The size of the sync basin resolved

Pablo Groisman,[1,2,3,*] Cecilia De Vita,[1,2] Julián Fernández Bonder,[1,4] and Yuanzhao Zhang[5,†]

[1] *Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires*
[2] *IMAS-UBA-CONICET, Buenos Aires, Argentina*
[3] *NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai*
[4] *Instituto de Cálculo UBA-CONICET, Buenos Aires, Argentina*
[5] *Santa Fe Institute, Santa Fe, NM, USA*

Sparsely coupled Kuramoto oscillators offer a fertile playground for exploring high-dimensional basins of attraction due to their simple yet multistable dynamics. For n identical Kuramoto oscillators on cycle graphs, it is well known that the only attractors are twisted states, whose phases wind around the circle with a constant gap between neighboring oscillators ($\theta_j = 2\pi qj/n$). It was conjectured in 2006 that basin sizes of these twisted states scale as e^{-kq^2} to the winding number q. Here, we provide new numerical and analytical evidence supporting the conjecture and uncover the dynamical mechanism behind the Gaussian scaling. The key idea is that, when starting with a random initial condition, the winding number of the solution stabilizes rapidly at $t \propto \log n$, before long-range correlation can develop among oscillators. This timescale separation allows us to model the winding number as a sum of weakly dependent variables, leading to a Central Limit Theorem derivation of the basin scaling.

Basins of attraction map initial conditions to attractors and are fundamental to the analysis of multistable dynamical systems [1–3]. Even simple equations can generate complicated basins [4–10], as exemplified by Wada basins [11], fractal basin boundaries [12–15], and riddled basins [16–21]. Given the intricate and often high-dimensional nature of basins, it is perhaps not surprising that even the most basic question—how big are the basins—still holds plenty of mystery [22–40].

One of the canonical systems for studying basins is Kuramoto oscillators on cycle graphs [22, 32, 41, 42]:

$$\dot{\theta}_j = \sin(\theta_{j+1} - \theta_j) + \sin(\theta_{j-1} - \theta_j), \quad j = 1, \dots, n, \tag{1}$$

where $\theta_j \in [0, 2\pi)$ is the phase of oscillator j. Note that we assume periodic boundary conditions to close the ring. The sync state $\theta_1 = \cdots = \theta_n$ is always an attractor of the system. For $n > 4$, Eq. (1) has additional attractors in the form of phase-locked configurations with the oscillator's phases making q full twists around the ring: $\theta_j = 2\pi jq/n + c$. Here, q is the winding number and c is a constant. Such twisted states are stable if and only if $|q| < n/4$ [32]. By varying the network size n, one can easily change the number of attractors in the system.

In 2006, based on numerical evidence and heuristic arguments, Wiley, Strogatz, and Girvan [22] conjectured that the basin size of q-twisted states follows a simple scaling law of e^{-kq^2}, where k is some constant. The conjecture was later challenged in the literature. For instance, based on semi-analytical calculations, Ref. [32] suggested that the correct scaling should be $e^{-k|q|}$. More recently, there was additional evidence supporting the original Gaussian scaling based on the geometries of the basins [41]. Because the basin size decreases rapidly with q, basins with $q > \sqrt{n}$ can be exceedingly difficult to sample and direct numerical simulations cannot conclusively resolve the debate. It is thus important to establish the scaling relation through analytical means.

In this Letter, we show that basin sizes of twisted states in Eq. (1) scale as e^{-kq^2}, with $k = 1/2\sigma^2 n$ and σ^2 being the variance of a one-dimensional random variable that we will define later on. We break the argument into three steps:

1. Show the existence of a region $\mathcal{I}$ that is flow-invariant under the dynamics of Eq. (1) and the winding number does not change once the system enters $\mathcal{I}$.

2. Show that up to $t \propto \log n$, there is no long-range dependence between the oscillators. Consequently, we can apply the Central Limit Theorem (CLT) to establish that the winding number (given by a sum of the phases) is Gaussian distributed at these times.

3. Show that when starting from a random initial condition, the system enters the region $\mathcal{I}$ quickly at $t \propto \log n$. This bounds the time window for which the winding number can change. Since the CLT holds for the winding number when entering $\mathcal{I}$ and it remains invariant after, the Gaussian scaling must hold for the final winding number at $t \to \infty$.

