-1}(\lambda K_p)^{-\frac{n-1}{p-1}} \quad \forall\, i \in I. \tag{2.2}$$

Now, by (2.1), (1.7) and (2.2), we arrive at

$$\begin{aligned}\lambda &= \int_\Omega |\nabla u_n|^p\,dx + \int_\Omega h(x)|u_n|^p\,dx + o(1) \geq \sum_{i\in I}\mu_i \\ &\geq card(I)K_p^{-1}(\lambda K_p)^{-\frac{n-1}{p-1}}.\end{aligned}$$

We deduce that if (1.8) holds, then I is empty. In that case, $u_n \to u$ strongly in $W^{1,p}(\Omega)$ and in $L^{p_*}(\partial\Omega)$. In particular u is a minimizer for λ.

This completes the proof □

2.2. **Proof of Theorem 1.2.** Arguing exactly as in the proof of Theorem 1.1 we obtain that a normalized minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subset X_A$ for λ_A converges, up to a subsequence, strongly in $W^{1,p}(\Omega)$ to some u_A as soon as

$$\inf_{u\in X_A} \frac{\displaystyle\int_\Omega |\nabla u|^p + |u|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |u|^{p_*}\,dS\right)^{p/p_*}} < K_p^{-1}. \tag{2.3}$$

Since there exists a "good point" $x \in \partial\Omega$ such that $B_r(x)\cap A = \emptyset$, we deduce from the computations in the next section, by choosing a cut-off function ϕ with support in $B_{r/2}(x)$ in the definition of the test function u_ϵ (3.1), that this strict inequality (2.3) holds. Hence $u_n \to u$ strongly in $W^{1,p}(\Omega)$ and $L^{p_*}(\partial\Omega)$ and also a.e.. In particular u is a minimizer for λ_A. □

2.3. **Proof of Theorem 1.3.** We begin by noticing that

$$\lambda(\alpha) = \inf\{\lambda_A,\ A\subset\Omega \text{ measurable},\ |A|\geq\alpha\}.$$

Hence

$$\lambda(\alpha) = \inf_{u\in X,\ |\{u=0\}|\geq\alpha} \frac{\displaystyle\int_\Omega |\nabla u|^p + |u|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |u|^{p_*}\,dS\right)^{p/p_*}}.$$

Since $\alpha < |\Omega|$ and there exists a "good point", it follows from the test functions computations of the next section, by choosing a function ϕ with support in a ball of radius small enough in the definition of u_ϵ (3.1), that $\lambda(\alpha) < K_p^{-1}$.

By the same argument as before, this implies the existence of a nonnegative $u_* \in X$, $|\{u_* = 0\}| \geq \alpha$, such that

$$\frac{\displaystyle\int_\Omega |\nabla u_*|^p + |u_*|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |u_*|^{p_*}\,dS\right)^{p/p_*}} = \lambda(\alpha).$$

We now conclude as in [17], Theorem 1.2, that in fact $|\{u_* = 0\}| = \alpha$ and so $A_* = \{u_* = 0\}$ is an optimal set for $\lambda(\alpha)$. □

3. Proof of Theorem 1.1

In this section we prove our main result. First we recall some very well known formulae and prove Proposition 1.1. Finally we prove Theorem 1.1.

In all the subsequent computations, the following well known formulae will be used frequently:

$$\omega_{N-1} = \text{volume of the standard unit sphere } S^{N-1} \text{ of } \mathbb{R}^N = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)},$$

$$\int_0^{+\infty} \frac{r^\alpha}{(1+r^2)^\beta} dr = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{2\beta-\alpha-1}{2}\right)}{2\Gamma(\beta)} \qquad \text{for } 2\beta-\alpha>1,$$

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z) \qquad \text{for } Re(z)>0.$$

We first compute the value of K_p:

Proof of Propostion 1.1. Let U be the function defined by (1.5). We first compute the L^{p_*}-norm of U restricted to $\mathbb{R}^{N-1}\times\{0\} = \partial\mathbb{R}^N_+$.

$$\begin{aligned}
\int_{\mathbb{R}^{N-1}} |U(y,0)|^{p_*}\,dy &= \int_{\mathbb{R}^{N-1}} \frac{dy}{(1+|y|^2)^{p(N-1)/2(p-1)}} \\
&= \omega_{N-2}\int_0^\infty \frac{r^{N-2}\,dr}{(1+r^2)^{p(N-1)/2(p-1)}} \\
&= \pi^{(N-1)/2}\frac{\Gamma\left(\frac{N-1}{2(p-1)}\right)}{\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)}
\end{aligned}$$

We now compute the L^p-norm of the gradient of U. First

$$\nabla U(y,t) = -\frac{N-p}{p-1}\frac{(y,t+1)}{[(1+t)^2+|y|^2]^{\frac{N-p}{2(p-1)}+1}}.$$

Using the change of variable $y=(1+t)z$ and passing to polar coordinates, we can then write

$$\begin{aligned}
\int_{\mathbb{R}^N_+} |\nabla U(y,t)|^p\,dydt &= \left(\frac{N-p}{p-1}\right)^p \int_{\mathbb{R}^N_+} \frac{dydt}{[(1+t)^2+|y|^2]^{\frac{p(N-1)}{2(p-1)}}} \\
&= \left(\frac{N-p}{p-1}\right)^p \int_0^{+\infty} \frac{dt}{(1+t)^{\frac{N-1}{p-1}}}\omega_{N-2}\int_0^{+\infty}\frac{r^{N-2}\,dr}{(1+r^2)^{\frac{p(N-1)}{2(p-1)}}} \\
&= \left(\frac{N-p}{p-1}\right)^{p-1} \pi^{\frac{N-1}{2}}\frac{\Gamma\left(\frac{N-1}{2(p-1)}\right)}{\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)}.
\end{aligned}$$

