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CLASSIFYING CANTOR SETS BY THEIR FRACTAL DIMENSIONS

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ABSTRACT. In this article we study Cantor sets defined by monotone sequences, in the sense of Besicovich and Taylor. We classify these Cantor sets in terms of their h-Hausdorff and h-packing measures, for the family of dimension functions h, and characterize this classification in terms of the underlying sequences.

1. INTRODUCTION

A natural way to classify compact subsets $E \subseteq \mathbb{R}$ of Lebesgue measure zero is by their Hausdorff or packing dimension. This is a crude measurement, however, which often does not distinguish salient features of the set. For a finer classification, one could consider the family of h-Hausdorff measures, H^h, and h-packing measures, P^h, where h belongs to the set of dimension functions $\mathcal{D}$, defined in Section 2.

Definition 1.1. By the *dimension partition* of a set E, we mean a partition of $\mathcal{D}$ into (six) sets $\mathcal{H}^E_\alpha \cap \mathcal{P}^E_\beta$, for $\alpha \leq \beta \in \{0, 1, \infty\}$, where

$$\begin{aligned} \mathcal{H}^E_\alpha &= \{h \in \mathcal{D} : H^h(E) = \alpha\} \text{ for } \alpha = 0, \infty, \\ \mathcal{H}^E_1 &= \{h \in \mathcal{D} : 0 < H^h(E) < \infty\}, \end{aligned}$$

and $\mathcal{P}^E_\beta$ is defined similarly, but with h-packing measure replacing h-Hausdorff measure.

Sets which have the same dimension partition will have the same Hausdorff and packing dimensions; however the converse is not necessarily true.

We call a compact, perfect, totally disconnected, measure zero subset of the real line a Cantor set. There is a natural way (see Section 3.1) to associate to each summable, non-increasing sequence $a = \{a_n\} \subseteq \mathbb{R}^+$ a unique Cantor set C_a having gaps with lengths corresponding to the terms a_n. The study of the dimension of Cantor sets by means of its gaps was initiated by Besicovich and Taylor [1]. In fact, the Hausdorff and packing dimensions of C_a can be calculated in terms of the tails of the sequence a, the numbers $r^{(a)}_n = \sum_{i \geq n} a_i$, for $n \in \mathbb{N}$. In this paper we show

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that the classification of the sets C_a according to their dimension partitions can be characterized in terms of properties of their tails.

We introduce a partial ordering, $\preceq$, on the set of dimension functions, which preserves the order of Hausdorff and packing h-measures. A set E is said to be h-regular if $h \in \mathcal{H}_1^E \cap \mathcal{P}_1^E$. If this holds for $h = x^s$, then E is said to be s-regular, and in that case the Hausdorff and packing dimensions of E are both s. Although not every Cantor set C_a of dimension s is an s-regular set, we show that one can always find a dimension function h such that C_a is an h-regular set. This completes arguments begun in [2] and [5]. The dimension functions for which a given Cantor set, C_a, is regular form an equivalence class under this ordering and are called the dimension functions associated to the sequence a.

We prove that two Cantor sets C_a, C_b have the same dimension partition if and only if their associated dimension functions, h_a, h_b, are equivalent. More generally, we prove that C_a and C_b have the same dimension partition if and only a and b are *weak tail-equivalent* (see Definition 4.2).

Furthermore, we show that the weak tail-equivalence can be replaced by the stronger *tail-equivalence* relation only when the associated dimension functions have inverse with the doubling property.

2. Dimension functions and measures

A function h is said to be *increasing* if $h(x) < h(y)$ for $x < y$ and *doubling* if there exists $\tau > 0$ such that $h(x) \geq \tau\, h(2x)$ for all x in the domain of h. We will say that a function $h : (0, A] \to (0, \infty]$ is a *dimension function* if it is continuous, increasing, doubling and $h(x) \to 0$ as $x \to 0$. We denote by $\mathcal{D}$ the set of dimension functions. A typical example of a dimension function is $h_s(x) = x^s$ for some $s > 0$.

