Γ−CONVERGENCE OF ENERGY FUNCTIONALS IN FRACTIONAL ORLICZ SPACES BEYOND THE Δ_2 CONDITION

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ABSTRACT. Given a Young function A, $n \geq 1$ and $s \in (0,1)$ we consider the energy functional

$$\mathcal{J}_s(u) = (1-s) \iint_{\mathbb{R}^n \times \mathbb{R}^n} A\left(\frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{dxdy}{|x-y|^n}.$$

Without assuming the Δ_2 condition on A not its conjugated function $\bar{A}$, we prove the following liminf inequality: if $u \in E^A(\mathbb{R}^n)$ and $\{u_k\}_{k \in \mathbb{N}} \subset E^A(\mathbb{R}^n)$ is such that $u_k \to u$ in $E^A(\mathbb{R}^n)$, and $s_k \to 1$, then

$$\mathcal{J}(u) \leq \liminf_{k \to \infty} \mathcal{J}_{s_k}(u_k),$$

where $\mathcal{J}$ is a limit functional related with the behavior of the fractional Orlicz-Sobolev spaces as $s \to 1^+$. As a direct consequence, we obtain the Γ−convergence of the functional $\mathcal{J}_s$.

Finally, we extend our result to the study of the so called *fractional peridynamic* case.

1. INTRODUCTION

Nonlocal energy functionals given in terms of fractional parameter $s \in (0,1)$ and having the form

$$(1.1) \qquad (1-s) \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dxdy$$

with $\Omega \subset \mathbb{R}^n$, $n \geq 1$, a smooth bounded domain and $1 \leq p < \infty$ play a significant role in nonlinear analysis. The behavior of (1.1) as $s \uparrow 1$ in the $W^{1,p}$ norm (resp. BV for $p = 1$) was addressed in the seminal work of Bourgain, Brezis and Mironescu [6] (see also [10]). In recent years, several extensions and variants of these result have been developed. For instance, see [25, 11, 4, 23, 8, 18, 13].

In the recent year, there has been increasing attention to operators that do not follow power-like laws. A prototype energy functional in this context can be described using a so-called Young function $A \colon \mathbb{R}^+ \to \mathbb{R}$ and a fractional parameter $s \in (0,1)$ as follows

$$(1.2) \qquad \mathcal{J}_s(u) = (1-s) \iint_{\mathbb{R}^n \times \mathbb{R}^n} A(|D^s u|) \, d\nu$$

where $d\nu = |x-y|^{-n}dxdy$ and $D^s u(x,y) = \frac{u(x)-u(y)}{|x-y|^s}$. The behavior of (1.2) as well as many possible extensions, was addressed recently in several articles, see for

instance [1, 2, 14, 15, 20, 7]. In particular, (1.2) converges to a limit functional as $s \uparrow 1$. When both A and its conjugated function $\bar{A}$ satisfy the Δ_2 condition, or which is equivalent, when there exist constants p^+ and p^- such that

$$(1.3) \qquad\qquad 1 < p^- \le \frac{tA'(t)}{A(t)} \le p^+ < \infty,$$

a proof of such convergence can be found in [14, Theorem 4.1]. For an arbitrary Young function the proof is contained in [2, Theorem 1.1]. More precisely, these results state that for any function $u \in E^A(\mathbb{R}^n)$ it holds that

$$(1.4) \qquad\qquad \lim_{s \uparrow 1} \mathcal{J}_s(u) = \mathcal{J}(u)$$

where the limit energy function $\mathcal{J}$ is defined as

$$\mathcal{J}(u) = \int_{\mathbb{R}^n} A_0(|\nabla u|)\, dx$$

being A_0 the Young function defined as follows

$$(1.5) \qquad\qquad A_0(t) = \int_0^t \int_{\mathbb{S}^{n-1}} A(r|\omega \cdot e_n|)\, d\mathcal{H}^{n-1}(\omega) \frac{dr}{r},$$

where ω denotes any fixed unit vector in $\mathbb{S}^{n-1}$. For the precise definition of the functional spaces see Section 2.

The Γ-convergence of functionals is a very useful tool in the study of minimum problems arising in the calculus of variations. This is due of the main features of this type of convergence which implies the convergence of minima (see for instance[9]).

