PERIDYNAMICS AND ANISOTROPIC FRACTIONAL SOBOLEV SPACES WITH VARIABLE EXPONENTS

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Abstract. In this paper, our primary objective is to develop the peridynamic fractional Sobolev space and establish novel BBM-type results associated with it. We also address the peridynamic fractional anisotropic $p-$Laplacian. A secondary objective is to explore anisotropic fractional Sobolev spaces with variable exponents, where we also derive new BBM-type results. Additionally, we address the eigenvalue problem in the isotropic case.

1. Introduction

Given a measurable function u on $\mathbb{R}^N$, and an index vector $\overrightarrow{p} = (p_1, p_2, \ldots, p_N)$, where $1 \leq p_j \leq \infty$ for each j, we can calculate the numbers $\|u\|_{\overrightarrow{p}}$ by first calculating the L^{p_1}-norm of $u(x_1, \ldots, x_N)$ with respect to x_1, and then the L^{p_2}-norm of the result with respect to x_2, and so on finishing with the L^{p_N}-norm with respect to x_N:

$$\|u\|_{\overrightarrow{p}} = \left\| \cdots \left\| \|u\|_{L^{p_1}(dx_1)} \right\|_{L^{p_2}(dx_2)} \cdots \right\|_{L^{p_N}(dx_N)}$$

where

$$\|f\|_{L^p(dt)} = \begin{cases} \left(\int_{-\infty}^{\infty} |f(\ldots, t, \ldots)|^p dt \right)^{1/p} & \text{if } 1 < p < \infty \\ \operatorname{ess\,sup}_t |f(\ldots, t, \ldots)| & \text{if } p = \infty. \end{cases}$$

If all the numbers p_j are finite, then

$$\|u\|_{\overrightarrow{p}} = \left(\int_{-\infty}^{\infty} \cdots \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |u(x_1, \ldots, x_n)|^{p_1} \, \mathrm{d}x_1 \right)^{p_2/p_1} \mathrm{d}x_2 \right)^{p_3/p_2} \ldots \mathrm{d}x_N \right)^{1/p_N}.$$

We will denote by $L^{\overrightarrow{p}} = L^{\overrightarrow{p}}\left(\mathbb{R}^N\right)$ the Banach space of set of functions u for which $\|u\|_{\overrightarrow{p}} < \infty$, where equivalence classes of almost everywhere equal. We refer to Benedek and Panzone [2] for general information on spaces $L^{\overrightarrow{p}}$.

We define the anisotropic Sobolev space as the class of function u in $L^{\overrightarrow{p}}$ whose partial derivatives $\partial_i u$ are elements of L^{p_i}, see [1] and references therein for more details.

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This type of anisotropy led to two possible generalizations: one for Sobolev spaces with a variable exponent and the other for fractional Sobolev spaces.
For Sobolev spaces with a variable exponent, variational problems have been extensively studied, with a particular focus on the isotropic variable exponent Sobolev spaces, as detailed in [7, 8, 9, 10]. In [3], Fan carried out a systematic investigation of anisotropic variable exponent Sobolev spaces, providing an important theoretical framework for studying anisotropic and parabolic equations involving the operator $-\Delta_{\overrightarrow{p}(x)}u := -\sum_{i=1}^{N}\partial_i\left(|\partial_i u|^{p_i(x)-2}\partial_i u\right)$, (see [4, 5, 6]).
While variable exponent Sobolev spaces extend the classical theory by allowing the integrability order to vary, the fractional framework further generalizes this concept by using non-integer orders of differentiability, leading to a richer and more nuanced analysis of function spaces. Fractional order Sobolev spaces have gained significant attention recently, particularly following the influential papers [11] and [12]. In the seminal paper [13], Bourgain, Brezis and Mironescu proved that for any smooth bounded domain $\Omega \subset \mathbb{R}^N, u \in W^{1,p}(\Omega), 1 \leq p < \infty$, the well-known fractional Gagliardo seminorm recovers its local counterpart as s goes to 1, in the sense that

$$\lim_{s\uparrow 1}(1-s)\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy = K(N,p)\int_{\Omega}|\nabla u|^p dx,$$

where the constant $K(N,p)$ is defined as

$$K(N,p) = \frac{1}{p}\int_{\mathbb{S}^{N-1}}|\boldsymbol{\omega}\cdot h|^p d\mathcal{H}^{N-1}(h).$$

Here $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ denotes the unit sphere, $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure and ω is an arbitrary unit vector of $\mathbb{R}^N$. See [14] for a survey on this and related results.
In [17], the authors studied the fractional anisotropic Sobolev space, where they considered different fractional regularity and integrability in each coordinate direction. They introduced N different fractional parameters $s_1, \cdots, s_N$ and N different integrability parameters $p_1, \cdots, p_N$, defining a space of functions $u(x) = u(x_1, \cdots, x_N)$ such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}}\frac{|u(x+he_i)-u(x)|^{p_i}}{|h|^{1+s_ip_i}}dhdx < \infty \quad \text{for} \quad i = 1, \cdots, N.$$

A Bourgain-Brezis-Mironescu (BBM) type result for an energy functional strongly related to the fractional anisotropic $p-$Laplacian has been recently proven in [15].

One of the primary goals of this paper is, based on [27, 28], to study the peridynamic problem associated with the fractional anisotropic p-Laplacian. Peridynamics is distinguished by its nonlocality, which allows points separated by a positive distance to exert forces on each other. This feature sets peridynamics apart from classical theories that rely on gradients. As a result, peridynamics is particularly effective for addressing problems involving discontinuities, such as fractures, dislocations,

or multi-scale materials. Hence, for a given real number $\delta > 0$, $0 < s < 1$, and $1 < p < \infty$, we will analyze the behavior when $\delta \to 0$ of the functional of the form

$$[u]^i_{s,p,\delta} = \int_{\mathbb{R}^N} \int_{|h|\leq\delta} \frac{|u(x+he_i) - u(x)|^p}{|h|^{1+sp}}\, dh\, dx, \tag{1}$$

and the associated nonlocal operator, namely

$$(-\Delta_p)^s_\delta u(x) = p.v. \int_{|h|\leq\delta} \frac{|u(x+he_i) - u(x)|^{p-2}(u(x+he_i) - u(x))}{|h|^{1+sp}}\, dh.$$

It can be seen as a *peridynamic fractional anisotropic p-laplacian* because $(-\Delta_p)^s_\delta$ represents a truncation of the fractional anisotropic p-laplacian analyzed in the paper [15].

Recently, a novel category of fractional Sobolev spaces with variable exponents, denoted as $W^{s,\bar{p},p(.,.)}(\Omega)$, was introduced by U. Kaufmann *et al.* [18]. Initially proposed in their work, this framework has subsequently undergone refinement in [19], where it is precisely defined as

$$W^{s,\bar{p},p(.,.)}(\Omega) = \left\{ u \in L^{\bar{p}}(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy < \infty \right\},$$

where Ω is an open subset of $\mathbb{R}^N$ $(N \geq 2)$, $s \in (0,1)$, $\bar{p}(x) = p(x,x)$ and p satisfying

$$\begin{aligned} &p \in C\left(\mathbb{R}^N \times \mathbb{R}^N, (1+\infty)\right), \\ &p(x,y) = p(y,x) \text{ for all } x, y \in \mathbb{R}^N. \end{aligned} \tag{P}$$

A second goal of this paper is to explore the anisotropic properties of fractional Sobolev spaces with variable exponents, which will be discussed in Section 4.

Definition 1.1. Given $i = 1, \cdots, N$, $p_0 \in C\left(\mathbb{R}^N, (1+\infty)\right)$, Ω an open subset of $\mathbb{R}^N$, and p_i satisfying (P). Let $\vec{s} = (s_1, \cdots, s_N)$, and $\vec{p}(\cdot,\cdot) = (p_1(\cdot,\cdot), \cdots, p_N(\cdot,\cdot))$. We introduce the *anisotropic fractional Sobolev spaces with variable exponent* $W^{\vec{s},\vec{p}(\cdot,\cdot)}_{p_0}(\Omega)$ as

$$W^{\vec{s},\vec{p}(\cdot,\cdot)}_{p_0}(\mathbb{R}^N) = \left\{ u \in L^{p_0}(\mathbb{R}^N) : \quad \sum_{i=1}^N J_{s_i,p_i}(u) < \infty \right\},$$

equipped with the norm

$$\|u\|_{\vec{s},p_0,\vec{p}} = \|u\|_{p_0} + \sum_{i=1}^N [u]_{s_i,p_i},$$

where

$$J_{s_i,p_i}(u) = \int_{\mathbb{R}^N} \int_{\mathbb{R}} \frac{|u(x+he_i) - u(x)|^{p_i(x,x+he_i)}}{|h|^{1+s_ip_i(x,x+he_i)}}\, dhdx,$$

and

$$[u]_{s_i,p_i} = \inf\left\{ \lambda > 0 : \quad J_{s_i,p_i}\left(\frac{u}{\lambda}\right)) \leq 1 \right\}.$$

The rest of the paper is organized as follows: Section 2 is a preliminary section to establish the notation. In Section 3, we address the Peridynamics case. In Section 4, we cover the anisotropic variable case. Note that in the last section, we sometimes do not specify the exponent as variable in the notations if it is clear from the context.

2. Preliminaries

In this section, we briefly review the definitions and list some basic properties of the Lebesgue spaces with variable exponent and the fractional Sobolev spaces with variable exponent.

2.1. **Variable exponent Lebesgue spaces.** Let Ω be a Lipschitz domain in $\mathbb{R}^N$ and $r \in C_+(\overline{\Omega})$. We define the variable exponent Lebesgue space

$$L^{r(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R}, \text{ measurable and } \int_\Omega |u(x)|^{r(x)} dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{r,\Omega} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{r(x)} \frac{dx}{r(x)} \leq 1 \right\}.$$

In the Luxemburg norm definition, we used $\frac{dx}{r(x)}$, recognizing that this choice is inconsequential; it remains equivalent to using dx alone and streamlines the equations slightly. When there is no confusion we shall often write $\|\cdot\|_r$ instead of $\|\cdot\|_{r,\Omega}$. Some basic properties of $L^{r(x)}(\Omega)$ are listed in the following two propositions.