Before giving details on these steps, we provide some rationale behind our strategy. It is more convenient to work with the phase differences between consecutive oscillators rather than directly with the phases θ_j. We consider the new variables $\eta_j = \theta_{j+1} - \theta_j \in (-\pi, \pi]$. For $j = n$ we define $\eta_n = \theta_1 - \theta_n$. It is important to note that we force η_j to be in the interval $(-\pi, \pi]$. In these new variables, Eq. (1) is equivalent to

$$\dot{\eta}_j(t) = \sin(\eta_{j+1}) - 2\sin(\eta_j) + \sin(\eta_{j-1}), \tag{2}$$

with the caveat that the equation has to be interpreted mod $(-\pi, \pi]$. With this convention, if all the phase differences $\eta_j \neq \pi$, we can compute the winding number as

$$q(t) = \frac{1}{2\pi} \sum_{j=1}^{n} \eta_j(t) = \left[\frac{1}{2\pi} \sum_{j=1}^{n-1} \eta_j(t) \right], \qquad (3)$$

where $[x]$ denotes the closest integer to x.

Because the phase differences at $t = 0$, $\eta_j(0)$, are independent random variables uniformly sampled from $(-\pi, \pi]$, by the CLT the winding number (their sum) follows a normal distribution when n is large. The mean of the winding number is zero and its variance is $(n-1)/12$, since random variables uniformly distributed in $[-1/2, 1/2]$ have variance $1/12$. To obtain a well-defined distribution in the limit of $n \to \infty$, we simply need to normalize q by $\sqrt{n}$. Observe that as $t \to \infty$ we lose the independence (in fact for $t = \infty$ we have $\eta_i = \eta_j$ for every i, j) and hence at this time the Gaussian scaling can not be obtained as a consequence of the Central Limit Theorem. Moreover, the winding number q is not conserved by the dynamics. So, how can we demonstrate that the distribution of q would remain Gaussian as $t \to \infty$?

Numerically, we found that q typically stabilizes very early on at a time t_s and remains unchanged for $t > t_s$. The magenta curve in Fig. 1 shows that the average stabilization time $\langle t_s \rangle$ grows slowly with the system size as $\log n$. The hope is that, at this early time, no long-range correlation has developed in the system and the CLT can still be applied to coarse-grained oscillator states. It is known that as long as the range of dependence is of order not larger than $n^{1/4}$, the CLT still holds [43]. Indeed, numerical evidence supports the no long-range dependence assumption (Fig. 2). Later, in Step 2, we will explain this observation by utilizing the local coupling in Eq. (1).

We now proceed with the three steps of the argument.

Step 1. Since it is not easy to estimate the stabilization time t_s directly, as a first step, we would like to find a region in the phase space where q would stay invariant. This would allow us to control the stabilization time t_s by estimating the time t_e it takes to enter the invariant region. Since $t_s \leq t_e$, if we can show that for most initial conditions $t_e \propto \log n$, it would establish the desired bound $t_s \leq \alpha \log n$, where α is a finite constant. Indeed, Fig. 1 provides numerical evidence that $\langle t_e \rangle \propto \log n$. We will also give analytical arguments for this in Step 3.

We denote $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)$ and consider the region $\mathcal{I} = \{\boldsymbol{\eta} \colon \eta_i \in (-\frac{\pi}{2}, \frac{\pi}{2}) \text{ for all } i\}$. We can establish its flow invariance through a maximum principle. Assume $\boldsymbol{\eta}(0) \in \mathcal{I}$ and let t_0 be the first time such that $\boldsymbol{\eta}(t_0) \in \partial\mathcal{I}$, the boundary of $\mathcal{I}$. Then, for some i we have $\eta_i(t_0) \in \{-\pi/2, \pi/2\}$. Without loss of generality, we can assume $\eta_i(t_0) = \pi/2$. From Eq. (2), we have $\dot{\eta}_i(t_0) \leq 0$ with strict inequality unless $\eta_{i-1}(t_0) = \eta_{i+1}(t_0) = \pi/2$. If we have strict inequality, we obtain a contradiction since η_i needs to increase at t_0 to exit $\mathcal{I}$. If $\dot{\eta}_i(t_0) = 0$, we have $\eta_{i-1}(t_0) = \eta_{i+1}(t_0) = \pi/2$, and proceeding inductively we obtain that either there is a node j with $\eta_j(t_0) = \pi/2$ and $\dot{\eta}_j(t_0) < 0$, or $\eta_j(t_0) = \pi/2$ for every j. The state at which all the phase differences are $\pi/2$ is an unstable equilibrium and hence cannot be reached in finite time. We conclude that there is no such t_0 at which $\boldsymbol{\eta}$ can exit