Regarding the energy functional (1.2), when (1.3) is assumed, in [14, Theorem 6.5] it is proved that

$$(1.6) \qquad\qquad \Gamma - \lim_{s \uparrow 1} \bar{\mathcal{J}}_s = \bar{\mathcal{J}}$$

where $\bar{\mathcal{J}}, \bar{\mathcal{J}}_s \colon L^A(\Omega) \to \bar{\mathbb{R}}$ are defined by

$$\bar{\mathcal{J}}_s(u) = \begin{cases} \mathcal{J}_s(u) & \text{if } u \in W_0^{s,A}(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \qquad \bar{\mathcal{J}}(u) = \begin{cases} \mathcal{J}(u) & \text{if } u \in W_0^{1,A_0}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ is open, $W_0^{s,A}(\Omega)$, $s \in (0, 1]$ stand for the fractional Orlicz Sobolev spaces defined in Section 2.

The proof of this result follows by proving the two necessary conditions in the definition of Γ−convergence, namely:

   (i) (lim inf inequality) For every sequence $\{u_k\}_{k \in \mathbb{N}} \subset L^A(\Omega)$ such that $u_k \to u$ in $L^A(\Omega)$ and $\{s_k\}_{k \in \mathbb{N}}$ such that $s_k \uparrow 1$,

$$\bar{\mathcal{J}}(u) \le \liminf_{k \to \infty} \bar{\mathcal{J}}_{s_k}(u_k).$$

   (ii) (lim sup inequality) For every $u \in L^A(\Omega)$ there exists a sequence $\{u_k\}_{k \in \mathbb{N}} \subset L^A(\Omega)$ converging to u such that

$$\bar{\mathcal{J}}(u) \ge \limsup_{k \to \infty} \bar{\mathcal{J}}_{s_k}(u_k)$$

   where $\{s_k\}_{k \in \mathbb{N}}$ is such that $s_k \uparrow 1$.

The proof of condition (ii) follows by choosing the constant sequence in (1.4). However, the challenging part lies in proving the liminf inequality, which cannot be adequately addressed solely by (1.4). A more refined convergence result is required, one that allows the description of the convergence of sequences of functions.

Under the assumption of (1.3) the liminf inequality is treated in [14, Theorem 5.1], where *the case of a sequence* is studied. The proof of that result heavily relies on the doubling condition, making highly non trivial to replicate the same proof without assuming the Δ_2 condition. Hence, to the best of our knowledge, we believe this limitation constitutes the primary obstacle preventing the treatment of the Γ-convergence of (1.2) without assuming the Δ_2 condition on A and its conjugate function $\bar{A}$.

The main aim of this article is to fill this gap. First, we prove that the following liminf inequality holds for the functional $\mathcal{J}_s$ for an arbitrary Young function A without any growth assumptions. More precisely, in Theorem 3.4 we prove that for any function $u \in E^A(\mathbb{R}^n)$ and any sequence $\{u_k\}_{k\in\mathbb{N}} \subset E^A(\mathbb{R}^n)$ such that $u_k \to u$ in $E^A(\mathbb{R}^n)$, if $\{s_k\}_{k\in\mathbb{N}}$ converges to 1 as $k \to \infty$, then

$$\mathcal{J}(u) \leq \liminf_{k\to\infty} \mathcal{J}_{s_k}(u_k).$$

As a direct consequence, in Corollary 3.5 we obtain our first main result:

$$(1.7) \qquad\qquad \Gamma - \lim_{s\uparrow 1} \bar{\mathcal{J}}_s = \bar{\mathcal{J}}$$

where $\bar{\mathcal{J}}, \bar{\mathcal{J}}_s \colon E^A(\Omega) \to \bar{\mathbb{R}}$ are defined by

(1.8)

$$\bar{\mathcal{J}}_s(u) = \begin{cases} \mathcal{J}_s(u) & \text{if } u \in W_0^s E^A(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \qquad \bar{\mathcal{J}}(u) = \begin{cases} \mathcal{J}(u) & \text{if } u \in W_0^1 E^A(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ is open and $s \in (0, 1]$.

Another interesting topic explored in this work concerns a new variant of the functional (1.2), specifically given an horizon $\delta > 0$, the *peridynamic fractional A-energy* is

$$(1.9) \qquad\qquad \mathcal{J}_\delta(u) = \int \int_{\mathbb{R}^n \times B(x,\delta)} A(|D^s u|) \, d\nu.$$

This operators was studied by Ortega in [24] and under certain hypotheses of regular variation at the origin and Δ_2 condition, the author was able to determine the limit as $\delta \to 0$. Moreover the author obtained Gamma convergence results.

