## ON A LOWER BOUND FOR THE CONNECTIVITY OF THE INDEPENDENCE COMPLEX OF A GRAPH

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ABSTRACT. Aharoni, Berger and Ziv proposed a function which is a lower bound for the connectivity of the independence complex of a graph. They conjectured that this bound is optimal for every graph. We give two different arguments which show that the conjecture is false

Given a finite simple graph G, its independence complex  $I_G$  is defined as the simplicial complex whose vertices are the vertices of G and whose simplices are the independent sets of G. The topology of independence complexes has been studied by a number of authors. In particular, the connectivity of independence complexes has shown to be of interest in the study of Tverberg graphs [5, Theorem 2.2], independent systems of representatives [3, Theorem 2.1] and other important problems.

In [3], Aharoni, Berger and Ziv proposed a function  $\psi$  defined on graphs which is a lower bound for the connectivity of  $I_G$  and conjectured that this bound is optimal. No explicit proof of this bound is given in that article, although the corresponding bound for the homological connectivity follows immediately from a result of Meshulam [9, Claim 3.1]. Moreover, a homological version of the conjecture has been considered, as well as reformulations taking into account the existence of counterexamples in which the independence complex is simply-connected or not [2].

In this note we give an explicit proof of the fact that  $\psi(G)$  is a lower bound for the connectivity of  $I_G$ , we prove that the conjecture is true in the cases where  $I_G$  is not simply-connected or where  $\psi(G) \leq 1$ , we show that there exist counterexamples to the conjecture with  $\psi(G) = 2$ , and that there are counterexamples in which  $\psi(G)$  and the connectivity of  $I_G$  take arbitrary values l, k with  $1 \leq l < k$ .

The connectivity  $\operatorname{conn}(X)$  of a topological space X is usually defined as follows:  $\operatorname{conn}(\emptyset) = -2$ ,  $\operatorname{conn}(X) = k$  if  $\pi_i(X) = 0$  for every  $0 \le i \le k$  and  $\pi_{k+1}(X) \ne 0$ , and  $\operatorname{conn}(X) = \infty$  if  $\pi_i(X) = 0$  for every  $i \ge 0$ . The homological connectivity  $\operatorname{conn}_H(X)$  is defined in the same way replacing the homotopy groups  $\pi_i(X)$  by the reduced homology groups with integer coefficients  $\tilde{H}_i(X)$ . In this context, however, in order to keep the notation of [3], we will use the shifted versions

$$\eta(X) = \text{conn}(X) + 2, \quad \eta_H(X) = \text{conn}_H(X) + 2.$$

Date: June 1, 2011.

<sup>2000</sup> Mathematics Subject Classification. 05C69, 55U10.

Key words and phrases. Independence complex, topological connectivity.

<sup>&</sup>lt;sup>†</sup> Supported by the Centre for Discrete Mathematics and its Applications (DIMAP), EPSRC award EP/D063191/1.

<sup>&</sup>lt;sup>‡</sup> Supported by grant KAW 2005.0098 from the Knut and Alice Wallenberg Foundation.

With this notation, X is non-empty when  $\eta(X) \geq 1$ , path-connected if  $\eta(X) \geq 2$  and simply-connected when  $\eta(X) \geq 3$ . By the Hurewicz theorem, connectivity and homological connectivity coincide for simply-connected spaces, while in general  $\eta(X) \leq \eta_H(X)$ .

All the graphs considered in this note are finite and simple (undirected, loopless and without multiple edges). If e is an edge of a graph G, G-e denotes the subgraph obtained by removing the edge e and  $G \setminus e$  is the subgraph obtained by removing the endpoints of e and all neighbours of each of those endpoints. We denote by E(G) the set of edges of G.

