THE CONCENTRATION-COMPACTNESS PRINCIPLE FOR ORLICZ SPACES AND APPLICATIONS

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Abstract. In this paper we extend the well-known concentration – compactness principle of P.L. Lions to Orlicz spaces. As an application we show an existence result to some critical elliptic problem with nonstandard growth.

1. Introduction

The study of Orlicz and Orlicz-Sobolev spaces is a subject that has a long history in analysis since the begginning of the 1930s starting with the work of Orlicz himself and the famous Polish school.

These spaces apear naturally in several applications in physics and engineering when one need to deal with the so-called *nonstandard growth* differential equations. Some prototypical examples of such problems are equations of the form

$$-\Delta_a u := -\operatorname{div}\left(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u\right) = f, \tag{1.1}$$

posed in a domain $\Omega \subset \mathbb{R}^n$ and complemented with some boundary conditions.

The study of these problems is connected to Orlicz and Orlicz-Sobolev spaces since the natural space for solutions is $W^{1,A}(\Omega)$ where $A'(t) = a(t)$.

The regularity problem for (1.1) was analyzed in the classical work of Lieberman [12] where it is shown, under adequate assumptions on A and f, that bounded solutions are Hölder continuous.

The existence problem of (1.1) is related to the growth of the source term f and therefore related to the integrability properties of functions in $W^{1,A}(\Omega)$ and for that purpose it is extremely relevant the study of the Sobolev immersions for these spaces. Namely, what is needed is to find all Young functions B (see next section for precise definitions) such that

$$W^{1,A}(\Omega) \subset L^B(\Omega). \tag{1.2}$$

As far as we know, the first article that treated this problem was [5] and then it was refined in [4] where the author finds the optimal Young function such that (1.2) holds. In [4] this optimal Young function is denoted by A_n (it depends only on A and n) and it is shown that (1.2) holds if and only if $B \leq A_n$ (in the sense of Young funcions). Moreover, if $B \ll A_n$ then the immersion (1.2) is compact. This *critical* Young function A_n has a precise formula given in (2.6).

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This compactness property of (1.2) for $B \ll A_n$ is crucial for the existence problem of (1.1). Namely, if $F'(t) = f(t)$ and $F \ll A_n$ then the standard variational methods (under adequate assumptions both on A and F) can yield existence results for

$$-\Delta_a u = f(u).$$

However, for *critical-type* problems, where $F \sim A_n$ the existence problem becomes much more delicate.

In the classical setting (when the differential operator is the Laplace operator), this problem is very much related with the so-called Yamabe problem in differential geometry and the literature is so vast that is impossible to give an extensive review in this short introduction.

One extremely important tool to deal with such critical problems was developed by P.L. Lions in his famous article [13]. P.L. Lions developed what is called as the *concentration-compactness principle* that consists in analyze the lack of compactness for bounded sequences in $W^{1,p}(\Omega)$. What Lions proved is that for bounded domains Ω the only possibility is the appearance of *concentration points*.

The concentration-compactness principle has been proved to be an extremely powerful tool and has been used by several authors in too many different problems and also it has been generalized to different settings. See [2, 7, 9, 10, 14] and references therein.

The main point of this article is therefore to generalized Lions' concentration-compactness principle to the context of Orlicz spaces. Then, as an application of the method, we give a proof of existence of solutions to

$$\begin{cases} -\Delta_a u = a_n(u) + \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where f is a subcritical nonlinearity in the sense that $F \ll A_n$ (here $A_n' = a_n$).

Organization of the paper. In section 2, we give a review of Young functions and Orlicz and Orlicz-Sobolev spaces that are needed in the course of the arguments. There are no new results there so any expert in the subject can safely skip this section.

In section 3, we prove some preliminary technical lemmas and in section 4 we prove our main result (Theorem 4.3) where we extend Lions' concentration-compactness principle to the Orlicz setting.

Finally, in section 5 we apply Theorem 4.3 to obtain some existence result for (1.3).

2. Young functions and Orlicz and Orlicz-Sobolev spaces: a review

This section is devoted to give a very short overview of some known results on Young functions and Orlicz and Orlicz-Sobolev spaces that will be needed in the rest of the paper. There are no new results in this section so if the reader is familiar with the topic, he or she can skip the section and go directly to section 3. An excellent source for these topics is the classical book [11].

2.1. **Young functions.** Let us begin with the definition of a Young function.

Definition 2.1. A function $A\colon \mathbb{R}^+ \to \mathbb{R}$ is said to be a Young function if it has the form

$$A(t) = \int_0^t a(\tau)\, d\tau, \qquad t \geq 0,$$

where $a : [0,\infty) \to [0,\infty)$ has the following properties:

(i) $a(0) = 0$,
(ii) $a(s) > 0$ for $s > 0$,
(iii) a is right continuous at any point $s \geq 0$,
(iv) a is nondecreasing on $(0,\infty)$.

Associated to any Young function A one define its *complementary function* (or Legendre conjugate) of A.

Definition 2.2. Let A be a Young function, we define its complementary function $\tilde{A}$ as

$$\tilde{A}(s) := \sup_{t\geq 0}\{st - A(t)\}$$

Observe that, by definition, $\tilde{A}$ is the optimal function in Young's inequality

$$st \leq A(t) + \tilde{A}(s).$$

It is a known fact that $\tilde{A}$ is also a Young function. Moreover, $\tilde{A}$ is given by

$$\tilde{A}(s) = \int_0^s \tilde{a}(\tau)\, d\tau,$$

where $\tilde{a}$ is the generalized inverse of a.

We need the notion of comparison between Young functions.

Definition 2.3. Given two Young functions A and B, we say that $A \leq B$ if there exists a constant $c > 0$ and $t_0 > 0$ such that $A(t) \leq B(ct)$, for every $t \geq t_0$.

Whenever $A \leq B$ and $B \leq A$ we say that A and B are equivalent Young functions and this fact will be denoted by $A \sim B$.

Finally, we say that B is essentially larger than A, denoted by $A \ll B$, if for any $c > 0$,

$$\lim_{t\to\infty} \frac{A(ct)}{B(t)} = 0.$$

A very important and useful property is the so-called Δ_2-condition. We recall this concept in the next definition.

Definition 2.4. We say that a Young function A satisfies the Δ_2-condition if

$$A(2t) \leq CA(t)$$

for all $t \geq 0$ for a fixed positive constant $C > 2$.

In [11, Theorem 4.1] it is shown that the Δ_2-condition is equivalent to

$$\frac{ta(t)}{A(t)} \leq p^+ \quad \text{for } t > 0, \tag{2.1}$$

for some $p^+ > 1$.

Moreover, from this inequality it is easy to deduce that both A and $\tilde{A}$ verify the Δ_2-condition if and only if

$$p^- \leq \frac{ta(t)}{A(t)} \leq p^+ \quad \text{for } t > 0, \tag{2.2}$$

where $1 < p^- \le p^+ < \infty$.

Let us now recall some simple inequalities for Young functions that will be helpful later on.

Lemma 2.5 ([8, Lemma 2.6]). *Let A be a Young function satisfying* (2.1)*. Then for every $\eta > 0$ there exists $C_\eta > 0$ such that*

$$A(s+t) \le C_\eta A(s) + (1+\eta)^{p_+} A(t) \quad s, t > 0.$$

Lemma 2.6 ([6, Lemma 2.1]). *Let A be a Young function satisfying* (2.2)*, $s, t > 0$. Then*

$$\min\{s^{p_-}, t^{p_+}\} A(t) \le A(st) \le \max\{s^{p_-}, s^{p_+}\} A(t).$$

In order to understand the behavior of a Young function A at infinity, it is very helpful to introduce the notion of the Matuszewska-Orlicz function and the Matuszewska-Orlicz index.

Definition 2.7. Given a Young function A, we define the associated Matuszewska-Orlicz function as

$$M(t, A) := \limsup_{s\to\infty} \frac{A(st)}{A(s)}.$$

When no confusion arises, we will simply denote $M(t) = M(t, A)$.

The Matuszewska-Orlicz index is then defined as

$$p_\infty(A) := \lim_{t\to\infty} \frac{\ln M(t, A)}{\ln t} = \inf_{t>0} \frac{\ln M(t, A)}{\ln t}.$$

Again, when no confusion arises, we will simply denote $p_\infty = p_\infty(A)$.