In this work, we demonstrate that without the hypotheses of Δ_2 and regular variation at the origin, the limit as $\delta \to 0$ becomes trivial, with the limit function being the Matuszewska function introduced in [22].

2. Preliminaries

2.1. **Young functions.** A function $A \colon \mathbb{R}_+ \to \mathbb{R}_+$ is called a *Young function* if it can be written as

$$A(t) = \int_0^t a(\tau) \, d\tau,$$

where $a \colon \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, right continuous, $a(t) > 0$ if $t > 0$, $a(0) = 0$ and $a(t) \to \infty$ as $t \to \infty$.

From this definition, it can readily be checked that A is convex, $A(0) = 0$ and it is superlinear at 0 and at ∞, i.e.

$$\lim_{t \to 0} \frac{A(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{A(t)}{t} = \infty.$$

Observe that the convexity of the Young function immediately gives that

(2.1) $A(\tau t) \le \tau A(t)$ for $0 < \tau < 1$, $A(\tau t) \ge \tau A(t)$ for $\tau > 1$.

Given a Young function A, an important associated function is the so-called *complementary function*, denoted by $\bar{A}$, that is defined as

$$\bar{A}(t) := \sup\{\tau t - A(\tau)\colon \tau \ge 0\}.$$

The complementary function $\bar{A}$ is also a Young function.

Observe that $\bar{A}$ is the optimal Young function in the *Young-type inequality*

$$\tau t \le \bar{A}(\tau) + A(t)$$

for all $\tau, t \ge 0$. It is also easy to see that $\bar{\bar{A}} = A$.

A Young function A is said to verify the Δ_2 condition if there exists a positive constant C such that

$$A(2t) \le CA(t) \quad \text{for } t \ge 0.$$

Similarly, a Young function A is said to verify the Δ_2^0 condition if and only if there exist $C_0 \ge 2$ and $T_0 > 0$ such that

$$A(2t) \le C_0 A(t) \quad \text{for } t \le T_0.$$

Given a Young function A it holds that

$$A(t) \le ta(t) \le A(2t)$$

which easily follows from the following expressions

$$A(2t) = \int_0^{2t} a(\tau)\, d\tau > \int_t^{2t} a(\tau)\, d\tau > ta(t), \qquad A(t) = \int_0^t a(\tau)\, d\tau \le ta(t).$$

2.2. Orlicz and Orlicz-Sobolev spaces.
For an introduction to Orlicz spaces we refer [19, 17].

Fractional order Orlicz-Sobolev spaces, as we will use them here, were introduced in [12] and then further analysed by several authors. The results used in this paper are found in [3, 2] and in [5].

2.2.1. *Orlicz spaces.* Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a Young function A, the Orlicz class is defined as

$$\mathcal{L}^A(\Omega) := \left\{ u \in L^1_{\mathrm{loc}}(\Omega)\colon \int_\Omega A(|u|)\, dx < \infty \right\}.$$

Then the Orlicz space $L^A(\Omega)$ is defined as the linear hull of $\mathcal{L}^A(\Omega)$ and is characterized as

$$L^A(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega)\colon \text{ there exists } k > 0 \text{ such that } \int_\Omega A\left(\frac{|u|}{k}\right) dx < \infty \right\}.$$

In general the Orlicz class is strictly smaller than the Orlicz space, and $\mathcal{L}^A(\Omega) = L^A(\Omega)$ if and only if A satisfies the Δ_2 condition.

The space $L^A(\Omega)$ is a Banach space when it is endowed, for instance, with the *Luxemburg norm*, i.e.

$$\|u\|_{L^A(\Omega)} = \|u\|_A := \inf\left\{k > 0\colon \int_\Omega A\left(\frac{|u|}{k}\right) dx \le 1\right\}.$$

This space $L^A(\Omega)$ turns out to be separable if and only if A satisfies the Δ_2 condition.

An important subspace of $L^A(\Omega)$ is $E^A(\Omega)$ that it is defined as the closure of the functions in $L^A(\Omega)$ that are bounded. This space is characterized as

$$E^A(\Omega) = \left\{u \in L^1_{\mathrm{loc}}(\Omega)\colon \int_\Omega A\left(\frac{|u|}{k}\right) dx < \infty \text{ for all } k > 0\right\}.$$

This subspace $E^A(\Omega)$ is separable, and we have the inclusions

$$E^A(\Omega) \subset \mathcal{L}^A(\Omega) \subset L^A(\Omega)$$

with equalities if and only if A satisfies the Δ_2 condition.

