

COP AND ROBBER ON FINITE SPACES

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ABSTRACT. A cop tries to capture a robber in a topological space X being unable to see him. For which spaces X does the cop have a strategy which allows him to capture the robber independently of his efforts to escape? In other words, when is there a curve $\gamma : \mathbb{R}_{\geq 0} \rightarrow X$ which has a coincidence with any other curve in X . We analyze in particular the case of finite topological spaces and discover general results and exotic examples about paths in these spaces.

1. INTRODUCTION

Let X be a topological space. By a curve in X we mean a continuous map $\gamma : \mathbb{R}_{\geq 0} \rightarrow X$. We say that the cop has a strategy in X if there exists a curve γ in X which has a coincidence with any other curve ρ in X , that is a point $t \in \mathbb{R}_{\geq 0}$, which depends on ρ , such that $\gamma(t) = \rho(t)$. Otherwise we say that the robber has a strategy in X . For instance, it is easy to see that the cop has a strategy in the interval $[0, 1] \subseteq \mathbb{R}$. He just moves from 0 to 1, and then stays still. Also, the existence of a strategy for the cop requires the space to be path-connected and to have the fixed point property: if $f : X \rightarrow X$ is fixed point free, then $f\gamma$ is a curve for the robber without coincidences with γ . Note also that a strategy for the cop is necessarily surjective.

Different variants of this problem have been studied: (a) If the cop can see the robber at any time, he can change his path depending on the robber moves (perfect information). (b) There could be more than one pursuer or evader. (c) If the players move in a metric space, restrictions can be put on their maximum speeds and also a positive radius of capture can be allowed, or a radius of visibility [2, 3, 14, 15, 18, 21, 26]. In this context the problem is many times referred as to *the lion and man problem*. The topological non-metric problem with perfect information has been analyzed in [7]. (d) There are discrete analogues of the problem. The players could move in discrete time, by turns, in a metric space, and even the space can be discrete (and finite). This is the case of the game played in a graph G , in which the players move from a vertex to a neighbor in each turn. When there is perfect information, the minimum number $c(G)$ of cops needed to capture the robber, is called the *cop number*, and is one of the most studied invariants in this area [5, 11], [13, Subsection 9.5.3]. See [17, 19] for a discrete variant with an invisible evader in which the pursuer can jump to any vertex in each turn.

Applications of pursuit-evasion games have been found in the study of Brownian motions, CAT(k) spaces and in a variety of fields such as Robotics, Control theory and Biology.

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When the cop has no information about the robber position, the continuous version has been extensively studied for graphs, considered as 1-dimensional complexes [5, 12, 18], [13, Subsection 9.5.1]. In this context the problem is known as *graph searching*. In few graphs the cop has a strategy (this was proved in [27], and we will see that the same is true for arbitrary CW-complexes), so again the minimum number of pursuers (the *search* or *sweeping number*) is the relevant invariant. Graphs are natural for speleological reasons [16, 27] (searchers looking for a lost man in a dark cave), and they are useful to model other situations, like clearing a system of pipes which is contaminated by a noxious gas. On the other hand, the particular case of finite topological spaces is interesting for several reasons. Recently, classical dynamical systems have been studied using finite spaces [9]. A real dynamical system can be investigated from a sampled point cloud by taking a cube grid and a multivalued map from the face poset of the grid, considered as a finite space. Although continuous time dynamical systems $\phi : X \times \mathbb{R}_{\geq 0} \rightarrow X$ on finite spaces are not interesting (contrary to discrete multivalued), some of the curves that we construct here share many properties with the trajectories $\phi_x : \mathbb{R}_{\geq 0} \rightarrow X$ of a general dynamical system. Secondly, our problem when posed for finite spaces, is analogous to a different pursuit-evasion problem involving regular CW-complexes. Namely, suppose that a watcher and a thief move continuously in an art gallery, whose rooms are some of the open cells of a regular cell complex K . The watcher cannot see the thief until they are in the same room, moment in which the watcher wins. We will show that the watcher has a strategy if and only if the cop has a strategy in the finite space of rooms ordered by the coface relation. Finally, the description of those finite spaces in which the cop has a strategy is a much more complex problem than the analogous for graphs, and requires a development of a whole bunch of new ideas.

Path-components of finite spaces are well understood since Stong's paper [28]. A T_0 topology in a finite set is equivalent to a partial order in the same set, and two points x, y in a finite T_0 space X are in the same path-component if and only if there is a sequence (called fence) $x = x_0 \leq x_1 \geq x_2 \leq \dots x_k = y$. However, this does not mean that we understand what a path $\gamma : [0, 1] \rightarrow X$ in a finite space looks like. It is a key part of this work to study properties of paths and curves in finite spaces and to exhibit examples of wild nature. Many articles about finite spaces deal only with the combinatorial interpretation they have as posets, or even less, with the simplicial complex associated to that poset (order complex). The results in this paper go in a very different direction and the essence of finite spaces is crucial in every proof.

We have attempted to characterize those finite spaces in which the cop has a strategy. In a first stage it seemed plausible that the existence of such a strategy in a finite T_0 space depended only on the subspace of extrema, which consists of minimal and maximal points (Corollary 14). Although we managed to describe the spaces of height 1 in which there is a strategy (Theorem 20), this does not give a direct answer to the general question (Example 22 and Corollary 25). Some examples (Section 8) which lie between the necessary and sufficient conditions obtained (Corollary 25 and Theorem 28), show that further analysis is needed.

2. GENERAL RESULTS

We know already that the cop has a strategy in the 1-dimensional disk. The 2-dimensional case is relevant being the space in which Rado formulated his original problem

of the lion and man in 1925, in which players can see each other and have equal maximum speeds [22]. We will understand this example and much more in this section.

Remark 1. If the cop has a strategy in a space X , so does he in any retract. Indeed, if $A \subseteq X$ is a retract, with $r : X \rightarrow A$ a retraction, and $\gamma : \mathbb{R}_{\geq 0} \rightarrow X$ is a strategy for the cop in X , then $r\gamma$ is a strategy in A , for if $\rho : \mathbb{R}_{\geq 0} \rightarrow A$ is another curve, then there exists $t \in \mathbb{R}_{\geq 0}$ such that $\gamma(t) = i\rho(t)$ ($i : A \rightarrow X$ the inclusion) and then $r\gamma(t) = \rho(t)$.

A *subdivision* of $\mathbb{R}_{\geq 0}$ is a collection of closed intervals of $\mathbb{R}_{\geq 0}$ such that: no member of the collection is a singleton, the collection covers $\mathbb{R}_{\geq 0}$, the intersection of two different members of the collection is empty or a singleton, and the collection is locally finite (for every $t \in \mathbb{R}_{\geq 0}$ there is a neighborhood intersecting only finitely many members). This last condition is equivalent to requiring that each $t \in \mathbb{R}_{\geq 0}$ is either in the interior of one interval or in the boundary of exactly two intervals. Note that a subdivision of $\mathbb{R}_{\geq 0}$ contains a unique unbounded interval of the form $[a, \infty)$ or otherwise all its members are bounded. A subdivision of a compact real interval $[a, b]$ is defined in the same way. The last condition implies that the collection in this case is finite. In this second case our definition is the standard definition of subdivision.

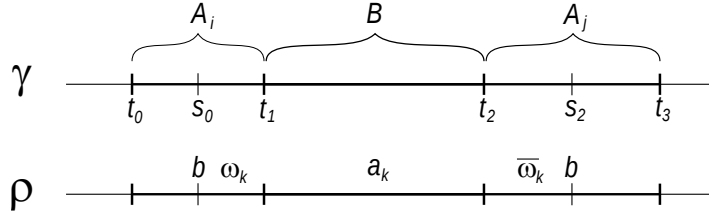
Let X be a topological space, \mathcal{U} an open cover of X , and $\gamma : \mathbb{R}_{\geq 0} \rightarrow X$ a curve (or $\gamma : [a, b] \rightarrow X$ a path). A subdivision of $\mathbb{R}_{\geq 0}$ (or $[a, b]$) is said to be γ -*admissible* if each interval is contained in the preimage $\gamma^{-1}(U)$ of some $U \in \mathcal{U}$. Every time we have a γ -admissible subdivision of $\mathbb{R}_{\geq 0}$ (or $[a, b]$) we will choose for each interval I an open set $U \in \mathcal{U}$ containing $\gamma(I)$ (this may be done in more than one way). A γ -admissible subdivision is *minimal* if there are no adjacent intervals which are contained in the same $\gamma^{-1}(U)$.

By a Lebesgue number argument it is easy to prove that for any space X , any open cover \mathcal{U} and any path $\gamma : [a, b] \rightarrow X$, there exists a γ -admissible subdivision of $[a, b]$. By gluing adjacent intervals, this can be turned into a minimal γ -admissible subdivision. If we now consider a curve $\gamma' : \mathbb{R}_{\geq 0} \rightarrow X$, then there exists a minimal γ' -admissible subdivision of $\mathbb{R}_{\geq 0}$, but this requires a more delicate argument. The details can be found in Theorem 29 in the appendix. In most of the results in this article we will only need the path version.

Lemma 2. *Let X be a topological space and let $\mathcal{U} = \{A_0, A_1, A_2, B\}$ be an open cover of X such that A_0, A_1, A_2 are pairwise disjoint. Moreover, suppose there exist $a_i \in A_i \setminus B$ for each $i = 0, 1, 2$, $b \in B$ and a path ω_i in X from b to a_i whose image does not intersect A_j for $j \neq i$. Then the robber has a strategy in X .*

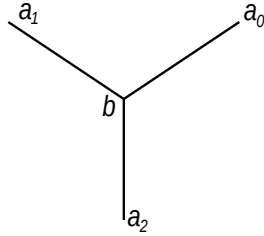
Proof. Let $\gamma : \mathbb{R}_{\geq 0} \rightarrow X$ be a curve. We must show there is another curve ρ having no coincidence with γ . By Theorem 29 there is a minimal γ -admissible subdivision of $\mathbb{R}_{\geq 0}$. Since the subdivision is locally finite, a function $\mathbb{R}_{\geq 0} \rightarrow X$ is continuous if and only if it is continuous in each interval. Assume $[t_0, t_1], [t_1, t_2], [t_2, t_3]$ are three adjacent intervals in the subdivision and suppose γ maps $[t_0, t_1]$ to some A_i . Since the A_j are pairwise disjoint, by minimality of the subdivision $\gamma([t_1, t_2]) \subseteq B$ and $\gamma([t_2, t_3]) \subseteq A_j$ for some j which could be equal to i or not. Let $k \in \{0, 1, 2\}$ be different from i, j . Let $s_0 = \frac{t_0+t_1}{2}$ be the center of the first interval, and s_2 the center of the third one. We define ρ in $[s_0, s_2]$ as follows: $\rho|_{[s_0, t_1]}$ is (a reparametrization of) ω_k , $\rho|_{[t_1, t_2]}$ is the constant path at a_k and $\rho|_{[t_2, s_2]}$ is (a reparametrization of) $\bar{\omega}_k$, the inverse of ω_k . See Figure 1.