Hence

$$K_p^{-1} = \frac{\displaystyle\int_{\mathbb{R}^N_+} |\nabla U(y,t)|\, dydt}{\left(\displaystyle\int_{\mathbb{R}^{N-1}} |U(y,0)|^{p_*} dy\right)^{\frac{p}{p_*}}} = \left(\frac{N-p}{p-1}\right)^{p-1} \pi^{\frac{p-1}{2}} \left(\frac{\Gamma\left(\frac{N-1}{2(p-1)}\right)}{\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)}\right)^{\frac{p-1}{N-1}}$$

and the proof is complete $\square$

We now turn our attention to the proof of Theorem 1.1. Let $x_0 \in \partial\Omega$ be a "good point". By taking an appropriate chart, we can assume that $x_0 = 0$ and that there exist $r > 0$ and $\lambda_1, \ldots, \lambda_{N-1} \in \mathbb{R}$ such that

$$\begin{aligned} B_r \cap \Omega =& \{(y,t) \in B_r,\ t > \rho(y)\} \\ B_r \cap \partial\Omega =& \{(y,t) \in B_r,\ t = \rho(y)\} \end{aligned}$$

where $y = (y_1, \ldots, y_{N-1}) \in \mathbb{R}^{N-1}$, B_r is the Euclidean ball centered at the origin and of radius r, and

$$\rho(y) = \frac{1}{2}\sum_{i=1}^{N-1} \lambda_i y_i^2 + \sum_{i,j,k} c_{ijk} y_i y_j y_k + O(|y|^4).$$

Since $x_0 = 0$ is a "good point", we have $\rho \geq 0$. Moreover, the λ_i's are the principal curvatures at 0 and thus

$$H(0) = \frac{1}{N-1}\sum_{i=1}^{N-1} \lambda_i.$$

Let ϕ be a smooth radial function with compact support in $B_{r/2}$ be such that $\phi \equiv 1$ in $B_{r/4}$. We consider the test functions

$$u_\epsilon(y,t) = \frac{\phi(y,t)}{[(t+\epsilon)^2 + |y|^2]^{\frac{N-p}{2(p-1)}}},\ \epsilon > 0. \tag{3.1}$$

In order to give the asymptotic development of the Rayleigh quotient for u_ϵ, we first compute the different terms involved:

Step 1. *We have the following estimates:*

$$\int_\Omega |\nabla u_\epsilon|^p\, dx = A_1 \epsilon^{-\frac{N-p}{p-1}} + \begin{cases} A_2 \epsilon^{1-\frac{N-p}{p-1}} + A_3 \epsilon^{2-\frac{N-p}{p-1}} \\ \qquad + \begin{cases} O(\epsilon^{3-\frac{N-p}{p-1}}) \text{ if } p < \frac{N+3}{4} \\ O(\ln(1/\epsilon)) \text{ if } p = \frac{N+3}{4} \\ O(1) \text{ if } \frac{N+3}{4} < p < \frac{N+1}{2} \end{cases} \\ A_2' \ln(1/\epsilon) \text{ if } p = \frac{N+1}{2} \\ O(1) \text{ if } p > \frac{N+1}{2} \end{cases} \tag{3.2}$$

$$\int_\Omega h(x)|u_\epsilon|^p\,dx = \begin{cases} D\epsilon^{-\frac{N-p^2}{p-1}} + \begin{cases} O(\epsilon^{1-\frac{N-p^2}{p-1}}) \text{ if } p < \frac{-1+\sqrt{4N+5}}{2} \\ O(\ln(1/\epsilon)) \text{ if } p = \frac{-1+\sqrt{4N+5}}{2} \\ O(1) \text{ if } \sqrt{N} > p > \frac{-1+\sqrt{4N+5}}{2} \end{cases} \\ O(\ln(1/\epsilon)) \text{ if } p = \sqrt{N} \\ O(1) \text{ if } p > \sqrt{N} \end{cases} \tag{3.3}$$

$$\begin{aligned}\int_{\partial\Omega} |u_\epsilon|^{p_*}\,dS =& B_1\epsilon^{-1-\frac{N-p}{p-1}} + B_2\epsilon^{-\frac{N-p}{p-1}} \\ &+ \begin{cases} B_3\epsilon^{1-\frac{N-p}{p-1}} + \begin{cases} O(\epsilon^{2-\frac{N-p}{p-1}}) \text{ if } p < \frac{N+2}{3} \\ O(\ln(1/\epsilon)) \text{ if } p = \frac{N+2}{3} \\ O(1) \text{ if } \frac{N+2}{3} < p < \frac{N+1}{2} \end{cases} \\ B_4\ln(1/\epsilon) \text{ if } p = \frac{N+1}{2} \\ O(1) \text{ if } p > \frac{N+1}{2} \end{cases}\end{aligned} \tag{3.4}$$

where

$$A_1 = \frac{1}{2}\left(\frac{N-p}{p-1}\right)^{p-1}\omega_{N-2}\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-1}{2(p-1)}\right)}{\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)}$$

$$A_2 = -\frac{H(0)\omega_{N-2}}{4}\left(\frac{N-p}{p-1}\right)^{p}\frac{\Gamma\left(\frac{N+1}{2}\right)\Gamma\left(\frac{N-2p+1}{2(p-1)}\right)}{\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)}$$

$$A_2' = -\frac{H(0)\omega_{N-2}}{2}\left(\frac{N-p}{p-1}\right)^{p}$$

$$A_3 = \frac{\omega_{N-2}}{16}\left(\frac{N-p}{p-1}\right)^{p}\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-2p+1}{2(p-1)}\right)}{\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)}\left(\frac{3}{2}\sum\lambda_i^2 + \sum_{i<j}\lambda_i\lambda_j\right)$$

$$B_1 = \omega_{N-2}\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-1}{2(p-1)}\right)}{2\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)}$$

$$B_2 = -\frac{\omega_{N-2}\sum\lambda_i}{8}\frac{p(N-1)}{p-1}\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-1}{2(p-1)}\right)}{\Gamma\left(1+\frac{p(N-1)}{2(p-1)}\right)}$$

$$\begin{aligned}B_3 =& \frac{\omega_{N-2}}{32}\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-2p+1}{2(p-1)}\right)}{\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)}\times \\ &\left\{\left(1+\frac{3(N-2p+1)}{p-1}\right)\sum\lambda_i^2 + \left(-2+\frac{2(N-2p+1)}{p-1}\right)\sum_{i<j}\lambda_i\lambda_j\right\}\end{aligned}$$

$$B_4 = \frac{\omega_{N-2}}{2}\left\{\left(\frac{1}{N-1} - \frac{p(N-1)}{4(p-1)}\right)\sum \lambda_i^2 - \frac{p(N-1)}{2(p-1)}\sum_{i<j}\lambda_i\lambda_j + o(1)\right\}$$

$$D = h(0)\frac{p-1}{N-p^2}\omega_{N-2}\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-p^2+p-1}{2(p-1)}\right)}{2\Gamma\left(\frac{p(N-p)}{2(p-1)}\right)}$$