Proposition 2.1 ([9, 10])**.** *The space $L^{r(\cdot)}(\Omega)$ is a separable, uniformly convex Banach space and its dual space is $L^{r'(\cdot)}(\Omega)$, where $\frac{1}{r(x)} + \frac{1}{r'(x)} = 1$. Furthermore, for any $u \in L^{r(\cdot)}(\Omega)$ and $v \in L^{r'(x)}(\Omega)$, we have*

$$\left| \int_\Omega uv \, dx \right| \leq 2\|u\|_{r,\Omega} \|v\|_{r',\Omega}.$$

Proposition 2.2 ([9])**.** *Define the modular $\rho_r : L^{r(\cdot)}(\Omega) \to \mathbb{R}$ by*

$$\rho_r(u) := \int_\Omega |u|^{r(x)} dx, \text{ for all } u \in L^{r(\cdot)}(\Omega).$$

Then, we have the following relations between norm and modular.

(i) $u \in L^{r(\cdot)}(\Omega) \setminus \{0\}$ *if and only if* $\rho_r\left(\frac{u}{\|u\|_{r,\Omega}}\right) = 1$.
(ii) $\rho_r(u) > 1$ $(= 1, < 1)$ *if and only if* $\|u\|_{r,\Omega} > 1$, $(= 1, < 1)$, *respectively.*
(iii) *If* $\|u\|_{r,\Omega} > 1$, *then* $\|u\|_{r,\Omega}^{r_-} \leq \rho_r(u) \leq \|u\|_{r,\Omega}^{r_+}$.
(iv) *If* $\|u\|_{r,\Omega} < 1$, *then* $\|u\|_{r,\Omega}^{r_+} \leq \rho_r(u) \leq \|u\|_{r,\Omega}^{r_-}$.
(v) *For a sequence* $(u_n) \subset L^{r(\cdot)}(\Omega)$ *and* $u \in L^{r(x)}(\Omega)$, *we have*

$$\lim_{n\to+\infty} \|u_n - u\|_{r,\Omega} = 0 \quad \text{if and only if} \quad \lim_{n\to+\infty} \rho_r(u_n - u) = 0.$$

2.2. **Fractional Sobolev spaces with variable exponents.** Let $s \in (0,1)$ and $p \in C(\overline{\Omega} \times \overline{\Omega})$ be such that p is symmetric, i.e., $p(x,y) = p(y,x)$ for all $x, y \in \overline{\Omega}$ and

$$1 < p^-(\Omega) := \inf_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \leq p^+(\Omega) := \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) < +\infty.$$

Let $\bar{p}(x) = p(x,x)$, then $\bar{p} \in C_+(\overline{\Omega})$, and we define

$$W^{s,\bar{p},p(.,.)}(\Omega) = \left\{ u \in L^{\bar{p}(x)}(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy < \infty \right\},$$

and for $u \in W^{s,\bar{p},p(.,.)}(\Omega)$, set

$$[u]_{s,p(.,.),\Omega} = \inf \left\{ \lambda > 0 : \ \rho_{p(.,.)}\left(\frac{u}{\lambda}\right) < 1 \right\},$$

where

$$\rho_{p(.,.)}(u) = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s\tilde{p}(x,y)}} \frac{dxdy}{p(x,y)}.$$

Then, $W^{s,\bar{p},p(.,.)}(\Omega)$ endowed with the norm

$$\|u\|_{s,\bar{p},p,\Omega} := \|u\|_{\bar{p},\Omega} + [u]_{s,p(.,.),\Omega},$$

is a separable and reflexive Banach space (see [18, 21, 22]).

Theorem 2.3. *([19]) Assume that $sp^+(\Omega) < N$. Then, it holds that*

(1) *the embedding $W^{s,\bar{p},p(.,.)}(\mathbb{R}^N) \hookrightarrow L^{r(x)}(\mathbb{R}^N)$ is continuous for any $r \in C_+(\mathbb{R}^N)$ satisfying $p(x) \leq r(x) \ll p_s^* := \frac{Np(x,x)}{N-sp(x,x)}$ for all $x \in \mathbb{R}^N$.*
(2) *if Ω is bounded, then the embedding $W^{s,p(.),p(.,.)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$ is compact for any $r \in C_+(\Omega)$ satisfying $r(x) < p_s^*$ for all $x \in \overline{\Omega}$.*

Invoking the continuity of p on $\bar{\Omega} \times \bar{\Omega}$ we extend p to $\mathbb{R}^N \times \mathbb{R}^N$ by using Tietze extension theorem, such that

$$1 < \inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y) \leq \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p(x,y) < \frac{N}{s}. \tag{P}$$

We now define the following space:

$$X = \left\{ u \in W^{s,\bar{p},p(\cdot,\cdot)}\left(\mathbb{R}^N\right) : \quad u = 0 \text{ on } \Omega^c \right\}$$

endowed with norm

$$\|u\|_X = [u]_{s,p(\cdot,\cdot),\mathbb{R}^N},$$

where $W^{s,\bar{p},p(\cdot,\cdot)}\left(\mathbb{R}^N\right)$ and $[u]_{s,p(\cdot,\cdot)\mathbb{R}^N}$ are defined in the same ways as $W^{s,\bar{p},p(\cdot,\cdot)}(\Omega)$ and $[u]_{s,p(\cdot,\cdot),\Omega}$ with Ω replaced by $\mathbb{R}^N$. Obviously, X is a closed subspace of $W^{s,\bar{p},p(\cdot,\cdot)}\left(\mathbb{R}^N\right)$ and hence, $(X, \|\cdot\|_X)$ is a reflexive and separable Banach space.

Lemma 2.4. *The functional* $\rho_{p(.,.)} : X \to \mathbb{R}$ *defined by*

$$\rho_{p(.,.)}(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \frac{dxdy}{p(x,y)},$$

has the following properties:

(i) *for* $\alpha > 0, \|u\|_X = (>,<)\alpha$ *if and only if* $\rho_{p(.,.)}\left(\frac{u}{\alpha}\right) = (>,<)1$,
(ii) *if* $\|u\|_X > 1$, *then* $\|u\|_X^{p^-} \le \rho_{p(.,.)}(u) \le \|u\|_X^{p^+}$,
(iii) *if* $\|u\|_X < 1$, *then* $\|u\|_X^{p^+} \le \rho_{p(.,.)}(u) \le \|u\|_X^{p^-}$.

Theorem 2.5 ([23])**.** *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded Lipschitz domain and let* $s \in (0,1)$. *Let* $p \in C\left(\mathbb{R}^N \times \mathbb{R}^N\right)$ *satisfy* (P). *Then, for any* $r \in C(\bar{\Omega})$ *satisfying*

$$r(x) < p_s^* \quad \text{for all} \quad x \in \overline{\Omega}, \tag{R}$$

there exists a constant $C > 0$ *such that*

$$\|u\|_{r,\Omega} \le C\|u\|_X, \quad \forall\ u \in X. \tag{2}$$

Moreover, the embedding $X \hookrightarrow L^{r(.)}(\Omega)$ *is compact.*

3. Peridynamics

In this section we will developed the theory for the peridynamic fractional anisotropic Sobolev space and the *peridynamic fractional anisotropic p-laplacian.*

3.1. **Peridynamic fractional anisotropic Sobolev space.** Given a real number $\delta > 0$, $0 < s < 1$ and $1 < p < \infty$. We will analyze, the behavior as $\delta \to 0$ of functional of the form

$$[u]^i_{s,p,\delta} = \int_{\mathbb{R}^n}\int_{|h|\le\delta} \frac{|u(x+he_i)-u(x)|^p}{|h|^{1+sp}}\, dh\, dx. \tag{3}$$

First, we need to define an appropriate functional space to work within; therefore, we introduce the anisotropic peridynamical Sobolev space as follows:

$$W_i^{s,p,\delta}(\mathbb{R}^n) := \{u \in L^p(\mathbb{R}^n),\ \text{such that } [u]^i_{s,p,\delta} < \infty\}. \tag{4}$$

This space is endowed with the norm

$$\|u\|^i_{s,p,\delta} = \left([u]^i_{s,p,\delta}\right)^{\frac{1}{p}} + \|u\|_p,$$

is a reflexive and separeble Banach space.
Now, in order to control every variable, we proceed as follows. Let $\vec{s} = (s_1,\dots,s_n)$, $\vec{p} = (p_1,\dots,p_n)$ and $\vec{\delta} = \{\delta_1,\dots,\delta_n\}$ with $s_i \in (0,1]$, $p_i \in (1,\infty)$ and $\delta_i > 0$ for $i = 1,\dots,n$. Then define

$$W^{\vec{s},\vec{p},\vec{\delta}}(\mathbb{R}^n) = \bigcap_{i=1}^n W_i^{s_i,p_i,\delta_i}(\mathbb{R}^n).$$

This anisotropic Sobolev space is then a separable and reflexive Banach space with norm

$$\|u\|_{\vec{s},\vec{p},\vec{\delta}} = \sum_{i=1}^{n} \|u\|^{i}_{s_i,p_i,\delta_i}.$$

Observe that the case when each δ_i tends to infinity corresponds to the Sobolev space studied in [17]. In that paper, the authors define the quantities

$$\overline{\vec{s}} = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{s_i}\right)^{-1}, \qquad \overline{\vec{s}\vec{p}} = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{s_i p_i}\right)^{-1}.$$

Then, assuming that $\overline{\vec{s}\vec{p}} < n$ the following critical Sobolev exponent is introduced

$$\vec{p}_{\vec{s}}{}^{\,*} = \vec{p}^{\,*} = \frac{n\overline{\vec{s}\vec{p}}/\overline{\vec{s}}}{n - \overline{\vec{s}\vec{p}}}. \tag{5}$$

The following theorem is a straightforward adaptation of Theorem 1.1 of [17] and the proof is omitted.