The main feature that we use in this article is the fact that, if A verifies the Δ_2-condition, then the index p_∞ is finite and for any $\varepsilon > 0$, there exists $t_0 > 0$ such that

$$t^{p_\infty} \le M(t, A) \le t^{p_\infty+\varepsilon}, \quad \text{for } t \ge t_0. \tag{2.3}$$

See [1] for this fact and more properties of this index.

Remark 2.8. It is also easy to check that if A satisfy (2.2), then $p^- \le p_\infty \le p^+$. and that

$$\min\{t^{p^+}, t^{p^-}\} M(s) \le M(st) \le \max\{t^{p^+}, t^{p^-}\} M(s).$$

We will need the following result regarding the function $M(\cdot, A)$.

Lemma 2.9. *If A verifies* (2.2)*, then M is a Young function.*

Proof. According to [11], we need to check that $M(\cdot, A)$ is convex, even and verifies

$$\lim_{t\to 0+} \frac{M(t, A)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{M(t, A)}{t} = \infty.$$

Note that if we call $A_s(t) := \frac{A(st)}{A(s)}$, then A_s is convex and even. So $M(t, A) = \sup_{s>0} A_s(t)$ is also convex and even.

Also, $M(t, A) \ge A_1(t) = A(t)$ and so $\lim_{t\to\infty} \frac{M(t,A)}{t} \ge \lim_{t\to\infty} \frac{A(t)}{t} = \infty$.

Finally, from Lemma 2.6, $A(st) \le t^{p^-} A(s)$ for $t \in (0, 1)$. Then it follows that $A_s(t) \le t^{p^-}$ for every $s > 0$. Hence $M(t, A) \le t^{p^-}$ and so $\frac{M(t,A)}{t} \le t^{p^- -1} \to 0$ as $t \to 0+$. □

2.2. **Orlicz and Orlicz-Sobolev spaces.** Given a Young function A and an open set $\Omega \subset \mathbb{R}^n$ we consider the spaces $L^A(\Omega)$ and $W^{1,A}(\Omega)$ defined as follows:

$$L^A(\Omega) := \{u \in L^1_{loc}(\Omega) \colon \Phi_{A,\Omega}(u) < \infty\},\ W^{1,A}(\Omega) := \{u \in W^{1,1}_{loc}(\Omega) \colon u, |\nabla u| \in L^A(\Omega)\},$$

where

$$\Phi_{A,\Omega}(u) = \int_\Omega A(|u|)\, dx.$$

These spaces are endowed with the so-called *Luxemburg norm* defined as

$$\|u\|_{L^A(\Omega)} = \|u\|_{A;\Omega} = \|u\|_A := \inf\left\{\lambda > 0 : \Phi_{A,\Omega}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

and

$$\|u\|_{W^{1,A}(\Omega)} = \|u\|_{1,A:\Omega} = \|u\|_{1,A} := \|u\|_{L^A(\Omega)} + \|\nabla u\|_{L^A(\Omega)}.$$

The spaces $L^A(\Omega)$ and $W^{1,A}(\Omega)$ are separable Banach spaces and both are reflexive if and only if A verifies (2.2). See [11].

From Lemma 2.6 we immediately get

Lemma 2.10. *Let A be a Young function satisfying* (2.2)*. Then the following inequlaity holds true for $u \in L^A(\Omega)$*

$$\min\{\|u\|_A^{p^-}, \|u\|_A^{p^+}\} \leq \int_\Omega A(|u|)\, dx \leq \max\{\|u\|_A^{p^-}, \|u\|_A^{p^+}\}.$$

In order for the Sobolev immersion theorem to hold, one need to impose some growth conditions on A. Following [4], we require A to verify

$$\int_K^\infty \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt = \infty. \tag{2.4}$$

$$\int_0^\delta \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt < \infty, \tag{2.5}$$

for some constants $K, \delta > 0$.

Given a Young function A that satisfies (2.4) and (2.5) its Orlicz-Sobolev conjugate is defined as

$$A_n(t) = A \circ H^{-1}(t), \tag{2.6}$$

where

$$H(t) = \left(\int_0^t \left(\frac{\tau}{A(\tau)}\right)^{\frac{1}{n-1}} d\tau\right)^{\frac{n-1}{n}}. \tag{2.7}$$

The following fundamental Orlicz-Sobolev embedding Theorem can be found in [4]

Theorem 2.11. *Let A be a Young function satisfying* (2.4) *and* (2.5) *and let A_n be defined in* (2.6)*. Then the embedding $W_0^{1,A}(\Omega) \hookrightarrow L^{A_n}(\Omega)$ is continuous. Moreover, the Young function A_n is optimal in the class of Orlicz spaces.*

Finally, given B any Young funcion, the embedding $W_0^{1,A}(\Omega) \hookrightarrow L^B(\Omega)$ is compact if and only if $B \ll A_n$.

From now on, we will denote by S_A the optimal constant in the embedding $W_0^{1,A}(\Omega) \subset L^{A_n}(\Omega)$. That is

$$S_A := \inf_{\phi \in C_c^\infty(\Omega)} \frac{\|\nabla \phi\|_A}{\|\phi\|_{A_n}}. \tag{2.8}$$

Remark 2.12. It is easy to see that $A \ll A_n$ and hence $W_0^{1,A}(\Omega) \subset L^A(\Omega)$ is compact.

3. Preliminary Lemmas

In this section we prove some technical lemmas that will be helpful in the sequel. Namely, we need to show that A_n defined in (2.6) verifies the Δ_2-condition whenever A does. Finally we need a version of the celebrated Brezis-Lieb Lemma to the Orlicz setting.

We begin with some preliminary estimates.

Lemma 3.1. *Let $H(t)$ be as in definition* (2.7)*, then the following inequality holds.*

$$C_1 + C_2 t^{\frac{n}{n-p^-}} \le H^{-1}(t) \le C_1 + C_2 t^{\frac{n}{n-p^+}}.$$

Proof. From the definition of H, (2.7), we get, for $t > 1$,

$$\begin{aligned} H(t)^{\frac{n}{n-1}} &= \int_0^1 \left(\frac{\tau}{A(\tau)}\right)^{\frac{1}{n-1}} d\tau + \int_1^t \left(\frac{\tau}{A(\tau)}\right)^{\frac{1}{n-1}} d\tau \\ &= C_0 + \int_1^t \left(\frac{\tau}{A(\tau)}\right)^{\frac{1}{n-1}} d\tau, \end{aligned}$$

Observe that C_0 depends on A and n.

Now, using Lemma 2.6, for $t > 1$ we obtain

$$A(1)^{\frac{-1}{n-1}} \int_1^t \tau^{\frac{1-p_+}{n-1}}\, d\tau \le \int_1^t \left(\frac{\tau}{A(\tau)}\right)^{\frac{1}{n-1}} d\tau \le A(1)^{\frac{-1}{n-1}} \int_1^t \tau^{\frac{1-p_-}{n-1}}\, d\tau.$$

Observe that, for $1 < p < n$,

$$\int_1^t \tau^{\frac{1-p}{n-1}}\, d\tau = \frac{n-1}{n-p}(t^{\frac{n-p}{n-1}} - 1).$$

Then

$$C_1 + C_2 t^{\frac{n-p^+}{n}} \le H(t) \le C_1 + C_2 t^{\frac{n-p^-}{n}}.$$

From this last estimate we obtain the desired result. □

Remark 3.2. Combining Lemma 2.6 with Lemma 3.1 is easy to conclude that $A_n(t)$ verifies

$$C_1 + C_2 t^{p_*^-} \le A_n(t) \le C_1 + C_2 t^{p_*^+},$$

for some constants $C_1, C_2 > 0$ depending only on A and n.

Recall that given an exponent $p \in (1, n)$ we denote by p^* the Sobolev conjugate, $p^* = \frac{np}{n-p}$.

Let us now check that the critical function A_n inherits the Δ_2-condition from A.

