THE ENERGY LANDSCAPE OF THE KURAMOTO MODEL IN ONE-DIMENSIONAL RANDOM GEOMETRIC GRAPHS WITH A HOLE

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Abstract. We study the energy function of the Kuramoto model in random geometric graphs defined in the unit circle as the number of nodes diverges. We prove the existence of at least one local minimum for each winding number $q \in \mathbb{Z}$ with high probability. Hence providing a large family of graphs that support patterns that are generic. These states are in correspondence with the explicit twisted states found in WSG and other highly symmetric networks, but in our situation there is no explicit formula due to the lack of symmetry. The method of proof is simple and robust. It allows other types of graphs like $k-$nn graphs or the boolean model and holds also for graphs defined in any simple closed curve or even a small neighborhood of the curve and for weighted graphs. It seems plausible that the method can be extended also to higher dimensions, but a more careful analysis is required.

1. Introduction

The study of local minima and the whole geometry of high-dimensional random non-convex functions is highly relevant in areas as diverse as deep-learning, statistical mechanics, complex networks and synchronicity.

Phase synchronization of systems of coupled oscillators is a phenomenon that has attracted the mathematical and scientific community because of its intrinsic mathematical interest [6, 8, 10, 15] and its ubiquity in technological, physical and biological models [3, 4, 7, 11, 18, 21, 22, 24].

One of the most popular models for describing synchronization of a system of coupled oscillators is the Kuramoto model. The model has been studied both by means of rigorous mathematical proofs and heuristics arguments and simulations in different families of graphs. Here we focus in the first class of evidence.

We consider graphs $G_n = G = (V, E)$ where the set of nodes $V = \{x_0, x_1, \dots, x_{n-1}\} \subset \mathbb{S}^1$ is a sample of n i.i.d uniform random variables. The distance between two nodes is given by the Euclidean distance in $\mathbb{C}$, which is equivalent to the geodesic distance in $\mathbb{S}^1$. For convenience, we assume that the nodes V are labeled counterclockwise with $\arg(x_0) = 0$ and we denote $x_n := x_0$.

The *random geometric graph* in the circle $\mathbb{S}^1$ with parameters n, ϵ_n is the graph that has V as the set of nodes in which we declare $\{x_i, x_j\} \in E$ (we denote this by $i \sim j$) if and only if $|x_i - x_j| < \epsilon_n$.

We can think of $x_0, \dots, x_{n-1}$ as points in $[0, 2\pi]$ and $|\cdot|$ as the absolute value function with the convention that everything is understood mod 2π.

For the sequence of random geometric graphs defined above, we are going to work in the regime

$$n\epsilon_n^2 \to 0, \qquad \frac{n\epsilon_n}{\log n} \to \infty, \qquad \text{as } n \to \infty. \tag{1.1}$$

The first condition implies $\epsilon_n \to 0$, which is important to obtain Proposition 3.1 below (this proposition does not hold if $\epsilon_n \not\to 0$). It will also be used for the conclusion of the main theorem. However, we expect the conclusion of our main theorem to hold even without Proposition 3.1 (but

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some bound from above is needed on ϵ_n to avoid high connectivity that leads to global synchronization [23]). The second condition is required to guarantee connectivity of the graph with high probability and so, it can't be removed without altering the behavior of the system. Observe that both conditions are verified for any sequence of the form $\epsilon_n \approx n^{-a}$ with $1/2 < a < 1$.

Let $u^n\colon [0,\infty)\times V \to \mathbb{R}$ be the unique solution to a system of n homogeneous Kuramoto equations

$$\begin{cases} \dfrac{d}{dt}u^n(t,x_i) = \dfrac{1}{n}\dfrac{1}{\epsilon_n^2 N_i}\displaystyle\sum_{j\sim i}\sin\left(u^n(t,x_j)-u^n(t,x_i)\right), \\ u^n(0,x_i) = \bar{u}^n(x_i), \qquad i = 0,1,...,n-1. \end{cases} \tag{1.2}$$

The random integer N_i denotes the number of neighbors of x_i. To lighten notation we call $u_i^n = u^n(t,x_i)$ and we also omit the dependence on n if it is not nessesary. Equations (1.2) define a gradient system. It is a direct computation to see that

$$\dot{u} = -\nabla E_n(u),$$

for

$$E_n(u) = E_n(u_0,\dots,u_{n-1}) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2\epsilon_n^2 N_i}\sum_{j\sim i}(1-\cos(u_j-u_i)).$$

The sine function in (1.2) can be replaced by an odd $2\pi-$periodic symmetric smooth function J with Taylor expansion $J(\theta) = \theta + o(\theta^2)$ without altering the conclusion of our main result.