Proof of Step 1. We have

$$[(t+\epsilon)^2+|y|^2]^{\frac{N-1}{p-1}}|\nabla u_\epsilon|^2 = \left(\frac{N-p}{p-1}\right)^2\phi^2+|\nabla\phi|^2-2\frac{N-p}{p-1}\phi(y\cdot\nabla_y\phi+(t+\epsilon)\partial_t\phi)$$

Hence in $B_{r/4}$,

$$|\nabla u_\epsilon|^p = \left(\frac{N-p}{p-1}\right)^p\frac{1}{[(t+\epsilon)^2+|y|^2]^{\frac{p(N-1)}{2(p-1)}}},$$

and then

$$\int_\Omega |\nabla u_\epsilon|^p\,dx = \left(\frac{N-p}{p-1}\right)^p(I_1 - I_2) + O(1)$$

with

$$I_1 = \int_{Q_a}\frac{1}{[(t+\epsilon)^2+|x|^2]^{\frac{p(n-1)}{2(p-1)}}} \quad \text{and} \quad I_2 = \int_{Q_a\setminus\Omega}\frac{1}{[(t+\epsilon)^2+|x|^2]^{\frac{p(n-1)}{2(p-1)}}},$$

where $Q_a := \{(y,t) \mid |y| \le a \text{ and } 0 \le t \le a\}$.

Changing variables $y = (1+t)z$ and passing to polar coordinates, we have

$$\begin{aligned}
I_1 &= \int_{Q_a}\frac{1}{[(t+\epsilon)^2+|y|^2]^{\frac{p(N-1)}{2(p-1)}}}\,dydt \\
&= \epsilon^{-\frac{N-p}{p-1}}\int_{\mathbb{R}^N_+}\frac{1}{[(1+t)^2+|y|^2]^{\frac{p(N-1)}{2(p-1)}}}\,dydt + O(1) \\
&= \epsilon^{-\frac{N-p}{p-1}}\omega_{N-2}\int_0^\infty\frac{dt}{(1+t)^{\frac{N-1}{p-1}}}\int_0^\infty\frac{r^{N-2}\,dr}{(1+r^2)^{\frac{p(N-1)}{2(p-1)}}} + O(1)
\end{aligned}$$

Hence

$$I_1 = \epsilon^{-\frac{N-p}{p-1}}\frac{p-1}{N-p}\omega_{N-2}\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-1}{2(p-1)}\right)}{2\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)} + O(1). \tag{3.5}$$

On the other hand, according to Taylor's formula,

$$
\begin{aligned}
I_2 &= \int_{|y|\le a}\int_0^{\rho(y)} \frac{1}{[(t+\epsilon)^2+|y|^2]^{\frac{p(N-1)}{2(p-1)}}}\,dtdy \\
&= \int_{|y|\le a} \frac{\rho(y)\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}} - \frac{p(N-1)}{2(p-1)}\epsilon \int_{|y|\le a} \frac{\rho(y)^2\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}+1}} \\
&\quad + O\left(\int_{|y|\le a} \frac{|y|^6\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}+1}}\right) \\
&= I_3 - \frac{p(N-1)}{2(p-1)}\epsilon I_4 + \begin{cases} O\left(\epsilon^{3-\frac{N-p}{p-1}}\right), \text{ if } p<\frac{N+3}{4} \\ O(\ln(1/\epsilon)), \text{ if } p=\frac{N+3}{4} \\ O(1), \text{ if } p>\frac{N+3}{4} \end{cases}
\end{aligned}
$$

As the sphere is symmetric, we have

$$
I_3 = \frac{1}{2}H(0)\int_{|y|\le a} \frac{|y|^2\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}} + O\left(\int_{|y|\le a} \frac{|y|^4\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}}\right)
$$

with

$$
\begin{aligned}
\int_{|y|\le a} \frac{|y|^2\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}} &= \epsilon^{1-\frac{N-p}{p-1}}\omega_{N-2}\int_0^{a/\epsilon} \frac{r^N\,dr}{(1+r^2)^{\frac{p(N-1)}{2(p-1)}}} \\
&= \begin{cases} \epsilon^{1-\frac{N-p}{p-1}}\omega_{N-2}\dfrac{\Gamma\left(\frac{N+1}{2}\right)\Gamma\left(\frac{N-2p+1}{2(p-1)}\right)}{2\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)} + O(1) \text{ if } p<\frac{N+1}{2} \\ \approx \omega_{N-2}\ln(1/\epsilon) \text{ if } p<\frac{N+1}{2} \\ O(1) \text{ if } p>\frac{N+1}{2} \end{cases}
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
\int_{|y|\le a} \frac{|y|^4\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}} &= \epsilon^{3-\frac{N-p}{p-1}}\omega_{N-2}\int_0^{a/\epsilon} \frac{r^{N+2}\,dr}{(1+r^2)^{\frac{p(N-1)}{2(p-1)}}} \\
&= \begin{cases} O(\epsilon^{3-\frac{N-p}{p-1}}) \text{ if } p<\frac{N+3}{4} \\ O(\ln(1/\epsilon)) \text{ if } p=\frac{N+3}{4} \\ O(1) \text{ if } p>\frac{N+3}{4} \end{cases}
\end{aligned}
\tag{3.7}
$$

Since $\frac{N+3}{4}<\frac{N+1}{2}$ we get

$$
I_3 = \begin{cases} \epsilon^{1-\frac{N-p}{p-1}}\omega_{N-2}H(0)\dfrac{\Gamma\left(\frac{N+1}{2}\right)\Gamma\left(\frac{N-2p+1}{2(p-1)}\right)}{4\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)} + \begin{cases} O(\epsilon^{3-\frac{N-p}{p-1}}) \text{ if } p<\frac{N+3}{4} \\ O(\ln(1/\epsilon)) \text{ if } p=\frac{N+3}{4} \\ O(1) \text{ if } \frac{N+3}{4}<p<\frac{N+1}{2} \end{cases} \\ \approx \frac{1}{2}H(0)\omega_{N-2}\ln(1/\epsilon) \text{ if } p=\frac{N+1}{2} \\ O(1) \text{ if } p>\frac{N+1}{2} \end{cases}
$$

Concerning I_4, we have

$$\begin{aligned} I_4 =& \frac{1}{4}\sum \lambda_i^2 \int_{|y|\le a} \frac{y_i^4\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}+1}} \\ &+ \frac{1}{2}\sum_{i<j}\lambda_i\lambda_j \int_{|y|\le a} \frac{y_i^2y_j^2\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}+1}} \\ &+ O\left(\int_{|y|\le a} \frac{|y|^5\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}+1}}\right). \end{aligned}$$