The following duality relation holds

$$(E^A(\Omega))^* = L^{\bar{A}}(\Omega),$$

where the equality is understood via the standard duality pairing. Observe that this automatically implies that $L^A(\Omega)$ is reflexive if and only if $A, \bar{A}$ satisfy the Δ_2 condition.

2.2.2. *Orlicz-Sobolev spaces.* Once the Orlicz spaces are defined, one can easily define the Orlicz-Sobolev spaces in the usual way, namely

$$W^1 L^A(\Omega) = \{u \in L^A(\Omega)\colon \partial_{x_i} u \in L^A(\Omega), i = 1, \dots, n\}$$

and

$$W^1 E^A(\Omega) = \{u \in E^A(\Omega)\colon \partial_{x_i} u \in E^A(\Omega), i = 1, \dots, n\}.$$

These two spaces coincide only if A satisfies the Δ_2 condition and in this case we will denote the Orlicz-Sobolev space as

$$W^{1,A}(\Omega) = W^1 L^A(\Omega) = W^1 E^A(\Omega).$$

These spaces are endowed with the natural norm

$$\|u\|_{W^1 L^A(\Omega)} = \|u\|_{1,A} = \|u\|_A + \|\nabla u\|_A.$$

With this norm, $W^1 L^A(\Omega)$ is a Banach space and $W^1 E^A(\Omega)$ is a closed subspace.

In order to work with Dirichlet boundary conditions it is necessary to define the Orlicz-Sobolev functions that vanish at the boundary. So, we define $W_0^1 L^A(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with respect to the topology $\sigma(W^1 L^A(\Omega), W^1 E^{\bar{A}}(\Omega))$ and $W_0^1 E^A(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in norm topology.

Again, when $A \in \Delta_2^\infty$ these spaces coincide and will be denoted as

$$W_0^{1,A}(\Omega) = W_0^1 L^A(\Omega) = W_0^1 E^A(\Omega).$$

These spaces are reflexive if and only if both $A, \bar{A} \in \Delta_2^\infty$. In any other case, what we have is that there exists a Banach space X such that $X^* = W_0^1 L^A(\Omega)$. This Banach space X can be easily characterized in terms of distributions, but it will not be used in this article.

2.2.3. *Fractional Orlicz-Sobolev spaces.* The construction of the fractional order Orlicz-Sobolev spaces is almost the same as the Orliz-Sobolev spaces. We will make only a sketch and the reader can consult with [5].

Given a fractional parameter $s \in (0,1)$, we define the Hölder quotient of a function $u \in L^A(\Omega)$ as

$$D^s u(x,y) = \frac{u(x) - u(y)}{|x - y|^s}.$$

Then, the fractional Orlicz-Sobolev space of order s is defined as

$$W^s L^A(\mathbb{R}^n) := \{u \in L^A(\mathbb{R}^n) \colon D^s u \in L^A(\mathbb{R}^{2n}, d\nu_n)\},$$

where $d\nu_n = |x - y|^{-n} dx dy$ and

$$W^s E^A(\mathbb{R}^n) := \{u \in E^A(\mathbb{R}^n) \colon D^s u \in E^A(\mathbb{R}^{2n}, d\nu_n)\}.$$

Since we are now working in the whole space, in order for these spaces to coincide we require that $A \in \Delta_2$. In this case, we denote

$$W^{s,A}(\mathbb{R}^n) = W^s L^A(\mathbb{R}^n) = W^s E^A(\mathbb{R}^n).$$

As in the previous case, we have that $W^s L^A(\mathbb{R}^n)$ is reflexive if and only if $A, \bar{A} \in \Delta_2$.

In these spaces the norm considered is

$$\|u\|_{W^s L^A(\mathbb{R}^n)} = \|u\|_{s,A} = \|u\|_A + \|D^s u\|_{A,d\nu_n}.$$

Again, with this norm, $W^s L^A(\mathbb{R}^n)$ is a Banach space and $W^s E^A(\mathbb{R}^n)$ is a closed subspace.

