

CALDERÓN-ZYGMUND ESTIMATES FOR HIGHER ORDER ELLIPTIC EQUATIONS IN ORLICZ-SOBOLEV SPACES

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ABSTRACT. In this paper we obtain Calderón-Zygmund estimates for the laplacian of the following fourth order quasilinear elliptic problem

$$\Delta(g(\Delta u)\Delta u) = \Delta(g(\Delta f)\Delta f).$$

where the primitive of $g(t)t$, $G(t)$, is an N -function. We prove that if $G(f) \in L^q$, then $G(\Delta u) \in L^q$ for $q \geq 1$.

1. INTRODUCTION

In this paper, we consider Calderón-Zygmund regularity results associated to quasilinear equations involving higher-order operators in the framework of Orlicz-Sobolev spaces. More precisely, we are concern with higher integrability properties on weak solutions to the problem

$$(1.1) \quad \Delta(g(\Delta u)\Delta u) = \Delta(g(\Delta f)\Delta f),$$

in a bounded domain $\Omega \subset \mathbb{R}^n$. Our results are of local nature so no regularity on the boundary of Ω nor boundary conditions are assumed.

This article can be seen as a first step in extending classical Calderón-Zygmund-type estimates to higher order nonlinear elliptic problems. For second order problems, this has been the subject of active research in the past two decades (or even more). For instance, in [17] the authors consider weak solutions to the problem

$$\operatorname{div}(g(|\nabla u|)\nabla u) = \operatorname{div}(g(|\mathbf{f}|)\mathbf{f})$$

and obtained the estimate

$$G(|\mathbf{f}|) \in L^q \Rightarrow G(|\nabla u|) \in L^q, \quad q \geq 1.$$

Here $G(t) = \int_0^t g(s)s \, ds$. The results in [17] can be thought as an extension of the classical estimates of Calderón-Zygmund type that for the case of the p -laplacian (i.e. $g(t) = t^{p-2}$) were obtained in [4] and [7].

So, the main result in this paper is to obtain the following estimate for weak solutions to (1.1):

$$G(|f|) \in L_{loc}^q \Rightarrow G(|\Delta u|) \in L_{loc}^q, \quad q \geq 1.$$

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One important motivation to analyze problem (1.1) came from the analysis of Lane-Emden type systems of the form

$$(1.2) \quad \begin{cases} -\Delta u = \phi(v) & \text{in } \Omega \\ -\Delta v = \psi(u) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us suppose that ψ is odd and one to one. Let $h = \psi^{-1}$. From the equation $-\Delta v = \psi(u)$, we may solve for u and get

$$u = -h(\Delta v).$$

Plugging into the first equation of the system (1.2), we deduce

$$\phi(v) = -\Delta u = \Delta(h(\Delta v)) = \Delta(g(\Delta v)\Delta v),$$

where

$$g(t) = \frac{h(t)}{t}.$$

The operator

$$\Delta_g^2 u = \Delta(g(\Delta u)\Delta u),$$

is called the biharmonic g -Laplacian and it was largely studied in [13].

Therefore, our main theorem is the following:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $G(t) = \int_0^t g(s)s ds$ be an N -function satisfying (2.5), (2.6) and (2.11)-(2.12), and let $u \in W_{loc}^{2,G}(\Omega)$ be a local weak solution of problem (1.1).*

Then, if

$$G(f) \in L_{loc}^q(\Omega), \quad \text{for some } q \geq 1, \text{ then we also have } G(\Delta u) \in L_{loc}^q(\Omega).$$

Moreover, the following estimate holds: there is $C > 0$ such that

$$\int_{B_r(x_0)} (G(\Delta u))^q dx \leq C \left[\left(\int_{B_{2r}(x_0)} G\left(\frac{1}{r^2}u\right) dx + \int_{B_{2r}(x_0)} G\left(\frac{1}{r}|\nabla u|\right) dx \right)^q + \int_{B_{2r}(x_0)} (G(f))^q dx \right],$$

for any $r > 0$ and $x_0 \in \Omega$ so that $B_{4r}(x_0) \subset \Omega$.

Let us mention that the strategy of the proof in obtaining Theorem 1.1 were used in [17] and that these ideas were already present in the seminal work [15].

To end this introduction, let us present an heuristic argument for the validity of Theorem 1.1.

Let $u \in W_{loc}^{2,G}(\Omega)$ be a local weak solution of (1.1) and assume that $G(f) \in L_{loc}^q(\Omega)$. Then, the function

$$w = g(\Delta u)\Delta u - g(f)f \in L_{loc}^1(\Omega)$$

is a very weak solution of $\Delta w = 0$ in Ω . Then, by [18], $w \in W_{loc}^{2,p}(\Omega)$ for any p , and consequently, w is indeed a weak solution of $\Delta w = 0$. This implies that

$$w \in C^\infty(\Omega).$$

So, letting h be the inverse of $g(t)t$,

$$(1.3) \quad \Delta u = h(g(f)f + w), \quad w \in C^\infty,$$

and if

$$h(g(f)f + w) \approx f + \text{smooth term},$$

we will certainly derive from (1.3) that

$$G(\Delta u) \approx G(f) \in L_{loc}^q(\Omega).$$

Organization of the paper. In Section 2 we collect the preliminaries needed in the work and state precisely the hypotheses needed for the nonlinearity g . In Section 3 we analyze Theorem 1.1 in the special and simpler case of $q = 1$. Finally, in Section 4 we prove Theorem 1.1.

2. PRELIMINARIES

In this section, we introduce basic definitions and preliminary results related to Orlicz spaces.

N-functions. We start recalling the definition of an N-function.

Definition 2.1. *A function $G: [0, \infty) \rightarrow \mathbb{R}$ is called an N-function if it admits the representation*

$$G(t) = \int_0^t g(\tau)\tau d\tau,$$

where the function g is right-continuous for $t \geq 0$, even, positive for $t > 0$, non-decreasing and satisfies the conditions

$$g(0) = 0, \quad \lim_{t \rightarrow \infty} g(t)t = \infty.$$

By [8, Chapter 1], an N-function has also the following properties:

- (1) G is continuous, convex, increasing, even and $G(0) = 0$.
- (2) G is super-linear at zero and at infinite, that is

$$\lim_{t \rightarrow 0^+} \frac{G(t)}{t} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{G(t)}{t} = \infty.$$

Indeed, the above conditions serve as an equivalent definition of N-functions.

An important class of N-functions is the following:

Definition 2.2. *We say that the N-function G satisfies the (global) Δ_2 condition if there exists $C > 2$ such that*

$$G(2t) \leq CG(t), \quad \text{for all } t > 0.$$

Examples of functions satisfying the Δ_2 condition are, for instance:

- $G(t) = t^p$, $t \geq 0$, $p > 1$;
- $G(t) = (1 + |t|) \log(1 + |t|) - |t|$;
- $G(t) = t^p \chi_{(0,1]}(t) + t^q \chi_{(1,\infty)}(t)$, $t \geq 0$, $p, q > 1$.
- $G(t) = t^p \log^\alpha(1 + t)$, $p > 1$, $0 < \alpha \leq 1$.