First we compute

$$\begin{aligned} \int_{|y|\le a} \frac{y_i^4\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}+1}} &= \epsilon^{1-\frac{N-p}{p-1}} \int_{|y|\le a/\epsilon} \frac{y_i^4\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}+1}} \\ &= \begin{cases} O(1) \text{ if } p>\frac{N+1}{2} \\ \approx \omega_{N-2}\ln(1/\epsilon) \text{ if } p=\frac{N+1}{2} \end{cases} \end{aligned}$$

and if $p<\frac{N+1}{2}$,

$$\begin{aligned} &\int_{|y|\le a} \frac{y_i^4\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}+1}} \\ &= 2\epsilon^{1-\frac{N-p}{p-1}}\omega_{N-3}\int_0^\infty \frac{r^{N-3}\,dr}{(1+r^2)^{\frac{p(N-1)}{2(p-1)}-\frac{3}{2}}}\int_0^\infty \frac{y^4\,dy}{(1+y^2)^{\frac{p(N-1)}{2(p-1)}+1}} + O(1). \end{aligned}$$

Hence

$$\begin{aligned} &\int_{|y|\le a} \frac{y_i^4\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}+1}} \\ &= \begin{cases} \epsilon^{1-\frac{N-p}{p-1}}\frac{\omega_{N-3}}{2}\frac{\Gamma\left(\frac{N-2}{2}\right)\Gamma\left(\frac{N-2p+1}{2(p-1)}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{p(N-1)}{2(p-1)}+1\right)} + O(1) \text{ if } p<\frac{N+1}{2} \\ \approx \omega_{N-2}\ln(1/\epsilon) \text{ if } p=\frac{N+1}{2} \\ O(1) \text{ if } p>\frac{N+1}{2} \end{cases} \end{aligned} \tag{3.8}$$

In the same way

$$\int_{|y|\le a} \frac{y_i^2y_j^2\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}+1}} = \begin{cases} \approx \omega_{N-2}\ln(1/\epsilon) \text{ if } p=\frac{N+1}{2} \\ O(1) \text{ if } p>\frac{N+1}{2} \end{cases}$$

and if $p < \frac{N+1}{2}$,

$$\begin{aligned}&\int_{|y|\le a} \frac{y_i^2 y_j^2\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}+1}}\\ =&\epsilon^{1-\frac{N-p}{p-1}} \int_{|y|\le a/\epsilon} \frac{y_i^2 y_j^2\,dy}{(1+|y|^2)^{\frac{p(N-1)}{2(p-1)}+1}}\\ =&4\omega_{N-4}\int_0^\infty \frac{r^{N-4}\,dr}{(1+r^2)^{\frac{p(N-1)}{2(p-1)}-2}}\int_0^\infty \frac{y_i^2\,dy_i}{(1+y_i^2)^{\frac{p(N-1)}{2(p-1)}-\frac12}}\int_0^\infty \frac{y_j^2\,dy_j}{(1+y_j^2)^{\frac{p(N-1)}{2(p-1)}+1}}\\ &+O(1)\end{aligned}$$

Hence

$$\begin{aligned}&\int_{|y|\le a} \frac{y_i^2 y_j^2\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}+1}}\\ &=\begin{cases}\epsilon^{1-\frac{N-p}{p-1}}\frac{\omega_{N-4}}{2}\frac{\Gamma\left(\frac{N-3}{2}\right)\Gamma\left(\frac32\right)^2\Gamma\left(\frac{N-2p+1}{2(p-1)}\right)}{\Gamma\left(\frac{p(N-1)}{2(p-1)}+1\right)}+O(1) \text{ if } p<\frac{N+1}{2}\\ \approx \omega_{N-2}\ln(1/\epsilon) \text{ if } p=\frac{N+1}{2}\\ O(1) \text{ if } p>\frac{N+1}{2}\end{cases}\end{aligned}$$

Once again,

$$\begin{aligned}\int_{|y|\le a}\frac{|y|^5\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}+1}} &= \epsilon^{2-\frac{N-p}{p-1}}\omega_{N-2}\int_0^{a/\epsilon}\frac{r^{N+3}\,dr}{(1+r^2)^{\frac{p(N-1)}{2(p-1)}+1}}\\ &=\begin{cases}O(\epsilon^{2-\frac{N-p}{p-1}}) \text{ if } p<\frac{N+2}{3}\\ O(\ln(1/\epsilon)) \text{ if } p=\frac{N+2}{3}\\ O(1) \text{ if } p>\frac{N+2}{3}\end{cases}\end{aligned}$$

Using the fact that $\Gamma(\frac32)=\frac{\sqrt\pi}{2}$, $\Gamma(\frac52)=\frac{3\sqrt\pi}{4}$, and

$$\omega_{N-3}=\frac{1}{\sqrt\pi}\frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N-2}{2}\right)}\omega_{N-2},\qquad \omega_{N-4}=\frac{1}{\pi}\frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N-3}{2}\right)}\omega_{N-2},$$

we eventually get that

$$I_4=\begin{cases}\frac{\omega_{N-2}}{16}\epsilon^{1-\frac{N-p}{p-1}}\frac{\Gamma\left(\frac{N-2p+1}{2(p-1)}\right)\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{p(N-1)}{2(p-1)}+1\right)}\left(\frac32\sum\lambda_i^2+\sum_{i<j}\lambda_i\lambda_j\right)\\ \qquad\qquad +\begin{cases}O(\epsilon^{2-\frac{N-p}{p-1}}) \text{ if } p<\frac{N+2}{3}\\ O(\ln(1/\epsilon)) \text{ if } p=\frac{N+2}{3}\\ O(1) \text{ if } \frac{N+2}{3}<p<\frac{N+1}{2}\end{cases}\\ \frac{\omega_{N-2}}{2}\ln(1/\epsilon)\left(\frac12\sum\lambda_i^2+\sum_{i<j}\lambda_i\lambda_j+o(1)\right) \text{ if } p=\frac{N+1}{2}\\ O(1) \text{ if } p>\frac{N+1}{2}\end{cases}\tag{3.9}$$

We thus obtain

$$I_2 = \begin{cases} \epsilon^{1-\frac{N-p}{p-1}} \frac{H(0)\omega_{N-2}}{4} \frac{\Gamma\left(\frac{N+1}{2}\right)\Gamma\left(\frac{N-2p+1}{2(p-1)}\right)}{\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)} \\ \qquad -\epsilon^{2-\frac{N-p}{p-1}} \frac{\omega_{N-2}}{16} \frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-2p+1}{2(p-1)}\right)}{\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)} \left(\frac{3}{2}\sum \lambda_i^2 + \sum_{i<j}\lambda_i\lambda_j\right) \\ \qquad + \begin{cases} O(\epsilon^{3-\frac{N-p}{p-1}}) \text{ if } p < \frac{N+3}{4} \\ O(\ln(1/\epsilon)) \text{ if } p = \frac{N+3}{4} \\ O(1) \text{ if } \frac{N+3}{4} < p < \frac{N+1}{2} \end{cases} \\ \frac{H(0)\omega_{N-2}}{2} \ln(1/\epsilon)(1+o(1)) \text{ if } p = \frac{N+1}{2} \\ O(1) \text{ if } p > \frac{N+1}{2} \end{cases}$$

So the proof of (3.2) is completed.

