# AUTOMORPHISM GROUPS OF FINITE POSETS

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Abstract. For any finite group $G$, we construct a finite poset (or equivalently, a finite $T_0$-space) $X$, whose group of automorphisms is isomorphic to $G$. If the order of the group is $n$ and it has $r$ generators, $X$ has $n(r+2)$ points. This construction improves previous results by G. Birkhoff and M.C. Thornton. The relationship between automorphisms and homotopy types is also analyzed.

## 1. Introduction

It is well known that any finite group $G$ can be realized as the automorphism group of a finite poset. In 1946 Birkhoff [1] proved that if the order of $G$ is $n$, $G$ can be realized as the automorphisms of a poset with $n(n+1)$ points. In 1972 Thornton [2] improved slightly Birkhoff's result: He obtained a poset of $n(2r+1)$ points, when the group is generated by $r$ elements. Following Birkhoff's and Thornton's ideas, we exhibit here a simple proof of the following fact which improves their results

**Theorem.** *Given a group $G$ of finite order $n$ with $r$ generators, there exists a poset $X$ with $n(r+2)$ points such that $Aut(X) \simeq G$.*

The proof of the theorem uses basic topology. Recall that there exists a one-to-one correspondence between finite posets and finite $T_0$-topological spaces. Given a finite poset $X$, the subsets $U_x = \{y \in X \mid y \leq x\}$ constitute a basis for a topology on the set $X$. Conversely, given a $T_0$-topology on the set $X$, one can define a partial order given by $x \leq y$ if $x$ is contained in every open set which contains $y$. It is easy to see that these applications are mutually inverse. Therefore we regard finite posets and finite $T_0$-spaces as the same objects. Order preserving functions correspond to continuous maps and lower sets to open sets. A finite poset is connected if and only if it is connected as a topological space. For further details see [3].

## 2. The proof

Let $\{h_1, h_2, \dots, h_r\}$ be a set of $r$ generators of $G$. We define the poset $X = G \times \{-1, 0, \dots, r\}$ with the following order

- $(g, i) \leq (g, j)$ if $-1 \leq i \leq j \leq r$
- $(gh_i, -1) \leq (g, j)$ if $1 \leq i \leq j \leq r$

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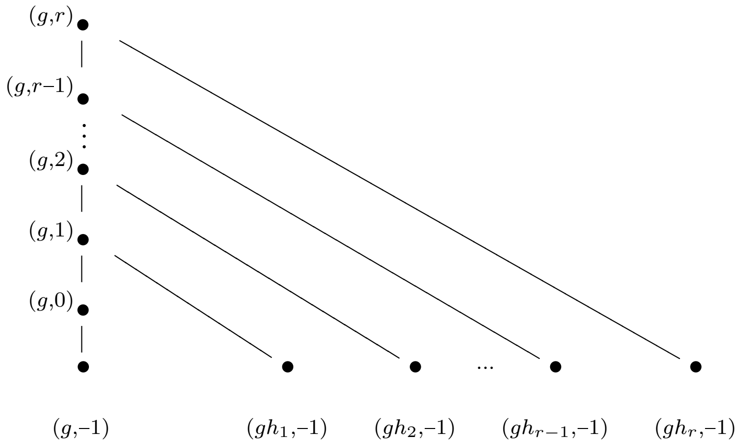


Fig. 1: $U_{(g,r)}$

Define $\phi : G \to Aut(X)$ by $\phi(g)(h,i) = (gh,i)$. It is easy to see that $\phi(g) : X \to X$ is order preserving and that it is an automorphism with inverse $\phi(g^{-1})$. Therefore $\phi$ is a well defined homomorphism. Clearly $\phi$ is a monomorphism since $\phi(g) = 1$ implies $(g,-1) = \phi(g)(e,-1) = (e,-1)$.

It remains to show that $\phi$ is an epimorphism. Let $f : X \to X$ be an automorphism. Since $(e,-1)$ is minimal in $X$, so is $f(e,-1)$ and therefore $f(e,-1) = (g,-1)$ for some $g \in G$. We will prove that $f = \phi(g)$.

Let $Y = \{x \in X \mid f(x) = \phi(g)(x)\}$. $Y$ is non-empty since $(e,-1) \in Y$. We prove first that $Y$ is an open subspace of $X$. Suppose $x = (h,i) \in Y$. Then the restrictions

$$f|_{U_x}, \phi(g)|_{U_x} : U_x \to U_{f(x)}$$

are isomorphisms. On the other hand, there exists a unique automorphism $U_x \to U_x$ since the unique chain of $i+2$ elements must be fixed by any such automorphism. Thus, $f|_{U_x}^{-1}\phi(g)|_{U_x} = 1_{U_x}$, and then $f|_{U_x} = \phi(g)|_{U_x}$, which proves that $U_x \subseteq Y$. Similarly we see that $Y \subseteq X$ is closed. Assume $x = (h,i) \notin Y$. Since $f \in Aut(X)$, it preserves the height $ht(y)$ of any point $y$. In particular $ht(f(x)) = ht(x) = i+1$ and therefore $f(x) = (k,i) = \phi(kh^{-1})(x)$ for some $k \in G$. Moreover $k \neq gh$ since $x \notin Y$. As above, $f|_{U_x} = \phi(kh^{-1})|_{U_x}$, and since $kh^{-1} \neq g$ we conclude that $U_x \cap Y = \emptyset$.