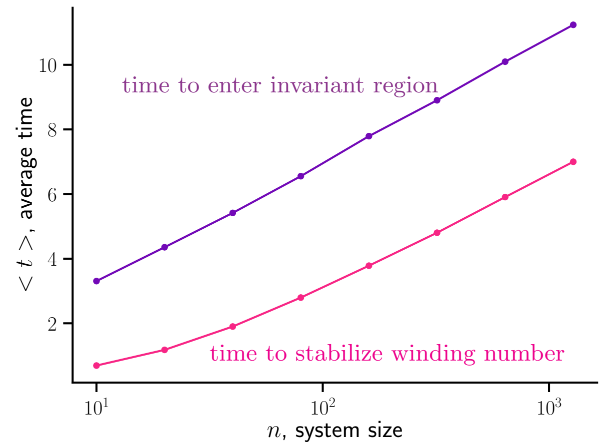

FIG. 1. Time till the winding number stops changing (t_s) and time till the system enters the invariant region $\mathcal{I}$ (t_e) both scale as $\log n$. This will be shown more rigorously in Steps 1 and 3. The magenta curve shows $\langle t_s \rangle$ and the purple curve shows $\langle t_e \rangle$. Each curve is averaged over 10^4 trajectories starting from random initial conditions. We always have $t_s \leq t_e$ for any individual trajectory, which is the point of Step 1.

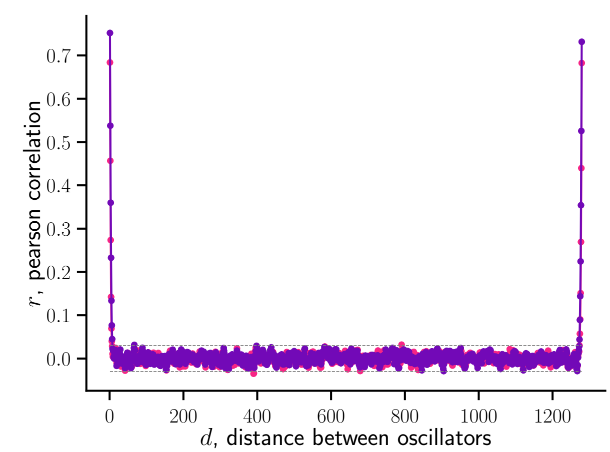

FIG. 2. No long-range correlation can develop before the winding number stops changing. Here, we calculate the Pearson correlation r between two oscillators that are distance d apart for $n = 1280$. The magenta curve shows r at $t = t_s$ (winding number stabilized), and the purple curve shows $t = t_e$ (entering the invariant region $\mathcal{I}$). In either case, the oscillators are essentially uncorrelated unless they are very close to each other ($d \leq 6$). We show the lack of long-range correlation analytically in Step 2.

$\mathcal{I}$.

With a slightly more involved argument, we can show something stronger still. Let $\mathcal{I}_i = \{\boldsymbol{\eta}\colon \eta_i \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$, it is immediate that $\mathcal{I} = \cap_i \mathcal{I}_i$. We will show that in fact each $\mathcal{I}_i$ is invariant. In other words, once a phase difference enters $(-\frac{\pi}{2}, \frac{\pi}{2})$, it will never leave.

We proceed with a perturbation argument. Instead of Eq. (2), consider the equations

$$\dot{\eta}_i^\varepsilon(t) = \sin\big(\eta_{i+1}^\varepsilon\big) - (2+\varepsilon)\sin\big(\eta_i^\varepsilon\big) + \sin\big(\eta_{i-1}^\varepsilon\big).$$

Assume $\eta_i^\varepsilon(0) \in \mathcal{I}_i$ and that at some finite time t_0 we have $\eta_i^\varepsilon(t_0) = \pi/2$ for the first time. Since

$$\dot{\eta}_i^\varepsilon(t_0) \le 2 - (2+\varepsilon) = -\varepsilon,$$

we have a contradiction. Thus, for every $\varepsilon > 0$, η_i^ε cannot leave $\mathcal{I}_i$. Now, assume that there is a time t_0 such that $|\eta_i(t_0)| > \pi/2$. By continuity of the solution at finite time t_0 with respect to the ODE parameters (see [44, Theorem 2 on page 84]), we have $\eta_i^\varepsilon(t_0) \to \eta_i(t_0)$ as $\varepsilon \to 0$. But $\eta_i^\varepsilon(t_0) \in (-\pi/2, \pi/2)$ for every ε. This is a contradiction. Hence, we conclude that there is no such t_0 and that η_i cannot leave $\mathcal{I}_i$ once inside.