Thus $\rho|_{[s_0, s_2]}$ is a continuous loop at b . By hypothesis $\rho|_{[s_0, s_2]}$ and $\gamma|_{[s_0, s_2]}$ are coincidence-free. We define ρ in this way for each triple of intervals as above. If the subdivision of

FIGURE 1. A curve avoiding γ in Lemma 2.

$\mathbb{R}_{\geq 0}$ contains no unbounded interval and the first interval $I_1 = [0, t_1]$ is mapped by γ into A_i , then ρ can be defined in the first half of I_1 as the inverse of the path already defined in the second half (or just as a constant path). This map is well-defined, continuous, and has no coincidence with γ . If there are no unbounded intervals, but the first interval I_1 is mapped to B , while the second I_2 is mapped to A_i , then we define ρ in the first half of I_2 as the inverse of the path defined in the second half, and we define ρ to be constant in I_1 . In the case that there is an unbounded interval $[t, \infty)$ in the subdivision (in particular when the subdivision has just one or two members), ρ can be defined with similar ideas, and details are left to the reader. \square

Example 3. Let Y be a union of three compact real intervals $[a_0, b_0]$, $[a_1, b_1]$, $[a_2, b_2]$ identifying b_0, b_1, b_2 (see Figure 2).

FIGURE 2. The space Y .

Define for each $i = 0, 1, 2$, $A_i = [a_i, b_i)$, and $B = Y \setminus \{a_0, a_1, a_2\}$. Let ω_i be a path from b_i to a_i in $[a_i, b_i]$. Then Lemma 2 applies, so the robber has a strategy in Y . This fact also follows from a general result for trees given by Parsons [27, Lemma 4].

In particular the robber has a strategy in the 2-dimensional disk D^2 , since the space Y is a retract. If we impose a bound for the maximum speed of the robber, and not for the cop, then even when the robber is invisible, the cop has a strategy in D^2 [1, Problema 7]. We have the following classification.

Theorem 4. *Let K be a CW-complex. Then the cop has a strategy in K if and only if K is homeomorphic to the interval $[0, 1]$ or it is the singleton.*

Proof. If $\dim(K) \geq 2$, the space Y in Example 3 is a retract of K , so the robber has a strategy by Remark 1. If $\dim(K) \leq 1$ and K has a cycle, then S^1 is a retract, and since it lacks the fixed point property, the robber has a strategy in K . If K is disconnected, the robber has a strategy. If K is a tree with a vertex of degree greater than or equal to 3,

then Y is a retract. If K is a tree with every vertex of degree smaller than or equal to 2, it is homeomorphic to $(0, 1)$, $[0, 1)$, $[0, 1]$ or the singleton. In the first two cases there is a fixed point free map, and in the latter two the cop has a strategy. \square

Recall that a space X satisfies the separation axiom T_0 if for any pair of different points there exists an open set containing exactly one of them.

Proposition 5. *If a topological space is not T_0 , then the robber has a strategy.*

Proof. Let X be a space with two different points x, y such that any open set containing one of them also contains the other. Then the map $X \rightarrow X$ which maps x to y and all the remaining points to x , is continuous and fixed point free. \square

The cop can have a strategy in non-compact spaces as the next example shows.

Example 6. Let $X = \mathbb{Z}_{\geq 0}$ with the topology generated by the basis $\{\{0\}\} \cup \{\{0, n\} : n \geq 1\}$. Then X is non-compact as the basis defined admits no finite subcover. Define $\gamma : \mathbb{R}_{\geq 0} \rightarrow X$ by $\gamma(n) = n$ for every $n \in \mathbb{Z}_{\geq 0}$ and $\gamma(t) = 0$ if $t \notin \mathbb{Z}_{\geq 0}$. Then γ is continuous and it is a strategy for the cop. If ρ is a curve in X which passes through 0, then $\rho^{-1}(0) \subseteq \mathbb{R}_{\geq 0}$ is open and non-empty. Since $\gamma^{-1}(0)$ is dense, ρ and γ have a coincidence point. If ρ does not go through 0, then it is constant as $\mathbb{Z}_{\geq 1}$ is a discrete subspace of X . Thus ρ and γ have a coincidence also in this case.

We say that the cop has a *strong strategy* in a space X if there exists a path $\gamma : [0, 1] \rightarrow X$ which has a coincidence with every other path in X . The path γ (and any linear reparametrization $[t, t'] \rightarrow X$) is then called a *strong strategy* in X . In this case the curve $\mathbb{R}_{\geq 0} \rightarrow X$ which coincides with γ in $[0, 1]$ and is constant in $\mathbb{R}_{\geq 1}$ is a (regular) strategy for the cop. On the other hand, the cop may have a strategy in a space X and not a strong strategy. Indeed, a necessary condition for the existence of a strong strategy is compactness, so we can take as X the space in Example 6.

Note that if the cop has a strong strategy in a Hausdorff space X , then X is necessarily metrizable by the Hahn-Mazurkiewicz Theorem.

Example 6 can be better understood in the language of finite (or more generally, Alexandroff) spaces. Finite spaces are rarely Hausdorff. In fact, a finite T_1 space is always discrete. Finite T_0 spaces are relevant in Homotopy Theory and Dynamical Systems. The rest of the paper will be focused on the cop and robber problem for finite spaces. Our first result is very elementary and does not require any previous knowledge about these spaces (cf. [27, Lemma 2]).

Proposition 7. *Let X be a finite topological space. If the cop has a strategy in X , then it also has a strong strategy in X .*

Proof. Assume there is no strong strategy. Let $\gamma : \mathbb{R}_{\geq 0} \rightarrow X$ be a curve in X . We prove that there exists another curve in X having no coincidence with γ .

For every $n \geq 1$, let γ_n be the restriction of γ to $[0, n]$. By assumption, for every n there exists a path $\rho_n : [0, n] \rightarrow X$ having no coincidence with γ_n . Since X is finite, there exists a subsequence $(\rho_{n_{k_1}}(1))_{k_1 \in \mathbb{N}}$ of $(\rho_n(1))_{n \geq 1}$ which is constant. There is a subsequence $(\rho_{n_{k_1 k_2}}(2))_{k_2 \in \mathbb{N}}$ of $(\rho_{n_{k_1}}(2))_{k_1 \geq 2}$ which is constant, and so on. We define $\rho : \mathbb{R}_{\geq 0} \rightarrow X$ to be equal to $\rho_{n_{k_1 k_2 \dots k_{m_1}}}$ in the interval $[m, m+1]$ for each $m \geq 0$. Clearly ρ is well-defined, continuous and has no coincidence with γ . \square

3. FINITE SPACES, THE LIFTING THEOREM AND THE ART GALLERY

If X is a finite poset, there is a topology in X given by the down-sets, i.e. subsets $U \subseteq X$ which satisfy that $x \leq y$ and $y \in U$ implies $x \in U$. This topology is T_0 . Conversely, if X is a finite T_0 space, then for every $x \in X$ we define the minimal open set U_x as the intersection of all the open sets containing x . The minimal open sets are open and in fact they are a basis for the topology. We define an order in X by $x \leq y$ if $x \in U_y$. These two maps which associate a topology to every order and an order to each topology are mutually inverse [4]. In view of this correspondence we consider finite T_0 spaces and finite posets as the same thing. A map $f : X \rightarrow Y$ between finite T_0 spaces is continuous if and only if it is order-preserving, that is $x \leq x'$ implies $f(x) \leq f(x')$. In finite spaces connected and path-connected components coincide. Two points x, y in a finite T_0 space are in the same component if and only if there exists a fence $x = x_0 \leq x_1 \geq \dots x_n = y$ from x to y .

Finite T_0 spaces will be represented by their Hasse diagram, a digraph whose vertices are the points of the space and with an arrow from x to y if x is covered by y , i.e. $x < y$ and there is no $x < z < y$. In the graphical representation of Hasse diagrams all arrows are assumed to point upwards.

Since for a point x in a finite space X , $U_x = \{y \in X \mid y \leq x\}$ is the smallest open set containing x , a sequence $(x_n)_{n \in \mathbb{N}}$ converges to a x if and only if for some $n_0 \in \mathbb{N}$, $x_n \leq x$ for every $n \geq n_0$.

Given a finite T_0 space X , we denote by $\mathcal{K}(X)$ its order complex. This is the simplicial complex whose simplices are the non-empty chains of X . Let $\mu : \mathcal{K}(X) \rightarrow X$ be the map which maps every point in the interior of the simplex $x_0 < x_1 < \dots < x_n$ to x_0 . Then μ is continuous and moreover it is a weak homotopy equivalence ([24, Theorem 2] and [29, Corollary 3.7]). If K is a finite simplicial complex, we denote by $\mathcal{X}(K)$ the poset of simplices of K ordered by inclusion (face poset). Since $\mathcal{K}(\mathcal{X}(K))$ is the first barycentric subdivision of K , there is also a weak homotopy equivalence $K \rightarrow \mathcal{X}(K)$.

If X is a finite T_0 space, we denote by X^{op} the space with the same underlying set and whose open sets are the closed sets of X . The order of the poset X^{op} is the opposite of the order in X .

Recall that a space X is said to be perfectly normal if for any disjoint closed sets $F, H \subseteq X$, there exists a map $f : X \rightarrow [0, 1]$ such that $f^{-1}(0) = F$ and $f^{-1}(1) = H$. For instance all metric spaces satisfy this axiom. In [29, Theorem 3.5] Wofsey proved a result which has the following particular case.

Theorem 8 (Wofsey's lifting theorem). *Let X be a finite T_0 space, Y a perfectly normal topological space and $f : Y \rightarrow X$ a continuous map. Then there exists $\tilde{f} : Y \rightarrow \mathcal{K}(X)$ such that $\mu\tilde{f} = f$.*

We have the following immediate consequence.

Corollary 9. *Let X be a finite T_0 space such that $\mathcal{K}(X)$ is a space in which the cop has a strategy. Then the cop has a strategy in X as well.*

Proof. Let $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathcal{K}(X)$ be a strategy for the cop. We claim that $\mu\gamma$ is a strategy for the cop in X . Indeed, if $\rho : \mathbb{R}_{\geq 0} \rightarrow X$ is a curve, by Theorem 8 there exists $\tilde{\rho} : \mathbb{R}_{\geq 0} \rightarrow \mathcal{K}(X)$ such that $\mu\tilde{\rho} = \rho$. Thus, there exists $t \in \mathbb{R}_{\geq 0}$ such that $\gamma(t) = \tilde{\rho}(t)$, and so $\mu\gamma(t) = \rho(t)$. \square

Remark 10. Unfortunately Theorem 4 says that not in many CW-complexes the cop has a strategy. Still, this corollary will be useful for the cases when $\mathcal{K}(X)$ is homeomorphic to $[0, 1]$, as in Figure 3.

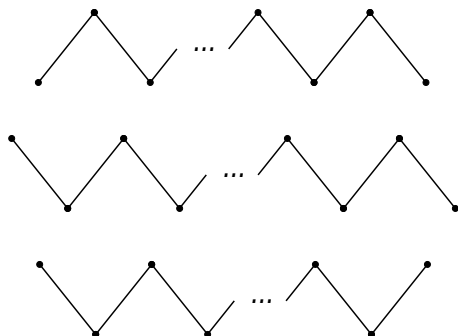


FIGURE 3. Spaces with associated complex homeomorphic to $[0, 1]$, in which the cop has a strategy.