We prove now that $X$ is connected. It suffices to prove that any two minimal elements of $X$ are in the same connected component. Given $h,k \in G$, we have $h = kh_{i_1}h_{i_2}\dots h_{i_m}$ for some $1 \leq i_1,\ i_2 \dots i_m \leq r$. On the other hand, $(kh_{i_1}h_{i_2}\dots h_{i_s},-1)$ and $(kh_{i_1}h_{i_2}\dots h_{i_{s+1}},-1)$ are connected via $(kh_{i_1}h_{i_2}\dots h_{i_s},-1) < (kh_{i_1}h_{i_2}\dots h_{i_s},r) > (kh_{i_1}h_{i_2}\dots h_{i_{s+1}},-1)$. This implies that $(k,-1)$ and $(h,-1)$ are in the same connected component.

Finally, since $X$ is connected and $Y$ is closed, open and nonempty, $Y = X$, i.e. $f = \phi(g)$. Therefore $\phi$ is an epimorphim, and then $G \simeq Aut(X)$. □

## 3. Homotopy types

If the generators $h_1, h_2, \dots, h_r$ are non-trivial, the open sets $U_{(g,r)}$ look as in Fig. 1. In that case it is not hard to prove that the finite space $X$ constructed above is weak homotopy equivalent to a wedge of $n(r-1)+1$ circles, or in other words, that the order complex of $X$ is homotopy equivalent to a wedge of $n(r-1)+1$ circles. The space $X$ deformation retracts to the subspace $Y = G \times \{-1, r\}$ of its minimal and maximal points. A retraction is given by the map $f : X \to Y$, defined as $f(g, i) = (g, r)$ if $i \geq 0$ and $f(g, -1) = (g, -1)$. Now the order complex $\mathcal{K}(Y)$ of $Y$ is a connected simplicial complex of dimension 1, so its homotopy type is completely determined by its Euler Characteristic. This complex has $2n$ vertices and $n(r+1)$ edges, which means that it has the homotopy type of a wedge of $1 - \chi(\mathcal{K}(Y)) = n(r-1)+1$ circles.

On the other hand, note that in general the automorphism group of a finite space, does not say much about its homotopy type as we state in the following

*Remark.* Given a finite group $G$ and a finite space $X$, there exists a finite space $Y$ which is homotopy equivalent to $X$ and such that $Aut(Y) \simeq G$.

We make this construction in two steps. First, we find a finite $T_0$-space $\tilde{X}$ homotopy equivalent to $X$ and such that $Aut(\tilde{X}) = 0$. To do this, assume that $X$ is $T_0$ and consider a linear extension $x_1, x_2, \ldots, x_n$ of the poset $X$. Now, for each $1 \leq k \leq n$ attach a chain of length $kn$ to $X$ with minimum $x_{n-k+1}$. The resulting space $\tilde{X}$ deformation retracts to $X$ and every automorphism $f : \tilde{X} \to \tilde{X}$ must fix the unique chain $C_1$ of length $n^2$ (with minimum $x_1$). Therefore $f$ restricts to a homeomorphism $\tilde{X} \smallsetminus C_1 \to \tilde{X} \smallsetminus C_1$ which must fix the unique chain $C_2$ of length $n(n-1)$ of $\tilde{X} \smallsetminus C_1$ (with minimum $x_2$). Applying this reasoning repeatidly, we conclude that $f$ fixes every point of $\tilde{X}$. On the other hand, we know that there exists a finite $T_0$-space $Z$ such that $Aut(Z) = G$.

Now the space $Y$ is constructed as follows. Take one copy of $\tilde{X}$ and of $Z$, and put every element of $Z$ under $x_1 \in \tilde{X}$. Clearly $Y$ deformation retracts to $\tilde{X}$. Moreover, if $f : Y \to Y$ is an automorphism, $f(x_1) \notin Z$ since $f(x_1)$ cannot be comparable with $x_1$ and distinct from it. Since there is only one chain of $n^2$ elements in $\tilde{X}$, it must be fixed by $f$. In particular $f(x_1) = x_1$, and then $f|_Z : Z \to Z$. Thus $f$ restricts to automorphisms of $\tilde{X}$ and of $Z$ and therefore $Aut(Y) \simeq Aut(Z) \simeq G$.

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