2.3. **Regularized functions.** As usual, we denote by $\rho \in C_c^\infty(\mathbb{R}^n)$ the standard mollifier where $\mathrm{supp}(\rho) = B_1(0)$ and $\rho_\varepsilon(x) = \varepsilon^{-n} \rho(\frac{x}{\varepsilon})$ is the approximation of the identity. It follows that $\{\rho_\varepsilon\}_{\varepsilon > 0}$ is a familiy of positive functions satisfying

$$(2.2) \qquad \rho_\varepsilon \in C_c^\infty(\mathbb{R}^n), \quad \mathrm{supp}(\rho_\varepsilon) = B_\varepsilon(0), \quad \int_{\mathbb{R}^n} \rho_\varepsilon \, dx = 1.$$

Given $u \in L^A(\mathbb{R}^n)$ we define the regularized functions $u_\varepsilon \in L^A(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ as

$$(2.3) \qquad u_\varepsilon(x) = u * \rho_\varepsilon(x).$$

The following estimate will be of use for our purposes.

Lemma 2.1. *Let* $u \in C_c^2(\mathbb{R}^n)$ *and* $\rho \in C_c^\infty(\mathbb{R}^n)$. *Then*

$$\|u * \rho\|_{C^2} \le C_\rho \|u\|_A$$

where C *depends only on* A, ρ *and their derivatives up to order 2.*

Proof. Let $u \in C_c^2(\mathbb{R}^n)$ and $\rho \in C_c^\infty(\mathbb{R}^n)$. Using Hölder's inequality we get that

$$|\partial^\alpha (u * \rho)(x)| = |u * \partial^\alpha \rho(x)| \le \int_{\mathbb{R}^n} |u(y) \partial^\alpha \rho(x - y)| \, dy$$
$$\le \|u\|_A \|\partial^\alpha \rho(x - \cdot)\|_{\bar{A}} \le C_{\rho,\alpha,A} \|u\|_A$$

since the Orlicz norm is invariant under translations, where $\partial^\alpha \rho$ denotes a partial derivative of ρ of order $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \le 2$.

This gives the lemma. $\square$

3. Proofs of the results

When G satisfies the Δ_2 condition, in [14, Lemma 2.13] it proved that the energy functional decreases under regularization of the function. We claim that the same holds without assuming any growth behavior on G.

Lemma 3.1. *Let G be a Young function and let $u \in L^A(\mathbb{R}^n)$ and $\{u_\varepsilon\}_{\varepsilon>0}$ be the family defined in* (2.3). *Then*

$$\mathcal{J}_s(u_\varepsilon) \leq \mathcal{J}_s(u)$$

for all $\varepsilon > 0$ and $0 < s < 1$.

Proof. Since A is convex and using the last property in (2.2) we can use Jensen's inequality to obtain that for each $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n \setminus \{0\}$ it holds that

(3.1)
$$A\left(\frac{|u_\varepsilon(x+h) - u_\varepsilon(x)|}{|h|^s}\right) \leq A\left(\int_{\mathbb{R}^n} \frac{|u(x+h-y) - u(x-y)|}{|h|^s} \rho_\varepsilon(y)\,dy\right)$$
$$\leq \int_{\mathbb{R}^n} A\left(\frac{|u(x+h-y) - u(x-y)|}{|h|^s}\right) \rho_\varepsilon(y)\,dy.$$

Integrating (3.1) over $\mathbb{R}^n$ and using again (2.2), for each $h \in \mathbb{R}^n \setminus \{0\}$ we get

$$\int_{\mathbb{R}^n} A\left(\frac{|u_\varepsilon(x+h) - u_\varepsilon(x)|}{|h|^s}\right) \frac{dx}{|h|^n} \leq \int_{\mathbb{R}^n} \left(\frac{|u(x+h-y) - u(x-y)|}{|h|^s}\right) \frac{dx}{|h|^n}$$
$$= \int_{\mathbb{R}^n} \left(\frac{|u(x+h-y) - u(x-y)|}{|h|^s} \frac{dx}{|h|^n}\right) \rho_\varepsilon(y)\,dy$$
$$= \int_{\mathbb{R}^n} A\left(\frac{|u(x+h) - u(x)|}{|h|^s}\right) \frac{dx}{|h|^n}.$$

Finally, integrating the last expression over $\mathbb{R}^n$ and performing the change of variables $y = x + h$ gives the result. $\qquad\square$

When assuming (1.3), in [14, Lemma 4.3] it is proved the convergence of the energy functional $\mathcal{J}_s$ for smooth functions. We claim that the same holds without assuming any growth condition on the Young function A, and moreover, the convergence is uniform on bounded sets of C^2.

Lemma 3.2. *Let A be a Young function and let $\mathcal{B} \subset C^2(\mathbb{R}^n)$ be such that there is $C > 0$ for which $\|u\|_{C^2} \leq C$ for all $u \in B$. Then it holds*

$$\mathcal{J}_s \to \mathcal{J}$$

uniformly in $\mathcal{B}$ as $s \uparrow 1$.