In fact in these spaces any path γ from one leaf in the Hasse diagram to the other leaf is a strong strategy for the cop. A lift $\tilde{\gamma} : [0, 1] \rightarrow \mathcal{K}(X)$ has a coincidence with any other path in $\mathcal{K}(X)$ by Bolzano's Theorem, so $\gamma = \mu\tilde{\gamma}$ has a coincidence with any path in X .

There is a relationship between the cop and robber game in finite spaces with another problem, already described in the introduction, played in polyhedra and some subspaces.

There is a thief hidden in an art gallery. When the doors of the gallery close, the thief starts walking around the different rooms. At the same time a watcher begins his route. If at some point both persons are in the same room, then the watcher automatically sees the thief, turns the alarm on, and the thief is caught. The watcher has no idea where the thief is until they are in the same room. If the thief manages to stay in a different room during the whole night, then the doors of the gallery open and he is free. Is there a route the watcher can make every night, so he can find the thief before morning, regardless the thief starting position and the way he moves? Or the thief can always remain unseen during the whole night? We will assume the gallery S is a collection of open cells of a regular CW-complex (with the subspace topology), each cell being a room. So there are rooms of different dimension, and the thief can get caught in any of them. A strategy for the watcher is a path $\gamma : [0, 1] \rightarrow S$ such that for any other path $\rho : [0, 1] \rightarrow S$ there exists t with $\gamma(t)$ and $\rho(t)$ being in the same open cell.

We extend the definition of face poset to S , so $\mathcal{X}(S)$ is the poset of open cells ordered by the face relation.

Proposition 11. *Let S be a collection of open cells of a finite regular CW-complex K . Then the watcher has a strategy in S if and only if the cop has a strategy in the finite space $\mathcal{X}(S)^{op}$.*

Proof. Let $\varsigma : S \rightarrow \mathcal{X}(S)^{op}$ be the map which maps every point in the open cell $e \subseteq S$ to $e \in \mathcal{X}(S)^{op}$. It is easy to see that ς is continuous (see [6, Theorem 11.3.2]). The polyhedron $\mathcal{K}(\mathcal{X}(S))$ is a subspace of S via the restriction $\iota : \mathcal{K}(\mathcal{X}(S)) \rightarrow S$ of the usual homeomorphism $K' \rightarrow K$. If $e_0 < e_1 < \dots < e_n$ is a chain in $\mathcal{X}(S)$, the map ι maps the open simplex $\{e_i\}_{0 \leq i \leq n}$ to the open cell e_n (see [23, Ch. III, Theorem 1.7]). In other

words, $\varsigma\iota = \mu : \mathcal{K}(\mathcal{X}(S)^{op}) \rightarrow \mathcal{X}(S)^{op}$. The rest of the proof is a direct application of the lifting Theorem.

Assume first that $\gamma : [0, 1] \rightarrow S$ is a strategy for the watcher. We claim that $\varsigma\gamma$ is a strong strategy for the cop. Let $\rho : [0, 1] \rightarrow \mathcal{X}(S)^{op}$ be any path. By Theorem 8, there exists $\tilde{\rho} : [0, 1] \rightarrow \mathcal{K}(\mathcal{X}(S))$ such that $\mu\tilde{\rho} = \rho$. Then $\iota\tilde{\rho}$ is a path in S and there exists $t \in [0, 1]$ such that $\iota\tilde{\rho}(t)$ and $\gamma(t)$ are in the same open cell. Thus $\rho(t) = \mu\tilde{\rho}(t) = \varsigma\iota\tilde{\rho}(t) = \varsigma\gamma(t)$.

Conversely, suppose now that the cop has a strategy in $\mathcal{X}(S)^{op}$, so there exists a strong strategy $\gamma : [0, 1] \rightarrow \mathcal{X}(S)^{op}$ by Proposition 7. There exists $\tilde{\gamma} : [0, 1] \rightarrow \mathcal{K}(\mathcal{X}(S))$ such that $\mu\tilde{\gamma} = \gamma$. We show that $\iota\tilde{\gamma}$ is a strategy for the watcher. Let $\rho : [0, 1] \rightarrow S$ be any path. Then $\varsigma\rho$ must have a coincidence with γ , say at $t \in [0, 1]$. Thus, $\varsigma\rho(t) = \gamma(t) = \mu\tilde{\gamma}(t) = \varsigma\iota\tilde{\gamma}(t)$. This means that $\rho(t)$ and $\iota\tilde{\gamma}(t)$ lie in the same open cell. \square

Example 12. In Figure 4 we see three art galleries. The 0-dimensional rooms are colored with red, the 1-dimensional with blue and the 2-dimensional with yellow. In the first one there are two rooms of dimension 2, three of dimension 1 and one of dimension 0. Here the watcher has a strategy (see Section 8). In the second gallery there are seven rooms of dimension 2, two of dimension 1 and two of dimension 0. Here the watcher has a strategy as well (see Theorem 20). In the third gallery there is one room of dimension 2, one of dimension 1 and two of dimension 0. Here the watcher does not have a strategy (see Example 22).

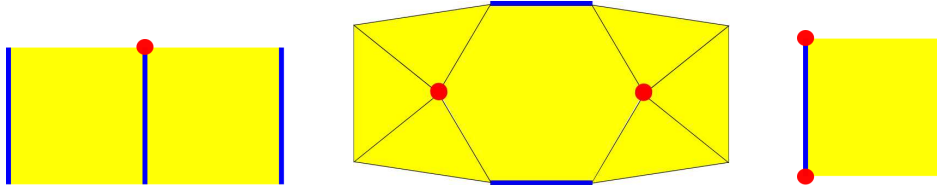


FIGURE 4. Two galleries in which the watcher has a strategy and another in which the thief has a strategy.

4. THE SUBSPACE OF EXTREMA

Let X be a finite T_0 space. We denote by $\mathfrak{E}(X)$ the subspace of extrema of X , formed by its maximal and its minimal points. The aim of this section is to prove that if the cop has a strategy in X , so does he in $\mathfrak{E}(X)$ (Corollary 14). An alternative proof is discussed in Remark 24. In general $\mathfrak{E}(X)$ is not a retract of X . However we have the following

Proposition 13. *Let X be a finite T_0 space and let $\gamma : \mathbb{R}_{\geq 0} \rightarrow X$ be a curve. Then there exists a curve $\mathfrak{E}(\gamma) : \mathbb{R}_{\geq 0} \rightarrow \mathfrak{E}(X)$ such that for every $t \in \mathbb{R}_{\geq 0}$, $\gamma(t) \in \mathfrak{E}(X)$ implies $\mathfrak{E}(\gamma)(t) = \gamma(t)$.*

Proof. The proof will be by induction in the cardinality of $X \setminus \mathfrak{E}(X)$. Assume $a \in X$ is a point which is not maximal nor minimal and it is covered only by maximal points. We are going to define a curve $\tilde{\gamma} : \mathbb{R}_{\geq 0} \rightarrow X \setminus \{a\}$ which coincides with γ at every $t \in \mathbb{R}_{\geq 0}$ such that $\gamma(t) \neq a$. Then the proof concludes by the induction assumption. In order to define $\tilde{\gamma}$, we consider the connected components of $\gamma^{-1}(a)$. They are intervals of type O: (t, t') (with $t' \in \mathbb{R}_{>0} \cup \{\infty\}$), and intervals of type not O: $[t, t']$ (possibly just one point), $[t, t']$ ($t' \in \mathbb{R}_{>0} \cup \{\infty\}$), $(t, t']$. The new map $\tilde{\gamma}$ will be constant in each of these intervals.

Let \hat{F}_a be the set of (maximal) elements in X which are greater than a . Let b be a minimal point of X which is smaller than a . In each interval of type O, $\tilde{\gamma}$ is defined to be constant with value b . The definition of $\tilde{\gamma}$ in the intervals of type not O will be given later.

Let $t < t' \in \mathbb{R}_{\geq 0}$ be such that $\gamma(t) = x$ and $\gamma(t') = y$ are two different elements in \hat{F}_a and that $(t, t') \cap \gamma^{-1}(\hat{F}_a) = \emptyset$. Then $[t, t']$ is called a change-interval and we will choose a point t'' in (t, t') as follows. If $\gamma|_{(t, t')}$ is not the constant map to a , we take t'' so that $\gamma(t'') \neq a$. Otherwise we take t'' to be any point in (t, t') . Note that in this second case (t, t') is an interval of type O. The point t'' is called a right margin for x and a left margin for y (see Figure 5). Note that if $t_0 < t_1 < t_2$ are margins (left and right), then there exists $t \in (t_0, t_1)$ and $t' \in (t_1, t_2)$ with $\gamma(t), \gamma(t')$ different elements of \hat{F}_a . Since \hat{F}_a is discrete and a closed subspace of X , the set of margins is discrete and closed.

If $\gamma(t) = x \in \hat{F}_a$ for some $t \in \mathbb{R}_{\geq 0}$ then either there is no margin smaller than t or the greatest margin smaller than t is a left margin for x . Symmetrically, either there is no margin greater than t or the smallest margin greater than t is a right margin for x . Suppose first that the image of the curve γ contains at least two different elements of \hat{F}_a . Then for each $x \in \hat{F}_a$ let G_x be the union of the following open intervals: 1. All the intervals (t_0, t_1) such that t_0 is a left margin for x , t_1 is a right margin for x , and there is no margin in the interval. 2. If the smallest margin t in $\mathbb{R}_{\geq 0}$ is a right margin for x , then we also add the interval $[0, t)$ to G_x . 3. If there is a greatest margin t' in $\mathbb{R}_{\geq 0}$, which is a left margin for x , then we add (t', ∞) to G_x .

If there is only one element $x \in \hat{F}_a$ in the image of γ , then there are no margins and we define $G_x = \mathbb{R}_{\geq 0}$ while $G_y = \emptyset$ for all $x \neq y \in \hat{F}_a$. If $\text{Im}(\gamma) \cap \hat{F}_a = \emptyset$, we choose an element $x \in \hat{F}_a$ and set $G_x = \mathbb{R}_{\geq 0}$, $G_y = \emptyset$ for all $y \neq x$.

Note that the open sets G_x for $x \in \hat{F}_a$, are pairwise disjoint and that $\gamma^{-1}(x) \subseteq G_x$ for every $x \in \hat{F}_a$. Also, each interval of type not O is contained in a G_x . Indeed, none of these intervals contains a margin, the intervals are connected, and the G_x cover all of $\mathbb{R}_{\geq 0}$ with exception of the margins.

We are now ready to define $\tilde{\gamma}$ in the intervals of type not O. If I is an interval of type not O, then it is contained in G_x for a unique $x \in \hat{F}_a$. We define $\tilde{\gamma}|_I$ to be the constant map to x .