Lemma 3.3. *Let A be a Young function satisfying* (2.2), (2.4) *and* (2.5) *and let A_n be the Young function defined in* (2.6)*. Then A_n verifies Δ_2-condition.*

Proof. By definition of H and using that $A(t)$ verifies the Δ_2-condition , we obtain

$$\begin{aligned} H(2t)^{\frac{n}{n-1}} &= \int_0^{2t} \left(\frac{\tau}{A(\tau)}\right)^{\frac{1}{n-1}} d\tau = 2^{\frac{n}{n-1}} \int_0^{t} \left(\frac{\tau}{A(2\tau)}\right)^{\frac{1}{n-1}} d\tau \\ &\geq 2^{\frac{n}{n-1}} \int_0^{t} \left(\frac{\tau}{CA(\tau)}\right)^{\frac{1}{n-1}} d\tau = \frac{2^{\frac{n}{n-1}}}{C^{\frac{1}{n-1}}} H(t)^{\frac{n}{n-1}}. \end{aligned}$$

Then

$$H(2t) \geq \frac{2}{C^{\frac{1}{n}}} H(t). \tag{3.1}$$

From (3.1) we easily get that

$$2H^{-1}(s) \geq H^{-1}\left(\frac{2}{C^{\frac{1}{n}}} s\right).$$

If we denote $2t = \frac{2}{C^{\frac{1}{n}}} s = \kappa s$, so

$$H^{-1}(2t) \leq 2H^{-1}\left(\frac{2}{\kappa} t\right).$$

Now, we are in position to prove that A_n verifies Δ_2-condition. In fact,

$$A_n(2t) = A(H^{-1}(2t)) \leq A\left(2H^{-1}\left(\frac{2}{\kappa}t\right)\right) \leq CA\left(H^{-1}\left(\frac{2}{\kappa}t\right)\right) = CA_n\left(\frac{2}{\kappa}t\right).$$

As we want to prove. □

To finish this section, we prove the Brezis-Lieb lemma in the Orlicz setting.

Lemma 3.4 (Brezis-Lieb Lemma)**.** *Let B be a Young function, $f_n \to f$ a.e and $f_n \rightharpoonup f$ in $L^B(\Omega)$ then, for every $\phi \in L^\infty(\Omega)$ it follows that*

$$\lim_{n\to\infty} \left(\int_\Omega B(|f_n|)\phi\, dx - \int_\Omega B(|f - f_n|)\phi\, dx \right) = \int_\Omega B(|f|)\phi\, dx.$$

Proof. First, by Lemma 2.5 we know that given $\varepsilon > 0$, there exists C_ε such that for every $a, b \in \mathbb{R}$, the following inequality holds

$$|B(|a+b|) - B(|a|)| \leq \varepsilon B(|a|) + C_\varepsilon B(|b|).$$

We define

$$W_{\varepsilon,n}(x) = (|B(|f_n(x)|) - B(|f(x) - f_n(x)|) - B(|f(x)|)| - \varepsilon B(|f_n(x)|))_+,$$

and note that $W_{\varepsilon,n}(x) \to 0$ as $n \to \infty$ a.e. On the other hand,

$$\begin{aligned} |B(|f_n(x)|) - B(|f(x) - f_n(x)|) - B(|f(x)|)| &\leq |B(|f_n(x)|) - B(|f(x) - f_n(x)|)| + B(|f(x)|) \\ &\leq \varepsilon B(|f_n(x)|) + C_\varepsilon B(|f(x)|) + B(|f(x)|), \end{aligned}$$

i.e.

$$|B(|f_n(x)|) - B(|f(x) - f_n(x)|) - B(|f(x)|)| - \varepsilon B(|f_n(x)|) \leq (C_\varepsilon + 1) B(|f(x)|),$$

therefore

$$0 \leq W_{\varepsilon,n}(x) \leq (C_\varepsilon + 1) B(|f(x)|).$$

By the dominated convergence Theorem, we conclude that

$$\lim_{n\to\infty} \int_\Omega W_{\varepsilon,n}(x)\phi(x)\, dx = 0.$$

On the other hand,

$$|B(|f_n(x)|) - B(|f(x) - f_n(x)|) - B(|f(x)|)| \leq W_{\varepsilon,n}(x) + \varepsilon B(|f_n(x)|).$$

Then, if we denote $I_n = \int_\Omega (B(|f_n(x)|) - B(|f(x) - f_n(x)|) - B(|f(x)|))\, \phi(x)\, dx$, we get

$$\begin{aligned} |I_n| &\leq \int_\Omega W_{\varepsilon,n}(x)|\phi(x)|\, dx + \varepsilon \int_\Omega B(|f_n|)|\phi(x)|\, dx \\ &\leq \int_\Omega W_{\varepsilon,n}(x)|\phi(x)|\, dx + \varepsilon \sup_{n\in\mathbb{N}} \int_\Omega B(|f_n|)|\phi|\, dx \\ &\leq \int_\Omega W_{\varepsilon,n}(x)|\phi(x)|\, dx + \varepsilon C\|\phi\|_\infty, \end{aligned}$$

for some constant $C > 0$. Hence, we can conclude that $\limsup I_n \leq \varepsilon C\|\phi\|_\infty$, for every $\varepsilon > 0$. □

4. Proof of the Concentration Compactness Principle

This is the principal section of the paper where we prove the concentration compactness principle in the context of Orlicz spaces.

In this section we assume that the Young function A satisfies condition (2.2).

Given A a young function satisfying (2.2), we denote by A_∞ the following Young function associated with A:

$$A_\infty(t) = \max\{t^{p^+}, t^{p^-}\}. \tag{4.1}$$

Observe that $A \leq A_\infty$ both in the sense of Young functions and also in the pointwise sense.

Remark 4.1. A_∞ verifies the Δ_2-condition.

In the sequel it will be helpful a comparison between the Orlicz functions A_∞ and M_n. This is the content of the next lemma.

Lemma 4.2. *With the same assumptions and notations of the section, assume that $p^+ < p_*^-$. Then*

$$A_\infty \ll M_n$$

Proof. This result is an immediate consequence of Remarks 2.8 and 3.2. □

This next theorem is our main result.

Theorem 4.3. *Let $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,A}(\Omega)$ be a sequence such that $u_k \rightharpoonup u$ weakly in $W^{1,A}(\Omega)$. Then there exists a countable set I, positive numbers $\{\mu_i\}_{i\in I}$ and $\{\nu_i\}_{i\in I}$ such that*

$$A_n(|u_k|)dx \rightharpoonup \nu = A_n(|u|)\, dx + \sum_{i\in I} \nu_i \delta_{x_i} \quad \textit{weakly-* in the sense of measures,} \tag{4.2}$$

$$A(|\nabla u_k|)\, dx \rightharpoonup \mu \geq A(|\nabla u|)\, dx + \sum_{i\in I} \mu_i \delta_{x_i} \quad \textit{weakly-* in the sense of measures,} \tag{4.3}$$

$$C\frac{1}{M_n^{-1}(\frac{1}{\nu_i})} \leq \frac{1}{A_\infty^{-1}(\frac{1}{\mu_i})}, \qquad \textit{for every } i \in I, \tag{4.4}$$

where C is a constant depending only on A and n and $M_n(t) = M(t, A_n)$ is the Matuszewska-Orlicz function associated to A_n.

The strategy of the proof is the same as in the original work of P.L. Lions [13].

Assume first that $u = 0$ and so, passing if necessary to a subsequence, $u_k \to 0$ a.e. in Ω. We will prove Theorem 4.3 in this case and then with the help of Lemma 3.4 we can easily extend it to the general case.

We divide the proof of Theorem 4.3 into a series of lemmas.The first one is a reverse-Hölder type inequality between the measures ν and μ.