Proof. Let A be an arbitrary Young function (without any assumption on its growth behavior). In [2, Lemma 3.4] it is proved that for any $u \in C_c^2(\mathbb{R}^n)$

$$\mathcal{J}_s(u) \leq \mathcal{J}(u) + \frac{1-s}{s} n\omega_n \int_{\mathbb{R}^n} A(2|u|)\,dx.$$

On the other hand, for any $x \in \mathbb{R}^n$ fixed, [14, Lemma 4.3] states that

(3.2)
$$\lim_{s\uparrow 1} \mathcal{F}_s(x) = A_0(|\nabla u(x)|)$$

where

$$\mathcal{F}_s(x) := (1-s) \int_{\mathbb{R}^n} A(|D^s u|) \frac{dy}{|x-y|^n}.$$

In light of (3.2), in order to prove that

$$(3.3) \qquad \lim_{s\uparrow 1} \mathcal{J}_s(u) = \mathcal{J}(u)$$

it only remains to show the existence of an integrable majorant for $\mathcal{F}_s(x)$. Moreover, an inspection of the proof of (3.2) reveals that

$$(3.4) \qquad \left| A(|D^s u|) - A\left(\left| \nabla u(x) \cdot \frac{x-y}{|x-y|^s} \right| \right) \right| \leq C|x-y|^{2-s}$$

where C only depends of the C^2-norm of u. This gives that the convergence in (3.3) is in fact uniform on bounded subsets of $C^2(\mathbb{R}^n)$.

As in [14, Theorem 4.1], we can assume without loss of generality that $\text{supp}(u) \subset B_R(0)$ with $R > 1$. Then, without any assumption on A, in [14, Theorem 4.1] it is proved that when $|x| < 2R$

$$(3.5) \qquad |\mathcal{F}_s(x)| \leq n\omega_n A(\|\nabla u\|_\infty) + \frac{1-s}{s} n\omega_n A(2\|u\|_\infty).$$

When $|x| \geq 2R$ the function u vanishes and

$$\mathcal{F}_s(x) = (1-s) \int_{B_R(0)} A\left(\frac{|u(y)|}{|x-y|^s} \right) \frac{dy}{|x-y|^n}.$$

Since $|x-y| \geq |x| - R \geq \frac{1}{2}|x|$, (and since $|x| \geq 2$) we get that

$$(3.6) \qquad \begin{aligned} |\mathcal{F}_s(x)| &\leq \frac{2^s}{|x|^n} \int_{B_R(0)} A\left(\frac{2^s|u(y)|}{|x|^s} \right) dy \leq \frac{2^s}{|x|^{n+s}} \int_{B_R(0)} A\left(2^s|u(y)|\right) dy \\ &\leq \frac{2^s}{|x|^{n+\frac{1}{2}}} \int_{B_R(0)} A\left(2^s|u(y)|\right) dy < \infty, \end{aligned}$$

since $u \in C_c^2(\mathbb{R}^n)$.

Then, from (3.5) and (3.6) we get that

$$|\mathcal{F}_s(x)| \leq C_1 \left(\chi_{B_R(0)}(x) + \frac{1}{|x|^{n+\frac{1}{2}}} \chi_{B_R(0)^c}(x) \right) \in L^1(\mathbb{R}^n)$$

with $C_1 > 0$ depending on n and u but independent of s. Then (3.3) follows from the Dominated convergence Theorem. $\square$

The following observation is the key in order to study the $\Gamma-$convergence of $\mathcal{J}_s$.

Remark 3.3. If $\{u_k\}_{k\in\mathbb{N}} \subset C^2(\mathbb{R}^n)$ is a sequence of functions uniformly bounded in the C^2-norm such that $u_k \to u \in C^2(\mathbb{R}^n)$ as $k \to \infty$, and $\{s_k\}_{k\in\mathbb{N}}$ converges to 1 as $k \to \infty$, then the uniform convergence established in Lemma 3.2 ensures that

$$\lim_{k\to\infty} \mathcal{J}_{s_k}(u_k) = \mathcal{J}(u).$$

Following carefully the ideas in [25], we prove the following liminf inequality.