We claim that the function $\tilde{\gamma} : \mathbb{R}_{\geq 0} \rightarrow X \setminus \{a\} \subseteq X$ is continuous. Let $(t_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_{\geq 0}$ which converges to some t . We want to prove that $\tilde{\gamma}(t_n) \rightarrow \tilde{\gamma}(t)$.

Case 1i: $\gamma(t) \neq a$ and eventually $\gamma(t_n) \neq a$ (that is for some $n_0 \in \mathbb{N}$ this happens for every $n \geq n_0$). In this case $\tilde{\gamma}(t) = \gamma(t)$ and eventually $\tilde{\gamma}(t_n) = \gamma(t_n)$, so by the continuity of γ , $\tilde{\gamma}(t_n) \rightarrow \tilde{\gamma}(t)$.

Case 1ii: $\gamma(t) \neq a$ and $\gamma(t_n) = a$ for infinitely many values of n . By assumption $a \rightarrow \gamma(t)$, so $\gamma(t) \geq a$, and then $\gamma(t) = x$ for some $x \in \hat{F}_a$. Then $t \in G_x$ and since G_x is open, t_n lies eventually in G_x . Therefore eventually $\tilde{\gamma}(t_n)$ is equal to x (if t_n is in an interval of type not O) or smaller than or equal to $\gamma(t_n)$ (if t_n is in an interval of type O or if $\gamma(t_n) \neq a$). Therefore $\tilde{\gamma}(t_n) \rightarrow x = \tilde{\gamma}(t)$.

Case 2: $\gamma(t) = a$. If t is in an interval of type O, then t_n lies in the same interval eventually, so $\tilde{\gamma}(t_n) = \tilde{\gamma}(t)$ eventually and the convergence is trivial. Otherwise, t lies in an interval of type not O. Thus, there exists $x \in \hat{F}_a$ such that $t \in G_x$ and $\tilde{\gamma}(t) = x > a = \gamma(t)$. Just as in Case 1ii, t_n lies eventually in G_x , so eventually $\tilde{\gamma}(t_n)$ is equal to x or smaller than or equal to $\gamma(t_n)$. Thus $\tilde{\gamma}(t_n) \rightarrow \tilde{\gamma}(t)$.

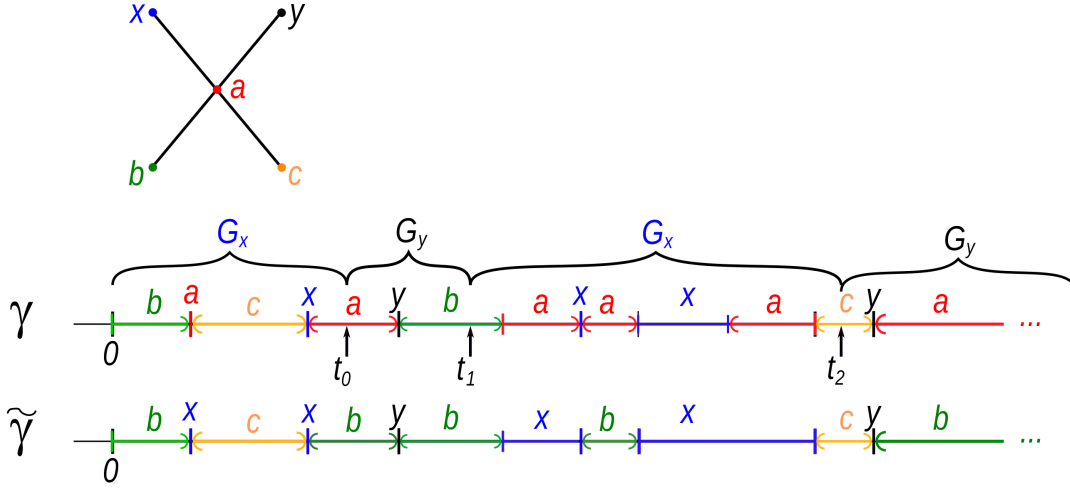


FIGURE 5. A space X of five points, a curve γ in X , the margins t_0, t_1, t_2 , the sets G_x, G_y , and the curve $\tilde{\gamma}$ constructed in the proof of Proposition 13.

□

Of course, the curve $\mathfrak{E}(\gamma)$ is not uniquely determined by γ in general. Proposition 13 can be compared with [27, Lemma 1], which is an analogue for graphs.

Corollary 14. *Let X be a finite T_0 space. If the robber has a strategy in $\mathfrak{E}(X)$, then so does he in X .*

Proof. Let $\gamma : \mathbb{R}_{\geq 0} \rightarrow X$ be a curve in X . By Proposition 13 there is a curve $\mathfrak{E}(\gamma) : \mathbb{R}_{\geq 0} \rightarrow \mathfrak{E}(X)$ which coincides with γ in $\gamma^{-1}(\mathfrak{E}(X))$. By hypothesis there exists a curve $\rho : \mathbb{R}_{\geq 0} \rightarrow \mathfrak{E}(X) \subseteq X$ having no coincidences with $\mathfrak{E}(\gamma)$. Then ρ is a curve in X which has no coincidences with γ . □

5. STRATEGIES AND OBSTRUCTIONS TO THEIR EXISTENCE

Motivated by Corollary 14 we study those finite T_0 spaces of height 1 (i.e. the longest chain has two elements) in which the robber has a strategy. We begin with examples which turn out to be crucial in the classification.

Example 15. Let $m \in \mathbb{N}$ and let X be the non-Hausdorff cone over the discrete space of m points. In other words, it is the space X of Figure 6. We claim that the cop has a (strong) strategy in X . Let $\gamma : [0, 1] \rightarrow X$ be the path defined by $\gamma(0) = a$, $\gamma(\frac{1}{n}) = a$ for every $n \in \mathbb{N}$, and in the interval $(\frac{1}{n+1}, \frac{1}{n})$, γ is constant with value $c_{\bar{n}}$ for every $n \in \mathbb{N}$, where \bar{n} denotes class modulo m .

It is easy to see that γ is indeed continuous. Let $\rho : [0, 1] \rightarrow X$ be a path. If $\rho(0) = a$, then $\rho(0) = \gamma(0)$. Otherwise $\rho(0) = c_k$ for some $0 \leq k \leq m-1$. Since $\{c_k\}$ is open in X , $\rho^{-1}(c_k)$ is an open neighborhood of 0 and then it intersects an interval of the form $(\frac{1}{n+1}, \frac{1}{n})$ for some $n \in \mathbb{N}$ with $\bar{n} = k$. Thus ρ and γ also have a coincidence point in this case. This proves that γ is a strong strategy for the cop. Note that if $U \subseteq [0, 1]$ is any

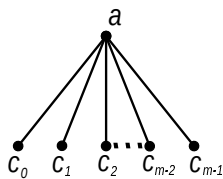
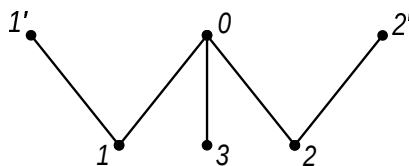


FIGURE 6. The space in Example 15.

neighborhood of 0 and $\rho : U \rightarrow X$ is a map, then ρ has a coincidence with $\gamma|_U$. This property will be used later and it will be generalized in Theorem 26.

Example 16. Let $X = S_{2,1}$ be the space depicted in Figure 7. We claim that the robber has a strategy in X .

FIGURE 7. The scorpion $S_{2,1}$.

Let $\gamma : [0, 1] \rightarrow X$ be a path. By Proposition 7 it suffices to show that there is a path $\rho : [0, 1] \rightarrow X$ which has no coincidence with γ (alternatively we can work with curves $\mathbb{R}_{\geq 0} \rightarrow X$ and use Theorem 29). Consider the open cover $\mathcal{U} = \{U_0, U_{1'}, U_{2'}\}$. There exists a minimal γ -admissible subdivision of $[0, 1]$. As always we choose for each interval in the subdivision an element of \mathcal{U} whose preimage contains the interval. If $[t_0, t_1]$ and $[t_1, t_2]$ are adjacent intervals of the subdivision, then by minimality and the fact that $U_{1'} \cap U_{2'} = \emptyset$, U_0 must have been chosen for one and only one of the intervals.

In the intervals of the subdivision for which U_0 was not chosen, we define ρ to be the constant map to 3. Since $3 \notin U_{i'}$ for $i = 1, 2$, ρ has no coincidences with γ in those intervals.

Let $[t_1, t_2]$ be an interval of the subdivision for which U_0 has been chosen. Suppose $[t_0, t_1]$ and $[t_2, t_3]$ are intervals of the subdivision adjacent to the first one. Let $U_{i'}$ be the member of \mathcal{U} chosen for the first interval and $U_{j'}$ for the second. Since $U_{i'} \cap U_0 = \{i\}$, $\gamma(t_1) = i$ and similarly $\gamma(t_2) = j$. We will define $\rho|_{[t_1, t_2]} : [t_1, t_2] \rightarrow X$. If $3 \notin \gamma([t_1, t_2])$, we define $\rho|_{[t_1, t_2]}$ to be the constant map to 3. Otherwise, since $\{3\} \in X$ is open, there exist $t_1 < s_1 < s_2 < t_2$ such that $\gamma((s_1, s_2)) = \{3\}$. Also, since $\{i\}$ and $\{j\}$ are open, there exist $s_0 > t_1$ and $s_3 < t_2$ such that $\gamma([t_1, s_0]) = \{i\}$ and $\gamma((s_3, t_2]) = \{j\}$. Note that $s_0 < s_1$ and $s_2 < s_3$ (see Figure 8).

Let $r_0, r_1, \dots, r_6 \in [0, 1]$ be such that $t_1 < r_0 < r_1 < s_0 < s_1 < r_2 < r_3 < r_4 < s_2 < s_3 < r_5 < r_6 < t_2$. Let \bar{i} be the element of $\{1, 2\}$ different from i , and define \bar{j} similarly. Define ρ to be 3 in $[t_1, r_0] \cup (r_6, t_2]$, 0 in $\{r_0, r_3, r_6\}$, \bar{i} in $(r_0, r_1) \cup (r_2, r_3)$, \bar{i}' in $[r_1, r_2]$, \bar{j} in $(r_3, r_4) \cup (r_5, r_6)$ and \bar{j}' in $[r_4, r_5]$. It is clear then that $\rho|_{[t_1, t_2]}$ is continuous and has no coincidence with $\gamma|_{[t_1, t_2]}$. There are two cases that we have not contemplated yet. When the interval for which U_0 has been chosen is the first or when it is the last in the

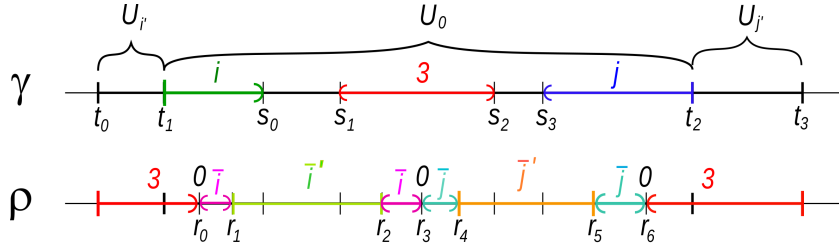


FIGURE 8. The robber has a strategy in $S_{2,1}$.

subdivision. In these cases the definition of ρ is simpler and left to the reader. The path $\rho : [0, 1] \rightarrow X$ is well-defined, continuous and has no coincidence with γ .