Theorem 3.1. *Let $s_1, \dots, s_n \in [s_0, 1)$ for some $s_0 \in (0,1)$ and $p_1, \dots, p_n > 1$ be such that $\overline{\vec{s}\vec{p}} < n$. Then $W^{\vec{s},\vec{p},\vec{\delta}}(\mathbb{R}^n) \subset\subset L^p_{loc}(\mathbb{R}^n)$ for every $1 \le p < \vec{p}^{\,*}$. Moreover, $W^{\vec{s},\vec{p},\vec{\delta}}(\mathbb{R}^n) \subset L^p_{loc}(\mathbb{R}^n)$ continuously for every $1 \le p \le \vec{p}^{\,*}$.*

3.2. **BBM-type results for $W_i^{s,p,\delta}(\mathbb{R}^n)$.** Let us start by revising several classical lemmas to establish the BBM-type results.
In the rest of this section we will use the following notation $D_i(u)(x) = |u(x+he_i) - u(x)|$.

Lemma 3.2. *Let $u \in W_i^{1,p}(\mathbb{R}^N)$. Therefore the following estimate holds*

$$[u]^{i}_{s,p,\delta} \le 2\frac{\delta^{p(1-s)}}{p(1-s)}\|\partial_i u\|_p^p.$$

Proof. Given $u \in C_c^1(\mathbb{R}^N)$ it is a well known fact that

$$\int_{\mathbb{R}^N} |D_i(u)(x)|^p\,dx \le |h|^p \|\partial_i u\|_p^p,$$

then

$$\begin{aligned}\int_{|h|<\delta}\int_{\mathbb{R}^N} \frac{|D_i(u,x)|^p}{|h|^{1+sp}}\,dxdh &= \int_{|h|<\delta} \frac{1}{|h|^{1+sp}}\left(\int_{\mathbb{R}^N}|D_i(u)(x)|^p\,dx\right)dh \\ &\le \|\partial_i u\|_p^p\, 2\int_0^\delta \frac{h^p}{|h|^{1+sp}}\,dh \\ &= 2\frac{\delta^{p(1-s)}}{p(1-s)}\|\partial_i u\|_p^p.\end{aligned}$$

We conclude the proof using the density of smooth functions with compact support in $W_i^{1,p}(\mathbb{R}^N)$. □

The next lemma shows that the functional $[\cdot]^i_{s,p,\delta}$ decreases when one regularizes a function u with a standard mollifier and shows the behavior with respect to truncations. The proof is in [15, Lemma 2.3].

Lemma 3.3. *On one hand, let $u \in L^p(\mathbb{R}^N)$, ρ_ε is the standard mollifier, then $u_\epsilon = \rho_\varepsilon * u$ verifies*

$$[u_\varepsilon]^i_{s,p,\delta} \leq [u]^i_{s,p,\delta}$$

for every $\varepsilon > 0$.
On the other hand, given $\eta \in C^1_c(\mathbb{R}^N)$ be such that $\eta = 1$ on B_1, $\eta = 0$ on B_2^c, $0 \leq \eta \leq 1$ and $\|\nabla\eta\|_\infty \leq 2$. Given $k \in \mathbb{N}$, we define $\eta_k(x) = \eta(x/k)$. If $u_k = \eta_k u$ is the truncation of u by η_k, then

$$[u_k]^i_{s,p,\delta} \leq 2^{p-1}[u]^i_{s,p,\delta} + \frac{2^{2p}\delta^{p(1-s)}}{k^p p(1-s)}\|u\|_p^p.$$

The following theorems are the main important in this section.

Theorem 3.4. *Let $u \in C^2_c(\mathbb{R}^N)$, $0 < s < 1$, and $\delta > 0$. Then*

$$\lim_{\delta\to 0}\frac{1}{\delta^{p(1-s)}}[u]^i_{s,p,\delta} = \frac{2}{p(1-s)}\|\partial_i u\|_p. \tag{6}$$

Proof. From the well-known estimate for smooth functions

$$\left|\frac{|D_i(u)(x)|^p}{|h|^p} - |\partial_i u|^p\right| \leq C|h|, \tag{7}$$

where C depends of the smoothness of u, we get that

$$\int_{|h|<\delta}\left|\frac{|D_i(u)(x)|^p}{|h|^{1+sp}} - \frac{|\partial_i u|^p}{|h|^{1+p(s-1)}}\right| dh \leq \frac{2C\delta^{p(1-s)+1}}{p(1-s)+1}.$$

Hence, when $\delta \to 0$ we conclude that

$$\lim_{\delta\to 0}\int_{|h|<\delta}\frac{|D_i(u)(x)|^p}{|h|^{1+sp}}\,dh = \lim_{\delta\to 0}\int_{|h|<\delta}\frac{|\partial_i u|^p}{|h|^{1+p(s-1)}}\,dh = \lim_{\delta\to 0} 2|\partial_i u|^p\frac{\delta^{p(1-s)}}{p(1-s)}.$$

Therefore

$$\lim_{\delta\to 0}\frac{1}{\delta^{p(1-s)}}\int_{|h|<\delta}\frac{|D_i(u)(x)|^p}{|h|^{1+sp}}\,dh = \frac{2|\partial_i u|^p}{p(1-s)}.$$

To end the proof we need to find an integrable majorant for

$$F_\delta(x) = \int_{|h|<\delta}\frac{|D_i(u)(x)|^p}{|h|^{1+sp}}\,dh.$$

Then since $u \in C^2_0(\mathbb{R}^N)$, there exists $R > \delta > 0$ such that $\mathrm{supp}(u) \subset Q_R$, where $Q_R = [-R, R]^n$. If $|x_i| \leq 2R$

$$|F_\delta(x)| = \int_{|h|<\delta}\frac{|u(x+he_i) - u(x)|^p}{|h|^{1+sp}}\,dh \leq \|\partial_i u\|^p_\infty\frac{\delta^{p(1-s)}}{p(1-s)}, \tag{8}$$

$$\left|\frac{1}{\delta^{p(s-1)}}F_\delta(x)\right| \leq C\chi_{Q_{2R}}(x),$$

where C is a constant depending of p, R and s. Finally if $|x_i| > 2R$, $F_\delta(x) = 0$. Hence by Lebesgue's Dominated Convergence Theorem

$$\lim_{\delta\to 0}\frac{1}{\delta^{p(1-s)}}[u]^i_{s,p,\delta} = \frac{2}{p(1-s)}\|\partial_i u\|_p \tag{9}$$

for all $u \in C^2_c(\mathbb{R}^N)$. □

Remark 3.5. From equation (7) the convergence in (27) is uniform on bounded subset of $C^2(\overline{\Omega}_\varepsilon)$.

Theorem 3.6. *Let $u \in L^p(\mathbb{R}^N)$, $0 < s < 1$, and $\delta > 0$. Then*

$$\lim_{\delta\to 0}\frac{1}{\delta^{p(1-s)}}[u]^i_{s,p,\delta} = \frac{2}{p(1-s)}\|\partial_i u\|_p.$$

Proof. On the one hand, let $u \in W^{1,p}_i(\mathbb{R}^N)$, $s \in (0,1)$, $p \in (1,\infty)$, and $\delta > 0$. Take a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C^2_0(\mathbb{R}^N)$ such that $u_k \to u$ in $L^p(\mathbb{R}^N)$ and $\partial_i u_k \to \partial_i u$ in $L^p(\mathbb{R}^N)$. Then

$$\begin{aligned}\left|\frac{1}{\delta^{p(1-s)}}[u]^i_{s,p,\delta} - \frac{2}{p(1-s)}\|\partial_i u\|_p\right| &\leq \delta^{(s-1)}\left|[u]^i_{s,p,\delta} - [u_k]^i_{s,p,\delta}\right| \\ &+ \left|\frac{1}{\delta^{p(1-s)}}[u_k]^i_{s,p,\delta} - \frac{2}{p(1-s)}\|\partial_i u_k\|_p\right| \\ &+ \frac{2}{p(1-s)}\left|\|(u_k)_{x_i}\|_p - \|u_{x_i}\|_p\right| \\ &= I + II + III.\end{aligned}$$

Note that $[\cdot]^i_{s,p,\delta}$ is a seminorm, which means it satisfies the triangle inequality. Therefore by Lemma 3.2, we have

$$I \leq \delta^{(s-1)}[u_k - u]^i_{s,p,\delta} \leq C(p,s)\|\partial_i u_k - \partial_i u\|_p,$$

which converges to zero if $k \to \infty$, uniformly in δ. The third term is bounded similarly, and the second term follows as a consequence of Theorem 3.4.
On the other hand, given $u \in L^p(\mathbb{R}^N)$ and assume that

$$\liminf_{\delta\to 0}\frac{1}{\delta^{p(1-s)}}[u]^i_{s,p,\delta} < \infty. \tag{10}$$

Let $u_{k,\varepsilon} \subset C^\infty_0(\mathbb{R}^N)$ be given by

$$u_{k,\epsilon} = \rho_\varepsilon * (u\eta_k),$$

and applying Lemma 3.3 together with (10), we obtain

$$\liminf_{\delta\to 0}\frac{1}{\delta^{p(1-s)}}[u_{k,\varepsilon}]^i_{s,p,\delta} < C,$$

where C is independent from k and $\varepsilon > 0$.
Next, for any fixed k and ϵ, we apply Theorem 3.4 and obtain

$$C \geq \lim_{\delta \to 0} \frac{1}{\delta^{p(1-s)}} [u_{k,\varepsilon}]^i_{s,p,\delta} = \frac{2}{p(1-s)} \|\partial_i u_{k,\varepsilon}\|_p.$$

Hence, there exist a subsequence $u_j = \{u_{k_j,\epsilon_j}\} \subset \{u_{k,\varepsilon}\}$ such that $\partial_i u_j \rightharpoonup \partial_i u$ weakly in $L^p(\mathbb{R}^N)$. Thus $u \in W^{1,p}_i(\mathbb{R}^N)$. This completes the proof of the Theorem. □

Finally, we are interested in proving the sequence case.