Given any dimension function h, one can define the *h-Hausdorff measure* of a set E, $H^h(E)$ in the same manner as the Hausdorff s-measure (see [6]): Let $|B|$ denote the diameter of a set B. Then

$$H^h(E) = \lim_{\delta \to 0^+} \left(\inf \left\{ \sum h(|E_i|) : \bigcup E_i \subseteq E,\ |E_i| \leq \delta \right\} \right).$$

The Hausdorff s-measure is the special case when $h = h_s$. In terms of our notation, the Hausdorff dimension of E is given by

$$\dim_H E = \sup\{s : h_s \in \mathcal{H}_\infty^E\} = \inf\{s : h_s \in \mathcal{H}_0^E\}.$$

The h-packing pre-measure is defined as in [7]. First, recall that a δ-packing of a given set E is a disjoint family of open balls centered at points of E with diameters less than δ. The *h-packing pre-measure* of E is defined by

$$P_0^h(E) = \lim_{\delta \to 0^+} \left(\sup \left\{ \sum h(|B_i|) : \{B_i\}_i \text{ is a } \delta\text{-packing of } E \right\} \right).$$

It is clear from the definition that the set function P_0^h is monotone, but it is not a measure because it is not σ-sub-additive. The *h-packing measure* is obtained by a standard argument:

$$P^h(E) = \inf \left\{ \sum_{i=1}^{\infty} P_0^h(E_i) : E = \bigcup E_i \right\}.$$

The pre-packing dimension is the critical index given by the formula

$$\dim_{P_0} E = \sup\{s : P_0^{h_s}(E) = \infty\} = \inf\{s : P_0^{h_s}(E) = 0\}$$

and is known to coincide with the upper box dimension [7]. Clearly, $P^h(E) \le P_0^h(E)$, and as h is a doubling function, $H^h(E) \le P^h(E)$ (see [8]). The packing dimension is defined analogously and, like the Hausdorff dimension, can be obtained as

$$\dim_P(E) = \sup\{s : h_s \in \mathcal{P}_\infty^E\} = \inf\{s : h_s \in \mathcal{P}_0^E\}.$$

Finally, we note that the set E is said to be *h-regular* if $0 < H^h(E) \le P^h(E) < \infty$ and s-regular if it is h_s-regular. The set $\mathcal{H}_1^E \cap \mathcal{P}_1^E$ consists of the h-regular functions of E.

We are interested in comparing different dimension functions.

Definition 2.1. Suppose $f, h : X \to (0, A]$. We will say $f \preceq h$ if there exists a positive constant c such that

$$f(x) \le c\,h(x) \text{ for all } x.$$

We will say f is *equivalent* to h, and write $f \equiv h$, if $f \preceq h$ and $h \preceq f$.

This defines a partial ordering that is consistent with the usual pointwise ordering of functions. It is not quite the same as the ordering defined in [5], but we find it to be more natural.

Note that the definition of equivalence of functions, when applied to sequences, implies that $x = \{x_n\}$ and $y = \{y_n\}$ are equivalent if and only if there exist $c_1, c_2 > 0$ such that $c_1 \le x_n/y_n \le c_2$ for all n.

The following easy result is very useful and also motivates the definition of $\preceq$.

Proposition 2.2. *Suppose $h_1, h_2 \in \mathcal{D}$ and $h_1 \preceq h_2$. There is a positive constant c such that for any Borel set E,*

$$H^{h_1}(E) \le cH^{h_2}(E) \text{ and } P_{(0)}^{h_1}(E) \le cP_{(0)}^{h_2}(E).$$

Proof. Suppose $h_1(x) \le ch_2(x)$ for all x. For any $\delta > 0$,

$$\begin{aligned} H_\delta^{h_1}(E) &= \inf\left\{\sum h_1(|U_i|), E \subseteq \bigcup_i U_i, |U_i| < \delta\right\} \\ &\le c \inf\left\{\sum h_2(|U_i|), E \subseteq \bigcup_i U_i, |U_i| < \delta\right\} = cH_\delta^{h_2}(E). \end{aligned}$$