Example 17. Let X be the opposite of the non-Hausdorff cone of the discrete space in $m + 1$ points (see Figure 9). We claim that the cop has a (strong) strategy in X . This is very similar to Example 6. We define $\gamma : [0, 1] \rightarrow X$ by $\gamma(\frac{k}{m}) = b_k$ for $0 \leq k \leq m$, and the value of γ is a in all the other points. Let $\rho : [0, 1] \rightarrow X$ be another path. If a is in the image of ρ , then since $\{a\} \subseteq X$ is open and $\gamma^{-1}(a)$ is dense in $[0, 1]$, ρ and γ have a coincidence point. Otherwise, ρ is constant, and since γ is surjective, there is also a coincidence. Note that $\gamma^{-1}(b_k)$ consists of just one point.

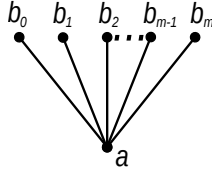


FIGURE 9. The space X in Example 17.

Example 18. Let $X = S_{3,0}^{op}$ be the space in Figure 10. Then the robber has a strategy in X .

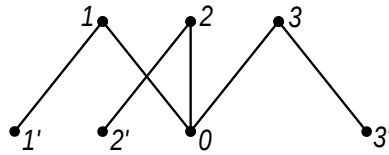


FIGURE 10. The scorpion $S_{3,0}^{op}$.

Let $\mathcal{U} = \{U_1, U_2, U_3\}$ and let $\gamma : \mathbb{R}_{\geq 0} \rightarrow X$ be a curve (alternatively we could work with a path). There is a minimal γ -admissible subdivision of $\mathbb{R}_{\geq 0}$, and for each interval I we choose a member of \mathcal{U} containing $\gamma(I)$. An interval I of the subdivision is called special if $\gamma(I) \cap \{1', 2', 3'\} \neq \emptyset$. If $I = [t_0, t_1]$ (or $I = [t_0, \infty)$) is a special interval and $i' \in \gamma(I)$ for some $i = 1, 2, 3$, then there is an open interval (s_0, s_1) contained in $I \cap \gamma^{-1}(i')$. We choose one such interval for each special interval.

Suppose now $[t_0, t_1]$ and $[t_2, t_3]$ are two special intervals with $t_1 < t_2$. Moreover, suppose there is no special interval contained in $[t_1, t_2]$. Let $(s_0, s_1) \subseteq \gamma^{-1}(i')$ be the interval chosen inside $[t_0, t_1]$. Similarly $(s_2, s_3) \subseteq \gamma^{-1}(j')$ is the interval chosen for $[t_2, t_3]$. Here $i, j \in \{1, 2, 3\}$ could be equal or not. Let $k \in \{1, 2, 3\}$ be different from i, j . We define $\rho|_{[s_1, s_2]}$ to be the constant function k' . Note that $\rho|_{[s_1, s_2]}$ has no coincidence with $\gamma|_{[s_1, s_2]}$ since there are no special intervals in $[t_1, t_2]$. We do this for every pair of consecutive special intervals as above.

Let $[t_0, t_1]$ be a special interval. Suppose (s_0, s_1) is the interval chosen inside $[t_0, t_1] \cap \gamma^{-1}(i')$, and that $\rho(s_0) = k', \rho(s_1) = l'$ have already been defined for $k, l \neq i$. We define $\rho|_{[s_0, s_1]}$ to be a path from k' to l' not touching i' (in fact there is a path going through $k', k, 0, l, l'$). Thus, $\rho|_{[s_0, s_1]}$ and $\gamma|_{[s_0, s_1]}$ are coincidence-free. This defines a curve ρ avoiding γ . We only have to pay attention to the first special interval and all the intervals at its left, and the last special interval if any. Also, there could be no special intervals at all. All these cases are simple and details are left to the reader.

6. THE CLASSIFICATION IN HEIGHT 1 AND AN OPEN QUESTION IN THE GENERAL CASE

If K is a compact polyhedron with nontrivial $b_1(K) = H_1(K; \mathbb{Q})$, then by a result of Borsuk, S^1 is a retract, and in particular K lacks the fixed point property. For finite spaces this is not true: a finite space X with the fixed point property may have $b_1(X) \neq 0$ [8]. For finite spaces of height 1, though, an analogue of Borsuk's result holds.

Proposition 19. *Let X be a finite T_0 space of height 1 which contains a cycle $x_0 < x_1 > x_2 < \dots < x_{n-1} > x_n = x_0$ ($n \geq 4$, $x_i \neq x_j$ if $0 \leq i < j \leq n-1$). Then the robber has a strategy in X .*

Proof. We can assume X is connected and the length n of the cycle is minimal. We claim that this cycle C is a retract of X . This follows almost directly from a well-known result in graph theory [20, Proposition 2.51]. Concretely, let G be the underlying undirected graph of the directed graph obtained from the Hasse diagram of X by removing the edge (x_0, x_{n-1}) . Define the map $r : X \rightarrow X$ by $r(x) = x_i$, where i is the minimum between $n-1$ and the distance between x and x_0 in G . Then r is a continuous retraction from X to C . Since C admits a fixed point free map, then so does X . Thus the robber has a strategy in X . \square

Note that for a connected finite T_0 space of height 1, having a cycle is equivalent to not being simply connected and also to having nontrivial first Betti number.

If a finite T_0 space X of height 1 has no cycle, then any non-empty connected subspace is a retract: just map every point in X (in the component of the subspace if not connected) to the closest point in the subspace in the Hasse diagram. In particular, if X has a subspace A homeomorphic to $S_{2,1}$ or $S_{3,0}^{op}$, then the robber has a strategy. Conversely, if X is of height 1 with no cycles and $S_{2,1}$ and $S_{3,0}^{op}$ are not subspaces, then the cop has a strategy.

Theorem 20. *Let X be a connected finite T_0 space of height 1. Then the cop has a strategy (equivalently a strong strategy) in X if and only if the following conditions are satisfied simultaneously:*

- X contains no cycle.
- No subspace of X is homeomorphic to $S_{2,1}$ nor $S_{3,0}^{op}$.

Proof. We have already established the necessity of the conditions. Conversely, suppose the conditions are satisfied. Let a_0, a_1, \dots, a_l be a fence of maximal length in X . That is, these $l + 1$ points are pairwise different, a_i and a_{i+1} are comparable in X for $0 \leq i \leq l - 1$ and there is no longer fence in X . We can assume $l \geq 2$. Since X contains no cycle, a_i and a_j are not comparable for $|i - j| > 1$. Suppose a_i is minimal in X . If $i = 0$, the unique point of X comparable with a_i is a_1 , by maximality of the fence. If $i = l$, a_i is only comparable with a_{i-1} . If $0 < i < l$, let $\{a_{i-1}, a_{i+1}, b_{i,1}, b_{i,2}, \dots, b_{i,m_i}\}$ be the set of points in X comparable with ($=$ greater than) a_i . Since $S_{3,0}^{op}$ is not a subspace of X , X contains no cycle, and the length of the fence is maximal, then the points $b_{i,j}$ are only comparable with a_i . Conversely, suppose $0 \leq i \leq l$ is such that a_i is maximal. Again, if $i = 0$ or $i = l$, a_i is only comparable with one point in X . If $2 \leq i \leq l - 2$, since $S_{2,1}$ is not a subspace of X , the unique points comparable with ($=$ smaller than) a_i in X are a_{i-1} and a_{i+1} . Let $\{a_0, a_2, c_1, c_2, \dots, c_{m_1}\}$ be the set of points comparable with a_1 . Let $\{a_{l-2}, a_l, d_1, d_2, \dots, d_{m_{l-1}}\}$ be the set of points which are smaller than a_{l-1} . By maximality of the length of the fence, the points c_j are only comparable with a_1 and the points d_j only comparable with a_{l-1} . Since X is connected, there are no other points in X than the a_i , the $b_{i,j}$, the c_j and the d_j .

If a_0 is maximal, we can define a new space by adding a point a_{-1} smaller than a_0 and incomparable with any other point in X . Similarly, if a_l is maximal, we can add a point a_{l+1} smaller than a_l and incomparable with any other point. The new space satisfies the hypothesis of the statement and contains X as a retract. Thus, we can assume a_0 and a_l are both minimal (so l is even), and X looks as the diagram in Figure 11. The case $l = 2$ follows directly from Example 15, so we will assume $l \geq 4$.

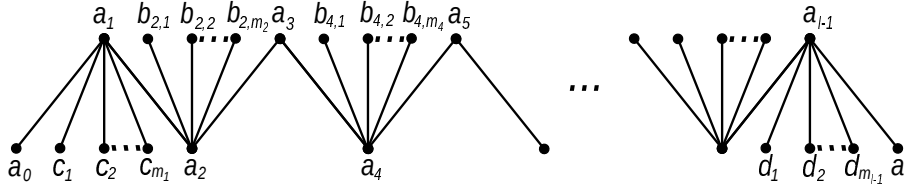


FIGURE 11. The space X in Theorem 20.

Essentially we glue the strategies in Examples 15 and 17 (see Remark 21 for comments in this direction). Let $\gamma : [0, l] \rightarrow X$ be the path defined as in Figure 12. Namely $\gamma(2i + 1) = a_{2i+1}$ for $0 \leq i \leq \frac{l}{2} - 1$. The interval $[2i - 1, 2i + 1]$ (for $1 \leq i \leq \frac{l}{2} - 1$) is subdivided in $m_{2i} + 1$ intervals by considering points $2i - 1 < t_{2i,1} < t_{2i,2} < \dots < t_{2i,m_{2i}} < 2i + 1$. We define $\gamma(t_{2i,j}) = b_{2i,j}$, while $\gamma(t) = a_{2i}$ for the remaining points in $(2i - 1, 2i + 1)$. In the interval $[0, 1)$, γ is defined as follows. The value of γ in $t_k = 1 - \frac{1}{k}$ is a_1 for every k , while γ in (t_k, t_{k+1}) is the constant map to $c_{\bar{k}}$, where \bar{k} is the class of k modulo $m_1 + 2$ and $c_0 = a_0$, $c_{m_1+1} = a_2$. In the interval $(l - 1, l]$, γ is defined with a symmetric idea by taking the value a_{l-1} in $s_k = l - 1 + \frac{1}{k}$ for every k , while γ in (s_{k+1}, s_k) is constant with value $d_{\bar{k}}$, being \bar{k} the class modulo $m_{l-1} + 2$ and $d_0 = a_l$, $d_{m_{l-1}+1} = a_{l-2}$. Clearly γ is continuous.

We claim γ is a strong strategy for the cop. Let $\rho : [0, l] \rightarrow X$ be another path. We want to prove that γ and ρ have a coincidence point. Let Y be the subspace of X which consists of the points a_1, a_2, \dots, a_{l-1} and let $r : X \rightarrow Y$ be the retraction that maps

FIGURE 12. The strong strategy γ .