Lemma 4.4. *Let A be a Young function satisfying* (2.2)*,* (2.4)*,* (2.5) *and let A_n be given by* (2.6)*. Then, for every $\phi \in C_c^\infty(\Omega)$ the following reverse Hölder inequality holds:*

$$\|\phi\|_{M_n,\nu} \leq C\|\phi\|_{A_\infty,\mu}, \tag{4.5}$$

where A_∞ is given by (4.1) *and $M_n(t) = M(t, A_n)$ is the Matuszewska-Orlicz function associated to A_n given in Definition 2.7.*

Proof. Let $\phi \in C_c^\infty(\Omega)$ and we apply Sobolev inequality to ϕu_k, to obtain

$$S_A\|\phi u_k\|_{A_n} \leq \|\nabla(\phi u_k)\|_A. \tag{4.6}$$

First, we can estimate the left hand side in the following way: given $\delta > 0$ let $K > 0$ be such that

$$A_n(st) \geq A_n(t)(M_n(s) - \delta), \qquad \text{for } t \geq K.$$

Then

$$\begin{aligned}
\liminf_{k\to\infty} \int_\Omega A_n\left(\frac{|\phi u_k|}{\lambda}\right) dx &\geq \liminf_{k\to\infty} \int_{\{u_k \geq K\}} A_n\left(\frac{|\phi u_k|}{\lambda}\right) \frac{A_n(|u_k|)}{A_n(|u_k|)}\, dx \\
&\geq \liminf_{k\to\infty} \int_{\{u_k\geq K\}} \left(M_n\left(\frac{|\phi|}{\lambda}\right) - \delta\right) A_n(u_k)\, dx \\
&= \liminf_{k\to\infty} \left(\int_\Omega - \int_{\{u_k<K\}}\right) \left(M_n\left(\frac{|\phi|}{\lambda}\right) - \delta\right) A_n(u_k)\, dx \\
&= \liminf_{k\to\infty} I - II.
\end{aligned}$$

From (4.2) it follows that

$$\lim_{k\to\infty} I = \int_\Omega \left(M_n\left(\frac{|\phi|}{\lambda}\right) - \delta\right) d\nu$$

and from the dominated convergence Theorem, since $u_k \to 0$ a.e., it follows that $\lim_{k\to\infty} II = 0$.

From these computations, since $\delta > 0$ is arbitrary, one immediately obtain that

$$\liminf_{k\to\infty} \int_\Omega A_n\left(\frac{|\phi u_k|}{\lambda}\right) dx \geq \int_\Omega M_n\left(\frac{|\phi|}{\lambda}\right) d\nu.$$

From this inequality it follows that

$$\liminf_{k\to\infty} \|\phi u_k\|_{A_n} \geq \|\phi\|_{M_n,d\nu}.$$

Now we deal with the right hand side of (4.6). First, we observe that

$$|\,\|\nabla(\phi u_k)\|_A - \|\phi\nabla u_k\|_A| \leq \|u_k\nabla\phi\|_A.$$

Then, we observe that the right side of the inequality converges to 0 since $u_k \to 0$ in $L^A(\Omega)$. Hence we can replace the right hand side of (4.6) by $\|\phi\nabla u_k\|_A$.

Now, using Lemma 2.6,

$$\begin{aligned}\limsup_{k\to\infty}\int_\Omega A\left(|\nabla u_k|\frac{\phi}{\lambda}\right)\,dx &\le \limsup_{k\to\infty}\int_\Omega \max\left\{\left(\frac{\phi}{\lambda}\right)^{p^+},\left(\frac{\phi}{\lambda}\right)^{p^-}\right\}A(|\nabla u_k|)\,dx\\ &= \limsup_{k\to\infty}\int_\Omega A_\infty\left(\frac{\phi}{\lambda}\right)A(|\nabla u_k|)\,dx\\ &= \int_\Omega A_\infty\left(\frac{\phi}{\lambda}\right)\,d\mu\end{aligned}$$

so

$$\limsup_{k\to\infty}\|\phi\nabla u_k\|_A \le \|\phi\|_{A_\infty,\mu}$$

This inequality completes the proof. □

The next lemma is an easy adaptation of the first part of [13, Lemma I.2]. We include the details for completeness.

Lemma 4.5. *Let ν be a non-negative, bounded Borel measure and let A, B be two Young functions such that $A \ll B$. If*

$$\|\phi\|_{B,\nu} \le C\|\phi\|_{A,\nu}, \tag{4.7}$$

for some constant $C > 0$ and for every $\phi \in C_c^\infty(\Omega)$. Then there exists $\delta > 0$ such that for all Borel sets $U \subset \overline{\Omega}$, either $\nu(U) = 0$ or $\nu(U) \ge \delta$.

Proof. It is easy to see that inequality (4.7) still holds for characteristic functions of Borel sets. Then we may take $\phi = \chi_U$ with $\nu(U) \neq 0$, so

$$\int_\Omega B\left(\frac{\chi_U}{\lambda}\right)\,d\nu = \int_U B\left(\frac{1}{\lambda}\right)\,d\nu = B\left(\frac{1}{\lambda}\right)\nu(U).$$

Then

$$\|\chi_U\|_{B,\nu} = \frac{1}{B^{-1}\left(\frac{1}{\nu(U)}\right)}. \tag{4.8}$$

Analogously,

$$\|\chi_U\|_{A,\nu} = \frac{1}{A^{-1}\left(\frac{1}{\nu(U)}\right)}.$$

Therefore we obtain the following inequality $A^{-1}\left(\frac{1}{\nu(U)}\right) \le CB^{-1}\left(\frac{1}{\nu(U)}\right)$.

Assume by contradiction that there exist U_k such that $\nu(U_k) = \varepsilon_k \to 0$, so

$$A^{-1}\left(\frac{1}{\varepsilon_k}\right) \le CB^{-1}\left(\frac{1}{\varepsilon_k}\right).$$

We choose t_k such that $B(t_k) = \frac{1}{\varepsilon_k}$. Then $t_k \to \infty$ and $A^{-1}(B(t_k)) \le Ct_k$. By composition with A we obtain that $B(t_k) \le A(Ct_k)$, which contradicts the fact that $A \ll B$. □

The next lemma is exactly as the end of [13, Lemma I.2].

Lemma 4.6. *Let ν be a non-negative bounded Borel measure on $\overline{\Omega}$. Assume that there exists $\delta > 0$ such that for every Borel set U we have that, $\nu(U) = 0$ or $\nu(U) \geq \delta$. Then, there exist a countable index set I, points $\{x_i\}_{i\in I} \subset \bar{\Omega}$ and scalars $\{\nu_i\}_{i\in I} \in (0,\infty)$ such that*

$$\nu = \sum_{i\in I} \nu_i \delta_{x_i}.$$

Now we need a lemma that plays a key role in the proof of Theorem 4.3.

Lemma 4.7. *Under the same assumptions of Lemma 4.4, there exist a countable index set I, points $\{x_i\}_{i\in I} \subset \bar{\Omega}$ and scalars $\{\nu_i\}_{i\in I} \subset (0,\infty)$, such that*

$$\nu = \sum_{i\in I} \nu_i \delta_{x_i}.$$

Proof. By the reverse Hölder inequality (4.5), the measure ν is absolutely continuous with respect to μ. In fact, if we choose $\phi = \chi_U$, by (4.8),

$$\|\chi_U\|_{A_\infty,\mu} = \begin{cases} 0 & \text{if } \mu(U) = 0 \\ \frac{1}{A_\infty^{-1}\left(\frac{1}{\mu(U)}\right)} & \text{if } \mu(U) > 0. \end{cases}$$

Also,

$$\|\chi_U\|_{M_n,\nu} = \begin{cases} 0 & \text{if } \nu(U) = 0 \\ \frac{1}{M_n^{-1}\left(\frac{1}{\nu(U)}\right)} & \text{if } \nu(U) > 0. \end{cases}$$

This facts together with (4.5) clearly imply that $\nu \ll \mu$.

As a consequence there exists $f \in L^1_\mu(\Omega)$, $f \geq 0$, such that $\nu = \mu\lfloor f$. Also by (4.5) we can conclude that $f \in L^\infty_\mu(\Omega)$. In fact,

$$\fint_U f\, d\mu = \frac{\nu(U)}{\mu(U)} \leq \frac{1}{\mu(U) M_n\left(CA_\infty^{-1}\left(\frac{1}{\mu(U)}\right)\right)}.$$

Observe that if we denote $t = A_\infty^{-1}\left(\frac{1}{\mu(U)}\right)$. We have the following equality

$$\mu(U) M_n\left(CA_\infty^{-1}\left(\frac{1}{\mu(U)}\right)\right) = \frac{M_n(Ct)}{A_\infty(t)},$$

and the last term goes to ∞ as $t \to \infty$ by Lemma 4.2.