Our interest in the Kuramoto model in graphs with this structure is threefold: on the one hand this kind of graphs is relevant to model several natural situations in which spatial considerations are important to determine the strength of the links between oscillators. On the other hand they form a large family of model networks with persistent behavior (robust to small perturbations) for which we expect to observe patterns.

Last but not least, there has been a recent interest to understand the behavior of the Kuramoto model on diverse models of random and non-random graphs [1, 2, 12, 13]. The main goal is to decide if the networks foster synchronization or not. In [2] the authors have recently shown that in expander graphs and in particular in Erdős-Rény graphs above the connectivity threshold, synchronization occurs with high probability as $n \to \infty$. Our results can be seen as a complement of those in the sense that we are exhibiting a class of random graphs that are not expanders for which global synchronization fails. Up to our knowledge this is the first rigorous proof of non-synchronization in random graphs.

Twisted states have been defined for particular classes of graphs as explicit equilibria of (1.2). They have been shown to be stable equilibria in rings in which each node is connected to its k nearest neighbors on each side [23], in Cayley graphs and in random graphs with a particular structure [16].They have also been studied in small-world networks [14] and in the continuum limit [17] among others.

Our notion of *twisted state* is a bit different since we don't expect to find explicit equilibria in our context besides complete synchronization. We think of them as stable equilibria that can be identified in some way to with the functions $u_q(x_i) = qx_i$. Precise definitions in the next section. We remark that we are considering functions that take values in $\mathbb{S}^1$ rather than $\mathbb{R}$. Alternatively, we can think of them as functions $u\colon [0,2\pi] \to \mathbb{R}$ with $u(2\pi) = u(0) + 2q\pi$ for some $q \in \mathbb{Z}$.

Situations in which the twisted states are explicit are not expected to be robust and persistent. Our interest is to find twisted states that are generic in some sense and for this same reason we do not expect us to be able to compute them explicitly.

Remark that as far as we know, in most of the literature that give rigorous proofs about existence of twisted states they are computed explicitly by exploding graph symmetries and the issue is to prove their stability. Here (and in most typical real situations with spatial structure and local interactions) the issue is to prove their existence. We are going to get the stability for free.

The Kuramoto model in random geometric graphs has been studied in [1]. In that work the authors are interested in the optimization landscape of the energy function determined by (1.2) as well as we are here, but they work on a different regime: in their setting the graphs are constructed on the sphere $\mathbb{S}^{d-1}$ rather than in the circle and $d \to \infty$ as $n \to \infty$. In that context, they obtain guarantees for global spontaneous synchronization (i.e. the global minimum $\theta_1 = \theta_2 = \cdots = \theta_n$ is the unique local minima of the energy). This is pretty different from our situation as we will see.

Besides the Kuramoto model, our work enters in the framework of random non-convex optimization, which is relevant not just in the study of dynamics of complex networks but also in deep-learning and statistical mechanics. In the first case due to the fact that most of modern learning algorithms (i.e., artificial neural networks) rely on the adequate optimization of a loss functions which is typically highly non-convex and random [5, 9, 19, 20].

In particular, our results show that for this kind of random energies, while the energy at a typical point diverges to infinity with the size of the graph, at local minima is of order one.

We are going to state our results for random geometric graphs but they can be exported straightforwardly to different kinds of graphs defined in any closed and simple curve like $k-$nn graphs or even deterministic graphs. We discuss this in Section 4.

For a continuous function $u\colon [0, 2\pi] \to \mathbb{R}$ with $u(2\pi) = u(0) + 2q\pi$ for some $q \in \mathbb{Z}$ we define its index by $I(u) = q$. If u is defined only in a discrete set $\{x_0, \dots, x_{n-1}\}$ we define its index as the index of its linear interpolation (a more precise definition is given below).

Our main result reads as follows.

Theorem 1.1. *For each $q \in \mathbb{Z}$ we have,*

$$\lim_{n\to\infty} \mathbb{P}\left((1.2) \text{ has an asymptotically stable equilibrium with index } q\right) = 1.$$

To prove this theorem we first consider in Section 2 a partition of the space $\mathcal{S}^n := (\mathbb{S}^1)^n$ where E_n is defined. Next, in Section 3 we prove Proposition 3.1 which is one of the main ingredients and then Theorem 1.1.

