Minimal Finite Models

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Abstract

We characterize the smallest finite spaces with the same homotopy groups as the spheres. Similarly, we describe the minimal finite models of any finite graph. We also develop new combinatorial techniques based on finite spaces to study classical invariants of general topological spaces.

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1 Introduction

This paper deals with finite topological spaces and their application to the homotopy theory of (general) topological spaces. One of its main goals is to characterize the *minimal finite models* of some spaces such as spheres and finite graphs. A minimal finite model of a topological space Y is a finite space with the smallest number of points that is weak homotopy equivalent to Y.
It is well known [3, 6] that finite spaces are related to simplicial complexes. Explicitly, following McCord [6] one can associate to any finite space X a finite simplicial complex $\mathcal{K}(X)$ and a weak homotopy equivalence $|\mathcal{K}(X)| \to X$. Moreover, given a finite simplicial complex K, there is a finite space $\mathcal{X}(K)$ and a weak homotopy equivalence $|K| \to \mathcal{X}(K)$. In this way, one can find *finite models* of topological spaces, i.e. finite topological spaces with the same weak homotopy type. McCord also exhibits in [6] finite models for the spheres S^n, denoted $\mathbb{S}^n S^0$, with only $2n+2$ points.

In his series of notes on finite spaces [2, 3, 4], J.P. May conjectures that $\mathbb{S}^n S^0$ is, in our terminology, a minimal finite model for the n-dimensional sphere. We prove that this conjecture is true. In fact, we prove the following stronger result:

Theorem 2.13. *Any space with the same homotopy groups as S^n has at least $2n+2$ points. Moreover, $\mathbb{S}^n S^0$ is the unique space with $2n+2$ points with this property.*

In particular, $\mathbb{S}^n S^0$ is a minimal finite model of S^n and it is unique.

We also obtain a similar result for minimal finite models of finite graphs. It is well known that finite connected graphs are homotopy equivalent to a wedge sum of finitely many copies of one-dimensional spheres $\bigvee_{i=1}^{m} S^1$. Their finite minimal models are characterized as follows.

Theorem 4.7. *Let $n \in \mathbb{N}$. A finite T_0-space X is a minimal finite model of $\bigvee_{i=1}^{n} S^1$ if and only if $h(X) = 2$, $\#X = min\{i + j \mid (i-1)(j-1) \geq n\}$ and $\#\mathtt{E}(\mathcal{H}(X)) = \#X + n - 1$.*

Here $h(X)$ denotes the height of X (viewed as a poset), $\#X$ denotes the number of points of X and $\#\mathtt{E}(\mathcal{H}(X))$ the number of edges of the Hasse diagram of X. In particular, one can compute explicitly the number of points of any minimal finite model of a finite graph.

Note that, in general, minimal finite models are not unique. If X is a finite model of a space, then so is X^{op}, which is the opposite preorder of X. Moreover, a space can have more than two minimal finite models. To illustrate this, we exhibit in the last section of this paper the three minimal models of $\bigvee_{i=1}^{3} S^1$, each of which with 6 points and 8 edges.

The main reason for investigating finite models of spaces with the same weak homotopy type instead of finite models with the same homotopy type is that the homotopy type of finite spaces rarely occurs in general spaces. More precisely, we prove in section 2 the following result:

Theorem 2.6. *If X is a T_1, connected and non contractible space, then it does not have the homotopy type of any finite space.*

In particular, finite spaces do not have the same homotopy type as any connected non contractible CW-complex.

In [7] Osaki introduces two methods of reduction which allow one to shrink a finite T_0-space to a smaller weak equivalent space. In that article, he asks whether any finite T_0-space X can be reduced to the smallest one with the same homotopy groups as X by a sequence of these two kinds of reductions. In section 2 of this paper, we exhibit an example which shows that the answer to his question is negative. Therefore, his methods of reduction are not always effective and could not be applied to prove Theorems 2.13 and 4.7 mentioned above.

We think that the methods and tools that we develop in this article are, in some cases, as important as the results that we obtain. We will show that these new tools, based on the combinatorics and the topology of finite spaces, are in many situations, even more efficient for investigating homotopy and homology theory of general topological spaces than the classical simplicial tools from simplicial complexes.

To illustrate this, consider the following result, proved in section 4, which is one of the key points in the solution of the problem of the minimal finite models of graphs:

Proposition 4.2. *Let X be a connected finite T_0-space and let $x_0, x \in X$, $x_0 \neq x$ such that x is neither maximal nor minimal in X. Then the inclusion map of the associated simplicial complexes $\mathcal{K}(X \smallsetminus \{x\}) \subseteq \mathcal{K}(X)$ induces an epimorphism*

$$i_* : E(\mathcal{K}(X \smallsetminus \{x\}), x_0) \to E(\mathcal{K}(X), x_0)$$

between their edge-path (fundamental) groups.

The conditions of maximality or minimality of points in a finite space, as well as the notion of *beat point* introduced by Stong [9], are hard to express in terms of simplicial complexes.

In this direction, we will show in section 3 how to compute combinatorially the fundamental group of a finite T_0-space from its Hasse diagram.