Theorem 3.7. *Let $\delta_k \to 0$ and $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ is a sequence such that*

$$\sup_k \|u_k\|_p < \infty, \quad u_k \to u \text{ in } L^p_{loc}(\mathbb{R}^N) \text{ and } \sup_k [u_k]_{s,p,\delta_k} < \infty.$$

Then

$$\frac{2}{p(1-s)} \|\partial_i u\|_p \leq \liminf_{\delta_k \to 0} \frac{1}{\delta_k^{p(1-s)}} [u_k]^i_{s,p,\delta_k}.$$

Proof. Based on Remark 3.5, we can apply Ponce's method [29]. Let $\varepsilon > 0$ and $R > 0$ be fixed. From Lemma 3.3 we get that

$$[u_k]^i_{s,p,\delta_k} \geq [u_{k,\varepsilon}]^i_{s,p,\delta_k} \geq \int_{B_R(0)} \int_{|h|\leq\delta_k} \frac{|u_{k,\varepsilon}(x+he_i) - u_{k,\varepsilon}(x)|^p}{|h|^{1+sp}} \, dh \, dx.$$

For each ε fixed we have that $u_{k,\varepsilon} \to u_\varepsilon$ in $C^2(B_R(0))$. From Remark 3.5 we get that

$$\frac{1}{\delta_k^{p(1-s)}} \int_{B_R(0)} \int_{|h|\leq\delta_k} \frac{|u_{k,\varepsilon}(x+he_i) - u_{k,\varepsilon}(x)|^p}{|h|^{1+sp}} \, dh \, dx \to \frac{2}{p(1-s)} \int_{B_R(0)} |\partial_i u_\varepsilon|^p \, dx$$

as $k \to \infty$. Therefore,

$$\liminf_{\delta_k \to 0} \frac{1}{\delta_k^{p(1-s)}} [u_k]^i_{s,p,\delta_k} \geq \frac{2}{p(1-s)} \int_{B_R(0)} |(\partial_i u)_\varepsilon|^p \, dx.$$

Taking $\varepsilon \to 0$ and $R \to \infty$, we get the desired result. □

3.3. **Applications.** In most applications, what turns out to be more useful than the norm or semi-norms are the energy functionals

$$J^i_\delta(u) := \frac{1}{\delta^{p(1-s)}} [u]^i_{s,p,\delta} \text{ and } J^i(u) := \frac{2}{p(1-s)} \|u_{x_i}\|^p_p.$$

Observe that we avoid using the notation $J^i_{s,p,\delta}(u)$ because it would be cumbersome. However, the reader should understand that the functional also depends on p and s. Theorems 3.6 and 3.7 immediately imply

$$J^i_\delta(u) \xrightarrow{\Gamma} J^i_0(u)$$

as $\delta \to 0$ in $L^p(\Omega)$ for every $\Omega \subset \mathbb{R}^N$ bounded. Just observe that Theorem 3.7 is exactly the lim-inf inequality and for the lim sup inequality one just has to take the constant sequence $u_k = u$ as a recovery sequence and apply Theorem 3.6.

Now, given $\vec{p} = (p_1, \dots, p_n)$ and $\vec{s} = (s_1, \dots, s_n)$, $s_i \in (0,1)$ and $\vec{\delta} = (\delta_1, \dots, \delta_n)$ We define the functional

$$J_{\vec{\delta}}(u) = \sum_{i=1}^{n} J^i_{\delta_i}(u).$$

In the context of the application, establishing the Gamma-convergence of the entire functional is crucial. The following corollary ensures this.

Corollary 3.8. *Let $\vec{p} = (p_1, \dots, p_n)$ and $\vec{s} = (s_1, \dots, s_n)$ with $p_i \in (1, \infty)$ and $s_i \in (0,1)$ for every $i = 1, \dots, n$. Let $\delta_i^k \to 0$ as $k \to \infty$ for $i = 1, \dots, j$ and let $\overline{\delta}_k = (\delta_1^k, \dots, \delta_j^k, \delta_{j+1}, \dots, \delta_n)$. Let $\vec{\delta}_0 = (\underbrace{0, \dots, 0}_{j}, \delta_{j+1}, \dots, \delta_n)$. Then $J_{\vec{\delta}_k}$ Gamma converges to $J_{\vec{\delta}_0}$ in $L^{p_{max}}(\Omega)$ for any $\Omega \subset \mathbb{R}^n$ bounded.*

Proof. The lim inf inequality for $J_{\vec{\delta}_k}$ follows from the lim inf inequality for every $J^i_{\delta_i^k}$ $(i = 1, \dots, j)$ and the continuity of $J^i_{\delta_i}$ for $i = j+1, \dots, n$.
The lim sup inequality follows since the constant sequence $u_k = u$ is a recovery sequence for *every* $J^i_{\delta_i^k}$, $i = 1, \dots, j$. □

3.3.1. *Peridynamic fractional anisotropic p-laplacian.* In this subsection, we will show the variational nature of the problem by defining the Euler-Lagrange equation for the functionals.

Definition 3.9. Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset. Then we define the anisotropic Sobolev space of functions vanishing at the boundary as

$$W_0^{\vec{s},\vec{p},\vec{\delta}}(\Omega) = \{u \in W^{\vec{s},\vec{p},\vec{\delta}}(\mathbb{R}^n) \colon u|_{\Omega^c} = 0\},$$
$$W_{i,0}^{s_i,p_i,\delta_i}(\Omega) = \{u \in W_i^{s_i,p_i,\delta_i}(\mathbb{R}^n) \colon u|_{\Omega^c} = 0\}.$$

The topological dual spaces of $W_0^{\vec{s},\vec{p},\vec{\delta}}(\Omega)$ and $W_{i,0}^{s_i,p_i,\delta_i}(\Omega)$ will be denoted by $W^{-\vec{s},\vec{p}',\vec{\delta}}(\Omega)$ and $W_i^{-s_i,p_i',\delta_i}(\Omega)$ respectively.

Now, it is easy to see that the functionals $J^i_{\delta_i}$ are Fréchet differentiable for every $i = 1, \dots, n$, every $s \in (0,1]$, $\delta_i > 0$ and every $p \in (1, \infty)$. Even more $(J^i_{\delta_i})' \colon W_{i,0}^{s_i,p_i,\delta_i}(\Omega) \to W_i^{-s_i,p_i',\delta_i}(\Omega)$ is continuous and is given by

$$\langle (J^i_{\delta_i})'(u), v\rangle = \frac{p}{\delta^{p(1-s)}} \int_{\mathbb{R}^n} \int_{|h| \le \delta} \frac{|D_i(u)(x)|^{p-2} D_i(u)(x) D_i(v)(x)}{|h|^{1+sp}}\, dh dx.$$

$$\langle (J^i)'(u), v\rangle = \frac{2}{(1-s)} \int_{\mathbb{R}^n} |\partial_{x_i} u|^{p-2} \partial_{x_i} u\ \partial_{x_i} v\, dx,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_{i,0}^{s_i,p_i,\delta}(\Omega)$ and its dual $W_i^{-s_i,p_i',\delta}(\Omega)$. The operator

$$J_{\vec{\delta}}(u) = \sum_{i=1}^{n} J^i_{\delta_i}(u).$$

is Fréchet differentiable with derivative given by

$$J'_{\vec{\delta}}(u) = \sum_{i=1}^{n} \frac{1}{p_i} (J^i_{\delta_i})'(u).$$

Finally, we define the *Peridynamic fractional anisotropic $\vec{p}$-laplacian* as

$$(-\widetilde{\Delta}_{\vec{p}})^{\vec{s}}_{\vec{\delta}} u := (J_{\vec{\delta}})'(u).$$

Therefore, we want to study the equation

$$\begin{cases} (-\widetilde{\Delta}_{\vec{p}})^{\vec{s}}_{\vec{\delta}} u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{11}$$

We say that $u \in W_0^{\vec{s},\vec{p}}(\Omega)$ is a weak solution of (11) if

$$\langle (-\widetilde{\Delta}_{\vec{p}})^{\vec{s}}_{\vec{\delta}} u, v \rangle = \int_\Omega f v \, dx$$

for all $v \in W_0^{\vec{s},\vec{p},\vec{\delta}}(\Omega)$.
For a weak solution to be well defined, as usual, we need to impose some integrability conditions on the source term f. Hence, we consider the case where $f = f(x)$ and $f \in L^{(\vec{p}\,^*)'}(\Omega)$.
The existence and uniqueness of weak solutions to (11) is then a direct consequence of the direct method in the calculus of variations since the solution is the unique minimizer of the functional

$$I(v) := J_{\vec{\delta}}(v) - \int_\Omega f v \, dx,$$

which is a strictly convex, coercive, and continuous functional in $W_0^{\vec{s},\vec{p},\vec{\delta}}(\Omega)$.
Let us summarize all of these facts into a single statement.