2. Geometry of the space $\mathcal{S}^n$

For a rectifiable closed curve $\gamma : [a, b] \to \mathbb{C}$ that does not contain the origin we define the index (or winding number) of γ (around the origin) as the total number of times that the curve travels counterclockwise around ω. More precisely,

$$I(\gamma) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{dz}{z}.$$

For each $\mathbf{z} = (z_0, \dots, z_{n-1}) \in \mathcal{S}^n \subset \mathbb{C}^n$, consider $\gamma_{\mathbf{z}} = \sum_{j=1}^n \gamma_j$ where γ_j is the geodesic from z_{j-1} to z_j in $\mathbb{S}^1$, $z_n = z_0$ and $\gamma_i + \gamma_j$ is the curve that results from concatenating γ_i and γ_j in the given order. Observe that $\gamma_{\mathbf{z}}$ is a well-defined picewise differentiable closed curve in $\mathbb{S}^1$ as far as $z_j \neq -z_{j-1}$ for every j (otherwise the geodesic from z_{j-1} to z_j is not unique). We abbreviate the notation by writing $I(\mathbf{z}) := I(\gamma_{\mathbf{z}})$.

If we write $z_j = e^{\mathrm{i}\theta_j}$ for some $\theta_j \in [0, 2\pi)$, then

$$\begin{aligned} I(\mathbf{z}) &= \frac{1}{2\pi\mathrm{i}} \int_{\gamma_{\mathbf{z}}} \frac{dz}{z} \\ &= \frac{1}{2\pi\mathrm{i}} \sum_{j=1}^{n} \int_{\gamma_j} \frac{dz}{z} \\ &= \frac{1}{2\pi} \sum_{j=1}^{n} \theta_j \ominus \theta_{j-1}. \end{aligned}$$

Here $\theta_j \ominus \theta_{j-1}$ is the signed length of the geodesic from z_{j-1} to z_j. If for each θ, θ' we choose $\bar{\theta}$ and $\bar{\theta}'$ such that $e^{\mathrm{i}\bar{\theta}} = e^{\mathrm{i}\theta}$, $e^{\mathrm{i}\bar{\theta}'} = e^{\mathrm{i}\theta'}$ with $-\pi < \bar{\theta} - \bar{\theta}' < \pi$, it can be computed as

$$\theta \ominus \theta' = \bar{\theta} - \bar{\theta}'.$$

Sometimes we will slightly abuse notation by writing $I(\theta_0, \dots, \theta_{n-1})$ instead of $I(z_0, \dots, z_{n-1})$. This is not a problem since the value of I is independent of the choice of $\theta_0, \dots, \theta_{n-1}$.

Observe that the set of points $\mathbf{z} \in \mathcal{S}^n$ for which the index $I(\mathbf{z})$ is well defined is open and that the function I is continuous in its domain and integer valued. Hence it is constant in each connected component and in fact the sets

$$K_q := \{\mathbf{z} \in \mathcal{S}^n : I(\mathbf{z}) = q\},$$

define the connected components of the domain of I. Note that each K_q is open and $\partial K_q = \{\mathbf{z} \in \overline{K_q} : I(\mathbf{z}) \text{ is not defined}\}$. So, we have the decomposition

$$\mathcal{S}^n = \bigcup_{q\in\mathbb{Z}} K_q \cup \left(\bigcup_{q\in\mathbb{Z}} \partial K_q \right).$$

Remark that for a given n, the sets $K_q = \emptyset$ for $|q| > \lfloor \frac{n-1}{2} \rfloor$. Also remark that for $|q| \le \lfloor \frac{n-1}{2} \rfloor$ we have $\partial K_q \cap \partial K_{q'} \neq \emptyset$. In fact the point $(0, \pi, 0, \pi, \dots) \in \partial K_q$ for every $q \le \lfloor \frac{n-1}{2} \rfloor$.

We will prove that for each $q \in \mathbb{Z}$, the energy E_n restricted to K_q attains a minimum with high probability as $n \to \infty$. Since the sets K_q are open, this minima are forced to be local minima of E_n.

3. Proof of the main theorem

To prove Theorem 1.1 we will need the following proposition applied to the functions $u_q(x) = qx$ but we state it for general smooth functions u due to its independent interest.