By [8, Theorem 4.1, Chapter 1], an N-function satisfies the Δ_2 condition if and only if there is $p^+ > 1$ such that

$$(2.1) \quad \frac{t^2 g(t)}{G(t)} \leq p^+, \quad \forall t > 0.$$

Associated to G is the N-function complementary to it which is defined as follows:

$$(2.2) \quad \tilde{G}(s) := \sup \{ts - G(t) : t > 0\}.$$

It is shown in [8] that \tilde{G} is also an N-function.

The definition of the complementary function gives the optimal function such that the following Young-type inequality holds

$$(2.3) \quad st \leq G(t) + \tilde{G}(s) \text{ for every } s, t \geq 0.$$

Remark 2.3. It is easy to check that the equality in (2.3) holds for $s = g(t)(t)$.

We also quote the following useful lemma.

Lemma 2.4. [?, Lemma 2.9] *Let G be an N-function. If G satisfies (2.1) then*

$$\tilde{G}(g(t)t) \leq (p^+ - 1)G(t).$$

By [8, Theorem 4.3, Chapter 1], a necessary and sufficient condition for the N-function \tilde{G} complementary to G to satisfy the Δ_2 condition is that there is $p^- > 1$ such that

$$(2.4) \quad p^- \leq \frac{t^2 g(t)}{G(t)}, \quad \forall t > 0.$$

From now on, we will assume the stronger condition that the N-function $G(t) = \int_0^t g(s)s \, ds$ satisfies the following estimate:

$$(2.5) \quad p^- - 3 \leq \frac{tg''(t)}{g'(t)} \leq p^+ - 3, \quad \forall t > 0,$$

for some $1 < p^- \leq p^+$. Moreover we assume that

$$(2.6) \quad G(\sqrt{t}) \text{ is convex in } \mathbb{R}.$$

We observe that condition (2.5) implies

$$(2.7) \quad p^- - 2 \leq \frac{tg'(t)}{g(t)} \leq p^+ - 2$$

and

$$(2.8) \quad 1 < p^- \leq \frac{t^2 g(t)}{G(t)} \leq p^+,$$

and so the Δ_2 -condition holds for both G and \tilde{G} . Also, observe that when $g(t) = |t|^{p-2}$, then assumption (2.6) implies $p \geq 2$.

Observe that the examples presented after Definition 2.2 all verify (2.5).

As a consequence of (2.8), it follows for $s, t \geq 0$ that

$$(2.9) \quad \min \left\{ s^{p^+}, s^{p^-} \right\} G(t) \leq G(st) \leq G(t) \max \left\{ s^{p^+}, s^{p^-} \right\}.$$

Hence, combining (2.3) and (2.9), we have the following Young's inequality with $\varepsilon \in (0, 1)$:

$$(2.10) \quad st = (\varepsilon s)(\varepsilon^{-1}t) \leq \tilde{G}(\varepsilon s) + G(\varepsilon^{-1}t) \leq \varepsilon \tilde{G}(s) + \varepsilon^{-p^+} G(t).$$

We quote the following useful estimate (see [12, Lemma 7.1]):

Lemma 2.5. *Suppose that G is an N-function satisfying (2.6) and (2.7). There exists a constant $C > 0$ such that for all $a, b \in \mathbb{R}$ we have*

$$(g(a)a - g(b)b)(a - b) \geq CG(|a - b|).$$

Definition 2.6. *Given two N-functions A and B , we say that A increases essentially more slowly than B , denoted by $A \ll B$, if for any $c > 0$,*

$$\lim_{t \rightarrow \infty} \frac{A(ct)}{B(t)} = 0.$$

Orlicz-Sobolev spaces. Given an N-function G , with $G'(t) = g(t)t$, we define the Orlicz-Lebesgue class $L^G(\Omega)$ as follows

$$L^G(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R}, \int_{\Omega} G(u) dx < \infty \right\}.$$

If G satisfies the Δ_2 condition, then L^G becomes a vector spaces, that it is a Banach space with respect, for instance, to the Luxemburg norm

$$\|u\|_G := \inf \left\{ \lambda > 0 : \int_{\Omega} G\left(\frac{u}{\lambda}\right) dx \leq 1 \right\}.$$

We will denote the convex modular associated to the norm by

$$\rho_G(u) := \int_{\Omega} G(u) dx.$$

For any positive integer $m \in \mathbb{N}$ we will also consider the Orlicz-Sobolev spaces

$$W^{m,G}(\Omega) := \{u \in L^G(\Omega), D^\alpha u \in L^G(\Omega), \text{ for all multi-index } |\alpha| \leq m\}.$$

The space $W^{m,G}(\Omega)$ equipped with the norm

$$\|u\|_{m,G} := \sum_{|\alpha| \leq m} \|D^\alpha u\|_G$$

is a Banach space.

In the case where also \tilde{G} satisfies the Δ_2 condition, these spaces are reflexive and separable.

In order to have the Sobolev immersions, we need to require some assumptions on G that are the analog of $p < n$ in the classical cases. So, we need to assume that G satisfies:

$$(2.11) \quad \int_0^1 \frac{G^{-1}(s)}{s^{(n+1)/n}} ds < \infty$$

$$(2.12) \quad \int_1^\infty \frac{G^{-1}(s)}{s^{(n+1)/n}} ds = \infty$$

For a given N -function G , define the first order Sobolev conjugate function G_1^* of G by means of

$$(G_1^*)^{-1}(t) := \int_0^t \frac{G^{-1}(s)}{s^{1+1/n}} ds.$$

Then G_1^* is an N -function (see [?]). Next, we define the m -th order conjugate Sobolev function of G recursively as follows

$$G_0^* := G, \quad G_j^* := (G_{j-1}^*)^*, \quad j = 1, \dots, m.$$

At each stage, we assume that

$$\int_0^1 \frac{(G_j^*)^{-1}(s)}{s^{(n+1)/n}} ds < \infty.$$

We obtain in this way a finite sequence of N -functions G_j^* , $j = 0, \dots, m_0$, where m_0 is such that

$$\int_1^\infty \frac{(G_{m_0-1}^*)^{-1}(s)}{s^{(n+1)/n}} ds = \infty$$

but

$$\int_1^\infty \frac{(G_{m_0}^*)^{-1}(s)}{s^{(n+1)/n}} ds < \infty.$$

Indeed, $m_0 \leq n$, since by induction it can be proved that (see [5])

$$(G_m^*)^{-1}(t) \leq K_m t^{(n-j)/n}.$$

Then, we have the following embedding theorem for higher-order Orlicz-Sobolev spaces stated in [5].

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the cone property. Let G be an N -function satisfying (2.8) and let m_0 be defined as before. Then*

- (1) *if $1 \leq m \leq m_0$, then $W^{m,G}(\Omega) \hookrightarrow L^{G_m^*}(\Omega)$. Moreover, if B is an N -function increasing essentially more slowly than G_m^* near infinity, then the embedding $W^{m,G}(\Omega) \hookrightarrow L^B(\Omega)$ is compact;*
- (2) *if $m > m_0$, then $W^{m,G}(\Omega) \hookrightarrow C(\Omega) \cap L^\infty(\Omega)$.*

In this paper we will consider the second order case $m = 2$.

For the construction of some auxiliary problems, we need to take into account some boundary conditions. To this end, we denote $W_0^{2,G}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{2,G}(\Omega)$.