Next, we establish the invariance of the winding number inside $\mathcal{I}$. Since for $\eta(t) \in \mathcal{I}$ formula (3) holds, we have

$$\frac{d}{dt} q(t) = \frac{1}{2\pi} \sum_{i=1}^{n} \frac{d}{dt} \eta_i(t) = 0.$$

To see this more intuitively, note that for the winding number (a discrete quantity) to change along a continuous flow, it can only happen when one of the phase differences η_i crosses π or $-\pi$, the boundary points on which q becomes ill-defined. Since $\mathcal{I}$ does not include any of the boundary points and is flow invariant, there can be no more change in q along the flow once inside $\mathcal{I}$.

Step 2. In this step, our goal is to establish that long-range correlations cannot develop in Eq. (1) at $t \propto \log n$, paving the way for the use of CLT. For continuous time, it is difficult to control the correlation between two oscillators that are far away from each other. So, we consider an Euler discrete scheme $\boldsymbol{\eta}^h$ with time step h that approximates Eq. (2),

$$\eta_i^h(t_{k+1}) = \eta_i^h(t_k) + hG(\boldsymbol{\eta}^h(t_k)),$$

with $G(\boldsymbol{\eta}^h(t_k)) = \sin\big(\eta_{i-1}^h(t_k)\big) - 2\sin\big(\eta_i^h(t_k)\big) + \sin\big(\eta_{i+1}^h(t_k)\big))$ and $t_k = kh$. For this discrete scheme, it is easy to see that the range of dependence of an oscillator increases by two (one on each side) at each time step. To reach time $t \propto \log n$, we need $h^{-1}\log n$ time steps. At that time, the range of dependence for each oscillator is at most $h^{-1}\log n$. We can apply CLT as long as the step size h approaches 0 not too fast as $n \to \infty$ ($h^{-1}\log n \le n^\kappa$ for $\kappa < 1/4$ is enough) [43].

Let us call $E_k = \max_{1\le i\le n} |\eta_i^h(t_k) - \eta_i(t_k)|$. An error analysis similar to Ref. [45] gives $E_k \le kh^2$. So, for $k = h^{-1}\log n$, we can control the error $E_k \le h\log n \to 0$ as long as h decays faster than $\log^{-1} n$.

To export the CLT to the continuous equation we need to control the difference between the winding number of the discrete approximation and the one of the continuous solution (divided by $\sqrt{n}$). To do that, observe that an error $E_{k,i}$ when $\eta_i(t_k)$ is at a distance larger than $E_{k,i}$ from $\{-\pi, \pi\}$ does not change the winding number. The difference between the winding number of the discrete approximation and the continuous solutions comes from those η_i that are close to π in the approximation and close to $-\pi$ in the solution of the ODE or vice versa (i.e. they contribute $\pm 2\pi$ to the difference of the sum involved in the computation of the winding number). The number of i in this situation and their contribution can be modeled as a sum of random variables that take values $-1, 0$ or 1. The probability of being 1 or -1 is bounded by E_k. So, a CLT holds for the total difference (i.e. the difference in the winding numbers of the discrete and the continuous solution). Then, we can approximate this total difference with a Gaussian variable with standard deviation $\sqrt{n}E_k$. When we divide by $\sqrt{n}$, we get an error of order E_k. In other words, the difference between the winding number of the discrete approximation given by the Euler method and the one of the continuous solution, when divided by $\sqrt{n}$, is also of order at most E_k. So, we get the same condition as before ($h < \log^{-1} n$). In this regime, the limiting distribution of the winding number of the approximation and the one of Eq. (2) (divided by $\sqrt{n}$), coincide.