Consider the function  $\psi$  defined for all finite simple graphs G with values in  $\{0, 1, \dots, \infty\}$ , as follows

$$\psi(G) = \begin{cases} 0 & \text{if } G = \emptyset \\ \infty & \text{if } G \neq \emptyset \text{ is discrete} \\ \max_{e \in E(G)} \{ \min\{\psi(G - e), \psi(G \setminus e) + 1 \} \} & \text{otherwise} \end{cases}$$

The join K\*L of two simplicial complexes K and L is the simplicial complex with simplices  $\sigma \sqcup \tau$  for  $\sigma \in K$  and  $\tau \in L$ . The (unreduced) suspension  $\Sigma K$  is the join of K with a 0-dimensional complex of two vertices.

If e is an edge of a graph G, we also consider e as a 1-dimensional simplicial complex and by  $\dot{e}$  we denote the 0-dimensional simplicial complex whose vertices are the endpoints of e. Meshulam [9] observed that  $I_{G-e} = I_G \cup (e * I_{G \setminus e})$  and that  $I_G \cap (e * I_{G \setminus e}) = \dot{e} * I_{G \setminus e} = \Sigma I_{G \setminus e}$ .

**Theorem 1.** For any graph G,  $\psi(G) \leq \eta(I_G)$ .

Proof. We prove first that  $\psi(G) \leq \eta_H(I_G)$ . This part of the proof is implicit in [3]. The inequality is trivial for discrete graphs. Assume then that G is non-discrete and let  $e \in E(G)$  be such that  $\psi(G) = \min\{\psi(G-e), \psi(G \setminus e) + 1\}$ . By induction  $\psi(G-e) \leq \eta_H(I_{G-e}) = \operatorname{conn}_H(I_{G-e}) + 2$  and  $\psi(G \setminus e) \leq \eta_H(I_{G\setminus e}) = \operatorname{conn}_H(I_{G\setminus e}) + 2$ , and therefore  $\tilde{H}_i(I_{G-e}) = 0$  for every  $0 \leq i \leq \psi(G) - 2$  and  $\tilde{H}_i(I_{G\setminus e}) = 0$  for every  $0 \leq i \leq \psi(G) - 3$ .

Following [9], since  $\tilde{H}_i(\dot{e}*I_{G\backslash e}) = \tilde{H}_{i-1}(I_{G\backslash e})$  and since  $e*I_{G\backslash e}$  is contractible, the Mayer-Vietoris sequence for the triple  $(I_{G-e}; I_G, e*I_{G\backslash e})$  gives a long exact sequence

$$\ldots \to \tilde{H}_{i-1}(I_{G \setminus e}) \to \tilde{H}_i(I_G) \to \tilde{H}_i(I_{G-e}) \to \tilde{H}_{i-2}(I_{G \setminus e}) \to \ldots$$

We deduce then that  $\tilde{H}_i(I_G) = 0$  for every  $0 \le i \le \psi(G) - 2$  or, in other words, that  $\psi(G) \le \eta_H(I_G)$ .

To prove the theorem it suffices to show that the condition  $\psi(G) \geq 3$  implies that  $I_G$  is simply-connected. If G is discrete,  $I_G$  is a simplex. Otherwise, by definition of  $\psi$ , there exists an edge e such that

$$\psi(G-e) \ge 3$$
 and  $\psi(G \setminus e) \ge 2$ .

By induction  $I_{G-e}$  is simply-connected and since  $\eta_H(I_{G\backslash e}) \geq \psi(G \backslash e) \geq 2$ ,  $I_{G\backslash e}$  is connected. The suspension  $\dot{e} * I_{G\backslash e}$  is then simply-connected and by van Kampen's theorem  $\pi_1(I_{G-e})$  is the free product of  $\pi_1(I_G)$  and  $\pi_1(e*I_{G\backslash e})$ . Since  $e*I_{G\backslash e}$  is contractible,  $\pi_1(I_G) = \pi_1(I_{G-e}) = 0$ .