By [16], the norm $\|u\|_{2,G}$ in $W_0^{2,G}(\Omega)$ is equivalent to

$$\|u\|_{2,G} \sim \|\Delta u\|_G.$$

From now, we will consider the norm $\|\Delta u\|_G$ in the space $W_0^{2,G}(\Omega)$. The relevant modular defined on $W_0^{2,G}(\Omega)$ is given by

$$\rho_{2,G}(u) := \rho_G(\Delta u) = \int_\Omega G(\Delta u) dx.$$

To close the section, we quote the following further useful relation between modulars and norms.

Lemma 2.8. *Let G be an N -function satisfying (2.8), and let $\xi^\pm: [0, \infty) \rightarrow \mathbb{R}$ be defined as*

$$\xi^-(t) := \min \{t^{p^-}, t^{p^+}\}, \quad \text{and} \quad \xi^+(t) := \max \{t^{p^-}, t^{p^+}\}.$$

Then

$$\xi^-(\|u\|_{2,G}) \leq \rho_{2,G}(u) \leq \xi^+(\|u\|_{2,G}).$$

Concept of solution. Let us state the definition of solution that will be used throughout the paper:

Definition 2.9. *We say that $u \in W_{loc}^{2,G}(\Omega)$ is a local weak solution of (1.1) in Ω if and only if*

$$\int_{\Omega} g(\Delta u) \Delta u \Delta \varphi \, dx = \int_{\Omega} g(f) f \Delta \varphi \, dx,$$

for any $\varphi \in C_0^\infty(\Omega)$.

Remark 2.10. Observe that by density, we may take φ in $W_0^{2,G}(\Omega)$ in Definition 2.9.

Maximal functions and a Vitali Lemma. In this section, we will include the properties of the maximal functions that we employ. Moreover, we state a modified Vitali Lemma from [15].

Definition 2.11. *Let $f \in L_{loc}^1(\Omega)$. The Hardy-Littlewood maximal function denoted by $\mathcal{M}(f)$ is defined by*

$$\mathcal{M}(x) = \sup_{r>0} \int_{B_r(x)} f(x) \, dx.$$

The following are standard properties of the maximal function: given $f \in L^q(\Omega)$, $q > 1$,

- $\|\mathcal{M}(f)\|_{L^q(\Omega)} \leq \|f\|_{L^q(\Omega)}$,
- for any $\lambda > 0$,

$$|\{x \in \Omega : \mathcal{M}(f)(x) > \lambda\}| \leq \frac{C}{\lambda} \int_{\Omega} |f(x)| \, dx,$$

We quote a modified version of the Vitali Lemma from [15]:

Lemma 2.12. *Let $\varepsilon \in (0, 1)$ and let $C \subset D \subset B_1$ be two measurable sets with $|C| < \varepsilon|B_1|$ and satisfying the following property:*

$$\text{for every } x \in B_1 \text{ with } |C \cap B_r(x)| \geq \varepsilon|B_r|, \text{ there holds } B_r(x) \cap B_1 \subset D.$$

Then,

$$|C| \leq 10^n \varepsilon |D|.$$

Finally, we state an iteration lemma from [6]:

Lemma 2.13. *Assume that φ is a non-negative, real-valued, bounded function defined on an interval $[r, R] \subset \mathbb{R}_+$. Assume further that for all $r \leq \rho < \sigma \leq R$ we have*

$$\varphi(\rho) \leq A_1(\sigma - \rho)^{-\alpha_1} + A_2(\sigma - \rho)^{-\alpha_2} + A_3 + \theta\varphi(\sigma),$$

for $\theta \in (0, 1)$ and positive constant A_1, A_2, A_3 and $\alpha_1 \leq \alpha_2$. Then, there is a constant $C(\alpha_2, \theta) > 0$ such that

$$\varphi(r) \leq C(\alpha_2, \theta) (A_1(R - r)^{-\alpha_1} + A_2(R - r)^{-\alpha_2} + A_3)$$

3. CALDERÓN-ZYGMUNG REGULARITY: THE CASE $q = 1$

We will start by proving the main result for $q = 1$.

Theorem 3.1. *Let G be an N -function satisfying (2.5) and (2.6). Let $u \in W_{loc}^{2,G}(\Omega)$ be a local weak solution of (1.1).*

Then, there is $C > 0$ such that

$$(3.1) \quad \int_{B_r(x_0)} G(\Delta u) dx \leq C \left(\int_{B_{2r}(x_0)} G\left(\frac{1}{r^2}u\right) dx + \int_{B_{2r}(x_0)} G\left(\frac{1}{r}|\nabla u|\right) dx + \int_{B_{2r}(x_0)} G(f) dx \right),$$

for any r and x_0 such that $B_{4r}(x_0) \subset \Omega$.

Proof. We will prove the estimate (3.1) for $r = 1$. The result for an arbitrary ball $B_r(x_0) \subset B_{4r}(x_0) \subset \Omega$ will follow by making the change of variables

$$u_r(x) = \frac{1}{r^2}u(rx).$$

Let $1 < \sigma < \rho < 2$ and take a cut-off function $\xi \in C_0^\infty(B_\rho)$, $\xi = 1$ in B_σ , $0 \leq \xi \leq 1$ in \mathbb{R}^n such that

$$(3.2) \quad |\nabla \xi| \leq \frac{C}{\rho - \sigma}, \quad |\Delta \xi| \leq \frac{C}{(\rho - \sigma)^2}.$$

Then $\xi u \in W_0^{2,G}(B_2)$ and so it can be used as a test function in Definition 2.9. Using that

$$\nabla(\xi u) = \xi \nabla u + u \nabla \xi \quad \text{and} \quad \Delta(u\xi) = \xi \Delta u + 2\nabla u \nabla \xi + u \Delta \xi,$$

we obtain

$$(3.3) \quad \begin{aligned} \int_{B_2} \xi g(|\Delta u|) |\Delta u|^2 dx &= - \int_{B_2} u g(\Delta u) \Delta u \Delta \xi dx - 2 \int_{B_2} g(\Delta u) \Delta u \nabla u \cdot \nabla \xi dx \\ &+ \int_{B_2} \xi g(f) f \Delta u dx + \int_{B_2} u g(f) f \Delta \xi dx + 2 \int_{B_2} g(f) f \nabla u \cdot \nabla \xi dx \\ &= I + II + III + IV + V. \end{aligned}$$

Since $\xi = 1$ in B_σ , $\xi \in C_0^\infty(B_\rho)$ and by (2.8), we have

$$p^- \int_{B_\sigma} G(\Delta u) dx \leq \int_{B_\rho} \xi g(|\Delta u|) |\Delta u|^2 dx.$$