Step 3. The purpose of this step is to show that, starting from random initial conditions, the oscillators enter the invariant region $\mathcal{I}$ quickly at $t_e \propto \log n$, thus establishing the Gaussian scaling of winding numbers through the CLT. If we look at each phase difference η_i separately, we can define the time $t_e^{(i)} \ge 0$ at which it enters the interval $(-\pi/2, \pi/2)$ (from Step 1 we know that once entered, it will never leave). Due to symmetry, we know these times are identically distributed. The maximum of n identically distributed random variables can be bounded by $C\log n$ if their tails are not too heavy [46]. In fact, for $t_e^{(1)}, \ldots, t_e^{(n)}$ with the tail distribution function $F(t) = \mathbb{P}(t_e^{(i)} > t)$, we have

$$\begin{aligned}\mathbb{P}\left(\max_{1\le i\le n} t_e^{(i)} > a_n\right) &= \mathbb{P}\left(\cup_i \{t_e^{(i)} > a_n\}\right) \\ &\le n\mathbb{P}(t_e^{(i)} > a_n) = nF(a_n).\end{aligned}$$

To bound the entering time t_e, we want $nF(a_n) \to 0$ for $a_n = C\log n$. This can be established if $F(t)$ has an exponential tail. In fact, it is enough to show $F(t) \le e^{-\lambda t}$ for $t \le (\log n)^2$ and some $\lambda > 0$. Combining with the fact that $\mathcal{I}_i$ is invariant, we have $t_e = \max_i t_e^{(i)} \le C\log n$.

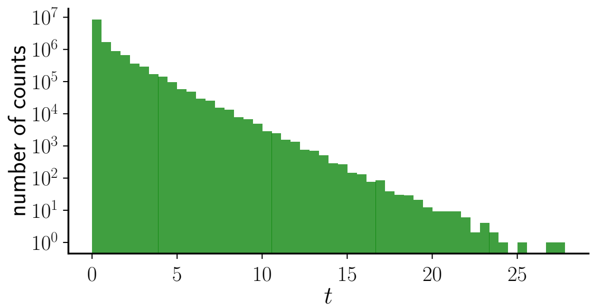

FIG. 3. Distribution of the times $t_e^{(i)}$ at which phase differences η_i enter $(-\pi/2, \pi/2)$. Half of the η_i are already inside $(-\pi/2, \pi/2)$ for random initial conditions (thus the spike at $t = 0$), while the nonzero entering times follow an exponential distribution. This suggests that $|\eta_i|$ becoming smaller than $\pi/2$ can be modeled as independent events happening at a constant rate. The data are collected from 10000 independent simulations of $n = 1280$ Kuramoto oscillators from random initial conditions. The exponential distribution of $t_e^{(i)}$ is a key ingredient for Step 3.

Note that we do not need to assume independence among $t_e^{(i)}$ for this to hold. In fact, strong correlation can quickly develop between η_i and η_j for neighboring i and j, as can be seen in Fig. 2. Figure 3 provides numerical support by showing that $\mathbb{P}(t_e^{(i)})$ follows an exponential distribution, which implies that $F(t) = \mathbb{P}(t_e^{(i)} > t)$ also decays exponentially.

Below, we show why the distribution $F(t)$ has an exponential tail. Eq. (1) is a gradient system, so its dynamics are fully determined by an energy function $E(\boldsymbol{\theta})$. It is easy to see that

$$E_n(\boldsymbol{\theta}) = n - \frac{1}{2}\sum_{j=1}^{n}\left(\cos(\theta_{j+1}-\theta_j) + \cos(\theta_{j-1}-\theta_j)\right),$$

which can also be written in terms of $\boldsymbol{\eta}$ as

$$E_n(\boldsymbol{\eta}) = n - \frac{1}{2}\sum_{j=1}^{n}\left(\cos(\eta_j) + \cos(\eta_{j-1})\right).$$