The arguments are similar for packing pre-measure. □

Corollary 2.3. *If $h_1, h_2 \in \mathcal{D}$ and $h_1 \equiv h_2$, then for any Borel set E, h_1 and h_2 belong to the same set $\mathcal{H}_\alpha^E \cap \mathcal{P}_\beta^E$.*

3. Cantor sets associated to sequences

3.1. **Cantor sets C_a.** Each Cantor set is completely determined by its gaps, the bounded convex components of the complement of the set. To each summable sequence of positive numbers, $a = \{a_n\}_{n=1}^\infty$, we can associate a unique Cantor set with gaps whose lengths correspond to the terms of this sequence.

To begin, let I be an interval of length $\sum_{n=1}^\infty a_n$. We remove from I an interval of length a_1. Then we remove from the left remaining interval an interval of length a_2 and from the right an interval of length a_3. Iterating this procedure, it is easy to see that we are left with a Cantor set which we will call C_a.

Observe that as $\sum a_k = |I|$, there is only one choice for the location of each interval to be removed in the construction. More precisely, the position of the first

gap we place (of length a_1) is uniquely determined by the property that the length of the remaining interval on its left should be $a_2 + a_4 + a_5 + a_8 + \dots$. Therefore, this construction defines the Cantor set unequivocally. As an example, if we take $a_n = 1/3^k$ for $n = 2^{k-1}, ..., 2^k - 1$, $k = 1, 2, \dots$, the classical middle-third Cantor set is produced. In this case the sequence $\{a_n\}$ is non-increasing.

This is also the case for any central Cantor set with fixed rate of dissection, those Cantor sets constructed in a similar manner as the classical 1/3 Cantor set but replacing 1/3 by a number $0 < a < 1/2$, where a is the ratio of the length of an interval of one step and the length of its parent interval.

We should remark that the order of the sequence is important. Different rearrangements could correspond to different Cantor sets, even of different dimensions; however, if two sequences correspond to the same Cantor set, one is clearly a rearrangement of the other. From here on we will assume that our sequence is positive, non-increasing and summable.

Given such a sequence, $a = \{a_n\}$, we denote by $r_n = r_n^{(a)}$ the *tail* of the series:

$$r_n = \sum_{j \geq n} a_j.$$

The Hausdorff and pre-packing dimensions of C_a are given by the formulas (see [2] and [5])

$$\dim_H C_a = \underline{\lim}_{n\to\infty} \frac{-\log n}{\log(r_n^{(a)}/n)} \text{ and } \dim_{P_0} C_a = \overline{\lim}_{n\to\infty} \frac{-\log n}{\log(r_n^{(a)}/n)}.$$

Motivated by the analogous result in [1] for s-Hausdorff measure, it was shown in [5] that for any dimension function h, the Hausdorff h-measure and h-packing premeasure of the Cantor set C_a are determined by the limiting behaviour of $h(r_n/n)$.

Theorem 3.1 ([5, Prop. 4.1, Thm. 4.2])**.** *For any $h \in \mathcal{D}$,*

$$\frac{1}{4} \underline{\lim}_{n\to\infty} nh(\frac{r_n^{(a)}}{n}) \leq H^h(C_a) \leq 4 \underline{\lim}_{n\to\infty} nh(\frac{r_n^{(a)}}{n}),$$

and similarly for $P_0^h(C_a)$, but with lim sup replacing lim inf.

This suggests that it will be of interest to study the following class of functions:

Definition 3.2. We will say that an increasing, continuous function $h : (0, A] \to (0, \infty]$ is *associated* to the sequence a (or to the Cantor set C_a) if the sequence $\{h(r_n^{(a)}/n)\}$ is equivalent to the sequence $\{1/n\}$.