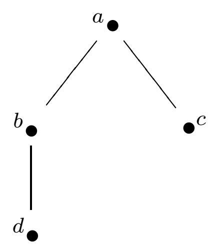
2 Preliminaries and the problem of the spheres

Let X be a finite space. For each $x \in X$ we denote by U_x the minimal open set of x, defined as the intersection of all open sets containing x (cf. [2, 6]).
Given a topology in a finite set X, the associated preorder in X is defined by $x \leq y$ if $x \in U_y$. Alexandroff [1] proved that this association is a one to one correspondence between topologies and preorders in X. Moreover, T_0-topologies correspond to (partial) orders.
Therefore we will regard finite spaces as finite preorders and viceversa.
Note that a function $f : X \to Y$ between finite spaces is continuous if and only if it is order preserving.

Given a finite space X we will denote X^{op} the space whose underlying set is X but with the opposite preorder.

We recall that the Hasse diagram of a finite poset P is the digraph whose set of vertices is P and whose edges are the ordered pairs $(x, y) \in P$ with $x < y$ such that there is no $z \in P$ with $x < z < y$. We define $\mathcal{H}(X) = (\mathtt{V}(\mathcal{H}(X)), \mathtt{E}(\mathcal{H}(X)))$, the *Hasse diagram* of a finite T_0-space X, as the Hasse diagram of the associated order of X.
In order to indicate the orientation of an edge of $\mathcal{H}(X)$ in a figure, we will put y over x if $(x, y) \in \mathtt{E}(\mathcal{H}(X))$.

Example 2.1. Let $X = \{a, b, c, d\}$ be the space whose open sets are $\emptyset$, $\{a, b, c, d\}$, $\{b, d\}$, $\{c\}$, $\{d\}$, $\{b, c, d\}$ and $\{c, d\}$. The Hasse diagram of X is



Following McCord [6], one can see that, in order to investigate homotopy types of finite spaces, it suffices to study T_0-spaces. Stong developed in [9] a very powerful tool to classify the homotopy types of T_0-spaces. We recall from [9] and May's notes [2] the following definitions and results:

Definition 2.2. (Stong) Let X be a finite T_0-space. A point $x \in X$ is called an *up beat point* if there exists $y \in X$, $y > x$ such that $z > x$ implies $z \geq y$. Analogously, a point $x \in X$ will be called a *down beat point* if there exists $y \in X$, $y < x$ such that $z < x$ implies $z \leq y$. A finite T_0-space X is called a *minimal finite space* if it has no beat points.

Note that if $x \in X$ is a beat point, there exists $y \in X,\ y \neq x$, with the following property: Given any $z \in X$, if z is comparable with x, then z is also comparable with y.
Moreover, it is not difficult to prove the following characterization of minimal finite spaces.

Proposition 2.3. *Let X be a finite T_0-space. Then X is a minimal finite space if and only if there are no $x, y \in X$ with $x \neq y$ such that if $z \in X$ is comparable with x, then so is it with y.*

For any finite space X, Stong defines its *core* as a minimal finite space which is a strong deformation retract of X. He shows that any finite space has a core, and that the only map $f : X \to X$ between minimal finite spaces which is homotopic to the identity is the identity itself. Then, the core X_c of a space X, is unique up to homeomorphism and it is the space of minimum cardinality that is homotopy equivalent to X.
Note that two finite spaces are homotopy equivalent if and only if they have homeomorphic cores. In particular, a finite space is contractible if and only if its core is a point.
Since the core of a finite space is the disjoint union of the cores of its connected components, we can deduce the following

Lemma 2.4. *Let X be a finite space such that X_c is discrete. Then X is a disjoint union of contractible spaces.*

As we pointed out in the introduction, finite spaces do not have in general the same homotopy type as T_1-spaces:

Theorem 2.5. *Let X be a finite space and let Y be a T_1-space homotopy equivalent to X. Then X is a disjoint union of contractible spaces.*

Proof. Since $X \simeq Y$, $X_c \simeq Y$. Let $f : X_c \to Y$ be a homotopy equivalence with homotopy inverse g. Then $gf = 1_{X_c}$ since X_c is a minimal finite space. Since f is a one to one map from X_c to a T_1-space, it follows that X_c is also T_1 and therefore discrete. Now the result follows from the previous lemma. □

Corollary 2.6. *Let Y be a connected and non contractible T_1-space. Then Y does not have the same homotopy type as any finite space.*

Proof. Follows immediately from the previous Theorem. □

For example, for any $n \geq 1$, the n-dimensional sphere S^n does not have the homotopy type of any finite space. Although, S^n does have, as any finite polyhedron, the same *weak* homotopy type as some finite space.

Definition 2.7. Let X be a space. We say that a finite space Y is a *finite model* of X if it is weak equivalent to X.
We say that Y is a *minimal finite model* if it is a finite model of minimum cardinality.

By weak (homotopy) equivalent, we mean a topological space Y such that there is a finite sequence $X = X_0, X_1, \ldots, X_r = Y$ and weak homotopy equivalences $X_i \to X_{i+1}$ or $X_{i+1} \to X_i$ for each $i = 0, \ldots, r-1$.