$a_0, c_1, c_2, \dots, c_{m_1}$ to a_1 , it maps $a_l, d_1, d_2, \dots, d_{m_{l-1}}$ to a_{l-1} , and for each even $2 \leq i \leq l-2$, it maps $b_{i,1}, b_{i,2}, \dots, b_{i,m_i}$ to a_i .

Let $\omega : [1, l-1] \rightarrow Y$ be the map defined by $\omega(2i+1) = a_{2i+1}$ for $0 \leq i \leq \frac{l}{2} - 1$ while in $(2i-1, 2i+1)$ is constant with value a_{2i} for $1 \leq i \leq \frac{l}{2} - 1$. Note that ω has a coincidence with any other path $[1, l-1] \rightarrow Y$ by Remark 10. In particular, there exists $t \in [1, l-1]$ such that $\omega(t) = r\rho|_{[1, l-1]}(t)$.

Case 1: $t = 1$. In this case $r\rho(1) = \omega(1) = a_1$. This implies that $\rho(1) \in \{a_1, a_0, c_1, c_2, \dots, c_{m_1}\} \subseteq U_{a_1}$. Thus, there is a neighborhood U of 1 in $[0, 1]$ such that $\rho(U) \subseteq U_{a_1}$. By Example 15, ρ and γ have a coincidence in U .

Case 2: $t = l-1$ is symmetric to case 1.

Case 3: $t = 2i+1$ for some $1 \leq i \leq \frac{l}{2} - 2$. In this case $r\rho(2i+1) = \omega(2i+1) = a_{2i+1}$, so $\rho(2i+1) = a_{2i+1} = \gamma(2i+1)$.

Case 4: $t \in (2i-1, 2i+1)$ for some $1 \leq i \leq \frac{l}{2} - 1$. In this case $r\rho(t) = \omega(t) = a_{2i}$, so $\rho(t) \in \{a_{2i}, b_{2i,1}, b_{2i,2}, \dots, b_{2i,m_{2i}}\}$. If $\rho(t) = a_{2i}$, since $\{a_{2i}\}$ is open and $\gamma|_{[2i-1, 2i+1]}^{-1}(a_{2i})$ is dense in $[2i-1, 2i+1]$, ρ and γ have a coincidence point. Suppose then that $\rho(t) = b_{2i,j}$ for some $1 \leq j \leq m_{2i}$. If $\rho|_{(2i-1, 2i+1)}$ is constant, then $\rho(t_{2i,j}) = b_{2i,j} = \gamma(t_{2i,j})$. If $\rho|_{(2i-1, 2i+1)}$ is not constant, necessarily a_{2i} is in its image. And by the reasoning at the beginning of this case, ρ and γ have a coincidence. \square

Remark 21. A slightly different proof of Theorem 20 may be given using a “horizontal” retraction instead of a “vertical” one. The proof would be by induction on l and would use the following result, whose proof is an easy exercise:

Let Z be a topological space and let X, Y be two subspaces of Z such that $Z = X \cup Y$. Let $z_0 \in X \cap Y$, and suppose there exists a retraction $r_X : Z \rightarrow X$ such that $r_X(Y \setminus X) = \{z_0\}$. Suppose $\gamma_X : [0, 1] \rightarrow X$ and $\gamma_Y : [0, 1] \rightarrow Y$ are strong strategies for the cop in X and Y respectively and that $\gamma_X^{-1}(z_0) = \{1\}$ while $\gamma_Y(0) = z_0$. Furthermore, suppose one of the following holds:

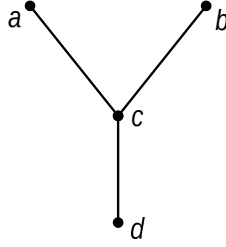
- i) There exists a retraction $r_Y : Z \rightarrow Y$ such that $r_Y(X \setminus Y) = \{z_0\}$ and $\gamma_Y^{-1}(z_0) = \{0\}$, or
- ii) Y is an open subspace of Z and for any neighborhood $U \subseteq [0, 1]$ of 0 and any map $\rho : U \rightarrow Y$, ρ and $\gamma_Y|_U$ have a coincidence point.

Then $\gamma = \gamma_X * \gamma_Y : [0, 1] \rightarrow Z$ is a strong strategy for the cop in Z .

Note that the property of having a strategy is not invariant under reversal of the order. Concretely the cop has a strategy in $S_{2,1}^{op}$ while the robber has a strategy in $S_{2,1}$.

Theorem 20 completes the classification of the finite T_0 spaces of height 1 in which the cop has a strategy. On the other hand, by Corollary 14 if X is any finite T_0 space and the robber has a strategy in $\mathfrak{E}(X)$, so does he in X . What if the cop has a strategy in $\mathfrak{E}(X)$? This does not mean that he has a strategy in X as we will see.

Example 22. The robber has a strategy in the space $X = \mathfrak{Y}$ in Figure 13.

FIGURE 13. The space \mathfrak{Q} .

Let $\gamma : [0, 1] \rightarrow X$ be a path. Let $\mathcal{U} = \{U_a, U_b\}$. We consider a minimal γ -admissible subdivision of $[0, 1]$. Suppose $[t_0, t_1], [t_1, t_2]$ are adjacent intervals in the subdivision. Without loss of generality we assume U_a has been chosen for $[t_0, t_1]$ (so in particular $\gamma([t_0, t_1]) \subseteq U_a$) and U_b has been chosen for $[t_1, t_2]$. Since $U_a \cap U_b = \{c, d\}$ is open, there exist $s_0, s_1 \in [0, 1]$ with $\frac{t_0+t_1}{2} < s_0 < t_1 < s_1 < \frac{t_1+t_2}{2}$ and $\gamma((s_0, s_1)) \subseteq \{c, d\}$. We will define $\rho|_{[\frac{t_0+t_1}{2}, \frac{t_1+t_2}{2}]}$. If $d \notin \gamma((s_0, s_1))$, we define ρ to be d in (s_0, s_1) , a in $[s_1, \frac{t_1+t_2}{2}]$, b in $[\frac{t_0+t_1}{2}, s_0]$. On the other hand, if $d \in \gamma((s_0, s_1))$, since $\{d\}$ is open in X , there exist $s_0 < r_0 < r_1 < s_1$ such that γ is identically d in (r_0, r_1) . Then we define ρ to be c in (r_0, r_1) , a in $[r_1, \frac{t_1+t_2}{2}]$, b in $[\frac{t_0+t_1}{2}, r_0]$. In any case, $\rho|_{[\frac{t_0+t_1}{2}, \frac{t_1+t_2}{2}]}$ is continuous. We do this for every pair of adjacent intervals in the subdivision of $[0, 1]$. As always, the definition of ρ in the first interval of the subdivision and the last one is simpler and left to the reader. The resulting map $\rho : [0, 1] \rightarrow X$ is continuous and has no coincidences with γ .

Corollary 23. *Let X be a finite T_0 space which contains a subspace homeomorphic to \mathfrak{Q} , which we identify with \mathfrak{Q} . Suppose $\{a, b\} \subseteq \mathfrak{Q}$ has no upper bound in X . Then \mathfrak{Q} is a retract of X , and in particular the robber has a strategy in X .*

Proof. Define $r : X \rightarrow \mathfrak{Q}$ as follows. Let $x \in X$. If $x \geq a$, $r(x) = a$. If $x \geq b$, $r(x) = b$. If $x \leq d$, $r(x) = d$. In any other case, $r(x) = c$. Note that r is well defined by hypothesis. We check that r is continuous. Let $x \leq y$ be two points in X . If $x \leq d$, then $r(x) = d \leq r(y)$. If $x \geq a$, then $r(x) = a = r(y)$. Similarly, if $x \geq b$, $r(x) = r(y)$. In any other case $r(x) = c$ and $r(y) \neq d$, so $r(x) \leq r(y)$. Since r fixes \mathfrak{Q} pointwise, it is a retraction. \square

Remark 24. In Corollary 14 we proved that if the cop has a strategy in a finite T_0 space X , then so does he in $\mathfrak{E}(X)$. We used Proposition 13. Corollary 23 can be used to give an alternative proof. If the cop has a strategy in X , then any non-minimal point is smaller than or equal to a unique maximal element. Otherwise, the hypotheses of Corollary 23 would be fulfilled. In this case $\mathfrak{E}(X)$ is a retract of X , with a retraction which maps every non-minimal point to the unique greater maximal point. Thus, the cop has a strategy in $\mathfrak{E}(X)$.

From Corollary 14, Theorem 20 and Corollary 23, we deduce the following result.

Corollary 25. *If X is a finite T_0 space in which the cop has a strategy, then*

- $\mathfrak{E}(X)$ is simply-connected,
- $S_{2,1}$ and $S_{3,0}^{op}$ are not subspaces of $\mathfrak{E}(X)$,
- There is no subspace A of X homeomorphic to \mathfrak{Q} with $\mathfrak{E}(A) \subseteq \mathfrak{E}(X)$.

We do not know if the converse of Corollary 25 holds. In the remaining of the paper we prove a converse in a particular case and analyze a delicate example.

7. THE MAX STRATEGY

Theorem 26. *Let X be a finite T_0 space with maximum x . Then the cop has a strategy in X . Moreover, there exists a loop $\zeta_X : [0, 1] \rightarrow X$ at x which has a coincidence with any other path in X and, furthermore, for every neighborhood U of 0 in $[0, 1]$ and every continuous map $\rho : U \rightarrow X$, there is a coincidence between $\zeta_X|_U$ and ρ . The map ζ_X has the following property: if X has a minimum y , then $\zeta_X^{-1}(y)$ is dense in $[0, 1]$.*

Proof. The proof is by induction on $n = \#X$. If $n = 1$, the unique path $\zeta_X : [0, 1] \rightarrow X$ satisfies all the requirements. Let X be any finite T_0 space with maximum x . Let x_0, x_1, \dots, x_{m-1} be the points covered by x . By induction there exist paths $\zeta_{U_{x_i}} : [0, 1] \rightarrow U_{x_i}$ satisfying the conditions in the statement for every $0 \leq i \leq m-1$. We define $\zeta = \zeta_X : [0, 1] \rightarrow X$ as follows. As required $\zeta(0) = \zeta(1) = x$. For each $k \geq 2$, $\zeta(\frac{1}{k}) = x_{\bar{k}}$, where \bar{k} is the class of k modulo m . Moreover ζ restricted to the interval $[\frac{1}{k}, \frac{1}{2}(\frac{1}{k} + \frac{1}{k-1})]$ is (the linear reparametrization of) $\zeta_{U_{x_{\bar{k}}}}$ for $k \geq 2$. Finally, ζ restricted to $[\frac{1}{2}(\frac{1}{k} + \frac{1}{k-1}), \frac{1}{k-1}]$ is a path in X from $x_{\bar{k}}$ to $x_{\overline{k-1}}$ if $k \geq 3$, and a path from $x_{\bar{2}}$ to x if $k = 2$. If X has a minimum y , we take this path to be constant with value y in $(\frac{1}{2}(\frac{1}{k} + \frac{1}{k-1}), \frac{1}{k-1})$ (see Figure 14). Note that ζ is continuous in 0 since x is the maximum of X . Note also that if X has minimum y , then y is the minimum of every U_{x_i} , so by induction $\zeta^{-1}(y)$ is dense in $[0, 1]$.