Proposition 3.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\vec{\delta} = (\delta_1, \dots, \delta_n)$, $\delta_i > 0$, $\vec{s} = (s_1, \dots, s_n)$, $s_i \in (0,1)$, $\vec{p} = (p_1, \dots, p_n)$, $p_i \in (1, \infty)$ and $\vec{p}\,^*$ the critical Sobolev exponent defined in the beginning of this section. Assume that $p_{max} < \vec{p}\,^*$ and $\overline{\vec{s}\vec{p}} < n$. Then, for every $f \in L^{(\vec{p}\,^*)'}(\Omega)$, there exists a unique weak solution $u \in W_0^{\vec{s},\vec{p},\vec{\delta}}(\Omega)$ of (11).*

Now, we will analyze the asymptotic behavior of the solution to (11) when (or some of the) $\delta_i \to 0$.
To this end assume that $\delta_1^k, \dots, \delta_j^k \to 0$ (as $k \to \infty$) and $\delta_{j+1}, \dots, \delta_n$ remain fixed and consider $\vec{\delta}_k := (\delta_1^k, \dots, \delta_j^k, \delta_{j+1}, \dots, \delta_n)$ and the functionals

$$I_k(v) := J_{\vec{\delta}_k}(v) - \int_\Omega f v \, dx.$$

By Proposition 3.10, there exists a unique minimizer u_k of I_k in $W_0^{\vec{s},\vec{p},\vec{\delta}}(\Omega)$. Hence we want to study the behavior of the sequence $\{u_k\}_{k\in\mathbb{N}}$ as $k\to\infty$.
To this end, recall that $\vec{\delta}_0 := (\underbrace{0,\dots,0}_{j},\delta_{j+1},\dots,\delta_n)$ and

$$I_0(v) := J_{\vec{\delta}_0}(v) - \int_\Omega f v\,dx,$$

where

$$J_{\vec{\delta}_0}(v) := \sum_{i=1}^{j} J^i(v) + \sum_{i=j+1}^{n} J^i_{\delta_i}(v).$$

Next, we make full use of the BBM results to conclude the following theorem:

Theorem 3.11. *With the preceding notation, the functionals $\bar{I}_k\colon L^{p_{min}}(\Omega)\to\bar{\mathbb{R}}$ defined as*

$$\bar{I}_k(v) = \begin{cases} I_k(v) & \text{if } v\in W_0^{\vec{s},\vec{p},\vec{\delta}}(\Omega),\\ \infty & \text{elsewhere.}\end{cases}$$

Gamma-converges to $\bar{I}\colon L^{p_{min}}(\Omega)\to\bar{\mathbb{R}}$ defined as

$$\bar{I}(v) = \begin{cases} I_0(v) & \text{if } v\in W_0^{\vec{s},\vec{p},\vec{\delta}}(\Omega),\\ \infty & \text{elsewhere.}\end{cases}$$

Proof. By Corollary 3.8 we have that $J_{\vec{\delta}_k}$ gamma converge to $J_{\vec{\delta}_0}$ as $k\to\infty$. So the result follows since the functional $u\mapsto\int_\Omega f u\,dx$ is continuous in $L^{p_{\min}}(\Omega)$. □

An immediate consequence of this result is the convergence of the solutions of (11) to the corresponding limiting problem.

Corollary 3.12. *Let $u_k\in W_0^{\vec{s},\vec{p},\vec{\delta}}(\Omega)$ be a weak solution to*

$$\begin{cases} (-\widetilde{\Delta}_{\vec{p}})^{\vec{s}}_{\vec{\delta}_k} u_k = f & \text{in } \Omega,\\ u_k = 0 & \text{in } \mathbb{R}^n\setminus\Omega.\end{cases}$$

Then $u_k\to u_0$ in $L^{p_{min}}(\Omega)$ where $u_0\in W_0^{\vec{s},\vec{p},\vec{\delta}_0}(\Omega)$ is the weak solution to

$$\begin{cases} (-\widetilde{\Delta}_{\vec{p}})^{\vec{s}_0}_{\vec{\delta}_k} u_0 = f & \text{in } \Omega,\\ u_0 = 0 & \text{in } \mathbb{R}^n\setminus\Omega.\end{cases}$$

4. Anisotropic fractional Sobolev Spaces with Variable Exponents

4.1. **Properties and Inequalities.** We define the fractional $i^{th}-$Sobolev variable exponent space as

$$W_i^{s_i,p_i(\cdot,\cdot)}(\mathbb{R}^N) := \left\{u\in L^{\bar{p}_i}(\mathbb{R}^N):\quad J_{s_i,p_i(\cdot,\cdot)}(u)<\infty\right\}.$$

We can see without difficulty that the space $W_i^{s_i,p_i(\cdot,\cdot)}(\mathbb{R}^N)$ constitutes a Banach space, characterized by the norm

$$\|u\|_{s_i,p_i} = \|u\|_{\bar{p}_i} + [u]_{s_i,p_i},$$

which is both separable and reflexive. Define

$$W^{\vec{s},\vec{p}(\cdot,\cdot)}(\mathbb{R}^N) = \bigcap_{i=1}^{N} W_i^{s_i,p_i(\cdot,\cdot)}(\mathbb{R}^N),$$

then $W^{\vec{s},\vec{p}(\cdot,\cdot)}(\mathbb{R}^N)$ is a Banach, separable and reflexive space.

Definition 4.1. A function $u \in L^1_{loc}(\mathbb{R}^N)$ belongs to the *homogeneous anisotropic fractional Sobolev spaces with variable exponent* $\dot{W}^{\vec{s},\vec{p}(\cdot,\cdot)}(\mathbb{R}^N)$ if $\sum_{i=1}^N [u]_{s_i,p_i} < \infty$.

Proposition 4.2. *If $u \in C_c^1(\mathbb{R}^N)$, then $u \in \dot{W}^{\vec{s},\vec{p}(\cdot,\cdot)}(\mathbb{R}^N)$.*

Proof. Let $u \in C_c^1(\mathbb{R}^N)$, there exists $R > 0$ such that $u(x) = 0$ for all $x \in \mathbb{R}^N$, $|x| \geq R$. By the mean value Theorem,

$$|u(x) - u(y)| \leq \|\nabla u\|_\infty |x - y|, \quad \forall\, x, y \in B_R(0).$$

By Tonelli's theorem and relabeling the variables of integration, we can write

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}} \frac{|u(x+he_i)-u(x)|^{p_i(x,x+he_i)}}{|h|^{1+s_ip_i(x,x_i(h))}}\,dhdx \\
&= \int_{B_R(0)}\int_{I_i} \frac{|u(x+he_i)-u(x)|^{p_i(x,x+he_i)}}{|h|^{1+s_ip_i(x,x+he_i)}}\,dhdx \\
&\leq \int_{B_R(0)}\int_{I_i} \frac{\|\nabla u\|_\infty^{p_i(x,x+he_i)}|h|^{p_i(x,x+he_i)}}{|h|^{1+s_ip_i(x,x+he_i)}}dhdx \\
&\leq \max\left(\|\nabla u\|_\infty^{p^-}, \|\nabla u\|_\infty^{p^+}\right)\int_{I_i} \frac{\max\left(|h|^{p^-}, |h|^{p^+}\right)}{\min\left\{|h|^{1+s_i(p_i)^-)}, |h|^{1+s_i(p_i)^+)}\right\}}dh \\
&:= C_i
\end{aligned}$$

where $C_i > 0$, and $I_i = \{h \in \mathbb{R} : \quad x + he_i \in B_R(0),\ \forall x \in B_R(x)\}$. It is not difficult to see that I_i will be a bounded subset of $\mathbb{R}$. This completes the proof. □

Definition 4.3. We define $W_{\bar{p}}^{\vec{s},\vec{p}(\cdot,\cdot)}(\Omega)$ as a $C_c^1(\mathbb{R}^N)$-module if, for any $u \in W_{\bar{p}}^{\vec{s},\vec{p}(\cdot,\cdot)}(\Omega)$ and $v \in C_c^1(\mathbb{R}^N)$, the product uv also belongs to $W_{\bar{p}}^{\vec{s},\vec{p}(\cdot,\cdot)}(\Omega)$.

Possessing the $C_c^1(\mathbb{R}^N)$-module property is considered advantageous. Let's investigate the conditions under which $W_{\bar{p}}^{\vec{s},\vec{p}(\cdot,\cdot)}(\Omega)$ is indeed a $C_c^1(\mathbb{R}^N)$-module.

Proposition 4.4. *Let Ω be a bounded subset of $\mathbb{R}^N$. $W_{\bar{p}}^{\vec{s},\vec{p}(\cdot,\cdot)}(\Omega)$ is a $C_c^1(\mathbb{R}^N)$-module if*

$$\bar{p}(x) \geq p_M(x) \quad \textit{for a.e. } x \in \Omega,$$

where $p_M(x) := \max\{p_1(x,x), \cdots, p_N(x,x)\}$.

Proof. Since $p_i(x) \leq \bar{p}(x)$, then $L^{\bar{p}}(\Omega) \hookrightarrow L^{p_i(\cdot)}(\Omega)$. Let $u \in W_{\bar{p}}^{\vec{s},\vec{p}(\cdot,\cdot)}(\Omega) \subset L^{\bar{p}}(\Omega)$ and $v \in C_c^1(\mathbb{R}^N) \subset L^{p_0(\cdot)}(\Omega)$, then, obviously $uv \in L^{p_0(\cdot)}(\Omega)$. From Proposition 4.2, $uv \in \dot{W}^{\vec{s},\vec{p}(\cdot,\cdot)}(\mathbb{R}^N)$. Thus $W_{p_0(\cdot)}^{\vec{s},\vec{p}(\cdot,\cdot)}(\Omega)$ is a $C_c^1(\mathbb{R}^N)$-module. □

The space $W_{p_M(\cdot)}^{\vec{s},\vec{p}(\cdot,\cdot)}(\Omega)$ is written simply by $W^{\vec{s},\vec{p}(\cdot,\cdot)}(\Omega)$, namely

$$W^{\vec{s},\vec{p}(\cdot,\cdot)}(\Omega) = \left\{ u \in L^{p_M(\cdot)}(\Omega) : \quad \sum_{i=1}^{N} J_{s_i,p_i,\Omega}(u) < \infty \right\}.$$

4.2. **BBM-type results for $W_i^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$.** In this section, we are operating with a constant direction and fixed parameters. Consequently, we will dispense with the use of the index "i" and simply represent them as s, p and $\overline{p}$ instead of s_i, p_i and $\overline{p}_i$ respectively, and we fix ourselves in the first direction.