In the following estimates we appeal to Young's inequality with $\varepsilon > 0$, see (2.10), the bounds (3.2) and Lemma 2.4:

$$\begin{aligned} |I| &= \left| \int_{B_\rho} u g(\Delta u) \Delta u \Delta \xi dx \right| \leq \varepsilon \int_{B_\rho} \tilde{G}(g(\Delta u) \Delta u) dx + C_\varepsilon (\rho - \sigma)^{-2p^+} \int_{B_\rho} G(u) dx \\ &\leq C\varepsilon \int_{B_\rho} G(\Delta u) dx + C_\varepsilon (\rho - \sigma)^{-2p^+} \int_{B_\rho} G(u) dx; \\ |II| &= \left| \int_{B_\rho} g(\Delta u) \Delta u \nabla u \cdot \nabla \xi dx \right| \leq C\varepsilon \int_{B_\rho} G(\Delta u) dx + C_\varepsilon (\rho - \sigma)^{-p^+} \int_{B_\rho} G(|\nabla u|) dx; \end{aligned}$$

$$\begin{aligned}
|III| &= \left| \int_{B_\rho} \xi g(f) f \Delta u \, dx \right| \leq C_\varepsilon \int_{B_\rho} G(f) \, dx + \varepsilon \int_{B_\rho} G(\Delta u) \, dx; \\
|IV| &= \left| \int_{B_\rho} u g(f) f \Delta \xi \, dx \right| \leq C_\varepsilon \int_{B_\rho} G(f) \, dx + C_\varepsilon (\rho - \sigma)^{-2p^+} \int_{B_\rho} G(u) \, dx; \\
|V| &= \left| \int_{B_\rho} g(f) f \nabla u \cdot \nabla \xi \, dx \right| \leq C_\varepsilon \int_{B_\rho} G(f) \, dx + C_\varepsilon (\rho - \sigma)^{-p^+} \int_{B_\rho} G(|\nabla u|) \, dx.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\int_{B_\sigma} G(\Delta u) \, dx &\leq C_\varepsilon \int_{B_\rho} G(\Delta u) \, dx + C_\varepsilon (\rho - \sigma)^{-2p^+} \int_{B_\rho} G(u) \, dx \\
&\quad + C_\varepsilon (\rho - \sigma)^{-p^+} \int_{B_\rho} G(|\nabla u|) \, dx + C_\varepsilon \int_{B_\rho} G(f) \, dx,
\end{aligned}$$

and so by Lemma 2.13, there is a constant $C(p^+, \varepsilon) > 0$ such that

$$\int_{B_1} G(\Delta u) \, dx \leq C(p^+, \varepsilon) \left(\int_{B_2} G(f) \, dx + \int_{B_2} G(u) \, dx + \int_{B_2} G(|\nabla u|) \, dx \right).$$

This ends the proof. \square

4. CALDERÓN-ZYGMUND REGULARITY: HIGHER ORDER INTEGRABILITY

In this section, we will prove Theorem 1.1. We will drive the proof into some steps.

Step 1: approximating sequence and preliminary estimates. Given $\varepsilon > 0$, define

$$g_\varepsilon(t) := g(\sqrt{\varepsilon + t^2}).$$

Then, for

$$G_\varepsilon(t) := \int_0^t g_\varepsilon(s) s \, ds,$$

we have that G_ε is an N-function and moreover

$$L^{G_\varepsilon}(\Omega) = L^G(\Omega)$$

and

$$W^{2, G_\varepsilon}(\Omega) = W^{2, G}(\Omega).$$

Let $B_2 \subset \Omega$. Consider the following boundary value problem:

$$(4.1) \quad \begin{cases} \Delta (g_\varepsilon(\Delta v^\varepsilon) \Delta v^\varepsilon) = 0 & \text{in } B_2 \\ v^\varepsilon - u \in W_0^{2, G}(B_2) \end{cases}$$

The existence of v^ε is proved in the next proposition.

Proposition 4.1. *Assume that $u \in W_{loc}^{2,G}(\Omega)$ is a local weak solution of (1.1) in Ω . Let $B_2 \subset \Omega$. Then, for any $\varepsilon > 0$, there is $v^\varepsilon \in W^{2,G}(B_2)$ solving (4.1) and such that*

$$(4.2) \quad \int_{B_2} G_\varepsilon(\Delta v^\varepsilon) dx \leq C \left(\int_{B_2} G(\Delta u) dx + 1, \right)$$

where the constant $C > 0$ is independent of ε .

Proof. For the proof we use the direct method of the calculus of variations. So we define the functional

$$\mathcal{F}_\varepsilon(v) = \int_{B_2} G_\varepsilon(\Delta v) dx$$

and we look for minima of \mathcal{F}_ε ,

$$\min_{v-u \in W_0^{2,G}(B_2)} \mathcal{F}_\varepsilon(v)$$

First, we show that \mathcal{F}_ε is coercive. Observe that

$$G_\varepsilon(t) = \int_0^t s g_\varepsilon(s) ds = G(\sqrt{\varepsilon + t^2}) - G(\sqrt{\varepsilon}),$$

so

$$G(t) \leq G(\sqrt{\varepsilon + t^2}) = G_\varepsilon(t) + G(\sqrt{\varepsilon}).$$

Suppose that $\|\Delta v\|_G > 1$, then

$$\mathcal{F}_\varepsilon(v) = \int_{B_2} G_\varepsilon(\Delta v) dx \geq \int_{B_2} G(\Delta v) dx \geq \|\Delta v\|_G^p.$$

On the other hand

$$\begin{aligned} \|\Delta v\|_G &= \|\Delta(v-u) + \Delta u\|_G \geq \|\Delta(v-u)\|_G - \|\Delta u\|_G \\ &\geq C\|v-u\|_{2,G} - C \geq C\|v\|_{2,G} - C. \end{aligned}$$

Observe that \mathcal{F}_ε is bounded below, since $G_\varepsilon(t) \geq 0$.

Now, we want to prove that \mathcal{F}_ε is weakly semi continuous. Suppose that $w_k \rightharpoonup w$ in $W^{2,G}(B_2)$. Using that the set $\mathcal{A} := \{v \in W^{2,G}(B_2) : v = u \text{ in } \partial B_2\}$ is convex and strongly closed we obtain that $w \in \mathcal{A}$. Also, up to subsequence, $w_k \rightarrow w$ in $L^G(\Omega)$ and $w_k \rightarrow w$ a.e. By Egoroff Theorem, for any $\delta > 0$ there exists $E_\delta > 0$ such that $w_k \rightarrow w$ uniformly in E_δ and $|B_2 \setminus E_\delta| < \delta$. We define

$$F_\delta = \left\{ x \in B_2 : |w(x)| + |\Delta w(x)| \leq \frac{1}{\delta} \right\}.$$

Then $|B_2 \setminus F_\delta| \rightarrow 0$ as $\delta \rightarrow 0$. Taking $H_\delta = E_\delta \cap F_\delta$, observe that $|B_2 \setminus H_\delta| \rightarrow 0$ as $\delta \rightarrow 0$. Now,

$$\mathcal{F}_\varepsilon(w_k) = \int_{B_2} G_\varepsilon(\Delta w_k) dx \geq \int_{H_\delta} G_\varepsilon(\Delta w_k) dx.$$

Using the convexity of $G_\varepsilon(t)$

$$\int_{H_\delta} G_\varepsilon(\Delta w_k) dx \geq \int_{H_\delta} G_\varepsilon(\Delta w) dx + \int_{H_\delta} G'_\varepsilon(\Delta w)(w_k - w) dx.$$

As the second term in the right hand side goes to 0 as $k \rightarrow \infty$. Then, taking limit

$$\liminf_{k \rightarrow \infty} \mathcal{F}_\varepsilon(w_k) \geq \int_{H_\delta} G_\varepsilon(\Delta w) dx \quad \text{for all } \delta > 0.$$

So, $\liminf_{k \rightarrow \infty} \mathcal{F}_\varepsilon(w_k) \geq \mathcal{F}_\varepsilon(w)$.