Proposition 3.1. *Assume $u \in C^2([0, 2\pi], \mathbb{R})$, then*

$$\lim_{n\to\infty} E_n(u) = \frac{1}{12\pi} \int_0^{2\pi} |u'(x)|^2 \, dx, \qquad \textit{in probability.}$$

Proof. We need to Poissonize. Consider an infinite sequence of independent uniform random variables in $\mathbb{S}^1$, $x_0, x_1, \dots$. Let $\mathbf{n}$ be an independent Poisson random variable with parameter n. Define for every $k \in \mathbb{N}$, $V_k = \{x_0, \dots, x_{k-1}\}$. Then $V = V_n$ and we denote $\mathbf{V} := V_{\mathbf{n}}$. The point process $\mathbf{V}$ is a Poisson Point Process in $\mathbb{S}^1$. Let us consider the Poissonized version of the energy,

$$\mathbf{E}_n(u) = \mathbf{E}_n(u_0, \dots, u_{\mathbf{n}-1}) = \frac{1}{n} \sum_{i=1}^{\mathbf{n}} \frac{1}{2\epsilon_n^2 \mathbf{N}_i} \sum_{\substack{j\sim i \\ x_j \in \mathbf{V}}} (1 - \cos(u_j - u_i)).$$

Here $\mathbf{N}_i$ is the number of neighbors of i in the graph constructed with $\mathbf{V}$ instead of V. We also need to consider versions of the energy for different sets of nodes. So, define for any V_k,

$$E_n^{V_k}(u) = E_n^{V_k}(u_0, \dots, u_{k-1}) = \frac{1}{n}\sum_{i=1}^{k} \frac{1}{2\epsilon_n^2 N_i^{V_k}} \sum_{\substack{j\sim i \\ x_j \in V_k}} (1-\cos(u_j - u_i)).$$

Similarly, $N_i^{V_k}$ denotes the number of neighbors of i in the graph constructed with V_k. With this notation we have $E_n = E_n^{V_n}$ and $\mathbf{E}_n = E_n^{\mathbf{V}}$. There is a random variable C independent of n and u such that for every integer $k \geq -n+2$,

$$|E_n^{V_{n+k}}(u) - E_n^{V_n}(u)| \leq C\frac{|k|}{n}\|(u')^2\|_\infty.$$

In particular,

$$|\mathbf{E}_n(u) - E_n(u)| \leq C\frac{|\mathbf{n}-n|}{n}\|(u')^2\|_\infty.$$

Since $\frac{|\mathbf{n}-n|}{n} \to 0$ a.s., it is enough to prove

$$\lim_{n\to\infty} \mathbf{E}_n(u) = \frac{1}{12\pi}\int_0^{2\pi} |u'(x)|^2\,dx, \qquad \text{in probability.}$$

We proceed to do that. First, observe that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{u(y)-u(x)}{\epsilon}\right)^2 \mathbf{1}\{|y-x|<\epsilon\}\,dy\,dx = \frac{2}{3}\int_0^{2\pi}|u'(x)|^2\,dx, \tag{3.1}$$

and moreover

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{1-\cos(u(y)-u(x))}{\epsilon^2}\right) \mathbf{1}\{|y-x|<\epsilon\}\,dy\,dx = \frac{1}{3}\int_0^{2\pi}|u'(x)|^2\,dx. \tag{3.2}$$

Let $\mathcal{N}(i)$ be the set of neighbors of i in the graph determined by $\mathbf{V}$. That is,

$$\mathcal{N}(i) := \{j\colon 0 \leq j \leq \mathbf{n}, |x_j - x_i| < \epsilon_n\}.$$

Conditional on $i \leq \mathbf{n}$, $x_i = x$ and $\mathcal{N}(i)$, the variables

$$\begin{aligned}\zeta_j^i &:= [1-\cos(u(x_j)-u(x_i))]\mathbf{1}\{x_j \sim x_i\}\\ &= [1-\cos(u(x+\epsilon z_j^i)-u(x))]\mathbf{1}\{|z_j^i|<1\}\end{aligned}$$

are i.i.d. and their absolute values are bounded by $\|(u')^2\|_\infty\epsilon_n^2$. Here $z_j^i := (x_j - x_i)/\epsilon$. Moreover, since the conditional distribution of z_j^i, $j \in \mathcal{N}(i)$ is uniform in $[-1,1]$, we have that

$$\begin{aligned}\mathbb{E}\left(\zeta_j^i \,|\, x_i = x, \mathcal{N}(i)\right) &= \frac{1}{2}\int_{-1}^{1}[1-\cos(u(x+\epsilon z)-u(x))]\,dz\\ &= \frac{1}{2\epsilon_n}\int_0^{2\pi}[1-\cos(u(y)-u(x))]\mathbf{1}\{|y-x|<\epsilon_n\}\,dy.\end{aligned}$$