Theorem 4.5. *Let $u \in C_c^2(\mathbb{R}^N)$. Then*

$$\lim_{s\to 1} J_{s,p(\cdot,\cdot)} = \int_{\mathbb{R}^N} \frac{2}{\overline{p}(x)} |\partial_1 u|^{\bar{p}(x)}\, dx.$$

Lemma 4.6. *Let $u \in C_c^2(\mathbb{R}^N)$. Then*

$$\int_{\mathbb{R}^N} \frac{2}{\overline{p}(x)} |\partial_1 u|^{\bar{p}(x)} dx \leq \liminf_{s\nearrow 1} (1-s) \int_{\mathbb{R}^N} \int_{\mathbb{R}} \frac{|u(x+he_1)-u(x)|^{p(x,x+he_1)}}{|h|^{1+sp(x,x+he_1)}}\, dh dx. \tag{12}$$

Proof. Since $u \in C_c^2(\mathbb{R}^N)$, there exists $C > 0$, depending on the C^2-norm of u, such that

$$|u(x+he_1) - u(x) - \partial_1 u(x).h| \leq C|h|^2, \quad \text{for all} \quad h \in \mathbb{R}. \tag{13}$$

Consider fixing $\varepsilon > 0$ and selecting $x \in \mathbb{R}^N$. The continuity of the function $h \mapsto p(x, x+he_1)$ at 0 implies the existence of a positive value r_1, this value satisfies the condition that

$$(1-\varepsilon)|\partial_1 u(x)|^{\bar{p}(x)} \leq |\partial_1 u(x)|^{p(x,x+he_1)} \leq (1+\varepsilon)|\partial_1 u(x)|^{\bar{p}(x)}, \tag{14}$$

for all $|h| < r_1$. Also we have the existence of $r_2 > 0$, such that

$$(1-\varepsilon)|h|^{\bar{p}(x)} \leq |h|^{p(x,x+he_1)} \leq (1+\varepsilon)|h|^{\bar{p}(x)}, \tag{15}$$

for all $|h| < r_2$. Again there exists $r_3 > 0$, such that

$$(1-\varepsilon)\frac{1}{|h|^{\bar{p}(x)}} \leq \frac{1}{|h|^{p(x,x+he_1)}} \leq (1+\varepsilon)\frac{1}{|h|^{\bar{p}(x)}}, \tag{16}$$

for all $0 < |h| < r_3$. We can derive from (13),

$$|\partial_1 u(x).h| \leq |u(x+he_1) - u(x)| + C|h|^2.$$

Using the inequality

$$(a+b)^q \leq (1+\varepsilon)a^q + C(q,\varepsilon)b^q, \quad a, b \geq 0,\ q > 1,$$

we get

$$|\partial_1 u(x).h|^{p(x,x+he_1)} \leq (1+\varepsilon)|u(x+he_1)-u(x)|^{p(x,x+he_1)} + C|h|^{2p(x,x+he_1)}.$$

If we consider $r_0 = \min\{1, r_1, r_2, r_3\}$ and take $0 < r < r_0$, then

$$\begin{aligned}&\int_{|h|<r} \frac{|\partial_1 u(x).h|^{p(x,x+he_1)}}{|h|^{1+sp(x,x+he_1)}} dh \\ &\leq (1+\varepsilon)\int_{|h|<r} \frac{|u(x+he_1)-u(x)|^{p(x,x+he_1)}}{|h|^{1+sp(x,x+he_1)}} dh + C\int_{|h|<r} \frac{|h|^{2p(x,x+he_1)}}{|h|^{1+sp(x,x+he_1)}} dh.\end{aligned}$$

Let's estimate from below the left-hand side of the previous inequality. Referring to (14), (15) and (16) we find that

$$\begin{aligned}\int_{|h|<r} \frac{|\partial_1 u(x).h|^{p(x,x+he_1)}}{|h|^{1+sp(x,x+he_1)}} dh &\geq (1-\varepsilon)^3 |\partial_1 u|^{\bar{p}(x)} \int_{|h|<r} |h|^{-1+(1-s)\overline{p}(x)} dh \\ &= \frac{(1-\varepsilon)^3}{(1-s)\overline{p}(x)} |\partial_1 u|^{\bar{p}(x)} r^{(1-s)\overline{p}(x)}.\end{aligned} \tag{17}$$

Therefore, integrating over $\mathbb{R}^N$, we get

$$\begin{aligned}&\frac{2}{p^-}(1-\varepsilon) r^{(1-s)p^-} \int_{\mathbb{R}^N} |\partial_1 u|^{\bar{p}(x)} dx \\ &\leq (1+\varepsilon)(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}} \frac{|u(x+he_1)-u(x)|^{p(x,x+he_1)}}{|h|^{1+sp(x,x+he_1)}} dh dx + C\frac{2(1-s)}{(2-s)p^-} r^{(2-s)p^-}.\end{aligned}$$

We pass to $\liminf_{s\nearrow 1}$, we obtain

$$\begin{aligned}&\frac{2}{p^-}(1-\varepsilon) r^{(1-s)p^-} \int_{\mathbb{R}^N} |\partial_1 u|^{\bar{p}(x)} dx \\ &\leq (1+\varepsilon)\liminf_{s\nearrow 1}(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}} \frac{|u(x+he_1)-u(x)|^{p(x,x+he_1)}}{|h|^{1+sp(x,x+he_1)}} dh dx.\end{aligned}$$

Since ε is arbitrarily chosen, we deduce (12). □

Lemma 4.7. *Let $u \in C_c^2(\mathbb{R}^N)$. Then*

$$\limsup_{s\nearrow 1}(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}} \frac{|u(x+he_1)-u(x)|^{p(x,x+he_1)}}{|h|^{1+sp(x,x+he_1)}} dh dx \leq \int_{\mathbb{R}^N} \frac{2}{\overline{p}(x)} |\partial_1 u|^{\bar{p}(x)} dx. \tag{18}$$

Proof. The proof of this lemma follows by combining the Dominated Convergence Theorem with the two subsequent lemmas. □

Lemma 4.8. *Let $u \in C_c^1(\mathbb{R}^N)$. Then there exists a function $F \in L^1(\mathbb{R}^N)$ such that*

$$F_s(x) := (1-s)\int_{\mathbb{R}} \frac{|u(x+he_1)-u(x)|^{p(x,x+he_1)}}{|h|^{1+sp(x,x+he_1)}} dh \leq F(x), \quad \textit{for all } x \in \mathbb{R}^N.$$

Proof. Since $u \in C_c^1(\mathbb{R}^N)$, there exists $R > 1$ such that $supp(u) \subset Q_R$, where $Q_R = [-R, R]^N$. If $|x_1| \leq 2R$,

$$|F_s(u)| = (1-s)\left(\int_{|h|\leq 1} + \int_{|h|>1}\right) \frac{|u(x+he_1) - u(x)|^{p(x,x+he_1)}}{|h|^{1+sp(x,x+he_1)}}\, dh := I_1 + I_2.$$

$$\begin{aligned}
|I_1| &\leq (1-s)\int_{|h|\leq 1} \|\partial_1 u\|_\infty^{p(x,x+he_1)} |h|^{-1+(1-s)p(x,x+he_1)} dh \\
&\leq (1-s)\left[\|\partial_1 u\|_\infty^{p^-} + \|\partial_1 u\|_\infty^{p^+}\right] \int_{|h|\leq 1} |h|^{-1+(1-s)p^-} dh \\
&= \frac{\alpha_N}{p^-}\left[\|\partial_1 u\|_\infty^{p^-} + \|\partial_1 u\|_\infty^{p^+}\right],
\end{aligned}$$

where $\alpha_N = |B(0,1)|$. We used in the previous inequality

$$|u(x+he_1)) - u(x)| \leq \int_0^1 |\partial_1 u(x+the_1).he_1| dt \leq \|\partial_1 u\|_\infty |h|.$$

Also

$$\begin{aligned}
|I_2| &\leq 2\left[\|u\|_\infty^{p^-} + \|u\|_\infty^{p^+}\right] \int_{|h|>1} |h|^{-1-sp^-} dh \\
&= \frac{2}{sp^-}\left[\|\partial_1 u\|_\infty^{p^-} + \|\partial_1 u\|_\infty^{p^+}\right].
\end{aligned}$$

If $|x_1| > 2R$, we denote by $x = (x_1, x')$ with $x' \in \mathbb{R}^{N-1}$, then

$$F_s(x) = (1-s)\int_{\mathbb{R}} \frac{|u(x_1+h, x') - u(x)|^{p(x,(x_1+h,x'))}}{|h|^{1+sp(x,(x_1+h,x'))}}\, dh.$$

Since $|x_1| > 2R$, we use that $|h| \geq |x_1| - |x_1+h| \geq |x_1| - R \geq \frac{1}{2}|x_1|$, we get

$$|F_s(x)| \leq \left(\frac{2}{|x_1|}\right)^{1+sp^-} \int_{\mathbb{R}} |u(x_1, x')|^{p_1(x,(x_1,x'))} dx_1.$$

Observe that

$$G(x') := \int_{\mathbb{R}} |u(x_1, x')|^{p_1(x,(x_1,x'))} dx_1 \in L^1(\mathbb{R}^{N-1}).$$

Therefore

$$|F_s(x)| \leq C\left(\chi_{Q_{2R}}(x) + \chi_{\{|x_1|\geq 2R\}}(x)\left(\frac{2}{|x_1|}\right)^{1+sp^-} G(x')\right) \in L^1(\mathbb{R}^N),$$

where C is a constant depending on N, p^-, u. □

Lemma 4.9. *Let $u \in C_c^2(\mathbb{R}^N)$. Then*

$$\lim_{s\nearrow 1} F_s(x) := \frac{2}{\overline{p}(x)} |\partial_1 u|^{\bar{p}(x)}.$$

Proof. Let u belongs to the space $C_c^2(\mathbb{R}^N)$. In this case, there exists a positive constant C that depends on the norm $|u|_{C^2(\mathbb{R}^N)}$, satisfying the condition:

$$|u(x+he_1)-u(x)|^{p(x,x+he_1)} \leq (1+\varepsilon)|\partial_1 u|^{p(x,x+he_1)}|h|^{p_1(x,x+he_1)} + C|h|^{2p(x,x+he_1)}$$

for all $x \in \mathbb{R}^N$ and $h \in \mathbb{R}$. From the preceding inequality, we obtain

$$\begin{aligned} A(h) := &\left| \frac{|u(x+he_1)-u(x)|^{p(x,x+he_1)}}{|h|^{1+sp(x,x+he_1)}} - (1+\varepsilon)|\partial_1 u|^{p(x,x+he_1)}|h|^{-1+(1-s)p(x,x+he_1)} \right| \\ &\leq C|h|^{-1+(2-s)p(x,x+he_1)}. \end{aligned}$$

By performing the integration over $\mathbb{R}$, we arrive at the following expression:

$$\int_{\mathbb{R}} A(h)\,dh = \left(\int_{|h|\leq 1} + \int_{|h|>1} \right) A(h)\,dh. \tag{19}$$