In other words, the function $r \mapsto \frac{1}{rM_n(CA_\infty^{-1}(\frac{1}{r}))}$ is bounded in $[0, \mu(\Omega))$, then $f \in L^\infty_\mu(\Omega)$.

On the other hand the Lebesgue decomposition of μ with respect to ν gives us

$$\mu = \nu\lfloor g + \sigma,$$

where $g \in L^1_\nu(\Omega)$, $g \geq 0$ and σ is a bounded positive measure, singular with respect to ν.

Let $\psi \in C^\infty_c(\Omega)$ and consider (4.5) applied to the test functions of the form $\varphi(g)\psi\chi_{\{g\leq k\}}$ where $\varphi(t)$ is to be determined, $\varphi(0) = 0$.

We obtain

$$\begin{aligned} C\|\varphi(g)\psi\chi_{\{g\le k\}}\|_{M_n,\nu} &\le \|\varphi(g)\psi\chi_{\{g\le k\}}\|_{A_\infty,\mu} \\ &\le \|\varphi(g)\psi\chi_{\{g\le k\}}\|_{A_\infty,g d\nu+d\sigma}. \end{aligned}$$

Since $\sigma\perp\nu$, A_∞ is sub-multiplicative (i.e. $A_\infty(st)\le A_\infty(s)A_\infty(t)$) and $A_\infty(\chi_U)=\chi_U$, we have that

$$\begin{aligned} \int_\Omega A_\infty\left(\frac{\varphi(g)\psi\chi_{\{g\le k\}}}{\lambda}\right) d\mu &= \int_\Omega A_\infty\left(\frac{\varphi(g)\psi\chi_{\{g\le k\}}}{\lambda}\right) g\,d\nu + \int_\Omega A_\infty\left(\frac{\varphi(g)\psi\chi_{\{g\le k\}}}{\lambda}\right) d\sigma \\ &= \int_\Omega A_\infty\left(\frac{\varphi(g)\psi\chi_{\{g\le k\}}}{\lambda}\right) g\,d\nu \\ &\le \int_\Omega A_\infty\left(\frac{\psi}{\lambda}\right) A_\infty(\varphi(g)) g\chi_{\{g\le k\}}\,d\nu. \end{aligned}$$

On the other hand, combining remarks 3.2 and 2.8, we have

$$\int_\Omega M_n\left(\frac{\varphi(g)\psi\chi_{\{g\le k\}}}{\lambda}\right) d\nu \ge \int_\Omega M_n\left(\frac{\psi}{\lambda}\right) \min\{\varphi(g)^{p_*^+},\varphi(g)^{p_*^-}\}\chi_{\{g\le k\}}\,d\nu$$

Hence, if we define

$$\varphi(t)=\begin{cases} t^{\frac{1}{p_*^+-p^-}} & \text{if } t<1\\ t^{\frac{1}{p_*^--p^+}} & \text{if } t\ge 1\end{cases}$$

then

$$\max\{\varphi(t)^{p^+},\varphi(t)^{p^-}\}t=\min\{\varphi(t)^{p_*^+},\varphi(t)^{p_*^-}\}$$

and we get that

$$h(x):=A_\infty(\varphi(g(x)))g(x)=\min\{\varphi(g(x))^{p_*^+},\varphi(g(x))^{p_*^-}\}$$

Hence if we denote $d\nu_k:=h(x)\chi_{\{g\le k\}}d\nu$ the following reverse Hölder inequality holds

$$C\|\psi\|_{M_n,\nu_k}\le\|\psi\|_{A_\infty,\nu_k}.$$

Now, by Lemmas 4.6 and 4.5, there exists $\{x_i^k\}_{i\in I^k}$ and $\nu_i^k>0$ such that $\nu_k=\sum_{i\in I^k}\nu_i^k\delta_{x_i^k}$. On the other hand, $\nu_k\nearrow h(x)\nu$. Then, we have

$$\nu=\sum_{i\in I}\nu_i\delta_{x_i}.$$

This finishes the proof. □

Now we are in position to prove Theorem 4.3.

Proof of Theorem 4.3. First we write $v_k=u_k-u$. Then, we can apply Lemmas 4.4–4.7 to conclude that

$$A_n(|v_k|)\,dx\rightharpoonup d\bar\nu=\sum_{i\in I}\nu_i\delta_{x_i}, \tag{4.9}$$

weakly star in the sense of measures.

Now, we use Lemma 3.4 to obtain

$$\lim_{k\to\infty}\left(\int_\Omega \phi A_n(|u_k|) - \int_\Omega \phi A_n(|v_k|)dx\right) = \int_\Omega \phi A_n(|u|)dx,$$

for any $\phi \in C_c^\infty(\Omega)$, from where the representation

$$A_n(|u_k|)\,dx \rightharpoonup d\nu = A_n(|u|)\,dx + d\bar\nu$$

follows.

It remains to analyze the measure μ and to estimate the weights ν_i and μ_i.

To this end, we consider again $v_k = u_k - u$ and denote by $\bar\mu$ the weak* limit of $A(|\nabla v_k|)\,dx$ as $k \to \infty$.

Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \le \phi \le 1$, $\phi(0) = 1$ and $\operatorname{supp}(\phi) \subset B_1(0)$. Now, for each $i \in I$ and $\varepsilon > 0$, we denote $\phi_{\varepsilon,i}(x) := \phi((x - x_i)/\varepsilon)$.

Now we apply (4.5) to the measures $\bar\nu$ and $\bar\mu$ to obtain

$$C\frac{1}{M_n^{-1}(\frac{1}{\nu_i})} \le C\|\phi_{\varepsilon,i}\|_{M_n,\bar\nu} \le \|\phi_{\varepsilon,i}\|_{A_\infty,\bar\mu},$$

On the one hand,

$$1 = \int_{B_\varepsilon(x_i)} A_\infty\left(\frac{|\phi_{\varepsilon,i}|}{\|\phi_{\varepsilon,i}\|_{A_\infty,\bar\mu}}\right) d\bar\mu \le A_\infty\left(\frac{1}{\|\phi_{\varepsilon,i}\|_{A_\infty,\bar\mu}}\right)\bar\mu(B_\varepsilon(x_i)),$$

hence,

$$\|\phi_{\varepsilon,i}\|_{A_\infty,\bar\mu} \le \frac{1}{A_\infty^{-1}(\frac{1}{\bar\mu(B_\varepsilon(x_i))})} \to \frac{1}{A_\infty^{-1}(\frac{1}{\bar\mu_i})} \quad \text{as } \varepsilon \to 0,$$

where

$$\bar\mu_i := \mu(\{x_i\}) = \lim_{\varepsilon\to 0}\bar\mu(B_\varepsilon(x_i)).$$

Therefore,

$$\bar\mu \ge \sum_{i\in I}\bar\mu_i\delta_{x_i} \qquad \text{and} \qquad C\frac{1}{M_n^{-1}(\frac{1}{\nu_i})} \le \frac{1}{A_\infty^{-1}(\frac{1}{\bar\mu_i})}$$

On the other hand, using Lemma 2.5, we have that for any $\delta > 0$ there exists a constant C_δ such that

$$A(|\nabla v_k|) \le (1+\delta)A(|\nabla u_k|) + C_\delta A(|\nabla u|).$$

This inequality implies, passing to the limit $k \to \infty$, that

$$d\bar\mu \le (1+\delta)d\mu + C_\delta A(|\nabla u|)\,dx,$$

from where it follows that

$$\bar\mu_i \le (1+\delta)\mu_i, \quad \text{where } \mu_i := \mu(\{x_i\}).$$

This shows that $\mu \ge \sum_{i\in I}\mu_i\delta_{x_i} = \tilde\mu$ and, since $\delta > 0$ is arbitrary, we get

$$C\frac{1}{M_n^{-1}(\frac{1}{\nu_i})} \le \frac{1}{A_\infty^{-1}(\frac{1}{\mu_i})}.$$

To end the proof it remains to show that $d\mu \ge A(|\nabla u|)\,dx$ since $\tilde\mu$ is orthogonal to the Lebesgue measure.

Now, the fact that $u_k \rightharpoonup u$ weakly in $W_0^{1,A}(\Omega)$ implies that $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^A(U)$ for all $U \subset \Omega$. Hence, since the modular is a convex and strongly continuous functional, by [3, Corollary

3.9] it follows that it is weakly lower semicontinuous. Hence we obtain that $d\mu \geq A(|\nabla u|)\,dx$ as we wanted to show.