By the law of large numbers, which holds when starting with i.i.d. initial conditions even for larger times than the CLT, we have that for $t \le \sqrt{n}$

$$\frac{1}{n}E_n(\boldsymbol{\eta}) \to 1 - \mathbb{E}(\cos(\eta_1(t))).$$

Similarly, for the derivative of the energy we have

$$\begin{aligned}
\frac{1}{n}\dot{E}_n(\boldsymbol{\eta}) &= -\frac{1}{n}|\nabla E_n(\boldsymbol{\eta}(t))|^2 \\
&= -\frac{1}{n}\sum_{i=1}^{n}[\sin(\eta_i(t)) - \sin(\eta_{i-1}(t))]^2 \\
&= -\frac{1}{n}\sum_{i=1}^{n}\sin^2(\eta_i(t)) + \sin^2(\eta_{i-1}(t)) \\
&- \frac{2}{n}\sum_{i=1}^{n}\sin(\eta_i(t))\sin(\eta_{i-1}(t)) \\
&\to -2\mathbb{E}(\sin^2(\eta_1(t))) - 2\mathbb{E}(\sin(\eta_1(t))\sin(\eta_2(t))) \\
&\le -2\mathbb{E}(\sin^2(\eta_1(t))),
\end{aligned}$$

where the last step follows from the fact that $\sin\eta_1(t)$ and $\sin\eta_2(t)$ have nonnegative correlation. Hence, if we show the existence of a positive constant c such that $\mathbb{E}(\sin^2(\eta_1(t))) \ge c\mathbb{E}(1-\cos(\eta_1(t)))$ for times of order up to $\sqrt{n}$, we obtain that for such times and large n, with high probability,

$$\dot{E}_n(t) \le -cE_n(t),$$

which implies

$$E_n(t) \le E_n(0)e^{-ct},$$

and consequently $\mathbb{E}(1-\cos(\eta_i(t))) \le e^{-ct}$. Finally,

$$\begin{aligned}
\mathbb{P}(t_e^{(i)} > t) &= \mathbb{P}(|\eta_i(t)| > \pi/2) \\
&= \mathbb{P}(1-\cos(\eta_i(t)) > 1) \\
&\le \mathbb{E}(1-\cos(\eta_i(t))) \\
&\le e^{-ct},
\end{aligned}$$

where we used Markov inequality to go from the second to the third line.

To show the existence of the positive constant c, observe that for any $\varepsilon > 0$, we can choose $c > 0$ such that $\sin^2(s) \ge c(1-\cos(s))$ for every $s \in (-\pi+\varepsilon, \pi-\varepsilon)$, with strict inequality except for $s = 0$. Also observe that since $\mathcal{I}$ is invariant and contains all stable equilibria, we have $\mathbb{P}(|\eta_i(t)| < \pi/2) \to 1$ as $t \to \infty$ (and $\mathbb{P}(|\eta_i(t)| > \pi/2) \to 0$). So, the only thing that can prevent the existence of the constant c is mass being lost at $|\eta(t)| = \pi$ at a slower rate than being gained at $\eta(t) = 0$. But, by symmetry, mass is grown at 0 and lost at π at the same rate. As a consequence,

$$\inf_{t>0}\frac{\mathbb{P}(\varepsilon < |\eta_i(t)| < \pi-\varepsilon)}{\mathbb{P}(|\eta_i(t)| > \pi-\varepsilon)} > 0.$$

This is because if $\mathbb{P}(\varepsilon < |\eta_i(t)| < \pi-\varepsilon)$ converges to zero faster than $\mathbb{P}(|\eta_i(t)| > \pi-\varepsilon)$, that would mean that mass is growing at 0 faster than the rate at which is lost at π. Thus,

$$\inf_{t>0}\frac{\mathbb{E}(\sin^2(\eta_i(t)))}{\mathbb{E}(1-\cos(\eta_i(t)))} = c > 0,$$

and we have, for $t \leq \sqrt{n}$, $E_n(t) \leq E_n(0)e^{-ct}$.

Now, combining all three steps, at time $\log n$ (when the winding number is stabilized), we can establish the independence for phase differences η_i and η_j that are at distance $h^{-1}\log n = \log^{2+\delta} n$ from each other. Because $\log^{2+\delta} n < n^{1/4}$, we can apply CLT to the phase differences to obtain the Gaussian scaling. Finally, the value σ^2 is given by

$$\lim_{n\to\infty} n^{-1}\mathrm{Var}\left(\left[\frac{1}{2\pi}\sum_{j=1}^{n-1}\eta_j(t_e)\right]\right).$$

In this Letter, we established that the basin sizes in Kuramoto oscillators with nearest-neighbor coupling scales with winding number q as e^{-kq^2}, contributing to a central debate on multistable dynamical systems spanning the past 20 years. Our results offer new insights into the dynamics of locally coupled Kuramoto oscillators (e.g., their winding number stabilizes early, before long-range correlations can develop), and the techniques developed here may also be applied to probe the basin sizes in other high-dimensional dynamical systems. Future work has the opportunity to extend our results to more general network structures (e.g., ring networks with higher density [22], signed networks [42], non-regular networks [47], higher-order networks [40], etc.) and dynamics beyond Kuramoto oscillators [26].