In [3, Conjecture 2.4] it was conjectured that  $\psi(G) = \eta(I_G)$ . This has been confirmed for some classes of graphs, e.g. chordal graphs [8], but, as we will show, it is not true in general. In view of Theorem 1 it is clear that the homological version of the conjecture, i.e. the equation  $\psi(G) = \eta_H(I_G)$ , does not hold in general since  $\eta_H(I_G)$  can be strictly greater than  $\eta(I_G)$ . This follows from the existence of a finite connected complex K with non-trivial fundamental group but such that  $H_1(K) = 0$  and the well-known fact that for every finite simplicial complex K

there is a graph G with  $I_G$  homeomorphic to K, for instance the complement graph of the 1-skeleton of the barycentric subdivision of K.

Proposition 2. Let G be a graph.

- a) If  $\psi(G) \in \{0, 1\}$ , then  $\psi(G) = \eta(I_G)$ .
- b) If  $I_G$  is not simply-connected, then  $\psi(G) = \eta(I_G)$ .

*Proof.* It is easy to see that  $\psi(G) = 0$  if and only if G is empty, so the only non-trivial case of a) is  $\psi(G) = 1$ .

Since the 1-skeleton of  $I_G$  is the complement  $\overline{G}$  of G, we have that  $\eta(I_G) = 1$  if and only if  $\overline{G}$  is disconnected. We will prove, by induction on the number of edges in G, that if  $\psi(G) = 1$  then  $\overline{G}$  is disconnected. By definition of  $\psi$ , G is non-discrete and for every edge e of G we have

$$\psi(G-e)=1 \text{ or } G \setminus e \text{ is empty.}$$

If there exists an edge  $e \in G$  such that  $\psi(G-e) = 1$  then, by induction,  $\overline{G-e}$  is disconnected and therefore so is  $\overline{G}$ . It suffices then to consider the case when for every edge  $e \in G$  the graph  $G \setminus e$  is empty. Translating this into a statement about complements we see that  $\overline{G}$  has the following property:

for every pair of non-adjacent vertices x, y we have  $N(x) \cap N(y) = \emptyset$ ,

where N(v) is the neighbourhood of v. It is easy to see that this property characterizes precisely the graphs in which every connected component is a clique. Since  $\overline{G}$  is not a clique itself, it must be disconnected, as we wanted to show.

To prove b) note that if  $I_G$  is not simply-connected, then  $\psi(G) \leq \eta(I_G) \leq 2$  by Theorem 1, and the result follows from part a).

We now prove that the conjecture is not true. The first argument we show is not constructive and reduces to the fact that it is algorithmically undecidable whether  $\eta(I_G) \geq 3$  or  $\eta(I_G) \leq 2$  for a given graph G, while  $\psi(G)$  is a computable function of G.

**Proposition 3.** There exists a graph G with  $\psi(G) = 2$  and  $\eta(I_G) \geq 3$ .

*Proof.* The truth of the implication

if 
$$\psi(G) = 2$$
 then  $\eta(I_G) = 2$ 

together with Theorem 1 and Proposition 2 would provide an algorithm (Turing machine) capable of determining if a given finite simplicial complex K is simply-connected. The algorithm would just find a graph G with  $I_G$  homeomorphic to K and check if  $\psi(G) \geq 3$ . However it is known that there can be no such algorithm. It is a consequence of the non-existence of an algorithm to determine whether a group  $\Gamma$  given by a finite presentation is trivial or not [1, 10] and a construction that associates to each presentation of  $\Gamma$  a finite 2-dimensional complex with fundamental group isomorphic to  $\Gamma$  (see [6] for example).

We will give more explicit counterexamples to the conjecture, all of them different from the one shown in Proposition 3. Their construction requires the next observation in which  $G \sqcup H$  denotes the disjoint union of graphs G and H.