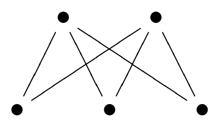
For example, the singleton is the unique minimal finite model of every contractible space. Moreover, it is the unique minimal finite model of every homotopically trivial space, i.e. with trivial homotopy groups.
Since every finite space is homotopy equivalent to its core, which is a smaller space, we have the following

Remark 2.8. Every minimal finite model is a minimal finite space.

In [6] McCord associates to each finite T_0-space X a simplicial complex $\mathcal{K}(X)$ whose simplices are the non empty chains of X and proves that $|\mathcal{K}(X)|$ is weak equivalent to X.

Since $\mathcal{K}(X) = \mathcal{K}(X^{op})$, if X is a minimal finite model of a space Y, then so is X^{op}.

Example 2.9. The 5-point T_0-space X, whose Hasse diagram is



has an associated polyhedron $|\mathcal{K}(X)|$, which is homotopy equivalent to $S^1 \vee S^1$. Therefore, X is a finite model of $S^1 \vee S^1$. In fact, it is a minimal finite model since every space with fewer than 5 points is either contractible, or non connected or weak equivalent to S^1. However, this minimal finite model is not unique since X^{op} is another minimal finite model not homeomorphic to X.

We will generalize this result later, when we characterize the minimal finite models of graphs.

Note that, by Whitehead Theorem, if X is a finite model of a (compact) polyhedron Y, then Y is homotopy equivalent to $|\mathcal{K}(X)|$.

Let X be a finite space. The *non-Hausdorff suspension* $\mathbb{S}X$ of X is the finite space $X \cup \{+, -\}$ whose open sets are those of X together with $X \cup \{+\}$, $X \cup \{-\}$ and $X \cup \{+, -\}$. The non-Hausdorff suspension of order n is defined recursively by $\mathbb{S}^n X = \mathbb{S}(\mathbb{S}^{n-1} X)$.

In [6], McCord proved that the $(2n+2)$-point space $\mathbb{S}^n S^0$ is a finite model of S^n. In [3] May conjectures that $\mathbb{S}^n S^0$ is a minimal finite model of the sphere. We will show that this conjecture is true. In fact, we prove a stronger result. Namely, we will see that any space with the same homotopy groups as S^n has at least $2n+2$ points. Moreover, if it has exactly $2n+2$ points then it has to be homeomorphic to $\mathbb{S}^n S^0$.

Before we proceed with the proof of the conjecture, we would like to make some remarks about Osaki's methods of reduction [7].
In [7] Osaki proves the following result.

Theorem 2.10. *(Osaki) Let X be a finite T_0-space. Suppose there exists $x \in X$ such that $U_x \cap U_y$ is either empty or homotopically trivial for all $y \in X$. Then the quotient map $p : X \to X/U_x$ is a weak homotopy equivalence.*

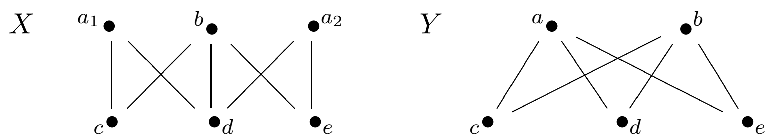
The process of obtaining X/U_x from X is called an *open reduction*. There is an analogous result for the *minimal closed sets* F_x, i.e. the closures of the one point spaces $\{x\}$.

Theorem 2.11. *(Osaki) Let X be a finite T_0-space. Suppose there exists $x \in X$ such that $F_x \cap F_y$ is either empty or homotopically trivial for all $y \in X$. Then the quotient map $p : X \to X/F_x$ is a weak homotopy equivalence.*

The process of obtaining X/F_x from X is called a *closed reduction*.
Osaki asserts in [7] that he does not know whether by a sequence of reductions, each finite T_0-space can be reduced to the smallest space with the same homotopy groups.

We show with the following example that the answer to this question is negative.

Let $X = \{a_1, b, a_2, c, d, e\}$ be the 6-point T_0-space with the following order: $c, d < a_1$; $c, d, e < b$ and $d, e < a_2$. Let $D_3 = \{c, d, e\}$ be the 3-point discrete space and $Y = \mathbb{S}D_3 = \{a, b, c, d, e\}$ the non-Haussdorf suspension of D_3.



The function $f : X \to Y$ defined by $f(a_1) = f(a_2) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$ and $f(e) = e$ is continuous because it preserves the order.
In order to prove that f is a weak homotopy equivalence we use Theorem 6 of [6].

The sets U_y form a basis-like cover of Y. Using the theory developed by Stong, it is easy to verify that $f^{-1}(U_y)$ is contractible for each $y \in Y$ and, since U_y is also contractible, the map $f|_{f^{-1}(U_y)} : f^{-1}(U_y) \to U_y$ is a weak homotopy equivalence for each $y \in Y$.
Applying Theorem 6 of [6], one proves that f is a weak homotopy equivalence. Therefore X and Y have the same homotopy groups.
Another way to show that X and Y are weak equivalent is considering the associated polyhedra $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ which are homotopy equivalent to $S^1 \vee S^1$.