Finally, we show the uniqueness of the solution. The uniqueness of minimizers follows from the strict convexity of G_ε , so it remains to see that all solutions to (4.1) are in fact minimizers of \mathcal{F}_ε . But this follows in a classical fashion, in fact using $u - v$ as a test function in the weak formulation of (4.1), we get

$$\begin{aligned} 0 &= \int_{B_2} \Delta(g(\Delta u)\Delta u)(u - v) dx = \int_{B_2} g(\Delta u)\Delta u(\Delta u - \Delta v) dx \\ &= \int_{B_2} g(\Delta u)\Delta u\Delta u dx - \int_{B_2} g(\Delta u)\Delta u\Delta v dx \\ &\geq \int_{B_2} g(\Delta u)\Delta u\Delta u dx - \int_{B_2} \tilde{G}(g(\Delta u)\Delta u) dx - \int_{B_2} G(\Delta v) dx \end{aligned}$$

where we have used the Young-type inequality (2.3) in the last step. Therefore, we arrive at

$$\int_{B_2} G(\Delta v) dx \geq \int_{B_2} g(\Delta u)\Delta u\Delta u - \tilde{G}(g(\Delta u)\Delta u) dx = \int_{B_2} G(\Delta u) dx,$$

where in the last equality we use Remark 2.3.

This completes the proof. \square

Remark 4.2. Observe that the function $w := g_\varepsilon(\Delta v^\varepsilon)\Delta v^\varepsilon$ is a very weak solution of the equation $\Delta w = 0$ in B_2 (see for instance [18]) and hence, by [18, Theorem 1.3], $w \in W_{loc}^{2,p}(B_2)$ for any $p \geq 1$. Hence, w is indeed a weak solution and therefore $w \in C^\infty(B_2)$. In addition, the function

$$t \rightarrow g_\varepsilon(t)t$$

is strictly monotone and of class $C^{2,\alpha}$. Then, by the inverse function Theorem, it has a differentiable inverse $h_\varepsilon \in C^{2,\beta}$ and since

$$\Delta v^\varepsilon = h_\varepsilon(g_\varepsilon(\Delta v^\varepsilon)\Delta v^\varepsilon),$$

we derive that v^ε is in $C^{4,\gamma}(B_2)$ for some $0 < \gamma < 1$. We will use this fact in the next result.

Proposition 4.3. *Assume that $u \in W_{loc}^{2,G}(\Omega)$ is a local weak solution of (1.1) in Ω . Let $B_2 \subset \Omega$. Then, for any $\varepsilon > 0$, the solution $v^\varepsilon \in W^{2,G}(B_2)$ of (4.1) satisfies for each $\delta \in (0, 2)$,*

$$(4.3) \quad \sup_{B_{2-\delta}} G(\Delta v^\varepsilon) \leq C \left(\int_{B_2} G(\Delta u) dx + 1 \right)$$

where $C > 0$ depends on δ but not on ε .

Proof. Let us consider first

$$w = G_\varepsilon(\Delta v),$$

where $v := v^\varepsilon$. Then, for each $1 \leq i \leq n$, and by Remark 4.2, we may differentiate w to get

$$D_i w = g(\sqrt{\varepsilon + (\Delta v)^2})\Delta v D_i(\Delta v)$$

and

$$(4.4) \quad \begin{aligned} D_{ii}w &= \frac{g'(\sqrt{\varepsilon + (\Delta v)^2})}{\sqrt{\varepsilon + (\Delta v)^2}} (\Delta v)^2 (D_i(\Delta v))^2 + g(\sqrt{\varepsilon + (\Delta v)^2}) (D_i(\Delta v))^2 \\ &\quad + g(\sqrt{\varepsilon + (\Delta v)^2}) \Delta v D_{ii}(\Delta v) \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

In the next lines, we will show that there is a uniformly elliptic coefficient $a = a(x)$ such that

$$(4.5) \quad a(x) \Delta w \geq 0, \text{ in } B_2,$$

that is, w is subharmonic.

Since

$$D_i(g_\varepsilon(\Delta v)\Delta v) = \left[\frac{g'(\sqrt{\varepsilon + (\Delta v)^2})}{\sqrt{\varepsilon + (\Delta v)^2}} (\Delta v)^2 + g(\sqrt{\varepsilon + (\Delta v)^2}) \right] D_i(\Delta v)$$

we obtain by multiplication of equation (4.1) by Δv that

$$0 = \Delta v \Delta (g_\varepsilon(\Delta v)\Delta v) = \Delta v \sum_{i=1}^n D_{ii}(g_\varepsilon(\Delta v)\Delta v),$$

but

$$(4.6) \quad \begin{aligned} \Delta v D_{ii}(g_\varepsilon(\Delta v)\Delta v) &= \left(g''(\sqrt{\varepsilon + (\Delta v)^2}) - \frac{g'(\sqrt{\varepsilon + (\Delta v)^2})}{\sqrt{\varepsilon + (\Delta v)^2}} \right) \frac{(\Delta v)^4}{\varepsilon + (\Delta v)^2} [D_i(\Delta v)]^2 \\ &\quad + 2 \frac{g'(\sqrt{\varepsilon + (\Delta v)^2})}{\sqrt{\varepsilon + (\Delta v)^2}} (\Delta v)^2 [D_i(\Delta v)]^2 + \frac{g'(\sqrt{\varepsilon + (\Delta v)^2})}{\sqrt{\varepsilon + (\Delta v)^2}} (\Delta v)^3 D_{ii}(\Delta v) \\ &\quad + \frac{g'(\sqrt{\varepsilon + (\Delta v)^2})}{\sqrt{\varepsilon + (\Delta v)^2}} (\Delta v)^2 [D_i(\Delta v)]^2 + g(\sqrt{\varepsilon + (\Delta v)^2}) \Delta v D_{ii}(\Delta v) \\ &=: J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

Observe that

$$(4.7) \quad J_2 = 2I_1, J_4 = I_1 \quad \text{and} \quad J_5 = I_3.$$

Moreover, if $\Delta v D_{ii}(\Delta v) \leq 0$, then $J_3 \leq 0$ and we regret this term. So we may assume that $\Delta v D_{ii}(\Delta v) > 0$.