To prove (3.3), we first observe that

$$\begin{aligned} \int_\Omega h(x)|u_\epsilon|^p\,dx &= h(0)\int_\Omega |u_\epsilon|^p\,dx + O\left(\int_\Omega |x||u_\epsilon|^p\,dx\right) \\ &= h(0)\int_{Q_a} |u_\epsilon|^p\,dx + O\left(\int_{Q_a\setminus\Omega} |u_\epsilon|^p\,dx + \int_{Q_a} |x||u_\epsilon|^p\,dx\right), \end{aligned} \tag{3.10}$$

where, as before, $Q_a = \{(y,t) \mid |y| \le a \text{ and } 0 \le t \le a\}$.

Now,

$$\begin{aligned} \int_{Q_a} |u_\epsilon|^p dx &= \int_{|y|\le a, 0<t\le a} \frac{dydt}{[(t+\epsilon)^2+|y|^2]^{\frac{p(N-p)}{2(p-1)}}} + O(1) \\ &= \epsilon^{-\frac{N-p^2}{p-1}} \int_{|y|\le a/\epsilon, 0<t\le a/\epsilon} \frac{dydt}{[(1+t)^2+|y|^2]^{\frac{p(N-p)}{2(p-1)}}} + O(1) \\ &= \begin{cases} O(\ln(1/\epsilon)) \text{ if } p^2 = N \\ O(1) \text{ if } p^2 > N \end{cases} \end{aligned}$$

If $p^2 < N$, using the change of variable $y = (1+t)z$ and then passing to polar coordinates, we get

$$\int_{Q_a} |u_\epsilon|^p dx = \epsilon^{-\frac{N-p^2}{p-1}} \omega_{N-2} \int_0^\infty \frac{dt}{(1+t)^{\frac{N-p^2}{p-1}+1}} \int_0^\infty \frac{r^{N-2}\,dr}{(1+r^2)^{\frac{p(N-p)}{2(p-1)}}} + O(1)$$

Hence

$$\int_{Q_a} |u_\epsilon|^p dx = \begin{cases} \epsilon^{-\frac{N-p^2}{p-1}} \frac{p-1}{N-p^2}\omega_{N-2} \frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-p^2+p-1}{2(p-1)}\right)}{2\Gamma\left(\frac{p(N-p)}{2(p-1)}\right)} + O(1) \text{ if } p^2 < N \\ O(\ln(1/\epsilon)) \text{ if } p^2 = N \\ O(1) \text{ if } p^2 > N \end{cases} \tag{3.11}$$

On the other hand, using Taylor's formula,

$$
\begin{aligned}
\int_{Q_a\setminus\Omega} |u_\epsilon|^p dx &= \int_{|y|\le a}\int_0^{\rho(y)} \frac{dt}{[(t+\epsilon)^2+|y|^2]^{\frac{p(N-p)}{2(p-1)}}}\,dy + O(1)\\
&= O\left(\int_{|y|\le a} \frac{|y|^2\,dy}{(\epsilon^2+|y|^2)^{\frac{p(N-p)}{2(p-1)}}}\,dy\right) + O(1)\\
&= \epsilon^{1-\frac{N-p^2}{p-1}} O\left(\int_0^{a/\epsilon} \frac{r^N\,dr}{(1+r^2)^{\frac{p(N-p)}{2(p-1)}}}\right) + O(1) \qquad (3.12)\\
&= \begin{cases} O(\epsilon^{1-\frac{N-p^2}{p-1}}) \text{ if } p<\frac{-1+\sqrt{4N+5}}{2}\\ O(\ln(1/\epsilon)) \text{ if } p=\frac{-1+\sqrt{4N+5}}{2}\\ O(1) \text{ if } p>\frac{-1+\sqrt{4N+5}}{2}\end{cases}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\int_{Q_a} |x||u_\epsilon|^p dx &= \int_{Q_a} \frac{|(y,t)|}{[(t+\epsilon)^2+|y|^2]^{\frac{p(N-p)}{2(p-1)}}}\,dydt + O(1)\\
&= \epsilon^{1-\frac{N-p^2}{p-1}} \int_{Q_{a/\epsilon}} \frac{|(y,t)|}{[(1+t)^2+|y|^2]^{\frac{p(N-p)}{2(p-1)}}}\,dydt + O(1) \qquad (3.13)\\
&= \begin{cases} O(\epsilon^{1-\frac{N-p^2}{p-1}}) \text{ if } p<\frac{-1+\sqrt{4N+5}}{2}\\ O(\ln(1/\epsilon)) \text{ if } p=\frac{-1+\sqrt{4N+5}}{2}\\ O(1) \text{ if } p>\frac{-1+\sqrt{4N+5}}{2}\end{cases}
\end{aligned}
$$

Combining (3.10), (3.11), (3.12) and (3.13), gives (3.3).