On the other hand, it is easy to see that Osaki reduction methods cannot be applied to the space X. Therefore his methods are not effective in this case since we cannot obtain, by a sequence of reductions, the smallest space with the same homotopy groups as X.

In order to achieve our goal, we must then choose a different approach.

We denote by $h(X)$ the height of a poset X, i.e. the maximum length of a chain in X.

Theorem 2.12. *Let $X \neq *$ be a minimal finite space. Then X has at least $2h(X)$ points. Moreover, if X has exactly $2h(X)$ points, then it is homeomorphic to $\mathbb{S}^{h(X)-1}S^0$.*

Proof. Let $x_1 < x_2 < \ldots < x_h$ be a chain in X of length $h = h(X)$. Since X is a minimal finite space, x_i is not an up beat point for any $1 \leq i < h$. Then, for every $1 \leq i < h$ there exists $y_{i+1} \in X$ such that $y_{i+1} > x_i$ and $y_{i+1} \ngeq x_{i+1}$. We assert that the points y_i (for $1 < i \leq h$) are all distinct from each other and also different from the x_j ($1 \leq j \leq h$).

Since $y_{i+1} > x_i$, it follows that $y_{i+1} \neq x_j$ for all $j \leq i$. But $y_{i+1} \neq x_j$ for all $j > i$ because $y_{i+1} \ngeq x_{i+1}$.
If $y_{i+1} = y_{j+1}$ for some $i < j$, then $y_{i+1} = y_{j+1} \geq x_j \geq x_{i+1}$, which is a contradiction.

Since finite spaces with minimum or maximum are contractible and $X \neq *$ is a minimal finite space, it cannot have a minimum. Then there exists $y_1 \in X$ such that $y_1 \ngeq x_1$. Therefore, y_1 must be distinct from the other $2h-1$ points and $\#X \geq 2h$.

Let us suppose now that X has exactly $2h$ points, i.e.

$$X = \{x_1, x_2, \ldots, x_h, y_1, y_2, \ldots, y_h\}.$$

Because of the maximality of the chain $x_1 < \ldots < x_h$, we get that x_i and y_i are incomparable for all i.
We show that $y_i < x_j$ and $y_i < y_j$ for all $i < j$ by induction in j.

For $j = 1$ there is nothing to prove.
Let $1 \leq k < h$ and assume the statement holds for $j = k$. As x_{k+1} is not a down beat point, there exists $z \in X$ such that $z < x_{k+1}$, and $z \nleq x_k$. Since x_{k+1} and y_{k+1} are incomparable, it follows that $z \neq y_{k+1}$. By induction we know that every point in X, with the exception of y_k and y_{k+1}, is greater than x_{k+1} or less than x_k. Then $z = y_k$ and so, $y_k < x_{k+1}$.
Analogously, y_{k+1} is not a down beat point and there exists $w \in X$ such that $w < y_{k+1}$ and $w \nleq x_k$. Again by induction, and because $y_{k+1} \ngeq x_{k+1}$, we deduce that w must be y_k and then $y_k < y_{k+1}$.
Furthermore, if $i < k$, then $y_i < x_k < x_{k+1}$ and $y_i < x_k < y_{k+1}$.

We proved that, for any $i < j$, we have that $y_i < x_j$, $y_i < y_j$, $x_i < x_j$ and $x_i < y_j$. Moreover, for any $1 \leq i \leq h$, x_i and y_i are incomparable.
This is exactly the order of $\mathbb{S}^{h-1}S^0$. Therefore X is homeomorphic to $\mathbb{S}^{h-1}S^0$. □

Theorem 2.13. *Any space with the same homotopy groups as S^n has at least $2n+2$ points. Moreover, $\mathbb{S}^n S^0$ is the unique space with $2n+2$ points with this property.*

Proof. The case $n = 1$ is trivial. In the other cases, let us suppose that X is a finite space with minimum cardinality such that $\pi_k(X, x) = \pi_k(S^n, s)$ for all $k \geq 0$. Then X must be a minimal finite space and so is T_0.
By the Hurewicz Theorem, $H_n(|\mathcal{K}(X)|) = \pi_n(|\mathcal{K}(X)|) = \pi_n(S^n) \neq 0$. This implies that the dimension of the simplicial complex $\mathcal{K}(X)$ must be at least n, which means that the height of X is at least $n+1$.
The result now follows immediately from the previous theorem. □

Corollary 2.14. *The n-sphere has a unique minimal finite model and it has $2n+2$ points.*

Remark 2.15. After concluding this paper, we found an old article of McCord (*Singular homology and homotopy groups of finite spaces*, Notices of the American Mathematical Society, vol. 12(1965)) with a result (Theorem 2) without proof, from which the first part of 2.13 could be deduced. McCord's result can be easily deduced from our stronger theorem 2.12 (which also implies the uniqueness of these minimal models).

Furthermore, we think that the proof of 2.12 itself is interesting because it relates the combinatorial methods of Stong's theory with McCord's point of view.

3 Loops in the Hasse diagram and the fundamental group

In this section we give a full description of the fundamental group of a finite T_0-space in terms of its Hasse diagram. This characterization is induced from the well known description of the fundamental group of a simplicial complex [8].