Now, by (2.5),

$$g''(\sqrt{\varepsilon + (\Delta v)^2}) - \frac{g'(\sqrt{\varepsilon + (\Delta v)^2})}{\sqrt{\varepsilon + (\Delta v)^2}} \leq \frac{(p^+ - 4)g'(\sqrt{\varepsilon + (\Delta v)^2})}{\sqrt{\varepsilon + (\Delta v)^2}}$$

and so

$$(4.8) \quad J_1 \leq (p^+ - 4) \frac{(\Delta v)^2}{\varepsilon + (\Delta v)^2} I_1.$$

Regarding J_3 , we have

$$(4.9) \quad \begin{aligned} J_3 &= \frac{g'(\sqrt{\varepsilon + (\Delta v)^2})}{\sqrt{\varepsilon + (\Delta v)^2}} (\Delta v)^3 D_{ii}(\Delta v) \leq (p^+ - 2) \frac{(\Delta v)^2}{\varepsilon + (\Delta v)^2} g(\sqrt{\varepsilon + (\Delta v)^2}) \Delta v D_{ii}(\Delta v) \\ &= (p^+ - 2) \frac{(\Delta v)^2}{\varepsilon + (\Delta v)^2} I_3. \end{aligned}$$

Therefore, combining (4.6), (4.7), (4.8) and (4.9), using that $I_2 \geq 0$ and $\frac{(\Delta v)^2}{\varepsilon + (\Delta v)^2} \leq 1$, we obtain

$$0 = \sum_{i=1}^5 J_i \leq \left(3 + (p^+ - 4) \frac{(\Delta v)^2}{\varepsilon + (\Delta v)^2} \right) \Delta w,$$

which proves (4.5).

Therefore, the estimate (4.3) follows by observing that $G(t) \leq G_\varepsilon(t)$ for all t , the weak maximum principle [2, Chapter IV, Lemma 1.2] applied to $w = G_\varepsilon(\Delta v^\varepsilon)$ and the uniform estimate (4.2). \square

Step 2: convergence of the approximating sequence. In the next proposition, we show that the approximating sequence v^ε converges to a biharmonic solution.

Proposition 4.4. *There exists $v \in W^{2,G}(B_2)$ which is a weak solution of*

$$(4.10) \quad \begin{cases} \Delta(g(\Delta v)\Delta v) = 0, & \text{in } B_2, \\ v = u, & \text{on } \partial B_2, \end{cases}$$

and satisfies, for each $\delta \in (0, 2)$, the estimate

$$(4.11) \quad \sup_{B_{2-\delta}} G(\Delta v) \leq C \left(\int_{B_2} G(\Delta u) dx + 1 \right).$$

Proof. By the estimate (4.2), there is a subsequence $\varepsilon_k \rightarrow 0$, with corresponding solutions $v_k := v^{\varepsilon_k}$ of (4.1) and a function $v \in W^{2,G}(B_2)$, with $u - v \in W_0^{2,G}(B_2)$ such that

$$v_k \rightharpoonup v \text{ weakly in } W^{2,G}(B_2).$$

Consequently,

$$v_k \rightarrow v \text{ strongly in } L^G(B_2)$$

and

$$\nabla v_k \rightarrow \nabla v \text{ strongly in } L^G(B_2).$$

Let $r_m > 0$ be an increasing sequence such that $r_m \rightarrow 2$ as $m \rightarrow \infty$. Fix m with $r_m > 1$. Then, by Proposition 4.3 and the fact that $G \leq G_\varepsilon$,

$$(4.12) \quad \|G(v_k)\|_{L^\infty(B_{r_m})} \leq \|G_{\varepsilon_k}(v_k)\|_{L^\infty(B_{r_m})} \leq M_m, \text{ for all } k.$$

Let $\varphi \in C_0^\infty(B_2)$, such that $0 \leq \varphi \leq 1$, $\varphi = 1$ in B_{r_m} . In what follows, we will show that Δv_k is a Cauchy sequence in measure in B_{r_m} . Hence, take k, l and consider the corresponding solutions v_k and v_l of (4.1). Then,

$$\Delta(g(\Delta v_k)\Delta v_k) = \Delta(g(\Delta v_k)\Delta v_k - g_{\varepsilon_k}(\Delta v_k)\Delta v_k)$$

and similarly,

$$\Delta(g(\Delta v_l)\Delta v_l) = \Delta(g(\Delta v_l)\Delta v_l - g_{\varepsilon_l}(\Delta v_l)\Delta v_l).$$

Next, testing with $(v_k - v_l)\varphi$, we get

$$\begin{aligned}
& \int_{B_2} [g(\Delta v_k)\Delta v_k - g(\Delta v_l)\Delta v_l] (\Delta v_k - \Delta v_l)\varphi \, dx \\
&= - \int_{B_2} [g(\Delta v_k)\Delta v_k - g(\Delta v_l)\Delta v_l] (v_k - v_l)\Delta\varphi \, dx \\
(4.13) \quad & - 2 \int_{B_2} [g(\Delta v_k)\Delta v_k - g(\Delta v_l)\Delta v_l] (\nabla v_k - \nabla v_l) \cdot \nabla\varphi \, dx \\
& + \int_{B_2} [g(\Delta v_k)\Delta v_k - g_{\varepsilon_k}(\Delta v_k)\Delta v_k]\Delta[(v_k - v_l)\varphi] \, dx \\
& - \int_{B_2} [g(\Delta v_l)\Delta v_l - g_{\varepsilon_l}(\Delta v_l)\Delta v_l]\Delta[(v_k - v_l)\varphi] \, dx \\
& =: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Regarding the term I_1 , observe that from Lemma 2.4 and Proposition 4.1, the term

$$g(\Delta v_k)\Delta v_k - g(\Delta v_l)\Delta v_l$$

belongs to $L^{\tilde{G}}(B_2)$ and it is uniformly bounded in norm. Hence, by Hölder inequality and the strong convergence of v_k , we obtain

$$(4.14) \quad I_1 \rightarrow 0 \text{ as } l, k \rightarrow \infty.$$

A similar reasoning shows that

$$(4.15) \quad I_2 \rightarrow 0 \text{ as } l, k \rightarrow \infty,$$

as well. Next, write

$$I_3 = I_{31} + I_{32} + I_{33},$$

where

$$\begin{aligned}
I_{31} &= \int_{B_2} [g(\Delta v_k)\Delta v_k - g_{\varepsilon_k}(\Delta v_k)\Delta v_k][\Delta v_k - \Delta v_l]\varphi \, dx, \\
I_{32} &= \int_{B_2} [g(\Delta v_k)\Delta v_k - g_{\varepsilon_k}(\Delta v_k)\Delta v_k](v_k - v_l)\Delta\varphi \, dx,
\end{aligned}$$

and finally,

$$I_{33} = 2 \int_{B_2} [g(\Delta v_k)\Delta v_k - g_{\varepsilon_k}(\Delta v_k)\Delta v_k](\nabla v_k - \nabla v_l) \cdot \nabla\varphi \, dx.$$

Since (4.12) holds, we obtain by Lemma 2.4 again,

$$(4.16) \quad |I_{32}| \leq C(M) \int_{B_2} G(v_k - v_l) \, dx \rightarrow 0 \text{ as } k, l \rightarrow \infty.$$

An analogous reasoning shows that $I_{33} \rightarrow 0$ as $k, l \rightarrow \infty$. Regarding the term I_{31} , in [17] it is proved that

$$h_\varepsilon(t) := |g(t) - g_\varepsilon(t)|t = O(\varepsilon),$$

for $t \in [0, M]$. Therefore, this fact together with Proposition 4.3 imply that $I_{31} \rightarrow 0$ as $k, l \rightarrow \infty$. The term I_4 in (4.13) is treated similarly. Hence, we derive that

$$\int_{B_{r_m}} [g(\Delta v_k)\Delta v_k - g(\Delta v_l)\Delta v_l] (\Delta v_k - \Delta v_l) dx \rightarrow 0, \quad \text{as } k, l \rightarrow \infty.$$