Finally, to prove (3.4), we first observe that

$$\int_{\partial\Omega} |u_\epsilon|^{p_*}\,dS = \int_{Q_a} |u_\epsilon|^{p_*}\,dS$$

for small ϵ and so

$$
\begin{aligned}
\int_{\partial\Omega} |u_\epsilon|^{p_*}\,dS &= \int_{|y|\le a} \frac{\sqrt{1+|\nabla\rho|^2}}{[(\epsilon+\rho(y))^2+|y|^2]^{\frac{p(N-1)}{2(p-1)}}}\,dy\\
&= \int_{|y|\le a} \frac{1+\frac12|\nabla\rho|^2+O(|y|^4)}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}}\Big[1-\frac{p(N-1)}{2(p-1)}\frac{\rho(2\epsilon+\rho)}{\epsilon^2+|y|^2}\\
&\qquad - c_{N,p}\frac{\rho^2(2\epsilon+\rho)^2}{(\epsilon^2+|y|^2)^2} + O\left(\frac{\rho^3(2\epsilon+\rho)^3}{(\epsilon^2+|y|^2)^3}\right)\Big]\,dy,
\end{aligned}
$$

where

$$c_{N,p} = -\frac{p(N-1)}{4(p-1)}\left[\frac{p(N-1)}{2(p-1)}+1\right].$$

Hence

$$\int_{\partial\Omega} |u_\epsilon|^{p_*}\, dS =$$
$$= \int_{|y|\le a} \frac{dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}}\, dy - \epsilon^{\frac{p(N-1)}{p-1}} \int_{|y|\le a} \frac{\rho(y)\, dy}{(\epsilon^2+|y|^2)^{1+\frac{p(N-1)}{2(p-1)}}}$$
$$+ \frac{1}{2}\int_{|y|\le a} \frac{|\nabla\rho|^2\, dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}} - \frac{p(N-1)}{2(p-1)} \int_{|y|\le a} \frac{\rho^2(y)\, dy}{(\epsilon^2+|y|^2)^{1+\frac{p(N-1)}{2(p-1)}}}$$
$$- 4\epsilon^2 c_{N,p} \int_{|y|\le a} \frac{\rho^2(y)\, dy}{(\epsilon^2+|y|^2)^{2+\frac{p(N-1)}{2(p-1)}}}$$
$$+ O\left(\int_{|y|\le a} \frac{|y|^4\, dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}}\, dy + \epsilon \int_{|y|\le a} \frac{|y|^4\, dy}{(\epsilon^2+|y|^2)^{1+\frac{p(N-1)}{2(p-1)}}}\, dy \right)$$
$$= I_5 - \epsilon^{\frac{p(N-1)}{p-1}} I_7 + \frac{1}{2} I_6 - \frac{p(N-1)}{2(p-1)} I_8 - 4\epsilon^2 c_{N,p} I_9 + O(I_{10}).$$

We first compute I_5 as follows:

$$\begin{aligned} I_5 &= \int_{|y|\le a} \frac{dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}} = \omega_{N-2}\epsilon^{-1-\frac{N-p}{p-1}} \int_0^{a/\epsilon} \frac{r^{N-2}\, dr}{(1+r^2)^{\frac{p(N-1)}{2(p-1)}}} \\ &= \omega_{N-2}\epsilon^{-1-\frac{N-p}{p-1}} \int_0^{\infty} \frac{r^{N-2}\, dr}{(1+r^2)^{\frac{p(N-1)}{2(p-1)}}} + O(1) \\ &= \omega_{N-2}\epsilon^{-1-\frac{N-p}{p-1}} \frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-1}{2(p-1)}\right)}{2\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)} + O(1). \end{aligned} \tag{3.14}$$

According to (3.6) and (3.7), using the relation $\Gamma\left(\frac{N+1}{2}\right) = \frac{N-1}{2}\Gamma\left(\frac{N-1}{2}\right)$, we have

(3.15)

$$I_6 = \int_{|y|\le a} \frac{|\nabla\rho|^2\, dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}}$$
$$= \sum \lambda_i^2 \int_{|y|\le a} \frac{|y_i|^2\, dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}} + O\left(\int_{|y|\le a} \frac{|y|^4\, dx}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}} \right)$$
$$= \frac{\sum \lambda_i^2}{N-1} \int_{|y|\le a} \frac{|y|^2\, dy}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}} + O\left(\int_{|y|\le a} \frac{|y|^4\, dx}{(\epsilon^2+|y|^2)^{\frac{p(N-1)}{2(p-1)}}} \right)$$
$$= \begin{cases} \frac{1}{4}\sum \lambda_i^2 \omega_{N-2}\epsilon^{1-\frac{N-p}{p-1}} \frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-2p+1}{2(p-1)}\right)}{\Gamma\left(\frac{p(N-1)}{2(p-1)}\right)} + \begin{cases} O(\epsilon^{3-\frac{N-p}{p-1}}) \text{ if } p < \frac{N+3}{4} \\ O(\ln(1/\epsilon)) \text{ if } p = \frac{N+3}{4} \\ O(1) \text{ if } \frac{N+1}{2} > p > \frac{N+3}{4} \end{cases} \\ \frac{\omega_{N-2}\sum\lambda_i^2}{N-1} \ln(1/\epsilon)) \text{ if } p = \frac{N+1}{2} \\ O(1) \text{ if } p > \frac{N+1}{2} \end{cases}$$

By radial symmetry, we have

$$
\begin{aligned}
I_7 =& \int_{|y|\le a} \frac{\rho(y)\,dy}{(\epsilon^2+|y|^2)^{1+\frac{p(N-1)}{2(p-1)}}} \\
=& \frac{\sum \lambda_i}{2(N-1)} \int_{|y|\le a} \frac{|y|^2\,dy}{(\epsilon^2+|y|^2)^{1+\frac{p(N-1)}{2(p-1)}}} + O\left(\int_{|y|\le a} \frac{|y|^4\,dy}{(\epsilon^2+|y|^2)^{1+\frac{p(N-1)}{2(p-1)}}} \right) \\
=& \frac{\omega_{N-2}\sum \lambda_i}{2(N-1)} \epsilon^{-1-\frac{N-p}{p-1}} \int_0^{a/\epsilon} \frac{r^N\,dr}{(1+r^2)^{1+\frac{p(N-1)}{2(p-1)}}} \\
&+ \epsilon^{-\frac{N-p}{p-1}} O\left(\int_0^{a/\epsilon} \frac{r^{N+2}\,dr}{(1+r^2)^{1+\frac{p(N-1)}{2(p-1)}}} \right) \\
=& \frac{\omega_{N-2}\sum \lambda_i}{2(N-1)} \epsilon^{-1-\frac{N-p}{p-1}} \int_0^{\infty} \frac{r^N\,dr}{(1+r^2)^{1+\frac{p(N-1)}{2(p-1)}}} + \begin{cases} O(\epsilon^{1-\frac{N-p}{p-1}}) \text{ if } p<\frac{N+1}{2} \\ O(\ln(1/\epsilon)) \text{ if } p=\frac{N+1}{2} \\ O(\epsilon^{-\frac{N-p}{p-1}}) \text{ if } p>\frac{N+1}{2} \end{cases}
\end{aligned}
$$

and so

$$
\begin{aligned}
I_7 =& \frac{\omega_{N-2}\sum \lambda_i}{8} \epsilon^{-1-\frac{N-p}{p-1}} \frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-1}{2(p-1)}\right)}{\Gamma\left(1+\frac{p(N-1)}{2(p-1)}\right)} \\
&+ \begin{cases} O(\epsilon^{1-\frac{N-p}{p-1}}) \text{ if } p<\frac{N+1}{2} \\ O(\ln(1/\epsilon)) \text{ if } p=\frac{N+1}{2} \\ O(\epsilon^{-\frac{N-p}{p-1}}) \text{ if } p>\frac{N+1}{2} \end{cases}
\end{aligned}
\tag{3.16}
$$