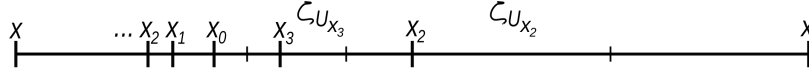


FIGURE 14. The path ζ in the case that the maximum covers $m = 4$ points.

Let U be a neighborhood of 0 in $[0, 1]$ and let $\rho : U \rightarrow X$ be a continuous map. We want to prove that $\zeta|_U$ and ρ have a coincidence point. If $\rho(0) = x$, then $\rho(0) = \zeta(0)$. Suppose then that $\rho(0) \neq x$. Then $\rho(0) \in U_{x_i}$ for some $0 \leq i \leq m-1$. Since U_{x_i} is open, there is a neighborhood V of 0 in U mapped into U_{x_i} by ρ . There exists $k \in \mathbb{N}$ such that $\bar{k} = i$ and $I = [\frac{1}{k}, \frac{1}{2}(\frac{1}{k} + \frac{1}{k-1})] \subseteq V$. Thus the restriction of ζ to I is (the linear reparametrization of) $\zeta_{U_{x_i}}$ and therefore it has a coincidence with $\rho|_I$. \square

Definition 27. Let X be a finite T_0 space with maximum. Any path $\zeta_X : [0, 1] \rightarrow X$ as constructed in the proof of Theorem 26 will be called *the max strategy* in X , even though it is not uniquely determined by X . Any linear reparametrization $[t_0, t_1] \rightarrow X$ of ζ_X will also be called the max strategy in X .

Theorem 28. *Let X be a finite T_0 space such that the following conditions are satisfied simultaneously:*

- (1) $\mathfrak{E}(X)$ is simply-connected,
- (2) $S_{2,1}$ and $S_{3,0}^{op}$ are not subspaces of $\mathfrak{E}(X)$,
- (3) There is no subspace A of X homeomorphic to \mathfrak{H} with $\mathfrak{E}(A) \subseteq \mathfrak{E}(X)$.
- (4) There is no subspace A of X homeomorphic to \mathfrak{H}^{op} with $\mathfrak{E}(A) \subseteq \mathfrak{E}(X)$

Then the cop has a strategy in X .

Proof. The fact of $\mathfrak{C}(X)$ being simply-connected is equivalent to $\mathfrak{C}(X)$ containing no cycles.

Just as in the proof of Theorem 20, we can assume $\mathfrak{C}(X)$ looks as in Figure 11. We may assume a_0 and a_l are minimal as well: adding points a_{-1} and a_{l+1} to X if needed, as described in the proof of Theorem 20, comparable with just one point of X each, produces a new space satisfying all four conditions and having X as a retract.

We define a path $\gamma : [0, l] \rightarrow X$. The first part of the definition looks very similar to the one of the strategy constructed in Theorem 20, but then, in some intervals we will use the max strategy of certain subspaces of X . We define $\gamma(2i + 1) = a_{2i+1}$ for $0 \leq i \leq \frac{l}{2} - 1$. The interval $[2i - 1, 2i + 1]$ (for $1 \leq i \leq \frac{l}{2} - 1$) is subdivided in $m_{2i} + 2$ intervals by taking points $2i - 1 < t_{2i,1} < t_{2i,2} < \dots < t_{2i,m_{2i}} < t_{2i,m_{2i}+1} < 2i + 1$. We define $\gamma(t_{2i,j}) = b_{2i,j}$ for $1 \leq j \leq m_{2i}$.

The intervals $[2i - 1, t_{2i,1}]$ for $1 \leq i \leq \frac{l}{2} - 1$ are divided in two parts by taking a point $2i - 1 < r_{2i-1} < t_{2i,1}$. The intervals $[t_{2i,j}, t_{2i,j+1}]$ for $1 \leq j \leq m_{2i}$ are divided in two by taking $t_{2i,j} < r_{2i,j} < t_{2i,j+1}$.

In the interval $[0, 1]$, γ is the inverse path of the max strategy of U_{a_1} . In $[l - 1, l]$, γ is the max strategy of $U_{a_{l-1}}$. In the intervals $[t_{2i,j}, r_{2i,j}]$ for $1 \leq j \leq m_{2i}$, γ is the max strategy of $U_{b_{2i,j}}$. In $(r_{2i,j}, t_{2i,j+1})$ for $1 \leq j \leq m_{2i}$, it is constant with value a_{2i} . In $[t_{2i,m_{2i}+1}, 2i + 1]$ it is the inverse path of the max strategy for $U_{a_{2i+1}} \cap F_{a_{2i}}$. Here F_x denotes the closure of $\{x\}$, i.e. the set of points which are greater than x . Finally, in $[2i - 1, r_{2i-1}]$ it is the max strategy of $U_{a_{2i-1}} \cap F_{a_{2i}}$ while in $(r_{2i-1}, t_{2i,1})$ it is the constant function a_{2i} . Clearly $\gamma : [0, l] \rightarrow X$ is continuous. We will prove that any other path $\rho : [0, l] \rightarrow X$ has a coincidence with γ .

Let $\rho : [0, l] \rightarrow X$ be a continuous map.

Let Y be the subspace of X formed by points a_1, a_2, \dots, a_{l-1} . We define a retraction $r : X \rightarrow Y$ as follows. For $0 \leq i \leq \frac{l}{2} - 1$, all the points of $U_{a_{2i+1}}$ not in Y are mapped to a_{2i+1} . For $1 \leq i \leq \frac{l}{2} - 1$ and $1 \leq j \leq m_{2i}$, all the points of $U_{b_{2i,j}}$ are mapped to a_{2i} . The map r is well-defined thanks to condition 3. It is clearly continuous.

Let $\omega : [1, l - 1] \rightarrow Y$ be the map defined in the proof of Theorem 20. By Remark 10 there exists $t \in [1, l - 1]$ such that $\omega(t) = r\rho|_{[1, l-1]}(t)$.

Case 1: $t = 1$. In this case $r\rho(1) = \omega(1) = a_1$, so $\rho(1) \in U_{a_1}$. Let $U = \rho^{-1}(U_{a_1}) \cap [0, 1]$. Then U is a neighborhood of 1 in $[0, 1]$. By Theorem 26, ρ and the inverse of $\zeta_{U_{a_1}}$ have a coincidence in U . Then so do ρ and γ .

Case 2: $t = l - 1$. This case is symmetric to case 1.

Case 3: $t = 2i - 1$ for some $2 \leq i \leq \frac{l}{2} - 1$. In this case $r\rho(t) = \omega(t) = a_{2i-1}$, so $\rho(t) \in U_{a_{2i-1}}$. If $\rho(t) = a_{2i-1}$, then $\rho(t) = \gamma(t)$. Otherwise $\rho(t) < a_{2i-1}$. Then either $a_{2i-2} \leq \rho(t)$ or $a_{2i} \leq \rho(t)$. In the first case $U_{\rho(t)} \subseteq U_{a_{2i-1}} \cap F_{a_{2i-2}}$ by condition 4. Therefore ρ and the inverse of $\zeta_{U_{a_{2i-1}} \cap F_{a_{2i-2}}}$ have a coincidence in $[t_{2i-2, m_{2i-2}+1}, 2i - 1]$. In the second case $U_{\rho(t)} \subseteq U_{a_{2i-1}} \cap F_{a_{2i}}$, and ρ and γ have a coincidence in $[2i - 1, r_{2i-1}]$.

Case 4: $t \in (2i - 1, 2i + 1)$ for some $1 \leq i \leq \frac{l}{2} - 1$. In this case $r\rho(t) = \omega(t) = a_{2i}$, so $\rho(t) = a_{2i}$ or $\rho(t) \in U_{b_{2i,j}}$ for some j . Note that by Theorem 26, $\gamma^{-1}(a_{2i}) \cap [2i - 1, 2i + 1]$ is dense in $[2i - 1, 2i - 1]$, so in the first case ρ and γ have a coincidence in $[2i - 1, 2i + 1]$. We can assume then that $\rho(t) \in U_{b_{2i,j}}$ for some j and that $\rho(t') \neq a_{2i}$ for every $t' \in (2i - 1, 2i + 1)$. By condition 3, $U_{b_{2i,j}} \setminus \{a_{2i}\}$ is a connected component of $X \setminus \{a_{2i}\}$, so $\rho((2i - 1, 2i + 1)) \subseteq U_{b_{2i,j}} \setminus \{a_{2i}\}$. In particular ρ and γ have a coincidence in $[t_{2i,j}, r_{2i,j}]$. \square

8. SELF-SIMILARITY

The fourth condition in the statement of Theorem 20 is not needed for the cop to have a strategy. This can be seen directly from the fact that the cop has a strategy in any space with maximum. In this section we describe an example in which the cop has a strategy, but any such strategy has to be more sophisticated than any we have constructed so far.

Self-similarity is a feature present in fractals, in which proper parts of an object look like the global object. This is a property our example will have.

Let X be the space in Figure 15. We describe a strong strategy (in fact a path $\gamma : [-1, 2] \rightarrow X$) for the cop in X . First we define $\sigma = \gamma|_{[0,1]} : [0, 1] \rightarrow X$. We divide $[0, 1]$ in four intervals of equal length. In the first interval $I_1 = [0, \frac{1}{4}]$, σ is defined as d in the extremes and as f in the interior. In the third interval $I_3 = [\frac{1}{2}, \frac{3}{4}]$, σ is defined as b in the left extreme, as d in the right extreme and as e in the interior. We also define $\sigma(1) = b$. This is stage 1 in the definition of σ . In the second and fourth intervals $I_2 = [\frac{1}{4}, \frac{1}{2}]$, $I_4 = [\frac{3}{4}, 1]$, $\sigma|_{I_2}$ and $\sigma|_{I_4}$ are the linear reparametrizations of $\sigma : [0, 1] \rightarrow X$. Concretely, to define σ in I_2 we repeat the construction above: we divide I_2 in four intervals of equal length and define σ in the first and third exactly as before. We do the same for I_4 . Note that $\sigma(\frac{1}{4}), \sigma(\frac{1}{2}), \sigma(\frac{3}{4})$ and $\sigma(1)$ are well-defined. Now, σ has been defined in all of $[0, 1]$ with exception of four (open) intervals of length $\frac{1}{16}$. This is the end of stage 2. In general, in stage n we define σ in 2^n closed intervals of length $\frac{1}{4^n}$. We repeat this process indefinitely. Now σ has been defined in all of $[0, 1]$ with exception of a G_δ set C of Lebesgue measure 0. We will not need to know much about C . It is easy to see that in stage n of the construction, σ has been defined in the complement of a union of 2^n open intervals, and $t = \frac{1}{3}$ always lies in the first of these. Thus $\frac{1}{3} \in C$, and in fact it is the minimum of C . The supremum of C is $1 \notin C$. Note also that $\sigma^{-1}(f) \cap [0, \frac{1}{3}]$ is dense in $[0, \frac{1}{3}]$. We define $\sigma|_C$ to be the constant function b . We claim that $\sigma : [0, 1] \rightarrow X$ is continuous.