In the initial integral on the right side, we observe

$$\begin{aligned} \int_{|h|\leq 1} A(h)\,dh &\leq C \int_{|h|\leq 1} |h|^{-1+(2-s)p(x,x+he_1)}\,dh \\ &\leq C(1+\varepsilon) \int_{|h|\leq 1} |h|^{-1+(2-s)\overline{p}(x)}\,dh = \frac{2C}{(2-s)\overline{p}(x)}, \end{aligned}$$

then

$$\lim_{s\nearrow 1}(1-s)\int_{|h|\leq 1} A(h)\,dh = 0,$$

that is

$$\begin{aligned} &\lim_{s\nearrow 1}(1-s)\int_{|h|\leq 1} \frac{|u(x+he_1)-u(x)|^{p(x,x+he_1)}}{|h|^{1+sp(x,x+he_1)}}\,dh \\ &= \lim_{s\nearrow 1}(1-s)(1+\varepsilon)|\partial_1 u|^{p(x,x+he_1)} \int_{|h|\leq 1} |h|^{-1+(1-s)p(x,x+he_1)}\,dh \\ &\leq \lim_{s\nearrow 1} 2(1-s)(1+\varepsilon)|\partial_1 u|^{p(x,x+he_1)} \int_{|h|\leq 1} |h|^{-1+(1-s)\overline{p}(x)}\,dh \\ &= (1+\varepsilon)\frac{2}{\overline{p}(x)}|\partial_1 u|^{p(x,x+he_1)}. \end{aligned} \tag{20}$$

For in the subsequent integral on the right side of (19), we find

$$\begin{aligned} \int_{|h|>1} A(h)\,dh &\leq C2^{p^+} \max\left(\|u\|_\infty^{p^-}, \|u\|_\infty^{p^+} \right) \int_{|h|>1} \frac{dh}{|h|^{1+sp(x,x+he_1)}} \\ &= \frac{C2^{p^+}}{sp^-} \max\left(\|u\|_\infty^{p^-}, \|u\|_\infty^{p^+} \right), \end{aligned}$$

then

$$\lim_{s\nearrow 1}(1-s)\int_{|h|>1} A(h)\,dh = 0,$$

and as illustrated in (20), we can see that

$$\lim_{s\nearrow 1}(1-s)\int_{|h|\leq 1}\frac{|u(x+he_1)-u(x)|^{p(x,x+he_1)}}{|h|^{1+sp(x,x+he_1)}}\,dh \leq (1+\varepsilon)\frac{2}{\overline{p}(x)}|\partial_1 u|^{p(x,x+he_1)}. \tag{21}$$

Therefore, by combining (20) and (21) and since ε is arbitrarily, we derive the following:

$$\lim_{s\nearrow 1}(1-s)F_s(x) \leq \frac{2}{\overline{p}(x)}|\partial_1 u|^{p(x,x+he_1)}. \tag{22}$$

Upon examining (17), we have also

$$\lim_{s\nearrow 1}(1-s)F_s(x) \geq \frac{2}{\overline{p}(x)}|\partial_1 u|^{p(x,x+he_1)}, \tag{23}$$

and the proof is completed. □

Proof of Theorem 4.5. The proof can be derived by combining both Lemma 4.6 and Lemma 4.7. □

4.3. **Eigenvalue problems: Isotropic case.** The fractional $p(x)$-Laplacian $(-\triangle)^s_{p(.,.)}$ is given by

$$(-\triangle)^s_{p(.,.)}u(x) := P.V.\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dy, \text{ for all } x\in\Omega.$$

P.V. is a commonly used abbreviation in the principal value sense.

The eigenvalue problem associated to $(-\triangle)^s_{p(.,.)}$ is the following:

$$\begin{cases}(-\triangle)^s_{p(.,.)}u = \Lambda|u|^{\bar{p}-2}u & \text{in } \Omega\\ u=0 & \text{in } \mathbb{R}^N\backslash\Omega.\end{cases} \tag{24}$$

The natural variational approach to study problem (24) is to consider critical points of the non-homogeneous Rayleigh type quotient

$$R_s(u) = \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy}{\displaystyle\int_\Omega |u(x)|^{\bar{p}(x)}dx}. \tag{25}$$

When looking in the literature, no prior studies have delved into the exploration of problems associated with the fractional $p(x)-$Laplacian, offering a comprehensive examination of its spectrum. However, we posit that addressing this issue may not be overly challenging, as a parallel investigation to that of the $p(x)-$Laplacian could

be undertaken. Nevertheless, a notable complication arises in tackling the eigenvalue problem (24) due to its lack of homogeneity. This difficulty manifests particularly when dealing with operators featuring nonstandard growth. Specifically, eigenvalues in equation (25) are contingent on the magnitude of $\int_\Omega |u(x)|^{\bar{p}(x)}dx = \mu$, wherein distinct constraints may yield disparate eigenvalues.
An innovative approach, as proposed by Franzina and Lindqvist in [24], involves incorporating norms into the Rayleigh-type quotient instead of modulars. The overarching objective is to introduce homogeneity into the analysis, thereby mitigating the challenges associated with nonstandard growth operators. The focus of their investigation centered on the local $p(x)-$Laplacian. In contrast, we introduce an exploration into a nonlocal, homogeneous Rayleigh-type quotient, expanding the scope of the study. We propose the study of the following quotient:

$$H_s(u) := \frac{K(u)}{k(u)}, \tag{26}$$

where $K(u) = [u]_{s,p(.,.)}$ and $k(u) = \|u\|_{\bar{p},\Omega}$ (Refer to Section 2 for definitions). Notice that the homogeneity of the quotient in (26) implies that eigenvalues are independent of the energy level chosen to perform the minimization, we can assume $\|u\|_{\bar{p},\Omega} = 1$.

We are led to consider the following operator

$$\mathcal{L}u(x) := P.V. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{K(u)^{p(x,y)-1}|x-y|^{N+sp(x,y)}}dy, \text{ for all } x \in \Omega.$$

We consider the homogeneous counterpart of (24):

$$\begin{cases} \mathcal{L}u = \Lambda \left|\frac{u}{k(u)}\right|^{\bar{p}-2} \frac{u}{k(u)} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{27}$$

The Euler-Lagrange equation:

We define

$$\Lambda_1 = \inf_{v \in X\backslash\{0\}} H_s(u).$$

Proposition 4.10. *There exists a non-negative minimizer $u \in X\backslash\{0\}$, of Λ_1.*

Proof. The Sobolev inequality (2) shows that $\Lambda_1 > 0$. Let $(v_n)_{n\in\mathbb{N}}$ be a minimizing sequence of Λ_1 such that $\|v_n\|_{\bar{p},\Omega} = 1$. Then

$$\Lambda_1 = \lim_{n\to\infty} [v_n]_{s,p(.,.)}.$$

By Theorem 2.5, we can extract a subsequence still denoted $(v_n)_{n\in\mathbb{N}}$ and find a function $u \in X$ such that

$$v_n \to u \quad \text{strongly in} \quad L^{\bar{p}(.)}(\Omega),$$
$$v_n \rightharpoonup u \quad \text{weakly in} \quad X.$$

Since the norm is weakly sequentially lower semi-continuous, we obtain

$$\frac{[u]_{s,p(.,.)}}{\|u\|_{\bar{p},\Omega}} \leq \liminf_{n\to\infty} \frac{[v_n]_{s,p(.,.)}}{\|v_n\|_{\bar{p},\Omega}} = \Lambda_1.$$

Therefore, u serves as a minimizer, and the same holds true for $|u|$. With this, the proof is complete.

□

To obtain the Euler-Lagrange equation for the minimizer, we select an arbitrary test function $\varphi \in C_0^\infty(\Omega)$. Next, we examine the competing function $v_\varepsilon(x) = u(x) + \varepsilon\varphi(x)$. We observe that a necessary condition for inequality

$$\frac{K(u)}{k(u)} \leq \frac{K(v_\varepsilon)}{k(v_\varepsilon)},$$

which gives that

$$\frac{d}{d\varepsilon}\left(\frac{K(v_\varepsilon)}{k(v_\varepsilon)}\right)_{\varepsilon=0} = 0.$$

Thus the necessary condition reads

$$\frac{1}{K(u)}\left(\frac{d}{d\varepsilon}K(v_\varepsilon)\right)_{\varepsilon=0} = \frac{1}{k(u)}\left(\frac{d}{d\varepsilon}k(v_\varepsilon)\right)_{\varepsilon=0}. \tag{28}$$

Before proceeding with our studies, let us compute the preceding identities.

Proposition 4.11. *Let $u \in X$ and $\varphi \in C_0^\infty(\Omega)$. Then,*

$$\left(\frac{d}{d\varepsilon}K(v_\varepsilon)\right)_{\varepsilon=0} = \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{\nabla_s u}{K(u)}\right|^{p(x,y)-2}\frac{\nabla_s u}{K(u)}\nabla_s\varphi\frac{dxdy}{|x-y|^N}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{\nabla_s u}{K(u)}\right|^{p(x,y)}\frac{dxdy}{|x-y|^N}}$$

and

$$\left(\frac{d}{d\varepsilon}k(v_\varepsilon)\right)_{\varepsilon=0} = \frac{\int_\Omega\left|\frac{u}{k(u)}\right|^{\bar{p}(x)-2}\frac{u}{k(u)}\varphi dx}{\int_\Omega\left|\frac{u}{k(u)}\right|^{\bar{p}(x)}dx},$$

where $\nabla_s u := \frac{u(x)-u(y)}{|x-y|^s}$.