To compute I_9 we proceed as in the computations of I_4, i.e.

$$
\begin{aligned}
I_9 =& \int_{|y|\le a} \frac{\rho^2(y)\,dy}{(\epsilon^2+|y|^2)^{2+\frac{p(N-1)}{2(p-1)}}} \\
=& \frac{1}{4}\sum \lambda_i^2 \int_{|y|\le a} \frac{y_1^4\,dy}{(\epsilon^2+|y|^2)^{2+\frac{p(N-1)}{2(p-1)}}} \\
&+ \frac{1}{2}\sum_{i<j} \lambda_i\lambda_j \int_{|y|\le a} \frac{y_i^2 y_j^2\,dy}{(\epsilon^2+|y|^2)^{2+\frac{p(N-1)}{2(p-1)}}} + O\left(\int_{|y|\le a} \frac{|y|^5\,dy}{(\epsilon^2+|y|^2)^{2+\frac{p(N-1)}{2(p-1)}}} \right).
\end{aligned}
$$

Now

$$
\begin{aligned}
\int_{|y|\le a} \frac{y_1^4\,dy}{(\epsilon^2+|y|^2)^{2+\frac{p(N-1)}{2(p-1)}}} &= \epsilon^{-\frac{N-1}{p-1}} \int_{\mathbb{R}^{N-1}} \frac{y_1^4\,dy}{(1+|y|^2)^{2+\frac{p(N-1)}{2(p-1)}}} + O(1) \\
&= 2\epsilon^{-\frac{N-1}{p-1}} \omega_{N-3} \int_0^\infty \frac{r^{N-3}\,dr}{(1+r^2)^{\frac{p(N-1)}{2(p-1)}-\frac{1}{2}}} \int_0^\infty \frac{s^4\,ds}{(1+s^2)^{2+\frac{p(N-1)}{2(p-1)}}} + O(1) \\
&= \frac{3\omega_{N-2}}{8} \epsilon^{-\frac{N-1}{p-1}} \frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-1}{2(p-1)}\right)}{\Gamma\left(2+\frac{p(N-1)}{2(p-1)}\right)} + O(1),
\end{aligned}
$$

$$\begin{aligned}
\int_{|y|\le a}\frac{y_i^2y_j^2\,dy}{(\epsilon^2+|y|^2)^{2+\frac{p(N-1)}{2(p-1)}}} &= \epsilon^{-\frac{N-1}{p-1}}\int_{\mathbb{R}^{N-1}}\frac{y_i^2y_j^2\,dy}{(1+|y|^2)^{2+\frac{p(N-1)}{2(p-1)}}}+O(1)\\
=&4\epsilon^{-\frac{N-1}{p-1}}\omega_{N-4}\int_0^\infty\frac{r^{N-4}\,dr}{(1+r^2)^{\frac{p(N-1)}{2(p-1)}-1}}\int_0^\infty\frac{y_i^2\,dy_i}{(1+y_i^2)^{\frac{1}{2}+\frac{p(N-1)}{2(p-1)}}}\\
&\times\int_0^\infty\frac{y_j^2dy_j}{(1+y_j^2)^{2+\frac{p(N-1)}{2(p-1)}}}+O(1)\\
=&\frac{\omega_{N-2}}{8}\epsilon^{-\frac{N-1}{p-1}}\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-1}{2(p-1)}\right)}{\Gamma\left(2+\frac{p(N-1)}{2(p-1)}\right)}+O(1),
\end{aligned}$$

and

$$\begin{aligned}
\int_{|y|\le a}\frac{|y|^5\,dy}{(\epsilon^2+|y|^2)^{2+\frac{p(N-1)}{2(p-1)}}} &= \epsilon^{-\frac{N-p}{p-1}}\omega_{N-2}\int_0^{a/\epsilon}\frac{r^{N+3}\,dr}{(1+r^2)^{2+\frac{p(N-1)}{2(p-1)}}}\\
&= O(\epsilon^{-\frac{N-p}{p-1}})
\end{aligned}$$

Hence

$$\begin{aligned}
I_9 =&\frac{\omega_{N-2}}{16}\epsilon^{-\frac{N-1}{p-1}}\frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-1}{2(p-1)}\right)}{\Gamma\left(2+\frac{p(N-1)}{2(p-1)}\right)}\left(\frac{3}{2}\sum\lambda_i^2+\sum_{i<j}\lambda_i\lambda_j\right)\\
&+O(\epsilon^{-\frac{N-p}{p-1}}).
\end{aligned}\tag{3.17}$$

Finally, for I_{10} we have,

$$\begin{aligned}
I_{10} =&\epsilon^{3-\frac{N-p}{p-1}}\omega_{N-2}\int_0^{a/\epsilon}\frac{r^{N+2}\,dr}{(1+r^2)^{\frac{p(N-1)}{2(p-1)}}}+\epsilon^{2-\frac{N-p}{p-1}}\omega_{N-2}\int_0^{a/\epsilon}\frac{r^{N+2}\,dr}{(1+r^2)^{1+\frac{p(N-1)}{2(p-1)}}}\\
=&\begin{cases}O(\epsilon^{3-\frac{N-p}{p-1}}) \text{ if } p<\frac{N+3}{4}\\ O(\ln(1/\epsilon)) \text{ if } p=\frac{N+3}{4}\\ O(1) \text{ if } p>\frac{N+3}{4}\end{cases}+\begin{cases}O(\epsilon^{2-\frac{N-p}{p-1}}) \text{ if } p<\frac{N+1}{2}\\ O(\epsilon\ln(1/\epsilon)) \text{ if } p=\frac{N+1}{2}\\ O(\epsilon) \text{ if } p>\frac{N+1}{2}\end{cases}
\end{aligned}$$

and so

$$I_{10}=\begin{cases}O(\epsilon^{2-\frac{N-p}{p-1}}) \text{ if } p\le\frac{N+2}{3}\\ O(1) \text{ if } p>\frac{N+2}{3}\end{cases}\tag{3.18}$$