**Lemma 4.** For any graphs G and H we have  $\psi(G \sqcup H) = \psi(G) + \psi(H)$ .

*Proof.* The result holds when both G and H are discrete. The general case now follows by induction on the number of edges in  $G \sqcup H$ . For every  $e \in E(G)$  we have  $(G \sqcup H) - e = (G - e) \sqcup H$  and  $(G \sqcup H) \setminus e = (G \setminus e) \sqcup H$ . If G is non-discrete, then by induction

$$\max_{e \in E(G)} \left\{ \min\{\psi((G \sqcup H) - e), \psi((G \sqcup H) \setminus e) + 1\} \right\} =$$

$$= \max_{e \in E(G)} \left\{ \min\{\psi((G - e) \sqcup H), \psi((G \setminus e) \sqcup H) + 1\} \right\} =$$

$$= \max_{e \in E(G)} \left\{ \min\{\psi(G - e), \psi(G \setminus e) + 1\} \right\} + \psi(H) =$$

$$= \psi(G) + \psi(H).$$

The same equation holds if H is non-discrete and the maximum is taken over the edges  $e \in E(H)$ . Then the result follows.

The lemma also follows immediately from the interpretation of  $\psi(G)$  as the maximal value achievable in a certain two-player game (see [3, p.257]).

Note that for any graphs G and H we have  $I_{G \sqcup H} = I_G * I_H$ . In particular, if H = e is just an edge,  $I_{G \sqcup e} = \Sigma I_G$ . Note also that  $\psi(e) = 1$ . Recall that for complexes K and L, the suspension  $\Sigma(K \vee L)$  of the wedge between K and L is homotopy equivalent to the wedge  $\Sigma K \vee \Sigma L$ .

**Proposition 5.** For any  $l, k \in \{3, 4, ..., \infty\}$  with  $l \leq k$  there exists a graph G such that  $\psi(G) = l$  and  $\eta(I_G) = k$ .

*Proof.* The case  $l=\infty$  is trivial. Assume then that l is finite. Note that if G is such that  $\psi(G)=l$  and  $\eta(I_G)=k\geq 3$ , then  $\psi(G\sqcup e)=\psi(G)+\psi(e)=l+1$  by Lemma 4, and  $\eta(I_{G\sqcup e})=\eta(\Sigma I_G)=\eta_H(\Sigma I_G)=\eta_H(I_G)+1=\eta(I_G)+1=k+1$ . Therefore, it suffices to prove the case l=3.

Let K be an acyclic finite simplicial complex with non-trivial fundamental group, i.e. with the properties

$$\pi_1(K) \neq 0$$
,  $\tilde{H}_i(K) = 0$  for all  $i$ .

(Such K can be obtained for example by triangulating the two-dimensional CW-complex of [7, Example 2.38]). Note that the suspension  $\Sigma K$  is simply-connected and acyclic, hence contractible.

Assume first that k is finite. Since every finite simplicial complex can be realized, up to homeomorphism, as an independence complex of some graph, we can choose a graph H such that we have a homeomorphism

$$I_H \cong K \vee S^{k-2}$$
.

Since  $\eta(K \vee S^{k-2}) = 2$ , we have  $\psi(H) = 2$  by Proposition 2.

Let  $G = H \sqcup e$ . Then  $I_G = \Sigma I_H$  is homotopy equivalent to  $\Sigma K \vee S^{k-1}$ , which in turn is homotopy equivalent to  $S^{k-1}$  since  $\Sigma K$  is contractible. It follows that  $\eta(I_G) = k$ . On the other hand  $\psi(G) = \psi(H) + \psi(e) = 3$  by Lemma 4. Therefore G has the desired property.

For the remaining case  $l=3, k=\infty$ , we consider a graph H such that  $I_H \cong K$  and define  $G=H\sqcup e$ . Then  $I_G\cong \Sigma K$  is contractible and  $\psi(G)=3$ .

Still, the study of the conjecture in special cases and for particular classes of graphs is an interesting problem and the bound provided by Theorem 1 can be useful even when it is not sharp.

**Acknowledgment.** We are grateful to the referee of the first version of the manuscript for his comments and suggestions.

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