Definition 3.1. Let (X, x_0) be a finite pointed T_0-space. An ordered pair of points $e = (x, y)$ is called an *$\mathcal{H}$-edge of X* if $(x, y) \in \mathtt{E}(\mathcal{H}(X))$ or $(y, x) \in \mathtt{E}(\mathcal{H}(X))$. The point x is called the *origin* of e and denoted $x = \mathfrak{o}(e)$, the point y is called the *end* of e and denoted $y = \mathfrak{e}(e)$. The *inverse* of an $\mathcal{H}$-edge $e = (x, y)$ is the $\mathcal{H}$-edge $e^{-1} = (y, x)$.
An *$\mathcal{H}$-path* in (X, x_0) is a finite sequence (possibly empty) of $\mathcal{H}$-edges $\xi = e_1e_2 \ldots e_n$ such that $\mathfrak{e}(e_i) = \mathfrak{o}(e_{i+1})$ for all $1 \leq i \leq n-1$. The *origin* of a non empty $\mathcal{H}$-path ξ is $\mathfrak{o}(\xi) = \mathfrak{o}(e_1)$ and its *end* is $\mathfrak{e}(\xi) = \mathfrak{e}(e_n)$. The origin and the end of the empty $\mathcal{H}$-path is $\mathfrak{o}(\emptyset) = \mathfrak{e}(\emptyset) = x_0$. If $\xi = e_1e_2 \ldots e_n$, we define $\overline{\xi} = e_n^{-1}e_{n-1}^{-1} \ldots e_1^{-1}$. If ξ, ξ' are $\mathcal{H}$-paths such that $\mathfrak{e}(\xi) = \mathfrak{o}(\xi')$, we define the product $\mathcal{H}$-path $\xi\xi'$ as the concatenation of the sequence ξ followed by the sequence ξ'.
An $\mathcal{H}$-path $\xi = e_1e_2 \ldots e_n$ is said to be *monotonic* if $e_i \in \mathtt{E}(\mathcal{H}(X))$ for all $1 \leq i \leq n$ or $e_i^{-1} \in \mathtt{E}(\mathcal{H}(X))$ for all $1 \leq i \leq n$.
A *loop at* x_0 is an $\mathcal{H}$-path that starts and ends in x_0. Given two loops ξ, ξ' at x_0, we say that they are *close* if there exist $\mathcal{H}$-paths $\xi_1, \xi_2, \xi_3, \xi_4$ such that ξ_2 and ξ_3 are monotonic and the set $\{\xi, \xi'\}$ coincides with $\{\xi_1\xi_2\xi_3\xi_4, \xi_1\xi_4\}$.
We say that two loops ξ, ξ' at x_0 are *$\mathcal{H}$-equivalent* if there exist a finite sequence of loops $\xi = \xi_1, \xi_2, \ldots, \xi_n = \xi'$ such that any two consecutive are close. We denote by $\langle \xi \rangle$ the $\mathcal{H}$-equivalence class of a loop ξ and $\mathscr{H}(X, x_0)$ the set of these classes.

Theorem 3.2. *Let (X, x_0) be a pointed finite T_0-space. Then the product $\langle \xi \rangle \langle \xi' \rangle = \langle \xi\xi' \rangle$ is well defined and induces a group structure on $\mathscr{H}(X, x_0)$.*

Proof. It is easy to check that the product is well defined, associative and that $\langle \emptyset \rangle$ is the identity. In order to prove that the inverse of $\langle e_1e_2 \ldots e_n \rangle$ is $\langle e_n^{-1}e_{n-1}^{-1} \ldots e_1^{-1} \rangle$ we need to show that for any composable $\mathcal{H}$-paths ξ, ξ' such that $\mathfrak{o}(\xi) = \mathfrak{e}(\xi') = x_0$ and for any $\mathcal{H}$-edge e, composable with ξ, one has that $\langle \xi e e^{-1} \xi' \rangle = \langle \xi\xi' \rangle$. But this follows immediately from the definition of close loops since e and e^{-1} are monotonic. □

Theorem 3.3. *Let (X, x_0) be a pointed finite T_0-space. Then the edge-path group $E(\mathcal{K}(X), x_0)$ of $\mathcal{K}(X)$ with base vertex x_0 is isomorphic to $\mathscr{H}(X, x_0)$.*

Proof. Let us define

$$\begin{aligned} \varphi : \mathscr{H}(X, x_0) &\longrightarrow E(\mathcal{K}(X), x_0), \\ \langle e_1e_2 \ldots e_n \rangle &\longmapsto [e_1e_2 \ldots e_n], \\ \langle \emptyset \rangle &\longmapsto [(x_0, x_0)], \end{aligned}$$

where $[\xi]$ denotes the class of ξ in $E(\mathcal{K}(X), x_0)$.