Putting these estimates together, we arrive at (3.4). This completes the proof of Step 1. □

Step 2. *We have, for any dimension $N\ge 2$,*

$$K_p^{-1}\frac{\int_\Omega|\nabla u_\epsilon|^p+|u_\epsilon|^p\,dx}{\left(\int_{\partial\Omega}|u_\epsilon|^{p_*}\,dS\right)^{p/p_*}}=\begin{cases}1+O(\epsilon^{\frac{N-p}{p-1}}) \textit{ if } p>\frac{N+1}{2}\\ 1-\frac{N-1}{2}H(0)\epsilon\ln(1/\epsilon)+o(\epsilon\ln(1/\epsilon)) \textit{ if } p=\frac{N+1}{2}\end{cases}$$

and, if $p < \frac{N+1}{2}$*, for dimension* $N = 2, 3, 4$

$$K_p^{-1}\frac{\displaystyle\int_\Omega |\nabla u_\epsilon|^p + |u_\epsilon|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |u_\epsilon|^{p_*}\,dS\right)^{p/p_*}} = 1 - \frac{(N-p)(p-1)}{N-2p+1}H(0)\epsilon + \begin{cases} \frac{D}{A_1}\epsilon^p + \begin{cases} E\epsilon^2 + O(\epsilon^{1+p}) \ \textit{if } p < \frac{N+2}{3} \\ O(\epsilon^{\frac{N-p}{p-1}}) \ \textit{if } \frac{N+2}{3} \le p < \sqrt{N} \end{cases} \\ O(\epsilon^{\frac{N-p}{p-1}}\ln(1/\epsilon)) \ \textit{if } p = \sqrt{N} \\ O(\epsilon^{\frac{N-p}{p-1}}) \ \textit{if } \sqrt{N} < p < \frac{N+1}{2} \end{cases}$$

where

$$E = \frac{(N-p)(p-1)}{4(N-1)(N-2p+1)}\left\{\frac{p+N-2}{N-1}\sum \lambda_i^2 - 2\sum_{i<j}\lambda_i\lambda_j\right\}.$$

Also, for dimensions $N \ge 5$*,*

$$K_p^{-1}\frac{\displaystyle\int_\Omega |\nabla u_\epsilon|^p + |u_\epsilon|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |u_\epsilon|^{p_*}\,dS\right)^{p/p_*}} = 1 - \frac{(N-p)(p-1)}{N-2p+1}H(0)\epsilon + \begin{cases} E\epsilon^2 + \begin{cases} \frac{D}{A_1}\epsilon^p + \begin{cases} o(\epsilon^2) \ \textit{if } p \le 2 \\ o(\epsilon^p) \ \textit{if } 2 \le p < \sqrt{N} \end{cases} \\ o(\epsilon^2) \ \textit{if } \sqrt{N} \le p < \frac{N+2}{3} \end{cases} \\ O(\epsilon^2) \ \textit{if } \frac{N+2}{3} \le p < \frac{N+1}{2} \end{cases}$$

Proof of Step 2. Noting that

$$\frac{A_1}{B_1^{\frac{N-p}{N-1}}} = K_p^{-1},$$

we have, when e.g. $n \ge 6$ and $p \le 2$, that

$$K_p^{-1}\frac{\displaystyle\int_\Omega |\nabla u_\epsilon|^p + |u_\epsilon|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |u_\epsilon|^{p_*}\,dS\right)^{p/p_*}} = 1 + \left(\frac{A_2}{A_1} - \frac{N-p}{N-1}\frac{B_2}{B_1}\right)\epsilon + \frac{D}{A_1}\epsilon^p$$
$$+ \left\{\frac{N-p}{N-1}\left[\frac{1}{2}\left(\frac{N-p}{N-1}+1\right)\left(\frac{B_2}{B_1}\right)^2 - \frac{B_3}{B_1} - \frac{B_2}{B_1}\frac{A_2}{A_1}\right] + \frac{A_3}{A_1}\right\}\epsilon^2 + o(\epsilon^2).$$

Using the fact that

$$\Gamma\left(\frac{N+1}{2}\right) = \Gamma\left(\frac{N-1}{2}+1\right) = \frac{N-1}{2}\Gamma\left(\frac{N-1}{2}\right)$$
$$\Gamma\left(\frac{N-1}{2(p-1)}\right) = \Gamma\left(\frac{N-2p+1}{2(p-1)}+1\right) = \frac{N-2p+1}{2(p-1)}\Gamma\left(\frac{N-2p+1}{2(p-1)}\right),$$

we get

$$\frac{A_2}{A_1} = -\frac{1}{2}\frac{N-p}{N-2p+1}\sum \lambda_i,$$

$$\frac{A_3}{A_1} = \frac{1}{4}\frac{N-p}{N-2p+1}\left\{\frac{3}{2}\sum \lambda_i^2 - 2\sum_{i<j}\lambda_i\lambda_j\right\},$$

$$\frac{B_2}{B_1} = -\frac{1}{2}\sum \lambda_i,$$

$$\frac{B_3}{B_1} = \frac{1}{8(N-2p+1)}\left\{(3N-5p+2)\sum \lambda_i^2 - 4(N-p)\sum_{i<j}\lambda_i\lambda_j\right\},$$

$$\frac{D}{A_1} = \begin{cases} \frac{2h(0)}{(N-3)(N-4)} \text{ if } p=2 \\ \text{has same sign as } \frac{h(0)}{N-p^2} \text{ otherwise.}\end{cases}$$

Hence

$$\frac{A_2}{A_1} - \frac{N-p}{N-1}\frac{B_2}{B_1} = -\frac{(N-p)(p-1)}{N-2p+1}H(0)$$

and

$$\frac{N-p}{N-1}\left[\frac{1}{2}\left(\frac{N-p}{N-1}+1\right)\left(\frac{B_2}{B_1}\right)^2 - \frac{B_3}{B_1} - \frac{B_2}{B_1}\frac{A_2}{A_1}\right] + \frac{A_3}{A_1}$$

$$= \frac{(N-p)(p-1)}{4(N-1)(N-2p+1)}\left\{\frac{p+N-2}{N-1}\sum \lambda_i^2 - 2\sum_{i<j}\lambda_i\lambda_j\right\},$$

which gives the result. We get the others equalities in much the same way. □

Proof of Theorem 1.1. At this point is just a combination of Steps 1 and 2. □

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