Then

$$\mathbb{E}\left[\frac{1}{\epsilon_n^2\mathbf{N}_i}\sum_{j\in\mathcal{N}(i)}\zeta_j^i\right] = \frac{1}{4\pi\epsilon_n^3}\int_0^{2\pi}\int_0^{2\pi}[1-\cos(u(y)-u(x))]\mathbf{1}\{|y-x|<\epsilon_n\}\,dydx.$$

and

$$\mathbb{E}\left[\frac{1}{\epsilon_n^2 \mathbf{N}_i}\sum_{j\in\mathcal{N}(i)}\zeta_j^i\right]^2 \leq \|(u')^2\|_\infty.$$

We conclude by applying the following two lemmas to the variables

$$Z_i^n := \frac{1}{\epsilon_n^2 \mathbf{N}_i}\sum_{j\in\mathcal{N}(i)}\zeta_j^i, \qquad 0\leq i\leq \mathbf{n}-1.$$

For the sake of well-definiteness we construct Z_i^n for $i\geq \mathbf{n}$ using independent copies of the process.

Lemma 3.2. *For Z_i^n defined as above, we have for $i\neq j$,*

$$|\mathrm{Cov}(Z_i^n, Z_j^n)| \leq \frac{\|(u')^2\|_\infty^2 \epsilon_n}{\pi}.$$

Proof. Observe that for every i, n, $|Z_i^n|\leq \|(u')^2\|_\infty$. We compute for $i\neq j$,

$$\begin{aligned}\mathbb{E}[Z_i^n Z_j^n | x_i, x_j, \mathcal{N}(i), \mathcal{N}(j)] &= \\ &= \mathbb{E}\left[Z_i^n Z_j^n \mathbf{1}\{|x_i-x_j|>2\epsilon_n\}|x_i,x_j,\mathcal{N}(i),\mathcal{N}(j)\right] \\ &+ \mathbb{E}\left[Z_i^n Z_j^n \mathbf{1}\{|x_i-x_j|\leq 2\epsilon_n\}|x_i,x_j,\mathcal{N}(i),\mathcal{N}(j)\right].\end{aligned}$$

But,

$$\mathbb{E}\left[Z_i^n Z_j^n \mathbf{1}\{|x_i-x_j|>2\epsilon_n\}|x_i,x_j,\mathcal{N}(i),\mathcal{N}(j)\right] = \mathbb{E}\left[Z_i^n|x_i,\mathcal{N}(i)\right]\mathbb{E}\left[Z_j^n|x_j,\mathcal{N}(j)\right]\mathbf{1}\{|x_i-x_j|>2\epsilon_n\}$$

and

$$\mathbb{E}\left[Z_i^n Z_j^n \mathbf{1}\{|x_i-x_j|\leq 2\epsilon_n\}\right] \leq \|(u')^2\|_\infty^2 \mathbb{P}(|x_i-x_j|\leq 2\epsilon_n) \leq \|(u')^2\|_\infty^2 \frac{2\epsilon_n}{\pi}.$$

Hence,

$$\begin{aligned}\left|\mathbb{E}(Z_i^n Z_j^n) - \mathbb{E}(Z_i^n)\mathbb{E}(Z_j^n)\right| &= \\ &= \left|\mathbb{E}(Z_i^n)\mathbb{E}(Z_j^n)\mathbb{P}(|x_i-x_j|>2\epsilon_n) + \mathbb{E}\left[Z_i^n Z_j^n \mathbf{1}\{|x_i-x_j|\leq 2\epsilon_n\}\right] - \mathbb{E}(Z_i^n)\mathbb{E}(Z_j^n)\right| \\ &= \left|-\mathbb{E}(Z_i^n)\mathbb{E}(Z_j^n)\mathbb{P}(|x_i-x_j|\leq 2\epsilon_n) + \mathbb{E}\left[Z_i^n Z_j^n \mathbf{1}\{|x_i-x_j|\leq 2\epsilon_n\}\right]\right| \\ &\leq \frac{4\|(u')^2\|_\infty^2 \epsilon_n}{\pi}.\end{aligned}$$

□

Lemma 3.3. *For $Z_1^n, Z_2^n, \ldots, Z_{\mathbf{n}}^n$ defined as above we have,*

$$\frac{1}{n}\sum_{i=1}^{\mathbf{n}} Z_i^n \to \mu := \frac{1}{12\pi}\int_0^{2\pi} |u'(x)|^2\,dx, \qquad \textit{in probability.}$$