To prove that φ is well defined, let us suppose that the loops $\xi_1\xi_2\xi_3\xi_4$ and $\xi_1\xi_4$ are close, where $\xi_2 = e_1e_2\dots e_n$, $\xi_3 = e'_1e'_2\dots e'_m$ are monotonic $\mathcal{H}$-paths. By induction, it can be proved that $[\xi_1\xi_2\xi_3\xi_4] = [\xi_1e_1e_2\dots e_{n-j}(\mathfrak{o}(e_{n-j+1}),\mathfrak{e}(e_n))\xi_3\xi_4]$ for $1\leq j\leq n$. In particular $[\xi_1\xi_2\xi_3\xi_4] = [\xi_1(\mathfrak{e}(\xi_1),\mathfrak{e}(e_n))\xi_3\xi_4]$.

Analogously,

$$[\xi_1(\mathfrak{e}(\xi_1),\mathfrak{e}(e_n))\xi_3\xi_4] = [\xi_1(\mathfrak{e}(\xi_1),\mathfrak{e}(e_n))(\mathfrak{o}(e'_1),\mathfrak{o}(\xi_4))\xi_4]$$

and then

$$[\xi_1\xi_2\xi_3\xi_4] = [\xi_1(\mathfrak{e}(\xi_1),\mathfrak{e}(e_n))(\mathfrak{o}(e'_1),\mathfrak{o}(\xi_4))\xi_4] = [\xi_1(\mathfrak{e}(\xi_1),\mathfrak{e}(e_n))(\mathfrak{e}(e_n),\mathfrak{e}(\xi_1))\xi_4] =$$

$$= [\xi_1(\mathfrak{e}(\xi_1),\mathfrak{e}(\xi_1))\xi_4] = [\xi_1\xi_4].$$

If $\xi = (x_0,x_1)(x_1,x_2)\dots(x_{n-1},x_n)$ is an edge path in $\mathcal{K}(X)$ with $x_n = x_0$, then x_{i-1} and x_i are comparable for all $1\leq i\leq n$. In this case, we can find monotonic $\mathcal{H}$-paths $\xi_1,\xi_2,\dots,\xi_n$ such that $\mathfrak{o}(\xi_i) = x_{i-1}$, $\mathfrak{e}(\xi_i) = x_i$ for all $1\leq i\leq n$. Let us define

$$\begin{aligned}\psi : E(\mathcal{K}(X),x_0) &\longrightarrow \mathscr{H}(X,x_0),\\ [\xi] &\longmapsto \langle\xi_1\xi_2\dots\xi_n\rangle.\end{aligned}$$

This definition does not depend on the choice of the $\mathcal{H}$-paths ξ_i since if two choices differ only for $i=k$ then $\xi_1\dots\xi_k\dots\xi_n$ and $\xi_1\dots\xi'_k\dots\xi_n$ are $\mathcal{H}$-equivalent because both of them are close to $\xi_1\dots\xi_k\xi_k^{-1}\xi'_k\dots\xi_n$.

The definition of ψ does not depend on the representative. Suppose that $\xi'(x,y)(y,z)\xi''$ and $\xi'(x,z)\xi''$ are simply equivalent edge paths in $\mathcal{K}(X)$ that start and end in x_0, where ξ and ξ' are edge paths and x,y,z are comparable.
In the case that y lies between x and z, we can choose the monotonic $\mathcal{H}$-path corresponding to (x,z) to be the juxtaposition of the corresponding to (x,y) and (y,z), and so ψ is equally defined in both edge paths.
In the case that $z\leq x\leq y$ we can choose monotonic $\mathcal{H}$-paths α, β from x to y and from z to x, and then α will be the corresponding $\mathcal{H}$-path to (x,y), $\overline{\alpha}\overline{\beta}$ that corresponding to (y,z) and $\overline{\beta}$ to (x,z). It only remains to prove that $\langle\gamma'\alpha\overline{\alpha}\overline{\beta}\gamma''\rangle = \langle\gamma'\overline{\beta}\gamma''\rangle$ for $\mathcal{H}$-paths γ' and γ'', which is trivial.
The other cases are analogous to the last one.

It remains to verify that φ and ψ are mutually inverses, but this is clear. □

Since $E(\mathcal{K}(X),x_0)$ is isomorphic to $\pi_1(|\mathcal{K}(X)|,x_0)$ (cf. [8]), we obtain the following result.

Corollary 3.4. *Let (X,x_0) be a pointed finite T_0-space, then $\mathscr{H}(X,x_0) = \pi_1(X,x_0)$.*

Remark 3.5. Since every finite space is homotopy equivalent to a finite T_0-space, this computation of the fundamental group can be applied to any finite space.

We finish this section with a couple of remarks on the Euler characteristic of finite spaces.

Since any finite T_0-space X is weak equivalent to the realization of $\mathcal{K}(X)$, whose simplices are the non empty chains in X, the Euler characteristic of X is

$$\chi(X) = \sum_{C \in \mathcal{C}(X)} (-1)^{\#C+1}$$

where $\mathcal{C}(X)$ is the set of non empty chains of X.