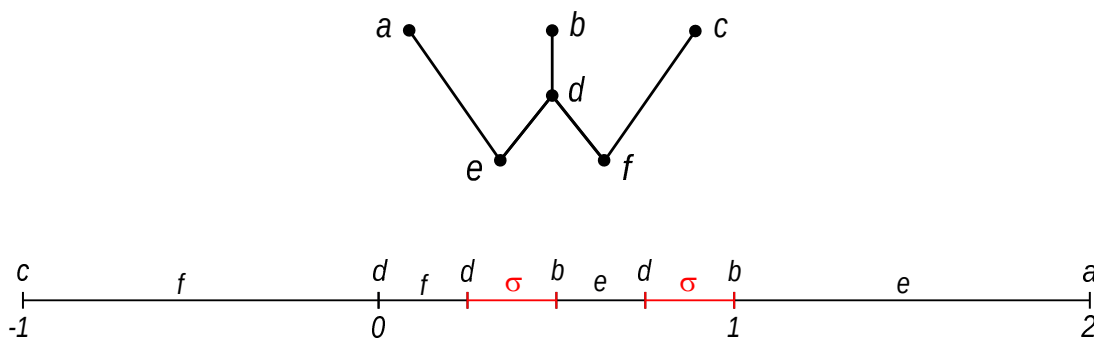


FIGURE 15. The space X and its fractal strategy γ .

Indeed, let $t \in [0, 1]$. We note that in each case there is a neighborhood of t which is mapped inside $U_{\sigma(t)}$. If $\sigma(t) = b$, we can take the neighborhood as all $[0, 1]$. If $\sigma(t)$ is equal to e or f , then there is an open interval containing t which is also mapped to $\sigma(t)$. If $\sigma(t) = d$, there is an open interval I containing t with $\sigma(I \setminus \{t\}) \subseteq \{e, f\} \subseteq U_d$.

Before we complete the definition of the strategy for the cop, we discuss some properties of σ .

Property (i): if $t \in \sigma^{-1}(b)$, then any neighborhood U of t contains an interval $[s, s']$, which in turn contains a point $s'' \in (s, s')$, such that $\sigma([s, s'']) = \{e\}$, $\sigma(s'') = d$ and $\sigma((s'', s']) = \{f\}$. Indeed, by definition there is some $n \geq 1$ such that in stage n one of the open intervals, say J , in which σ has not been defined yet lies inside U . Thus, in stage $n + 1$, the closure \bar{J} of J is divided in four closed intervals J_1, J_2, J_3, J_4 and σ is defined in J_1 and J_3 . Then, in stage $n + 2$, J_4 is divided in four intervals J'_1, J'_2, J'_3, J'_4 and σ is defined in the first and the third one. If $J_3 = [r, r']$ and $J'_1 = [r', r'']$, the points $s = \frac{r+r'}{2}$, $s' = \frac{r'+r''}{2}$, $s'' = r'$ satisfy the required condition. Note in particular that for any neighborhood U of t , $\sigma(U) = U_b$.

Property (ii): If $r_0 < r_1 \in [0, 1]$ and $\sigma(r_0) = f, \sigma(r_1) = e$, then there exists $r_0 < r < r_1$ such that $\sigma(r) = b$. This is easy to prove by considering the stages n, m in which $\sigma(r_0)$ and $\sigma(r_1)$ have been defined and analyzing the cases $n = m, n < m$ and $n > m$.

Property (iii): $\sigma^{-1}(\{e, f\})$ is dense in $[0, 1]$. For $t \in [0, 1]$ it is easy to see that if $\sigma(t)$ was defined in stage n , then any neighborhood of t already contains a point s such that $\sigma(s)$ was also defined in stage n and it is equal to e or f . For $t = 1$, $\sigma(t) = b$ so the assertion follows directly from Property (i).

We define now the strong strategy $\gamma : [-1, 2] \rightarrow X$ for the cop: $\gamma(-1) = c$, $\gamma|_{(-1,0)}$ is constant with value f , $\gamma|_{[0,1]} = \sigma$, $\gamma|_{(1,2)}$ is constant with value e , and $\gamma(2) = a$. Clearly γ is continuous. Let $\rho : [-1, 2] \rightarrow X$. We want to prove that there exists a point in which γ and ρ coincide. Assume there is no such coincidence.

Let $t \in [0, 1]$ be such that $\sigma(t) = b$. We claim then that $\rho(t) \in \{a, c\}$. If $\rho(t) \in \{e, f\}$, there is a neighborhood U of t mapped to $\rho(t)$ by ρ . But since $\sigma(t) = b$, by Property (i), $\sigma(U \cap [0, 1]) = U_b \ni \rho(t)$. Thus, γ and ρ have a coincidence in $U \cap [0, 1]$. If $\rho(t) = d$, there is a neighborhood U of t mapped to U_d by ρ . By Property (i), there exists an interval $[s, s'] \subseteq U$ and a point $s'' \in (s, s')$ such that $\sigma|_{[s, s'']}$ is e , $\sigma(s'') = d$, and $\sigma|_{(s'', s']}$ is f . But then $\sigma|_{[s, s']}$ is a strong strategy in the space U_d , so it must have a coincidence with $\rho|_{[s, s]}$. Thus, $\rho(t)$ is equal to a or c .

In particular, when $t = 1$, we deduce $\rho(1) = c$, since if $\rho(1) = a$, there would be a coincidence in $(1, 2]$. Let $t_1 = \min\{t \in [0, 1] : \rho(t) = c\}$. We claim that $t_1 > \frac{1}{3}$. Otherwise, since $\gamma^{-1}(f) \cap [-1, \frac{1}{3}]$ is dense in $[-1, \frac{1}{3}]$, $\rho|_{[-1, t_1]}$ could not go through f , so it should be constant, and we would have $\gamma(-1) = c = \rho(-1)$. Since $\frac{1}{3} < t_1$ and $\sigma(\frac{1}{3}) = b$, then $\rho(\frac{1}{3}) = a$. Let $t_0 = \max\{t \in [0, t_1] : \rho(t) = a\}$.

Since $U_a = \{a, e\}$ and $U_c = \{c, f\}$ are open, by definition of t_0 and t_1 there exist $t_0 < s_0 < s_1 < t_1$ such that $\sigma|_{(t_0, s_0)}$ is constant with value e and $\sigma|_{(s_1, t_1)}$ is constant with value f . Since $\sigma^{-1}(\{e, f\})$ is dense in $[0, 1]$ by Property (iii), and γ and ρ are coincidence-free, there exist $r_0 \in (t_0, s_0)$ and $r_1 \in (s_1, t_1)$ such that $\sigma(r_0) = f$ and $\sigma(r_1) = e$. By Property (ii) there exists $r \in (r_0, r_1)$ such that $\sigma(r) = b$. By Property (i), $\rho(r) \in \{a, c\}$, which contradicts the definition of t_0 or t_1 .

APPENDIX

Theorem 29. *Let X be a topological space, \mathcal{U} a finite open cover of X and $\gamma : \mathbb{R}_{\geq 0} \rightarrow X$ a curve. Then there exists a minimal γ -admissible subdivision of $\mathbb{R}_{\geq 0}$.*

Proof. By a Lebesgue number argument, each interval $[n, n + 1] \subseteq \mathbb{R}_{\geq 0}$ ($n \in \mathbb{Z}_{\geq 0}$) can be subdivided into finitely many closed intervals, each of which is mapped by γ into one of the members of \mathcal{U} . This is a γ -admissible subdivision which could be non-minimal. The poset of γ -admissible subdivisions of $\mathbb{R}_{\geq 0}$ is ordered by refinement. If $\{S_\alpha\}_{\alpha \in \Lambda}$ is a non-empty

chain of γ -admissible subdivisions, then there is a lower bound defined as follows. Fix $\alpha_0 \in \Lambda$. For each interval I in S_{α_0} let \tilde{I} be the union of the intervals containing I from among all the S_α with $\alpha \in \Lambda$. It is clear that \tilde{I} is an interval and that it is not a singleton. Let $t' \notin \tilde{I}$. Then t' is in the interior of an interval in S_{α_0} or in the boundary of two intervals in S_{α_0} . In the first case, suppose $t' \in (a, b)$, with $[a, b]$ a member of S_{α_0} (or $t' = 0$ and $a = 0$). Then we claim that $(a, b) \cap \tilde{I} = \emptyset$. Otherwise, there would be an interval J in some S_α containing I and intersecting (a, b) . Since S_{α_0} and S_α are comparable, either $[a, b] \subseteq J$ or $J \subseteq [a, b]$. In either case, $[a, b] \subseteq \tilde{I}$, so $t' \in \tilde{I}$, a contradiction. Similarly, if t' is in the boundary of two intervals of S_{α_0} , say $[a, t']$ and $[t', b]$, we claim that $(a, b) \cap \tilde{I} = \emptyset$. Otherwise, there is an interval J in some S_α containing I and intersecting (a, t') without loss of generality. As before this implies that $[a, t'] \subseteq \tilde{I}$, a contradiction. This proves that there is a neighborhood of t' disjoint with \tilde{I} , so \tilde{I} is closed. It is clear that the collection S of all the \tilde{I} with $I \in S_{\alpha_0}$ covers $\mathbb{R}_{\geq 0}$. Suppose \tilde{I} and \tilde{J} have at least two points in common, say a, b , for some $I, J \in S_{\alpha_0}$. Using that the family of subdivisions is a chain, there is an interval in some S_α containing both I and J , so $\tilde{I} = \tilde{J}$. This proves that the intersection of two different members of S is empty or one point. Since S_{α_0} refines S , the later is locally finite. Finally, we show that S is γ -admissible. Suppose \tilde{I} is not contained in $\gamma^{-1}(U)$ for any $U \in \mathcal{U}$. Then for every $U \in \mathcal{U}$ there exists $\alpha_U \in \Lambda$ and $J_U \in S_{\alpha_U}$ such that $I \subseteq J_U \subsetneq \gamma^{-1}(U)$. Let $V \in \mathcal{U}$ be such that S_{α_V} is the minimum among the (finitely many) S_{α_U} . Then J_V contains all the J_U , so $J_V \subsetneq \gamma^{-1}(U)$ for every $U \in \mathcal{U}$. This contradicts the fact that S_{α_V} is γ -admissible. It is clear that $S \leq S_\alpha$ for every $\alpha \in \Lambda$. Indeed, if $J \in S_\alpha$, there is $I \in S_{\alpha_0}$ intersecting J in at least two points, so either $I \subseteq J$ or $J \subseteq I$. In any case $J \subseteq \tilde{I}$, so S_α is a refinement of S . By Zorn's Lemma there is a minimal element in the poset of γ -admissible subdivisions of $\mathbb{R}_{\geq 0}$, which is then a minimal subdivision. \square

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