Theorem 3.4. *Let $u \in E^A(\mathbb{R}^n)$ and $\{u_k\}_{k\in\mathbb{N}} \subset E^A(\mathbb{R}^n)$ be such that $u_k \to u$ in $E^A(\mathbb{R}^n)$. Then, for any sequence $\{s_k\}_{k\in\mathbb{N}}$ satisfying that $s_k \nearrow 1$ as $k \to \infty$, we have that*

$$\mathcal{J}(u) \leq \liminf_{k\to\infty} \mathcal{J}_{s_k}(u_k).$$

Proof. Consider $u \in E^A(\mathbb{R}^n)$ and $\{u_k\}_{k \in \mathbb{N}} \subset E^A(\mathbb{R}^n)$ such that $u_k \to u$ in $E^A(\mathbb{R}^n)$, and let $\varepsilon > 0$ be fixed.

Following the notation of (2.3), for each $k \in \mathbb{N}$ we consider the regularized function $u_{k,\varepsilon}$ and a sequence $\{s_k\}_{k \in \mathbb{N}}$ such that $s_k \nearrow 1$ as $k \to \infty$. In light of Lemma 3.1 we have that

$$(3.7) \qquad\qquad \mathcal{J}_{s_k}(u_{k,\varepsilon}) \le \mathcal{J}_{s_k}(u_k).$$

On the other hand, by using Lemma 2.1 we get that

$$\|u_{k,\varepsilon}\|_{C^2} = \|u_k * \rho_\varepsilon\|_{C^2} \le C_\varepsilon \|u_k\|_{L^A(\mathbb{R}^n)} \le \tilde{C}_\varepsilon \qquad \text{for all } k \in \mathbb{N},$$

with $\tilde{C}_\varepsilon > 0$ independent of k since by hypothesis we have that $\{u_k\}_k$ converges.

Therefore, for each $\varepsilon > 0$ fixed, using Lemma 3.2 and (3.7), we have that

$$\mathcal{J}(u_\varepsilon) = \lim_{k \to \infty} \mathcal{J}_{s_k}((u_k)_\varepsilon) \le \liminf_{k \to \infty} \mathcal{J}_{s_k}(u_k)$$

hence, taking limit as $\varepsilon \downarrow 0$, we get the result.   □

As a direct consequence of Lemma 3.2 and Theorem 3.4 we get our main result:

Corollary 3.5. *With the notation introduced in* (1.8) *we have that*

$$\Gamma - \lim_{s \uparrow 1} \bar{\mathcal{J}}_s = \bar{\mathcal{J}}.$$

4. PERYDINAMICAL OPERATOR

A new variant of the functional 1.2 aims to analyze the behavior of these functionals under localization. In Ortega [24], given a Young function A and real numbers $\delta > 0$ and $0 < s < 1$ they define

$$(4.1) \qquad\qquad \mathcal{J}_\delta(u) = \int \int_{\mathbb{R}^n \times B(x,\delta)} A(|D^s u|) \, d\nu.$$

One of the main theorems by Ortega is dedicated to study the behavior as $\delta \to 0$. For this purpose, the author assumes certain properties about the function A. Specifically, A is required to be a Young function that satisfies the Δ_2 condition and exhibits some regular variation at the origin, that is

$$(4.2) \qquad\qquad \lim_{t \to 0^+} \frac{A(\lambda t)}{t} = \lambda^p \quad p \in \mathbb{R}_+.$$

Under these hypotheses and some others, Ortega can state and prove the punctual convergence to a certain functional

$$\lim_{\delta \to 0^+} \frac{p(1-s)}{A(\delta^{1-s})} \iint_{\mathbb{R}^n \times B(x,\delta)} A(|D^s u|) \, d\nu = K_{n,p} \int_{\mathbb{R}^n} |\nabla u(x)|^p dx.$$

In context of Gamma convergence the author also provide the lim inf inequality when $\{\delta_k\}_{k \in \mathbb{N}}$ is a sequence such that $\delta_k \searrow 0$.

The aim of this section is to show that in absence of the previous hypothesis the problem became trivial.