This finishes the proof. □

5. Application

In this section, we study the existence problem for the following elliptic equation

$$\begin{cases} -\Delta_a u = \frac{A_n'(|u|)}{|u|}u + \lambda\frac{F'(|u|)}{|u|}u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{5.1}$$

where $\Delta_a u = \operatorname{div}(\frac{a(|\nabla u|)\nabla u}{|\nabla u|})$ and $F \ll A_n$.

In this case, the associated functional reads

$$\mathcal{F}_\lambda(u) = \int_\Omega A(|\nabla u|) - A_n(|u|) - \lambda F(|u|)\,dx.$$

In this application, for technical reasons we need to impose a somewhat restrictive hypothesis on A, namely, we required that A verifies that there exists constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 \leq \frac{A(t)}{t^p} \leq c_2, \tag{5.2}$$

for $t \geq t_0$ and for some exponent $p_- \leq p \leq p_+$.

This hypothesis is only required at the end of the proof of the Palais-Smale condition (Lemma 5.4) and we believe that is only technical and could be removed.

Remark 5.1. Condition (5.2) implies that, since Ω is bounded, that

$$L^A(\Omega) = L^p(\Omega) \quad \text{and} \quad W^{1,A}(\Omega) = W^{1,p}(\Omega),$$

with equivalent norms

For this problem we can prove the following result

Theorem 5.2. *Assume that A and F are Young functions that satisfy the hypotheses of Theorem 4.3. Moreover, assume that*

$$p^- \leq \frac{tA'(t)}{A(t)} \leq p^+, \quad r^- \leq \frac{tF'(t)}{F(t)} \leq r^+, \quad p^+ < r^- < p_n^- \quad \text{and} \quad F \ll A_n.$$

Finally, assume that A satisfies (5.2)*. Then, there exists $\lambda_0 > 0$, such that if $\lambda > \lambda_0$ problem* (5.1) *has at least one nontrivial solution in $W_0^{1,A}(\Omega)$.*

We begin by proving that Palais-Smale sequences are bounded.

Lemma 5.3. *Let $\{u_j\}_{j\in\mathbb{N}} \subset W_0^{1,A}(\Omega)$ be a Palais-Smale sequence, then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W_0^{1,A}(\Omega)$.*

Proof. Let $\{u_j\}_{j\in\mathbb{N}} \subset W_0^{1,A}(\Omega)$ be a Palais-Smale sequence for $\mathcal{F}_\lambda$. Then, by definition

$$\mathcal{F}_\lambda(u_j) \to c \quad \text{and} \quad \mathcal{F}_\lambda'(u_j) \to 0.$$

Now, we have

$$c + 1 \geq \mathcal{F}_\lambda(u_j) = \mathcal{F}_\lambda(u_j) - \frac{1}{r-}\langle \mathcal{F}_\lambda'(u_j), u_j\rangle + \frac{1}{r-}\langle \mathcal{F}_\lambda'(u_j), u_j\rangle,$$

where

$$\langle \mathcal{F}_\lambda'(u_j), u_j\rangle = \int_\Omega \frac{A'(|\nabla u_j|)\nabla u_j \nabla u_j}{|\nabla u_j|} - \frac{A_n'(|u_j|)u_j u_j}{|u_j|} - \lambda \frac{F'(|u_j|)}{|u_j|} u_j u_j \, dx$$
$$\leq \int_\Omega p^+ A(|\nabla u_j|) - p_n^- A_n(|u_j|) - \lambda r^- F(|u_j|)\, dx.$$

Then, if $p^+ < r^- < p_n^-$ we conclude that

$$c+1 \geq \left(1 - \frac{p^+}{r-}\right) \int_\Omega A(|\nabla u_j|)\, dx - \frac{1}{r-} |\langle \mathcal{F}_\lambda'(u_j), u_j\rangle|.$$

We can assume that $\|\nabla u_j\|_{A,\Omega} \geq 1$, if not the sequence is bounded. As $\|\mathcal{F}'(u_j)\|_{-1,\tilde{A}}$ is bounded, using Lemma 2.10 we have that

$$c+1 \geq \left(1 - \frac{p^+}{r-}\right) \|\nabla u_j\|_A^{p^-} - \frac{C}{r-} \|\nabla u_j\|_A.$$

We deduce that u_j is bounded.

This finishes the proof. □

From the fact that $\{u_j\}_{j\in\mathbb{N}}$ is a Palais-Smale sequence it follows, from Lemma 5.3, that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W_0^{1,A}(\Omega)$. Hence passing to a subsequence if necessary, by Theorem 4.3, we have that

$$A_n(|u_j|) \rightharpoonup \nu = A_n(|u|) + \sum_{i\in I} \nu_i \delta_{x_i} \quad \nu_i > 0, \tag{5.3}$$

$$A(|\nabla u_j|) \rightharpoonup \mu \geq A(|\nabla u|) + \sum_{i\in I} \mu_i \delta_{x_i} \quad \mu_i > 0, \tag{5.4}$$

$$C \frac{1}{M_n^{-1}\left(\frac{1}{\nu_i}\right)} \leq \frac{1}{A_\infty^{-1}\left(\frac{1}{\mu_i}\right)}. \tag{5.5}$$

Note that if $I = \emptyset$, from (5.3) it is the easy to see that

$$\int_\Omega A_n\left(\frac{|u_j|}{\lambda}\right) dx \to \int_\Omega A_n\left(\frac{|u|}{\lambda}\right) dx,$$

for every $\lambda > 0$, from where it follows that $\|u_j\|_{A_n,\Omega} \to \|u\|_{A_n,\Omega}$ and since $L^{A_n}(\Omega)$ is uniformly convex (see [11]), then $u_j \to u$ strongly in $L^{A_n}(\Omega)$.

Now we can prove the Palais-Smale condition for small energy levels

Lemma 5.4. *Under the above assumptions on A and F, there exists a constant $C_0 > 0$, depending on A and n, such that every Palais-Smale sequence $\{u_j\}_{j\in\mathbb{N}}$ of $\mathcal{F}_\lambda$ with energy level $c < C_0$ verify that $I = \emptyset$ in* (5.3).

Proof. In fact, suppose that $I \neq \emptyset$. Then let $\phi \in C_c^\infty(\mathbb{R}^n)$ with support in the unit ball of $\mathbb{R}^n$ and $\phi(0) = 1$. Consider the rescaled functions $\phi_{i,\varepsilon}(x) = \phi(\frac{x-x_i}{\varepsilon})$ that are supported in $B_\varepsilon(x_i)$.

As $\mathcal{F}_\lambda'(u_j) \to 0$ in $W^{-1,\tilde{A}}(\Omega)$, we obtain that

$$\lim_{j\to\infty} \langle \mathcal{F}_\lambda'(u_j), \phi_{i,\varepsilon} u_j\rangle = 0.$$

On the other hand,

$$\langle \mathcal{F}_\lambda'(u_j), \phi_{i,\varepsilon} u_j\rangle = \int_\Omega \frac{A'(|\nabla u_j|)}{|\nabla u_j|}\nabla u_j \nabla(\phi_{i,\varepsilon} u_j) - \lambda \frac{F'(|u_j|)}{|u_j|} u_j \phi_{i,\varepsilon} u_j - \frac{A_n'(|u_j|)}{|u_j|} u_j \phi_{i,\varepsilon} u_j\, dx.$$

Then, passing to the limit as $j \to \infty$, we get, since $F \ll A_n$ implies that $W_0^{1,A}(\Omega) \subset L^F(\Omega)$ compactly,

$$\begin{aligned} 0 &\geq \lim_{j\to\infty} \left(\int_\Omega \frac{A'(|\nabla u_j|)}{|\nabla u_j|} \nabla u_j \nabla(\phi_{i,\varepsilon}) u_j\, dx \right) \\ &\quad + p^- \int_\Omega \phi_{i,\varepsilon}\, d\mu - p_n^+ \int_\Omega \phi_{i,\varepsilon}\, d\nu - r^+ \int_\Omega \lambda F(|u|)\phi_{i,\varepsilon}\, dx. \end{aligned} \tag{5.6}$$