By Lemma 2.5, this implies that $\{\Delta v_k\}$ is a Cauchy sequence in $L^G(B_{r_m})$ and hence $v_k \rightarrow v$ strongly in $W^{2,G}(B_{r_m})$. By considering the balls B_{r_m} , the assumption $r_m \rightarrow 2$ and extracting a diagonal subsequence, we obtain that

$$(4.17) \quad \Delta v_k \rightarrow \Delta v \text{ a.e. in } B_2.$$

Hence, taking $k \rightarrow \infty$ in (4.3), the estimate (4.11) follows. Finally, for any test function $\varphi \in C_0^\infty(B_2)$, and letting $K := \text{supp}(\varphi)$, we have by (4.3)

$$\sup_K G(\Delta v^\varepsilon) \leq C.$$

Consequently,

$$|g(\Delta v^\varepsilon)\Delta v^\varepsilon\Delta\varphi| \leq C(G(\Delta v^\varepsilon) + 1) \leq C$$

and by (4.17)

$$g(\Delta v^\varepsilon)\Delta v^\varepsilon\Delta\varphi \rightarrow g(\Delta v)\Delta v\Delta\varphi \quad \text{a.e. in } K \text{ as } \varepsilon \rightarrow 0^+.$$

Therefore, by Lebesgue's dominated convergence theorem, v solves (4.10). \square

Proposition 4.5. *Let $u \in W_{loc}^{2,G}(\Omega)$ be a weak solution of (1.1). For $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that if*

$$(4.18) \quad \int_{B_2} G(\Delta u) dx \leq 1 \text{ and } \int_{B_2} G(f) dx \leq \delta,$$

then there is $v \in W^{2,G}(B_2)$ a weak solution of

$$(4.19) \quad \begin{cases} \Delta(g(\Delta v)\Delta v) = 0, & \text{in } B_2 \\ v - u \in W_0^{2,G}(B_2), & \text{on } \partial B_2. \end{cases}$$

such that

$$(4.20) \quad \int_{B_2} G(\Delta u - \Delta v) dx \leq \varepsilon$$

and

$$(4.21) \quad \sup_{B_{3/2}} G(\Delta v) \leq N_0,$$

for some $N_0 > 1$.

Proof. The estimate (4.21) follows from (4.11) and the assumption (4.18).

Since u is a weak solution of (1.1) and v is a solution of (4.19), we test both equations with $\varphi = u - v \in W_0^{2,G}(B_2)$ and we get

$$(4.22) \quad \int_{B_2} (g(\Delta u)\Delta u - g(\Delta v)\Delta v) (\Delta u - \Delta v) dx = \int_{B_2} (g(f)f) (\Delta u - \Delta v) dx.$$

By Lemma 2.5 and Young inequality (2.10) with $\eta > 0$, we get

$$(4.23) \quad \int_{B_2} (g(\Delta u)\Delta u - g(\Delta v)\Delta v) (\Delta u - \Delta v) dx \geq C \int_{B_2} G(\Delta u - \Delta v) dx$$

and

$$(4.24) \quad \int_{B_2} (g(f)f) (\Delta u - \Delta v) dx \leq C_\eta \int_{B_2} G(f) dx + \eta \int_{B_2} G(\Delta u - \Delta v) dx.$$

Hence, combining (4.23) and (4.24) with (4.22) and appealing to (4.18), we get

$$(4.25) \quad C \int_{B_2} G(\Delta u - \Delta v) dx \leq \delta C_\eta + \eta \int_{B_2} G(\Delta u - \Delta v) dx.$$

Choosing $\eta < C/2$ and, for $\varepsilon > 0$ arbitrary, $\delta < \varepsilon/(2C_\eta)$ give (4.20). \square

4.1. Step 3: estimates for the level sets of the Hardy-Littlewood maximal function.

Proposition 4.6. *There exists $N_2 > 1$ such that for any $\varepsilon > 0$, there is $\delta > 0$ so that if u is a local weak solution of (1.1) in Ω with $B_{4r} \subset \Omega$ and*

$$|\{x \in B_r : \mathcal{M}(G(\Delta u))(x) > N_2\}| \geq \varepsilon |B_r|,$$

then there holds

$$B_r \subset \{x \in B_r : \mathcal{M}(G(\Delta u))(x) > 1\} \cup \{x \in B_r : \mathcal{M}(G(f))(x) > \delta\}.$$

Proof. By rescaling

$$u_r(x) := \frac{1}{r^2} u(rx), \quad h(x) := f(rx)$$

we obtain that v is a local weak solution of

$$\Delta(g(\Delta w)\Delta w) = \Delta(g(h)h),$$

in the set $\Omega' = \frac{1}{r}\Omega$. Observe that by assumption $B_4 \subset \Omega'$. Therefore, it is enough to show that

$$|\{x \in B_1 : \mathcal{M}(G(\Delta u))(x) > N_2\}| \geq \varepsilon |B_1|,$$

implies

$$B_1 \subset \{x \in B_1 : \mathcal{M}(G(\Delta u))(x) > 1\} \cup \{x \in B_1 : \mathcal{M}(G(f))(x) > \delta\}.$$

We will prove the counter reciprocal. Assume that $x_1 \in B_1$ satisfies

$$(4.26) \quad \mathcal{M}(G(\Delta u))(x_1) \leq 1 \quad \text{and} \quad \mathcal{M}(G(f))(x_1) \leq \delta.$$

Then,

$$\int_{B_2} G(\Delta u) dx = \frac{1}{\omega_n 2^n} \int_{B_2} G(\Delta u) dx \leq \left(\frac{3}{2}\right)^n \int_{B_3(x_1)} G(\Delta u) dx \leq \left(\frac{3}{2}\right)^n,$$

where ω_n is the measure of the unit ball in \mathbb{R}^n . Similarly,

$$(4.27) \quad \int_{B_2} G(f) dx \leq \left(\frac{3}{2}\right)^n \delta.$$

By Proposition 4.5, for any $\eta > 0$, there is $\delta > 0$ and $v \in W^{2,G}(B_2)$ a local weak solution of (4.10) such that

$$(4.28) \quad \int_{B_2} G(\Delta u - \Delta v) dx \leq \eta$$

and

$$(4.29) \quad \sup_{B_{3/2}} G(\Delta v) \leq N_0,$$

for some $N_0 > 1$. Let

$$A := \{x \in B_1 : \mathcal{M}(G(\Delta u))(x) > N_2\}$$

and

$$B = \{x \in B_1 : \mathcal{M}(G(\Delta u - \Delta v))(x) > N_0\},$$

for some $N_2 > 1$ to be chosen. We will show that $A \subset B$. Let

$$(4.30) \quad x_2 \in \{x \in B_1 : \mathcal{M}(G(\Delta u - \Delta v))(x) \leq N_0\}.$$