Proof. Call $\bar{Z}_n = \frac{1}{n}\sum_{i=1}^{\mathbf{n}} Z_i^n$. We compute,

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{\mathbf{n}} Z_i^n\right]^2 = \frac{\mathbb{E}(\mathbf{n})}{n^2}\mathbb{E}(Z_1^n)^2 + \frac{1}{n^2}\mathbb{E}\left[\sum_{i\neq j}^{\mathbf{n}} Z_i^n Z_j^n\right]$$
$$= \frac{1}{n}\mathbb{E}(Z_1^n)^2 + \frac{\mathbb{E}(\mathbf{n}(\mathbf{n}-1))}{n^2}\mathrm{Cov}(Z_1^n, Z_2^n) + \frac{\mathbb{E}(\mathbf{n}(\mathbf{n}-1))}{n^2}\mathbb{E}^2(Z_1^n).$$

We have that $\mathbb{E}(Z_1^n)^2 \leq \|(u')^2\|_\infty$ and $\mathrm{Cov}(Z_1^n, Z_2^n) \to 0$. Then $\mathbb{E}(\bar{Z}_n^2) \to \mu^2$. Since $\mathbb{E}(\bar{Z}_n) \to \mu$, the variance $\mathrm{Var}(\bar{Z}_n) \to 0$. By means of Tchebychev inequality, $\bar{Z}_n \to \mu$ in probability. □

Lemma 3.3 proves that $\mathbf{E}_n(u) \to \mu$ and hence the same holds for $E_n(u)$.

□

We are ready to prove our main theorem.

Proof of Theorem 1.1. First, by Bernstein's inequality and union bound, we have that

$$\mathbb{P}\left(\sup_{i=1}^{n} |N_i - \frac{\epsilon_n n}{\pi}| > \lambda\right) \leq 2ne^{-\frac{\frac{1}{2}\lambda^2}{\epsilon_n n/\pi + \lambda/3}}, \quad \lambda > 0. \tag{3.3}$$

For $\lambda = \frac{\epsilon_n n}{\pi}$ we obtain

$$\mathbb{P}\left(\sup_{i=1}^{n} N_i \geq \frac{2\epsilon_n n}{\pi}\right) \leq 2ne^{-c\epsilon_n n}. \tag{3.4}$$

Similarly, if we call $N_{ij} = |\{k\colon |x_i - x_k| < \epsilon_n, |x_j - x_k| < \epsilon_n\}|$ the number of common neighbors of i and j, we have $\mathbb{E}(N_{ij}|i \sim j) \geq \frac{n\epsilon_n}{2\pi}$ and

$$\mathbb{P}\left(N_{ij} \leq \frac{\epsilon_n n}{4\pi}\right) \leq e^{-c\epsilon_n n}.$$

Hence,

$$\mathbb{P}\left(\inf_{i\sim j} N_{ij} < \frac{n\epsilon_n}{4\pi}\right) \leq \sum_{i,j} \mathbb{P}\left(N_{ij} < \frac{n\epsilon_n}{4\pi}\middle| i \sim j\right)\mathbb{P}(i \sim j) \leq n^2 e^{-c\epsilon_n n}(\epsilon_n/\pi). \tag{3.5}$$

Let $\mathbf{z} \in \mathcal{S}^n$, $\mathbf{z} = (z_0, \ldots, z_{n-1}) = (e^{\mathrm{i}\theta_0}, \ldots, e^{\mathrm{i}\theta_{n-1}})$ such that $I(\mathbf{z})$ is not defined. Then there is k with $z_k = -z_{k-1}$ and hence we have $\cos(\theta_{k-1} - \theta_k) = -1$. For any $\theta \in [0, 2\pi)$ we have $\cos(\theta_{k-1} - \theta) \wedge \cos(\theta_k - \theta) \leq 0$. If G is connected and r is a neighbor of both k and $k-1$ we have

$$\sum_{j\sim r}(1 - \cos(\theta_j - \theta_r)) \geq 1.$$

Hence

$$E_n(\theta_0, \ldots, \theta_{n-1}) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{2\epsilon_n^2 N_i}\sum_{j\sim i}(1 - \cos(\theta_j - \theta_i)) \geq \frac{N_{k,k-1}}{2n\epsilon_n^2(N_k \vee N_{k-1})}. \tag{3.6}$$

Due to (3.4) and (3.5) we have for every $q \in \mathbb{Z}$,

$$\mathbb{P}\left(\inf_{\boldsymbol{\theta}\in\partial K_q} E(\boldsymbol{\theta}) \leq \frac{1}{16n\epsilon_n^2}\right) \leq \mathbb{P}\left(\sup_i N_i \geq \frac{2\epsilon_n n}{\pi}\right) + \mathbb{P}\left(\inf_{i,j} N_{ij} \leq \frac{\epsilon_n n}{4\pi}\right) + \mathbb{P}\left(\bigcup_{i=1}^{n}\{i \nsim i-1\}\right)$$
$$\leq 2ne^{-c\epsilon_n n} + n^2 e^{-c\epsilon_n n} + n\left(\frac{\pi - \epsilon_n}{\pi}\right)^{n-1}.$$