We thank Steven Strogatz for the insightful discussions.

* pgroisma@dm.uba.ar
† yzhang@santafe.edu

[1] J. Milnor, On the concept of attractor, Commun. Math. **99**, 177 (1985).
[2] J. Aguirre, R. L. Viana, and M. A. Sanjuán, Fractal structures in nonlinear dynamics, Rev. Mod. Phys. **81**, 333 (2009).
[3] E. Ott, *Chaos in dynamical systems* (Cambridge university press, 2002).
[4] J. P. Crutchfield, Subbasins, portals, and mazes: Transients in high dimensions, Nucl. Phys. B Proc. Suppl. **5**, 287 (1988).
[5] J. P. Crutchfield and K. Kaneko, Are attractors relevant to turbulence?, Phys. Rev. Lett. **60**, 2715 (1988).
[6] K. Wiesenfeld and P. Hadley, Attractor crowding in oscillator arrays, Phys. Rev. Lett. **62**, 1335 (1989).
[7] K. Kaneko, Clustering, coding, switching, hierarchical ordering, and control in a network of chaotic elements, Physica D **41**, 137 (1990).
[8] K. Kaneko, Dominance of Milnor attractors and noise-induced selection in a multiattractor system, Phys. Rev. Lett. **78**, 2736 (1997).
[9] M. Timme, F. Wolf, and T. Geisel, Prevalence of unstable attractors in networks of pulse-coupled oscillators, Phys. Rev. Lett. **89**, 154105 (2002).
[10] Á. Daza, A. Wagemakers, and M. A. Sanjuán, Multistability and unpredictability, Phys. Today **77**, 44 (2024).
[11] H. E. Nusse and J. A. Yorke, Basins of attraction, Science **271**, 1376 (1996).
[12] S. W. McDonald, C. Grebogi, E. Ott, and J. A. Yorke, Fractal basin boundaries, Physica D **17**, 125 (1985).
[13] C. Grebogi, E. Ott, and J. A. Yorke, Chaos, strange attractors, and fractal basin boundaries in nonlinear dynamics, Science **238**, 632 (1987).
[14] A. E. Motter, M. Gruiz, G. Károlyi, and T. Tél, Doubly transient chaos: Generic form of chaos in autonomous dissipative systems, Phys. Rev. Lett. **111**, 194101 (2013).
[15] Y. Zhang and S. P. Cornelius, Catch-22s of reservoir computing, Phys. Rev. Research **5**, 033213 (2023).
[16] J. Alexander, J. A. Yorke, Z. You, and I. Kan, Riddled basins, Int. J. Bifurcation Chaos **2**, 795 (1992).
[17] J. C. Sommerer and E. Ott, A physical system with qualitatively uncertain dynamics, Nature **365**, 138 (1993).
[18] E. Ott, J. C. Sommerer, J. C. Alexander, I. Kan, and J. A. Yorke, Scaling behavior of chaotic systems with riddled basins, Phys. Rev. Lett. **71**, 4134 (1993).
[19] J. F. Heagy, T. L. Carroll, and L. M. Pecora, Experimental and numerical evidence for riddled basins in coupled chaotic systems, Phys. Rev. Lett. **73**, 3528 (1994).
[20] P. Ashwin, J. Buescu, and I. Stewart, From attractor to chaotic saddle: A tale of transverse instability, Nonlinearity **9**, 703 (1996).
[21] Y. Zhang, Z. G. Nicolaou, J. D. Hart, R. Roy, and A. E. Motter, Critical switching in globally attractive chimeras, Phys. Rev. X **10**, 011044 (2020).
[22] D. A. Wiley, S. H. Strogatz, and M. Girvan, The size of the sync basin, Chaos **16**, 015103 (2006).
[23] N. Xu, D. Frenkel, and A. J. Liu, Direct determination of the size of basins of attraction of jammed solids, Phys. Rev. Lett. **106**, 245502 (2011).
[24] M. Monteforte and F. Wolf, Dynamic flux tubes form reservoirs of stability in neuronal circuits, Phys. Rev. X **2**, 041007 (2012).
[25] P. J. Menck, J. Heitzig, N. Marwan, and J. Kurths, How basin stability complements the linear-stability paradigm, Nat. Phys. **9**, 89 (2013).
[26] P. J. Menck, J. Heitzig, J. Kurths, and H. Joachim Schellnhuber, How dead ends undermine power grid stability, Nat. Commun. **5**, 3969 (2014).
[27] Y. Zou, T. Pereira, M. Small, Z. Liu, and J. Kurths, Basin of attraction determines hysteresis in explosive synchronization, Phys. Rev. Lett. **112**, 114102 (2014).
[28] S. Martiniani, K. J. Schrenk, J. D. Stevenson, D. J. Wales, and D. Frenkel, Turning intractable counting into sampling: Computing the configurational entropy of three-dimensional jammed packings, Phys. Rev. E **93**, 012906 (2016).
[29] E. A. Martens, M. J. Panaggio, and D. M. Abrams, Basins of attraction for chimera states, New J. Phys. **18**, 022002 (2016).
[30] S. Leng, W. Lin, and J. Kurths, Basin stability in delayed dynamics, Sci. Rep. **6**, 21449 (2016).
[31] C. Mitra, A. Choudhary, S. Sinha, J. Kurths, and R. V. Donner, Multiple-node basin stability in complex dynamical networks, Phys. Rev. E **95**, 032317 (2017).
[32] R. Delabays, M. Tyloo, and P. Jacquod, The size of the sync basin revisited, Chaos **27**, 103109 (2017).
[33] P. Schultz, P. J. Menck, J. Heitzig, and J. Kurths, Po-