Let assume that A is a Young function, with no Δ_2 and no regular variation hypothesis. Given $u \in C_c^2(\mathbb{R}^n)$, the equation (3.4) is still valid. Therefore

$$\lim_{\delta \to 0+} \frac{1}{A(\delta^{1-s})} \int_{B(x,\delta)} A\left(\frac{|u(x) - u(y)|}{|x-y|^s}\right) \frac{dy}{|x-y|^n} =$$
$$\lim_{\delta \to 0+} \frac{1}{A(\delta^{1-s})} \int_{B(x,\delta)} A\left(\frac{\nabla u(x) \cdot (x-y)}{|x-y|^s}\right) \frac{dy}{|x-y|^n}.$$

As

$$\int_{B(x,\delta)} A\left(\left|\nabla u(x) \cdot \frac{x-y}{|x-y|^s}\right|\right) \frac{dy}{|x-y|^n} = \int_{B(x,\delta)} A\left(|\nabla u(x)| \left|e_x \cdot \frac{x-y}{|x-y|^s}\right|\right) \frac{dy}{|x-y|^n}$$
$$= \int_{B(0,\delta)} A\left(|\nabla u(x)| \left|e \cdot \frac{h}{|h|^s}\right|\right) \frac{dh}{|h|^n}$$
$$= \int_{B(0,1)} A\left(|\nabla u(x)|\delta^{1-s} \left|e \cdot \frac{w}{|w|^s}\right|\right) \frac{dw}{|w|^n},$$

where e_x and e are unitary vectors. Our goal is to study the following limit

(4.3) $$\lim_{\delta \to 0^+} \frac{1}{A(\delta^{1-s})} \int_{B(0,1)} A\left(|\nabla u(x)|\delta^{1-s} \left|e \cdot \frac{w}{|w|^s}\right|\right) \frac{dw}{|w|^n}.$$

In [22, 21], the authors define the function Matuszewska of a function A at 0 as

$$\limsup_{\delta \to 0} \frac{A(t\delta)}{A(\delta)} = M_0(t).$$

When $A \in \Delta_2^0$ it follows that $M_0(t) < \infty$ for all $t > 0$, however without Δ_2^0 condition the Matuszewska function is defined as

$$\limsup_{\delta \to 0} \frac{A(t\delta)}{A(\delta)} = M_0(t) = \begin{cases} 1 \text{ if } t = 1, \\ \infty \text{ if } t > 1. \end{cases}$$

Moreover, under the hypothesis

(4.4) $$\liminf_{t \to 0} \frac{A(2t)}{A(t)} = \infty,$$

is proved in [16] that $M_0(t) = 0$ for all $t \in [0,1)$.

In order to analyze the limit (4.3), observe that if $0 \le |\nabla u(x)| < 1$

$$\lim_{\delta \to 0^+} \int_{B(0,1)} \frac{A\left(|\nabla u(x)|\delta^{1-s} \left|e \cdot \frac{w}{|w|^s}\right|\right)}{A(\delta^{1-s})} \frac{dw}{|w|^n} \le \limsup_{\delta \to 0^+} \int_{B(0,1)} \frac{A\left(|\nabla u(x)|\delta^{1-s} \left|e \cdot \frac{w}{|w|^s}\right|\right)}{A(\delta^{1-s})} \frac{dw}{|w|^n}$$
$$\le \int_{B(0,1)} \limsup_{\delta \to 0^+} \frac{A\left(|\nabla u(x)|\delta^{1-s} \left|e \cdot \frac{w}{|w|^s}\right|\right)}{A(\delta^{1-s})} \frac{dw}{|w|^n} = 0.$$

If $|\nabla u(x)| = 1$ the limit is trivially related to the measure of the unitary ball. Finally if $|\nabla u(x)| > 1$, by (4.4) and Fatou's lemma we have that

$$\infty = \int_{B(0,1)} \liminf_{\delta \to 0^+} \frac{A\left(|\nabla u(x)|\delta^{1-s} \left|e \cdot \frac{w}{|w|^s}\right|\right)}{A(\delta^{1-s})} \frac{dw}{|w|^n} \le \liminf_{\delta \to 0^+} \int_{B(0,1)} \frac{A\left(|\nabla u(x)|\delta^{1-s} \left|e \cdot \frac{w}{|w|^s}\right|\right)}{A(\delta^{1-s})} \frac{dw}{|w|^n}.$$

Therefore the limit (4.3) becomes trivial with the hypothesis (4.4) and without the hypotheses Δ_2 and (4.2). We can summarize this results in the following theorem.

Theorem 4.1. *Let A be an Orlicz function satisfying* (4.4) *and $u \in C_c^2(\mathbb{R}^n)$. Then, for every $x \in \mathbb{R}^n$, we have*

$$\lim_{\delta \to 0^+} \frac{1}{A(\delta^{1-s})} \int_{B(x,\delta)} A(|D^s u|) \frac{dy}{|x-y|^n} = K_n M_0(|\nabla u(x)|),$$

where K_n is a constant related to the measure of the unitary ball of $\mathbb{R}^n$.

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