One can check that any function h_a associated to the sequence a is a doubling function and thus belongs to $\mathcal{D}$. Indeed, if

$$c_1 h_a(\frac{r_n}{n}) \leq \frac{1}{n} \leq c_2 h_a(\frac{r_n}{n}) \text{ for all } n$$

and $r_n/n \leq x \leq r_{(n-1)}/(n-1)$, then by monotonicity,

$$h_a(2x) \leq h_a\left(\frac{2r_{n-1}}{n-1}\right) \leq h_a\left(\frac{r_{[(n-1)/2]}}{(n-1)/2}\right) \leq \frac{4}{c_1 n} \leq \frac{4c_2}{c_1} h_a(x).$$

In the special case that $a = \{n^{-1/s}\}$ for some $0 < s < 1$, it is known (see [3]) that C_a has Hausdorff dimension s and $0 < H^{h_s}(C_a) < \infty$. One can easily see that any function associated to C_a is equivalent to x^s. This generalizes to arbitrary

associated (dimension) functions, so we can speak of 'the' associated dimension function.

Lemma 3.3. *If h is associated to the sequence a and $g \in \mathcal{D}$, then $h \equiv g$ if and only if g is also associated to a.*

Proof. Suppose g is associated to a. Let $b_n = r_n^{(a)}/n$ and $b_{n+1} \le x \le b_n$. As $\{h(b_n)\} \equiv \{g(b_n)\} \equiv \{1/n\}$ and g is monotonic,

$$c_1 h(b_{n+1}) \le g(b_{n+1}) \le g(x) \le g(b_n) \le c_2 h(b_n)$$

for all n and for suitable constants c_1, c_2. Thus

$$c_1' \le c_1 \frac{h(b_{n+1})}{h(b_n)} \le \frac{g(x)}{h(x)} \le c_2 \frac{h(b_n)}{h(b_{n+1})} \le c_2'.$$

The other implication is straightforward. □

3.2. **Packing dimension of Cantor sets C_a.** The pre-packing dimension and packing pre-measure always majorizes the packing dimension and packing measure, and the strict inequality can hold. For example, it is an easy exercise to see that the packing dimension of the countable set $\{1/n\}_{n=1}^{\infty}$ is 0, but the pre-packing dimension equals 1/2. This phenomena does not happen for the Cantor sets C_a. To prove this, we begin with a technical result which generalizes [4, Prop. 2.2].

Lemma 3.4. *Let μ be a finite, regular, Borel measure and let $h \in \mathcal{D}$. If*

$$\varlimsup_{r \to 0} \frac{\mu(B(x_0, r))}{h(r)} < c \text{ for all } x_0 \in E,$$

then

$$P^h(E) \ge \frac{\mu(E)}{c}.$$

Proof. We need to prove that for any partition $\bigcup_{i=1}^{\infty} E_i = E$, we have $\sum_{i=1}^{\infty} P_0^h(E_i) \ge \mu(E)/c$. Since $\mu(E) \le \sum \mu(E_i)$, it is enough to prove that $P_0^h(E_i) \ge \mu(E_i)/c$ for each i.

Without loss of generality assume $E_i = E$, and we will show that for each $\delta > 0$,

$$P_{0,\delta}^h(E) \equiv \sup\Big\{\sum h(|B_i|) : \{B_i\}_i \text{ is a } \delta\text{-packing of } E\Big\} \le \frac{\mu(E)}{c}.$$

Consider the collection of balls $B(x, r)$, with $x \in E$ and $\mu(B(x, r)) < ch(r)$, where $r \le \delta$. The hypothesis ensures that for each $x \in E$ there are balls $B(x, r)$ in the collection, with r arbitrarily small. By the Vitali covering lemma, there are disjoint balls from this collection, $\{B_i\}_{i=1}^{\infty}$, with $\mu(E \setminus \bigcup B_i) = 0$. Thus

$$P_{0,\delta}^h(E) \ge \sum h(|B_i|) \ge \frac{1}{c} \sum \mu(B_i) = \frac{1}{c} \mu\Big(\bigcup B_i\Big) = \frac{1}{c} \mu(E). \qquad \square$$

We now specialize to the case of Cantor sets C_a associated to a non-increasing, summable sequence $a = \{a_j\}$. If we use the notation $\{I_j^{(k)}\}_{1 \le j \le 2^k}$ for the (remaining) intervals at step k in the Cantor set construction, then the sequence of lengths of these intervals, $\{|I_j^{(k)}|\}_{(k,j)}$, with $1 \le j \le 2^k, k \ge 1$ is (lexicographically) non-increasing. Hence the length of any Cantor interval of step k is at least the length of any Cantor interval of step $k+1$. This observation, together with the lemma above, is the key idea needed to prove that infinite (or positive) pre-packing measure implies infinite (respectively, positive) packing measure for the sets C_a. Of course, the other implication holds for all sets.