Although it is very well known that the Euler characteristic is a homotopy invariant, we exhibit a basic proof of this fact in the case of finite spaces:

Theorem 3.6. *Let X and Y be finite T_0-spaces with the same homotopy type. Then $\chi(X) = \chi(Y)$.*

Proof. Following [9], there exist two sequences of finite T_0-spaces $X = X_0 \supseteq \ldots \supseteq X_n = X_c$ and $Y = Y_0 \supseteq \ldots \supseteq Y_m = Y_c$, where X_{i+1} is constructed from X_i by removing a beat point and Y_{i+1} is constructed from Y_i, similarly.
Since X and Y are homotopy equivalent, X_c and Y_c are homeomorphic. Thus, $\chi(X_c) = \chi(Y_c)$.
It suffices to show that the Euler characteristic does not change when a beat point is removed.
Let P be a finite poset and let $p \in P$ be a beat point. Then there exists $q \in P$ such that r comparable with p implies r comparable with q.
Hence we have a bijection

$$\begin{aligned} \varphi : \{C \in \mathcal{C}P \mid p \in C,\ q \notin C\} &\longrightarrow \{C \in \mathcal{C}P \mid p \in C,\ q \in C\}, \\ C &\longmapsto C \cup \{q\}. \end{aligned}$$

Therefore

$$\chi(P) - \chi(P \smallsetminus \{p\}) = \sum_{p \in C \in \mathcal{C}P} (-1)^{\#C+1} = \sum_{q \notin C \ni p} (-1)^{\#C+1} + \sum_{q \in C \ni p} (-1)^{\#C+1} =$$

$$= \sum_{q \notin C \ni p} (-1)^{\#C+1} + \sum_{q \notin C \ni p} (-1)^{\#\varphi(C)+1} = \sum_{q \notin C \ni p} (-1)^{\#C+1} + \sum_{q \notin C \ni p} (-1)^{\#C} = 0.$$

□

4 Minimal finite models of graphs

Remark 4.1. If X is a connected finite T_0-space of height two, $|\mathcal{K}(X)|$ is a connected graph, i.e. a CW complex of dimension one. Therefore, the weak homotopy type of X is completely determined by its Euler characteristic. More precisely, if $\chi(X) = \#X - \#\mathtt{E}(\mathcal{H}(X)) = n$, then X is a finite model of $\bigvee_{i=1}^{1-n} S^1$.

Proposition 4.2. *Let X be a connected finite T_0-space and let $x_0, x \in X$, $x_0 \neq x$ such that x is neither maximal nor minimal in X. Then the inclusion map of the associated simplicial complexes $\mathcal{K}(X \smallsetminus \{x\}) \subseteq \mathcal{K}(X)$ induces an epimorphism*

$$i_* : E(\mathcal{K}(X \smallsetminus \{x\}), x_0) \to E(\mathcal{K}(X), x_0)$$

between their edge-path groups.

Proof. We have to check that every closed edge path in $\mathcal{K}(X)$ with base point x_0 is equivalent to another edge path that does not go through x.
Let us suppose that $y \leq x$ and $(y,x)(x,z)$ is an edge path in $\mathcal{K}(X)$.
If $x \leq z$ then $(y,x)(x,z) \equiv (y,z)$. In the case that $z < x$, since x is not maximal in X, there exists $w > x$. Therefore $(y,x)(x,z) \equiv (y,x)(x,w)(w,x)(x,z) \equiv (y,w)(w,z)$.
The case $y \geq x$ is analogous.
In this way, one can eliminate x from the writing of any closed edge path with base point x_0. □

Note that the space $X \smallsetminus \{x\}$ of the previous proposition is also connected.

The result above shows one of the advantages of using finite spaces instead of simplicial complexes. The conditions of maximality or minimality of points in a finite space are hard to express in terms of simplicial complexes.

Remark 4.3. If X is a finite T_0-space, then $h(X) \leq 2$ if and only if every point in X is maximal or minimal.

Corollary 4.4. *Let X be a connected finite space. Then there exists a connected T_0-subspace $Y \subseteq X$ of height at most two such that the fundamental group of X is a quotient of the fundamental group of Y.*

Proof. We can assume that X is T_0 because X has a core. Since the edge-path group is isomorphic to the fundamental group, the result follows immediately from the previous proposition. □

Remark 4.5. Note that the fundamental group of a connected finite T_0-space of height at most two is finitely generated by 4.1. Therefore, path-connected spaces whose fundamental group does not have a finite set of generators do not admit finite models.

Corollary 4.6. *Let $n \in \mathbb{N}$. If X is a minimal finite model of $\bigvee\limits_{i=1}^{n} S^1$, then $h(X) = 2$.*

Proof. Let X be a minimal finite model of $\bigvee\limits_{i=1}^{n} S^1$. Then there exists a connected T_0-subspace $Y \subseteq X$ of height two, $x \in Y$ and an epimorphism from $\pi_1(Y,x)$ to $\pi_1(X,x) = \mathop{*}\limits_{i=1}^{n} \mathbb{Z}$.
Since $h(Y) = 2$, Y is a model of a graph, thus $\pi_1(Y,x) = \mathop{*}\limits_{i=1}^{m} \mathbb{Z}$ for some integer m.
Note that $m \geq n$.