Now, we want to prove that

$$\lim_{j\to\infty} \left(\int_\Omega \frac{A'(|\nabla u_j|)}{|\nabla u_j|} \nabla u_j \nabla(\phi_{i,\varepsilon}) u_j\, dx \right) = 0. \tag{5.7}$$

In fact, by Hölder inequality

$$\begin{aligned} \left| \int_\Omega \frac{A'(|\nabla u_j|)}{|\nabla u_j|} \nabla u_j \nabla(\phi_{i,\varepsilon}) u_j\, dx \right| &\leq \int_\Omega A'(|\nabla u_j|)|\nabla \phi_{i,\varepsilon}||u_j|\, dx \\ &\leq \int_\Omega \frac{\|\nabla\phi\|_\infty}{\varepsilon} A'(|\nabla u_j|)|u_j|\, dx \\ &\leq 2\frac{\|\nabla\phi\|_\infty}{\varepsilon} \|A'(|\nabla u_j|)\|_{\widetilde{A},B_\varepsilon} \|u_j\|_{A,B_\varepsilon} \end{aligned}$$

It is easy to see that $\|A'(|\nabla u_j|)\|_{\widetilde{A},B_\varepsilon}$ is bounded. In fact, see [8, Lemma 2.9],

$$\int_{B_\varepsilon} \widetilde{A}(A'(|\nabla u_j|))\, dx \leq (p^+ - 1) \int_\Omega A(|\nabla u_j|)\, dx \leq C$$

Moreover

$$\lim_{j\to\infty} \|u_j\|_{A,B_\varepsilon} = \|u\|_{A,B_\varepsilon}$$

It is then enough to prove that $\frac{1}{\varepsilon}\|u\|_{A,B_\varepsilon}$ goes to 0 as ε goes to 0 and this is the only part of the proof where we need to use hypothesis (5.2).

From Remark 5.1, we know that there exists a constant $C > 0$ such that

$$\|u\|_{A,B_\varepsilon} \leq C\|u\|_{p,B_\varepsilon}.$$

Moreover, since $u \in W_0^{1,A}(\Omega) = W_0^{1,p}(\Omega)$, it follows that $u \in L^{p^*}(\Omega)$.

But then,

$$\|u\|_{p,B_\varepsilon} \leq \|u\|_{p^*,B_\varepsilon} |B_\varepsilon|^{\frac{1}{n}} = C\varepsilon \|u\|_{p^*,B_\varepsilon},$$

from where the proof of (5.7) is complete.

On the other hand,

$$\lim_{\varepsilon\to 0} \int_\Omega \phi_{i,\varepsilon}\, d\mu = \mu_i, \quad \lim_{\varepsilon\to 0} \int_\Omega \phi_{i,\varepsilon}\, d\nu = \nu_i \quad \text{and} \quad \lim_{\varepsilon\to 0} \int_\Omega \lambda F(|u|)\phi_{i,\varepsilon}\, dx = 0$$

and comming back to (5.6) we discover that

$$p^- \mu_i \leq p_n^+ \nu_i.$$

Then either $\nu_i = 0$ or, using (5.5), there exists a constant $\tilde{C}$ depending only on A and n such that

$$\tilde{C} \leq \nu_i$$

On the other hand, as $p^+ < r^- < p_n^-$,

$$\begin{aligned} c = \lim_{j\to\infty} \mathcal{F}_\lambda(u_j) &= \lim_{j\to\infty} \mathcal{F}_\lambda(u_j) - \frac{1}{p^+}\langle \mathcal{F}_\lambda(u_j), u_j\rangle \\ &\geq \lim_{j\to\infty} \int_\Omega \left(\frac{p_n^-}{p^+} - 1\right) A_n(|u_j|)\,dx + \lambda \int_\Omega \left(\frac{r^-}{p^+} - 1\right) F(|u_j|)\,dx \\ &\geq \lim_{j\to\infty} \int_\Omega \left(\frac{p_n^-}{p^+} - 1\right) A_n(|u_j|)\,dx. \end{aligned}$$

But

$$\begin{aligned} \lim_{j\to\infty} \int_\Omega \left(\frac{p_n^-}{p^+} - 1\right) A_n(|u_j|)\,dx &= \left(\frac{p_n^-}{p^+} - 1\right)\left(\int_\Omega A_n(|u|)\,dx + \sum_{l\in I} \nu_l\right) \\ &\geq \left(\frac{p_n^-}{p+} - 1\right)\nu_i \geq \left(\frac{p_n^-}{p^+} - 1\right)\tilde{C}. \end{aligned}$$

We get $c \geq C_0 := \left(\frac{p_n^-}{p^+} - 1\right)\tilde{C}$.

Therefore, if

$$c < C_0,$$

the index set I is empty. □

Now we are ready to prove the Palais-Smale condition below level c.

Theorem 5.5. *The functional $\mathcal{F}_\lambda$ verifies the Palais-Smale condition for every energy level $c < C_0$. That is, given $\{u_j\}_{j\in\mathbb{N}} \subset W_0^{1,A}(\Omega)$ a Palais-Smale sequence for $\mathcal{F}_\lambda$, with energy level $c < C_0$, then there exist $u \in W_0^{1,A}(\Omega)$ and $\{u_{j_k}\}_{k\in\mathbb{N}} \subset \{u_j\}_{j\in\mathbb{N}}$ a subsequence such that $u_{j_k} \to u$ strongly in $W_0^{1,A}(\Omega)$.*

Proof. Let $\{u_j\}_{j\in\mathbb{N}} \subset W_0^{1,A}(\Omega)$ be a Palais-Smale sequence for $\mathcal{F}_\lambda$. Then, by Lemma 5.3, we know that $\{u_j\}_{j\in\mathbb{N}}$ is bounded. Then, for a subsequence that we still denote $\{u_j\}_{j\in\mathbb{N}}$, , from Lemma 5.4 we have that $u_j \to u$ strongly in $L^{A_n}(\Omega)$.

We define $\phi_j := \mathcal{F}'(u_j)$. Since $\{u_j\}_{j\in\mathbb{N}}$ is a Palais-Smale sequence, we have $\phi_j \to 0$ in $W^{-1,\tilde{A}}(\Omega)$.

Observe that, by definition of ϕ_j, it follows that u_j is a weak solution of the following equation

$$\begin{cases} -\Delta_A u_j = \frac{A_n'(|u_j|)u_j}{|u_j|} + \lambda \frac{F'(|u_j|)}{|u_j|} u_j + \phi_j =: f_j & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.8}$$

We define $T\colon W^{-1,\tilde{A}}(\Omega) \to W_0^{1,A}(\Omega)$ to be the solution operator of $-\Delta_A$. That is, $T(f) := u$ where u is the weak solution of the following equation.

$$\begin{cases} -\Delta_A u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.9}$$

Then T is a continuous invertible operator.

To finish the proof, it is sufficient to show that f_j converges in $W^{-1,\tilde{A}}(\Omega)$. Since $\phi_j \to 0$ and $F \ll A_n$, we only need to prove that $\frac{A_n'(|u_j|)u_j}{|u_j|} \to \frac{A_n'(|u|)u}{|u|}$ strongly in $W^{-1,\tilde{A}}(\Omega)$.