Condition (1.1) guarantees $n\epsilon_n^2 \to 0$ and that $\sum_n n^2 e^{-c\epsilon_n n} < \infty$. Thus,

$$\lim_{n\to\infty} \inf_{(e^{i\theta_0},\dots,e^{i\theta_{n-1}})\in\partial K_q} E(\theta_0,\dots,\theta_{n-1}) = +\infty, \qquad \text{a.s.} \tag{3.7}$$

For $q \in \mathbb{Z}$ we consider the function $u_q(x) = qx$. Observe that $(u_q(x_0),\dots,u_q(x_{n-1})) \in K_q$ and compute

$$\lim_{n\to\infty} E_n(u_q) = \frac{1}{12\pi}\int_0^{2\pi} |u_q'(x)|^2\,dx = \frac{q^2}{6}, \qquad \text{in probability.}$$

Define the event

$$A_{n,q} := \left\{ \inf_{\mathbf{z}\in\partial K_q} E_n(\mathbf{z}) > \frac{q^2}{4} \text{ and } E_n(u_q) < \frac{q^2}{5} \right\}.$$

Due to Proposition 3.1 and (3.7), $\mathbb{P}(A_{n,q}) \to 1$ as $n \to \infty$. Finally, observe that since $\overline{K_q}$ is compact and E_n is continuous, it attains a minimum at $\overline{K_q}$. If $A_{n,q}$ occurs this minimum can not be attained in ∂K_q and hence there is a point $u_q^* \in K_q$ with

$$E_n(u_q^*) \le E_n(u), \qquad \text{for every } u \in K_q.$$

Since K_q is open, u_q^* is a local minimum of E_n and hence a stable equilibrium for (1.2). We have proved that for every $q \in \mathbb{Z}$

$$\mathbb{P}\left((1.2) \text{ has a stable equilibrium with index } q\right) \ge \mathbb{P}(A_{n,q}) \to 1.$$

To ensure that u_q^* is a strict local minima and hence asymptotically stable, we verify a well-known condition that implies that the Hessian $D^2E_n(u_q^*)$ is positive definite, namely

$$|u_q^*(x_i) - u_q^*(x_j)| < \frac{\pi}{2}, \qquad \text{for every } i \sim j, \tag{3.8}$$

(see [13]). Let $B_{n,q} := \{\omega\colon \omega \in A_{n,q} \text{ and (3.8) does not hold}\}$. If there is $k \sim \ell$ with $|u_q^*(x_k) - u_q^*(x_\ell)| \ge \frac{\pi}{2}$, proceeding as in (3.6) we bound from below

$$E_n(u_q^*) = \frac{1}{n}\sum_{i=1}^n \frac{1}{2\epsilon_n^2 N_i}\sum_{j\sim i}(1-\cos(u_q^*(x_i)-u_q^*(x_j))) \ge \frac{N_{k,\ell}}{n\epsilon_n^2(N_k \vee N_\ell)}.$$

Since in $A_{n,q}$ we have $E_n(u_q^*) \le E_n(u_q) \le q^2/5$, using Bernstein's inequality again we obtain for n large enough,

$$\begin{aligned}
\mathbb{P}(B_{n,q}) \le & \\
&\le \mathbb{P}\left(A_{n,q} \cap \left\{E_n(u_q^*) \ge \inf_{i\sim j} \frac{N_{i,j}}{n\epsilon_n^2(N_i \vee N_j)}\right\}\right) \\
&\le \mathbb{P}\left(\sup_i N_i \ge \frac{2\epsilon_n n}{\pi}\right) + \mathbb{P}\left(\inf_{i\sim j} N_{ij} \le \frac{\epsilon_n n}{4\pi}\right) \\
&\le (2n+n^2)e^{-c\epsilon_n n}.
\end{aligned}$$

Since $\mathbb{P}(A_{n,q}) \to 1$, we get that

$$\mathbb{P}(A_{n,q} \text{ and (3.8) holds}) \to 1.$$

In particular,

$$\lim_{n\to\infty} \mathbb{P}\left((1.2) \text{ has an asymptotically stable equilibrium with index } q\right) = 1.$$

□

4. Discussion

In this section we discuss other models for which our results still holds, possible extensions and other considerations.