Then, for $r \leq 1/2$, we have $B_r(x_2) \subset B_{3/2}$ and so by (4.30) and (4.29), we obtain

$$(4.31) \quad \begin{aligned} \int_{B_r(x_2)} G(\Delta u) dx &\leq \frac{1}{2} \left(\int_{B_r(x_2)} G(2(\Delta u - \Delta v)) dx + \int_{B_r(x_2)} G(2\Delta v) dx \right) \\ &\leq C \left(\int_{B_r(x_2)} G(\Delta u - \Delta v) dx + \int_{B_r(x_2)} G(\Delta v) dx \right) \\ &\leq CN_0, \end{aligned}$$

If now $r > 1/2$, then

$$\int_{B_r(x_2)} G(\Delta u) dx \leq 5^n \int_{B_{5r}(x_1)} G(\Delta u) dx \leq 5^n,$$

by (4.26). Therefore, we conclude that there is $N_2 > 1$ such that $x_2 \notin A$. Thus, $A \subset B$. Hence,

$$\begin{aligned} &|\{x \in B_1 : \mathcal{M}(G(\Delta u))(x) > N_2\}| \\ &= |A| \leq |B| \\ &= |\{x \in B_1 : \mathcal{M}(G(\Delta u - \Delta v))(x) > N_0\}| \\ &\leq C \int_{B_1} G(\Delta u - \Delta v) dx \leq C\eta \leq \varepsilon, \end{aligned}$$

taking η small enough. This ends the proof of the proposition. \square

Proposition 4.7. *Assume that u is a local weak solution of (1.1) and let δ and N_2 as in the previous proposition. Assume that*

$$(4.32) \quad |\{x \in B_1 : \mathcal{M}(G(\Delta u))(x) > N_2\}| < \varepsilon|B_1|.$$

Then, for any $\lambda > 0$, there holds

$$(4.33) \quad \begin{aligned} &|\{x \in B_1 : \mathcal{M}(G(\Delta u))(x) > \lambda N_2\}| \\ &\leq 10^n \varepsilon (|\{x \in B_1 : \mathcal{M}(G(\Delta u))(x) > \lambda\}| + |\{x \in B_1 : \mathcal{M}(G(f))(x) > \lambda\delta\}|). \end{aligned}$$

Proof. The proof follows as in [15] and [17, Lemma 2.15], by applying the Vitali Lemma 2.12. Indeed, consider first the case $\lambda = 1$ and define the sets

$$C = \{x \in B_1 : \mathcal{M}(G(\Delta u))(x) > N_2\}$$

and

$$D := \{x \in B_1 : \mathcal{M}(G(\Delta u))(x) > 1\} \cup \{x \in B_1 : \mathcal{M}(G(f))(x) > \delta\}.$$

Clearly,

$$C \subset D \subset B_1$$

and by assumption,

$$|C| < \varepsilon |B_1|.$$

Hence, by Proposition 4.6, we may apply Lemma 2.12 to obtain that inequality (4.33) holds. The general case for any $\lambda > 0$ follows by considering that u is a weak solutions of

$$\frac{1}{\lambda} \Delta(g(\Delta u)\Delta u) = \frac{1}{\lambda} \Delta(g(f)f),$$

and hence there is $N_2 > 1$ such that for any $\varepsilon > 0$ and $r > 1$, there is $\delta > 0$ such that

$$\left| \left\{ x \in B_r : \frac{1}{\lambda} \mathcal{M}(G(\Delta u))(x) > N_2 \right\} \right| \geq \varepsilon |B_r|,$$

then there holds

$$B_r \subset \left\{ x \in B_r : \frac{1}{\lambda} \mathcal{M}(G(\Delta u))(x) > 1 \right\} \cup \left\{ x \in B_r : \frac{1}{\lambda} \mathcal{M}(G(f))(x) > \delta \right\}$$

and the proof is complete. \square

In the final step, we will finish the proof of Theorem 1.1.

Step 4: proof of Theorem 1.1. For $\delta > 0$ small to be chosen, let

$$\lambda_0 := \frac{1}{\delta} \left\{ \int_{B_2} G(u) dx + \int_{B_2} G(|\nabla u|) dx + \left(\int_{B_2} G(f)^q dx \right)^{1/q} \right\}.$$

Then, by Theorem 3.1

$$(4.34) \quad \int_{B_1} \frac{1}{\lambda_0} G(\Delta u) dx \leq C \left(\int_{B_2} \frac{1}{\lambda_0} G(u) dx + \int_{B_2} \frac{1}{\lambda_0} G(|\nabla u|) dx + \int_{B_2} \frac{1}{\lambda_0} G(f) dx \right).$$

Therefore, by Hölder's inequality,

$$(4.35) \quad \begin{aligned} \int_{B_1} \frac{1}{\lambda_0} G(\Delta u) dx &\leq C \left(\int_{B_2} \frac{1}{\lambda_0} G(u) dx + \int_{B_2} \frac{1}{\lambda_0} G(|\nabla u|) dx + \int_{B_2} \frac{1}{\lambda_0} G(f) dx \right) \\ &\leq C \left(\delta + \frac{1}{\lambda_0} \left(\int_{B_2} G(f)^q dx \right)^{1/q} \right) \leq C\delta. \end{aligned}$$

Moreover,

$$(4.36) \quad \left| \left\{ x \in B_1 : \frac{1}{\lambda_0} \mathcal{M}(G(\Delta u))(x) > N_2 \right\} \right| \leq \frac{1}{N_2} \int_{B_1} \frac{1}{\lambda_0} G(\Delta u) dx \leq \varepsilon |B_1|,$$

choosing $\delta < \varepsilon N_2/C$ in (4.35). Also, by definition of λ_0 ,

$$(4.37) \quad \int_{B_1} \frac{1}{\lambda_0^q} G(f)^q dx \leq C\delta^q.$$

Hence, by the previous proposition, we derive

$$(4.38) \quad \left| \left\{ x \in B_1 : \frac{1}{\lambda_0} \mathcal{M}(G(\Delta u))(x) > \lambda N_2 \right\} \right| \leq 10^n \varepsilon \left(\left| \left\{ x \in B_1 : \frac{1}{\lambda_0} \mathcal{M}(G(\Delta u))(x) > \lambda \right\} \right| + \left| \left\{ x \in B_1 : \frac{1}{\lambda_0} \mathcal{M}(G(f))(x) > \lambda \delta^q \right\} \right| \right).$$

Thus, as $q > 1$, we deduce as in [17, page 1550], that

$$(4.39) \quad \int_{B_1} \left(\frac{1}{\lambda_0} \mathcal{M}(G(\Delta u)) \right)^q dx \leq C_1 \varepsilon \int_{B_1} \left(\frac{1}{\lambda_0} \mathcal{M}(G(\Delta u)) \right)^q dx + C_2 \int_{B_1} \left(\frac{1}{\lambda_0} G(f) \right)^q dx.$$

So, choosing ε small and recalling (4.37), we obtain

$$\int_{B_1} \left(\frac{1}{\lambda_0} \mathcal{M}(G(\Delta u)) \right)^q dx \leq C$$

which certainly implies

$$\int_{B_1} \left(\frac{1}{\lambda_0} G(\Delta u) \right)^q dx \leq C$$

and consequently,

$$\int_{B_1} G(\Delta u)^q dx \leq C \left\{ \left(\int_{B_2} G(u) dx + \int_{B_2} G(|\nabla u|) dx \right)^q + \int_{B_2} G(f)^q dx \right\}.$$

This completes the proof of the Theorem. □

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