In fact,

$$\left\langle \frac{A_n'(|u_j|)u_j}{|u_j|} - \frac{A_n'(|u|)u}{|u|}, \psi \right\rangle = \int_\Omega \left(\frac{A_n'(|u_j|)u_j}{|u_j|} - \frac{A_n'(|u_j|)u_j}{|u_j|} \right) \psi\, dx$$
$$\leq 2\|\psi\|_{A_n,\Omega} \left\| \frac{A_n'(|u_j|)u_j}{|u_j|} - \frac{A_n'(|u|)u}{|u|} \right\|_{\widetilde{A_n},\Omega}.$$

Therefore,

$$\left\| \frac{A_n'(|u_j|)u_j}{|u_j|} - \frac{A_n'(|u|)u}{|u|} \right\|_{W^{-1,\tilde{A}}(\Omega)} = \sup_{\substack{\psi\in W_0^{1,A}(\Omega)\\ \|\psi\|_{W_0^{1,A}(\Omega)}=1}} \int_\Omega \left(\frac{A_n'(|u_j|)u_j}{|u_j|} - \frac{A_n'(|u|)u}{|u|} \right) \psi\, dx$$
$$\leq C \left\| \frac{A_n'(|u_j|)u_j}{|u_j|} - \frac{A_n'(|u|)u}{|u|} \right\|_{\widetilde{A_n},\Omega}$$

Hence, to finish the proof we need to show that

$$\lim_{j\to\infty} \int_\Omega \widetilde{A_n} \left(\left| \frac{A_n'(|u_j|)u_j}{|u_j|} - \frac{A_n'(|u|)u}{|u|} \right| \right) dx = 0.$$

We can assume, passing to a further subsequence if necessary, that $u_j \to u$ a.e. in Ω. Hence

$$\widetilde{A_n} \left(\left| \frac{A_n'(|u_j|)u_j}{|u_j|} - \frac{A_n'(|u|)u}{|u|} \right| \right) \to 0 \quad \text{a.e. in } \Omega,$$

hence we need to find an integrable majorant for this integrand.

To this end, we observe that

$$\widetilde{A_n} \left(\left| \frac{A_n'(|u_j|)u_j}{|u_j|} - \frac{A_n'(|u|)u}{|u|} \right| \right) \leq C \left(\widetilde{A_n}(A_n'(|u_j|)) + \widetilde{A_n}(A_n'(|u|)) \right)$$
$$\leq C \left(A_n(|u_j|) + A_n(|u|) \right).$$

Now, since $u_j \to u$ strongly in $L^{A_n}(\Omega)$ and $u_j \to u$ a.e. in Ω, a straightforward modification of [3, Theorem 4.9], gives us the existence of a function $g \in L^1(\Omega)$ such that

$$A_n(|u_j(x)|) \leq g(x) \quad \text{a.e. in } \Omega.$$

This fact concludes the proof of the result. □

Now we are in condition to give the proof of the main result of the section.

Proof of Theorem 5.2. In view of the previous result, we seek for critical values of level $c < C_0$. For that purpose, we want to use the Mountain Pass Theorem. Hence we have to check the following condition:

(1) There exist constants $R, r > 0$ such that when $\|u\|_{1,A} = R$, then $\mathcal{F}_\lambda(u) > r$.
(2) There exist $v_0 \in W^{1,A}(\Omega)$, $\|v_0\|_{1,A} > R$, such that $\mathcal{F}_\lambda(v_0) < r$.

Let us first check (1). We suppose that $\|\nabla u\|_A \leq 1$ and $\|u\|_A \leq 1$. The other cases can be treated similarly.

By Poincaré inequality we have,

$$\begin{aligned}\mathcal{F}_\lambda(u) = \int_\Omega A(|\nabla u|) - A_n(|u|) - \lambda F(|u|)\,dx &\geq \|\nabla u\|_A^{p^+} - \|u\|_{A_n}^{p_n^-} - \lambda\|u\|_F^{r^-} \\ &\geq \|\nabla u\|_A^{p^+} - C\|\nabla u\|_A^{p_n^-} - \lambda C\|\nabla u\|_A^{r^-}.\end{aligned}$$

Let $g(t) = t^{p+} - Ct^{q-} - C\lambda t^{r-}$, then as $p^+ < r^- < p_n^-$ it is easy to check that $g(R) > r$ for some $R, r > 0$. This proves (1).

Now, in order to prove (2), for a fixed $u \in W_0^{1,A}(\Omega)$ such that $\|\nabla u\|_{A,\Omega} \geq 1, \|u\|_{A,\Omega} \geq 1$ and given $t > 1$ we have

$$\begin{aligned}\mathcal{F}_\lambda(tu) &= \int_\Omega A(|\nabla tu|) - A_n(|tu|) - \lambda F(|tu|)\,dx \\ &\leq t^{p^+}\int_\Omega A(|\nabla u|)\,dx - t^{p_n^-}\int_\Omega A_n(|u|)\,dx.\end{aligned}$$

So, since $p^+ < p_n^-$, it is easy to see that $\lim_{t\to\infty} \mathcal{F}_\lambda(tu) = -\infty$.

Now the candidate for a critical value according to the Mountain Pass Theorem is

$$c = \inf_{g\in\mathcal{C}} \sup_{t\in[0,1]} \mathcal{F}_\lambda(g(t)),$$

where $\mathcal{C} = \{g : [0,1] \to W_0^{1,A}(\Omega)\colon g \text{ continuous and } g(0) = 0, g(1) = v_0\}$.

We will show that $c < C_0$ if λ is chosen to be big enough and therefore Theorem 5.5 can be applied.

Let $g \in \mathcal{C}$ be given by $g(t) = tv_0$. Then

$$c \leq \sup_{t\in[0,1]} \mathcal{F}_\lambda(g(t)) = \sup_{t\in[0,1]} \mathcal{F}_\lambda(tv_0).$$

Now,

$$\mathcal{F}_\lambda(tv_0) \leq t^{p^-}\int_\Omega A(|\nabla v_0|)\,dx - \lambda t^{r^+}\int_\Omega F(|v_0|)\,dx$$

If we denote by $\phi(t) := t^{p^-}a_1 - \lambda t^{r^+}a_2$, then the maximum of ϕ is attained at $t_\lambda = \left(\frac{a_1}{\lambda a_2}\right)^{\frac{1}{r^+-p^-}}$, from where it follows that

$$\lim_{\lambda\to\infty} \sup_{t\in[0.1]} \mathcal{F}_\lambda(g(t)) = 0.$$

So, we conclude that there exists $\lambda_0 > 0$ such that if $\lambda \geq \lambda_0$ then $c < C_0$ and this finishes the proof. □

References

[1] Waldo Arriagada. Matuszewska–orlicz indices of the sobolev conjugate young function. *Partial Differential Equations in Applied Mathematics*, 3:100029, 2021.

[2] Julián Fernández Bonder, Nicolas Saintier, and Analía Silva. The concentration-compactness principle for fractional order Sobolev spaces in unbounded domains and applications to the generalized fractional Brezis-Nirenberg problem. *NoDEA Nonlinear Differential Equations Appl.*, 25(6):Paper No. 52, 25, 2018.

[3] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.
[4] Andrea Cianchi. Optimal Orlicz-Sobolev embeddings. *Rev. Mat. Iberoamericana*, 20(2):427–474, 2004.
[5] Thomas K. Donaldson and Neil S. Trudinger. Orlicz-Sobolev spaces and imbedding theorems. *J. Functional Analysis*, 8:52–75, 1971.
[6] J. Fernández Bonder, M. Pérez-Llanos, and A. M. Salort. A hölder infinity laplacian obtained as limit of orlicz fractional laplacians. *Rev Mat Complut.*, 2021.
[7] Julián Fernández Bonder, Nicolas Saintier, and Analia Silva. On the Sobolev embedding theorem for variable exponent spaces in the critical range. *J. Differential Equations*, 253(5):1604–1620, 2012.
[8] Julián Fernández Bonder and Ariel M. Salort. Fractional order Orlicz-Sobolev spaces. *J. Funct. Anal.*, 277(2):333–367, 2019.
[9] Julián Fernández Bonder and Analía Silva. Concentration-compactness principle for variable exponent spaces and applications. *Electron. J. Differential Equations*, pages No. 141, 18, 2010.
[10] Yongqiang Fu. The principle of concentration compactness in $L^{p(x)}$ spaces and its application. *Nonlinear Anal.*, 71(5-6):1876–1892, 2009.
[11] M. A. Krasnoselskiĭ and Ja. B. Rutickiĭ. *Convex functions and Orlicz spaces*. P. Noordhoff Ltd., Groningen, 1961. Translated from the first Russian edition by Leo F. Boron.
[12] Gary M. Lieberman. The natural generalization of the natural conditions of Ladyzhenskaya and Uraltseva for elliptic equations. *Comm. Partial Differential Equations*, 16(2-3):311–361, 1991.
[13] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana*, 1(1):145–201, 1985.
[14] Sunra Mosconi, Kanishka Perera, Marco Squassina, and Yang Yang. The Brezis-Nirenberg problem for the fractional p-Laplacian. *Calc. Var. Partial Differential Equations*, 55(4):Art. 105, 25, 2016.

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