tentials and limits to basin stability estimation, New J. Phys. **19**, 023005 (2017).

[34] S. Rakshit, B. K. Bera, M. Perc, and D. Ghosh, Basin stability for chimera states, Sci. Rep. **7**, 2412 (2017).

[35] S. Martiniani, K. J. Schrenk, K. Ramola, B. Chakraborty, and D. Frenkel, Numerical test of the Edwards conjecture shows that all packings are equally probable at jamming, Nat. Phys. **13**, 848 (2017).

[36] I. Belykh, D. Carter, and R. Jeter, Synchronization in multilayer networks: When good links go bad, SIAM J. Appl. Dyn. Syst. **18**, 2267 (2019).

[37] A. Townsend, M. Stillman, and S. H. Strogatz, Dense networks that do not synchronize and sparse ones that do, Chaos **30**, 083142 (2020).

[38] R. G. Andrzejak, G. Ruzzene, E. Schöll, and I. Omelchenko, Two populations of coupled quadratic maps exhibit a plentitude of symmetric and symmetry broken dynamics, Chaos **30**, 033125 (2020).

[39] R. G. Andrzejak, Chimeras confined by fractal boundaries in the complex plane, Chaos **31**, 053104 (2021).

[40] Y. Zhang, P. S. Skardal, F. Battiston, G. Petri, and M. Lucas, Deeper but smaller: Higher-order interactions increase linear stability but shrink basins, Sci. Adv. **10**, eado8049 (2024).

[41] Y. Zhang and S. H. Strogatz, Basins with tentacles, Phys. Rev. Lett. **127**, 194101 (2021).

[42] A. Díaz-Guilera, D. Marinelli, and C. J. Pérez-Vicente, Exploring the interplay of excitatory and inhibitory interactions in the Kuramoto model on circle topologies, Chaos **34** (2024).

[43] C. Francq and J.-M. Zakoïan, A central limit theorem for mixing triangular arrays of variables whose dependence is allowed to grow with the sample size, Econom. Theory **21**, 1165 (2005).

[44] L. Perko, *Differential equations and dynamical systems*, 3rd ed., Texts Appl. Math., Vol. 7 (New York, NY: Springer, 2001).

[45] K. W. Morton and D. F. Mayers, *Numerical solution of partial differential equations*, 2nd ed. (Cambridge University Press, Cambridge, 2005) pp. xiv+278, an introduction.

[46] L. de Haan and A. Ferreira, *Extreme Value Theory: An Introduction*, Springer Series in Operations Research and Financial Engineering (Springer New York, 2007).

[47] M. Kassabov, S. H. Strogatz, and A. Townsend, A global synchronization theorem for oscillators on a random graph, Chaos **32**, 093119 (2022).