There are m edges of $\mathcal{H}(Y)$ which are not in a maximal tree of the underlying non directed graph of $\mathcal{H}(Y)$ (i.e. $\mathcal{K}(Y)$). Therefore, we can remove $m - n$ edges from $\mathcal{H}(Y)$ in such a

way that it remains connected and the new space Z obtained in this way is a model of $\bigvee_{i=1}^{n} S^1$.
Note that $\#Z = \#Y \leq \#X$, but since X is a minimal finite model, $\#X \leq \#Z$ and then $X = Y$ has height two. □

If X is a minimal finite model of $\bigvee_{i=1}^{n} S^1$ and we call $i = \#\{y \in X \mid y \text{ is maximal}\}$, $j = \#\{y \in X \mid y \text{ is minimal}\}$, then $\#X = i + j$ and $\#\mathtt{E}(\mathcal{H}(X)) \leq ij$. Since $\chi(X) = 1 - n$, we have that $n \leq ij - (i + j) + 1 = (i - 1)(j - 1)$.

We can now state the main result of this section.

Theorem 4.7. *Let $n \in \mathbb{N}$. A finite T_0-space X is a minimal finite model of $\bigvee_{i=1}^{n} S^1$ if and only if $h(X) = 2$, $\#X = min\{i + j \mid (i - 1)(j - 1) \geq n\}$ and $\#\mathtt{E}(\mathcal{H}(X)) = \#X + n - 1$.*

Proof. We have already proved that if X is a minimal finite model of $\bigvee_{i=1}^{n} S^1$, then $h(X) = 2$ and $\#X \geq min\{i + j \mid (i - 1)(j - 1) \geq n\}$.
If i and j are such that $n \leq (i-1)(j-1)$, we can consider $Y = \{x_1, x_2, \dots, x_i, y_1, y_2, \dots y_j\}$ with the order $y_k \leq x_l$ for all k, l, which is a model of $\bigvee_{k=1}^{(i-1)(j-1)} S^1$. Then we can remove $(i - 1)(j - 1) - n$ edges from $\mathcal{H}(X)$ to obtain a connected space of cardinality $i + j$ which is a finite model of $\bigvee_{k=1}^{n} S^1$. Therefore $\#X \leq \#Y = i + j$.
This is true for any i, j with $n \leq (i-1)(j-1)$, then $\#X = min\{i+j \mid (i-1)(j-1) \geq n\}$.
Moreover, $\#\mathtt{E}(\mathcal{H}(X)) = \#X + n - 1$ because $\chi(X) = 1 - n$.

In order to show the converse of the theorem we only need to prove that the conditions $h(X) = 2$, $\#X = min\{i + j \mid (i - 1)(j - 1) \geq n\}$ and $\#\mathtt{E}(\mathcal{H}(X)) = \#X + n - 1$ imply that X is connected, because in this case, by 4.1, the first and third conditions would say that X is a model of $\bigvee_{i=1}^{n} S^1$, and the second condition would say that it has the right cardinality.

Suppose X satisfies the conditions of above and let X_l, $1 \leq l \leq k$, be the connected components of X.
Let us denote by M_l the set of maximal elements of X_l and let $m_l = X_l \smallsetminus M_l$. Let $i = \sum_{r=1}^{k} \#M_l$, $j = \sum_{r=1}^{k} \#m_l$.
Since $i + j = \#X = min\{s + t \mid (s - 1)(t - 1) \geq n\}$, it follows that $(i - 2)(j - 1) < n = \#\mathtt{E}(\mathcal{H}(X)) - \#X + 1 = \#\mathtt{E}(\mathcal{H}(X)) - (i + j) + 1$. Hence $ij - \#\mathtt{E}(\mathcal{H}(X)) < j - 1$.

This means that $\mathcal{K}(X)$ differs from the complete bipartite graph $(\cup m_l, \cup M_l)$ in less than $j - 1$ edges.

Since there are no edges from m_r to M_l if $r \neq l$,

$$j - 1 > \sum_{l=1}^{k} \#M_l(j - \#m_l) \geq \sum_{l=1}^{k} (j - \#m_l) = (k-1)j.$$

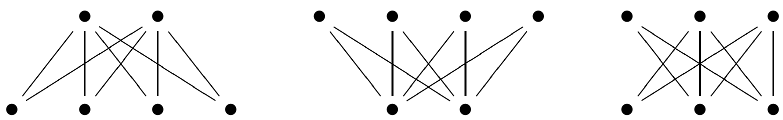
Therefore $k = 1$ and the proof is complete. □

Remark 4.8. The cardinality of a minimal finite model of $\bigvee_{i=1}^{n} S^1$ is

$$min\{2\lceil\sqrt{n}+1\rceil, 2\left\lceil\frac{1+\sqrt{1+4n}}{2}\right\rceil + 1\}.$$

Note that a space may admit many minimal finite models as we can see in the following example.

Example 4.9. Any minimal finite model of $\bigvee_{i=1}^{3} S^1$ has 6 points and 8 edges. So, they are, up to homeomorphism



In fact, it is not hard to prove, using our characterization, that $\bigvee_{i=1}^{n} S^1$ has a unique minimal finite model if and only if n is a square.

Note that since any graph is a $K(G,1)$, the minimal finite models of a graph X are, in fact, the smallest spaces with the same homotopy groups as X.

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