Proof. The proof for the derivation of k has been established in [24, Lemma A.1], and now let's address the derivation for K. Observe that for all $a, b \in \mathbb{R}$ and all $p > 1$ we have that

$$|b|^p - |a|^p = \int_0^1 \frac{d}{dt}|a + t(b-a)|^p dt = p(b-a)\int_0^1 |a + t(b-a)|^{p-2}[a + t(b-a)]dt.$$

By the definition of the Luxemburg norm, we know that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{\nabla_s v_\varepsilon}{K(v_\varepsilon)}\right|^{p(x,y)}\frac{dxdy}{p(x,y)|x-y|^N} = 1, \quad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{\nabla_s u}{K(u)}\right|^{p(x,y)}\frac{dxdy}{p(x,y)|x-y|^N} = 1.$$

Subtracting the preceding two expressions and employing the variables b for $\frac{\nabla_s v_\varepsilon}{K(v_\varepsilon)}$, a for $\frac{\nabla_s u}{K(u)}$ and $p = p(x,y)$ results in:

$$0 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[\frac{\nabla_s v_\varepsilon}{K(v_\varepsilon)} - \frac{\nabla_s u}{K(u)}\right] A(x,\varepsilon)\frac{dxdy}{|x-y|^N}, \tag{29}$$

where

$$A(x,\varepsilon) = \int_0^1 \left|\frac{\nabla_s u}{K(u)} + t\left(\frac{\nabla_s v_\varepsilon}{K(v_\varepsilon)} - \frac{\nabla_s u}{K(u)}\right)\right|^{p(x,y)-2}\left(\frac{\nabla_s u}{K(u)} + t\left(\frac{\nabla_s v_\varepsilon}{K(v_\varepsilon)} - \frac{\nabla_s u}{K(u)}\right)\right)dt.$$

By employing the definition of v_ε, we express (29) in the following manner:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[\frac{\nabla_s u}{K(v_\varepsilon)} - \frac{\nabla_s u}{K(u)}\right] A(x,\varepsilon)\frac{dxdy}{|x-y|^N} = -\varepsilon\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\nabla_s\varphi}{K(v_\varepsilon)}A(x,\varepsilon)\frac{dxdy}{|x-y|^N},$$

from where

$$\frac{K(v_\varepsilon)-K(u)}{\varepsilon}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\nabla_s u}{K(u)}A(x,\varepsilon)\frac{dxdy}{|x-y|^N} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\nabla_s\varphi A(x,\varepsilon)\frac{dxdy}{|x-y|^N}. \tag{30}$$

Since $K(v_\varepsilon)$ depends continuously on ε, then

$$\frac{\nabla_s v_\varepsilon}{K(v_\varepsilon)} - \frac{\nabla_s u}{K(u)} \to 0 \text{ a.e. as } \varepsilon \to 0^+,$$

and we deduce that

$$A(x,\varepsilon) \to \left|\frac{\nabla_s u}{K(u)}\right|^{p(x,y)-2}\frac{\nabla_s u}{K(u)} \text{ a.e. as } \varepsilon \to 0^+.$$

Letting $\varepsilon \to 0^+$ in (30), we get

$$\left(\frac{d}{d\varepsilon}K(v_\varepsilon)\right)_{\varepsilon=0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{\nabla_s u}{K(u)}\right|^{p(x,y)}\frac{dxdy}{|x-y|^N} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{\nabla_s u}{K(u)}\right|^{p(x,y)-2}\frac{\nabla_s u}{K(u)}\nabla_s\varphi\frac{dxdy}{|x-y|^N}.$$

That is

$$\left(\frac{d}{d\varepsilon}K(v_\varepsilon)\right)_{\varepsilon=0} = \frac{K\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{\nabla_s u}{K(u)}\right|^{p(x,y)-2}\frac{\nabla_s u}{K(u)}\frac{\nabla_s\varphi}{K(u)}\frac{dxdy}{|x-y|^N}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{\nabla_s u}{K(u)}\right|^{p(x,y)}\frac{dxdy}{|x-y|^N}}$$

□

Now, let's advance toward the primary objective of this part, which is to deduce the Euler-Lagrange equation that characterizes the minimum of Λ_1. By combining the Proposition 4.11 and the identity labeled as (28) , it becomes clear that:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{\nabla_s u}{K(u)}\right|^{p(x,y)-2}\frac{\nabla_s u}{K(u)}\nabla_s\varphi\frac{dxdy}{|x-y|^N} = \Lambda_1 S(u)\int_\Omega\left|\frac{u}{k(u)}\right|^{\bar{p}(x)-2}\frac{u}{k(u)}\varphi dx, \quad \forall\varphi\in C_0^\infty(\Omega), \tag{31}$$

where

$$S(u) := \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{\nabla_s u}{K(u)}\right|^{p(x,y)}\frac{dxdy}{|x-y|^N}}{\int_\Omega\left|\frac{u}{k}\right|^{\bar{p}(x)}dx}. \tag{32}$$

The preceding characterization presents the weak form of problem (27), and subsequently, we introduce the following definition.

Definition 4.12. We say that Λ is an eigenvalue of (27) with eigenfunction $u \in X$ if u is a nontrivial weak solution to (27), that is,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{\nabla_s u}{K(u)}\right|^{p(x,y)-2}\frac{\nabla_s u}{K(u)}\nabla_s\varphi\frac{dxdy}{|x-y|^N} = \Lambda S(u)\int_\Omega\left|\frac{u}{k(u)}\right|^{\bar{p}(x)-2}\frac{u}{k(u)}\varphi dx, \quad \forall\varphi\in X. \tag{33}$$

Theorem 4.13. *Λ_1 is the first eigenvalue and the minimum u of Λ_1 which is nontrivial is the corresponding first eigenfunction.*

Proof. In light of identity (31), it becomes clear that Λ_1 serves as an eigenvalue for (27), with u representing the corresponding eigenfunction. Now, let's verify that Λ_1 holds the position of the first eigenvalue. Let Λ be an eigenvalue of (27) and v its corresponding eigenfunction. By setting $\varphi = v = u$ in equation (33) and considering the definition of $S(v)$ as given in (32), we can derive:

$$\begin{aligned} K(v)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{\nabla_s v}{K(v)}\right|^{p(x,y)}\frac{dxdy}{|x-y|^N} &= \Lambda S(v)k(v)\int_\Omega\left|\frac{v}{k}\right|^{\bar{p}(x)}dx \\ &= \Lambda k(v)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{\nabla_s v}{K(v)}\right|^{p(x,y)}\frac{dxdy}{|x-y|^N}. \end{aligned}$$

Then

$$\Lambda = \frac{K(v)}{k(v)} \geq \Lambda_1.$$

Hence λ_1 is the first eigenvalue.

□

Sequence of eigenvalues:

Put

$$\mathcal{M} := \{u \in X : \ k(u) = 1\}.$$

The first eigenvalue can be defined as

$$\Lambda_1 = \inf_{u\in\mathcal{M}} K(u).$$

$\mathcal{M}$ is a submanifold of class C^1 in X.

Lemma 4.14. *Let $u \in X\backslash\{0\}$. For all $v \in X$ the following inequalities hold*

$$\left|\langle K^{'}(u), v\rangle\right| \leq K(v) \quad \text{and} \quad \left|\langle k^{'}(u), v\rangle\right| \leq k(v). \tag{34}$$

Proof. The second inequality of (34) has been established in [25, Proposition 3.6]. Employing analogous techniques, we extend this proof to establish the first one. □

Proposition 4.15. *The functional $\tilde{K}|_{\mathcal{M}}$ satisfies the (PS) condition, i.e., every sequence $(u_k) \subset \mathcal{M}$ such that $\tilde{K}(u_k) \to c$ for some $c \in \mathbb{R}$ and $\tilde{K}'(u_k) \to 0$ in X' admits a convergent subsequence.*

Proof. Let $c \in \mathbb{R}$ and $(c_k) \subset \mathbb{R}$ a sequence such that

$$K(u_k) \to c \quad \text{and} \quad K'(u_k) - c_k k'(u_k) \to 0 \text{ in } X'. \tag{35}$$

On one hand, we can see that

$$\langle K'(u_k) - c_k k'(u_k), u_k \rangle = \langle K'(u_k), u_k \rangle - c_k \langle k'(u_k), u_k \rangle = K(u_k) - c_k.$$

On the other hand

$$\Big| \langle K'(u_k) - c_k k'(u_k), u_k \rangle \Big| \leq \|K'(u_k) - c_k k'(u_k)\|_{X^*} K(u_k) \to 0.$$

Then $c_k \to c$. Since (u_k) is bounded in X, up to subsequence, $u_k \rightharpoonup u$ in X and $u_k \to u$ in $L^{\bar{p}(x)}(\Omega)$. By (34), we get

$$\Big| \langle k'(u_k), u_k - u \rangle \Big| \leq k(u_k - u) = \|u_k - u\|_{\bar{p},\Omega} \to 0.$$

from this and (35) we deduce that

$$\langle K'(u_k), u_k - u \rangle \to 0.$$

Since

$$\langle K'(u_k), u_k - u \rangle = K(u_k) - \langle K'(u_k), u \rangle \geq K(u_k) - K(u),$$

then, by the weak lower semicontinuity of the norm, we get

$$\limsup_{k \to \infty} K(u_k) \leq K(u) \leq \liminf_{k \to \infty} K(u_k).$$

□

As $\mathcal{M}$ is a closed symmetric submanifold with C^1 regularity in X, the previously mentioned result empowers us to derive a sequence of eigenvalues for equation (27) by employing a minimax procedure.

Definition 4.16. For $n \in \mathbb{N}$, we define the $n-$th variational eigenvalue Λ_n of (27) as

$$\Lambda_n := \inf_{A \in \mathcal{C}_m} \sup_{u \in A} \tilde{K}(u), \tag{36}$$

where $\mathcal{C}_m := \{C \subset \mathcal{M} : C \text{ is compact}, C = -C, \gamma(C) \geq m\}$ and γ is the Krasnoselskii genus.

Proposition 4.17. *The sequence (Λ_n) is nondecreasing, divergent, and for every $n \geq 1$ there exists $u_n \in \mathcal{M}$ solving (27), with $\Lambda = \Lambda_n$.*

Proof. Under Proposition 4.15, we can employ [26, Corollary 4.1] to establish that the values specified in (36) indeed constitute eigenvalues of (27) as per Definition 4.12. Additionally, given that $C_{m+1} \subset C_m$ holds for all m, the sequence (Λ_n) is characterized by a divergent, nondecreasing pattern. □

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