4.1. **Other graph models.** Using the same arguments, Theorem 1.1 also holds for the following families of graphs. All are based on nodes $V = \{x_0, \dots, x_{n-1}\}$ i.i.d uniformly distributed on the unit circle. Different models correspond to different sets of edges.

a. $k-$**nn graphs** Two vertices x_i and x_j are connected by an edge if the distance between x_i and x_j is among the k_n-th smallest distances from x_i to other nodes from x_i or vice versa. Condition (1.1) becomes

$$\frac{k_n^2}{n} \to 0, \qquad \frac{k_n}{\log n} \to \infty.$$

b. Boolean model. For each node x_i we consider a random radius r_i. We assume the radii are i.i.d. We declare two nodes x_i, x_j neighbors if

$$(x_i - r_i, x_i + r_i) \cap (x_j - r_j, x_j + r_j) \neq \emptyset.$$

The role of r_i is similar to the one of $\epsilon_n/2$ in the original model but now they are random. Condition (1.1) becomes

$$n\mathbb{E}(r_i^2) \to 0, \qquad \frac{n\mathbb{E}(r_i)}{\log n} \to \infty, \qquad \text{as } n \to \infty.$$

c. Random $N-$**nn.** This is similar to the $k-$nn graph but instead of considering a deterministic k we choose a random number N_i for each x_i. The variables $(N_i)_{0 \le i \le n-1}$ are i.i.d.

d. Weighted graphs. In any of the previous models or even in WSG networks (with small k) we can consider (random or deterministic) weights as far as they don't degenerate as $n \to \infty$. To get a tractable model it is better to consider a kernel $k : \mathbb{R} \to \mathbb{R}_{\ge 0}$ to be a symmetric, smooth function with compact support in $(-1, 1)$ and $\int k(z)dz = 1$. Then we consider the weighted graph $G = (V, E)$, where the weights are given by $w_{ij} = k\left(\epsilon_n^{-1}(x_j - x_i)\right)$. For these graphs, condition (1.1) remains unchanged.

e. Random geometric graphs in an ϵ_n-**neighborhood of a simple closed curve.** Consider a simple closed curve γ and its ϵ_n neighborhood

$$\gamma^{\epsilon_n} := \{x \in \mathbb{R}^d \colon d(x, \gamma) < \epsilon_n\}.$$

Here $d(x, \gamma) = \inf_{y \in \gamma} |x - y|$. For ϵ_n small enough γ^{ϵ_n} is homeomorphic to an ϵ_n-neighborhood of the unit circle C^{ϵ_n} and we can work on that setting without loss of generality. So, consider in $\mathbb{R}^d$ the set C^{ϵ_n} with

$$C = \{(x, y, 0, \dots, 0) \in \mathbb{R}^d \colon x^2 + y^2 = 1\}.$$

We consider as in the whole manuscript a sample $V = \{x_0, \dots, x_{n-1}\}$ of n i.i.d. uniform points in C^{ϵ_n} and we declare $x_i \sim x_j$ if and only if their projections in the unit circle are at distance less than ϵ_n. Observe that this implies that the distance between them is less than $3\epsilon_n$. By working with the projections, we obtain a random geometric graph in the circle and hence we can apply Theorem 1.1.

4.2. **Bounds for the existence of u_q.** In the course of the proof of Theorem 1.1 we saw that with high probability the infimum of the energy on the boundary of any K_q is bounded below by $(16n\epsilon_n^2)^{-1}$. This bound is sharp. Then we expect the event $A_{n,q}$ to have small probability for $(16n\epsilon_n^2)^{-1} < q^2/4$ and large probability when $(16n\epsilon_n^2)^{-1} > q^2/4$, which is equivalent to

$$|q| < \frac{1}{2\sqrt{n}\epsilon_n} \to \infty.$$

Hence, the larger the $|q|$, the larger the n we need to get the existence of a $q-$twisted state with high probability.

In fact, following the same arguments it can be proved that if $q_n < \frac{1}{2\sqrt{n}\epsilon_n}$ for n large enough, then

$$\lim_{n\to\infty} \mathbb{P}\left((1.2) \text{ has an asymptotically stable equilibrium with index } q_n\right) = 1.$$

4.3. **The role of the scaling factor.** Equation (1.2) is scaled according to the factor $1/n\epsilon_n^2$. The goal of this factor is to obtain Proposition 3.1, but once we obtain the existence of $q-$twisted states for a specific value of n, the scaling factor plays no role and the same conclusion can be obtained for any other constant used to normalize the energy E_n.

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