Theorem 3.5. *Suppose $a = \{a_j\}$ is a summable, non-increasing sequence with associated Cantor set C_a and $h \in \mathcal{D}$. If $P_0^h(C_a) = \infty$, then $P^h(C_a) = \infty$, while if $P_0^h(C_a) > 0$, then $P^h(C_a) > 0$.*

Proof. Let $b_n = r_n^{(a)}/n$. By Theorem 3.1, $P_0^h(C_a) = \infty$ implies $\overline{\lim}\, nh(b_n) = \infty$ and $P_0^h(C_a) > 0$ implies $\overline{\lim}\, nh(b_n) > 0$. Since h is increasing, if $2^k \le n \le 2^{k+1}$, then

$$nh(b_n) \le nh(b_{2^k}) \le 2^{k+1} h(b_{2^k}).$$

Therefore, $\overline{\lim}_{k\to\infty} 2^k h(b_{2^k}) = \infty$ in the first case and is (strictly) positive in the second.

The tail term, $r_{2^k}^{(a)}$, is the sum of gaps created at level $k+1$ or later, which in turn is equal to the sum of the lengths of the step k intervals. Thus $r_{2^k}/2^k$, the average length of a step k interval, is at most the length of the shortest interval of step $k-1$ and at least the length of longest interval of step $k+1$. As h is increasing,

$$h(|I_1^{(k)}|) \ge h(b_{2^k}) \ge h(|I_1^{(k+1)}|).$$

Let μ be the (uniform) Cantor measure on C_a (constructed as a limiting process, assigning at each step k the measure μ_k such that $\mu_k(I_j^k) = 2^{-k}$ and then taking the weak*-limit).

Fix $x_0 \in C_a$ and $r > 0$. Suppose k is the minimal integer such that $B(x_0, r)$ contains a step k interval. The minimality of k ensures that $B(x_0, r)$ can intersect at most 5 step k intervals. Thus $\mu(B(x_0, r)) \le 5\, 2^{-k}$. Also, if $I_j^{(k)}$ is a step k interval contained in $B(x_0, r)$, then $2r \ge |I_j^{(k)}|$. Since h is a doubling function,

$$h(r) \ge \tau\, h(2r) \ge \tau\, h(|I_j^{(k)}|) \ge \tau\, h(b_{2^{k+1}})$$

for some $\tau > 0$.

Combining these facts, we see that

$$\frac{\mu(B(x_0, r))}{h(r)} \le \frac{5 \cdot 2^{-k}}{\tau\, h(b_{2^{k+1}})} = \frac{10}{\tau\, 2^{k+1} h(b_{2^{k+1}})}.$$

Thus if $P_0^h(C_a) = \infty$, then

$$\underline{\lim}_{r\to 0} \frac{\mu(B(x_0, r))}{h(r)} = 0,$$

while if $P_0^h(C_a) > 0$, then

$$c_0 = \underline{\lim}_{r\to 0} \frac{\mu(B(x_0, r))}{h(r)} < \infty.$$

Applying Lemma 3.4 we conclude that in the first case, $P^h(C_a) \ge \mu(C_a)/c$ for every $c > 0$ and therefore $P^h(C_a) = \infty$, while $P^h(C_a) \ge \mu(C_a)/c_0 > 0$ in the second case. □

Corollary 3.6. *(i) For any dimension function h, $P_0^h(C_a) = 0$ (or ∞) if and only if $P^h(C_a) = 0$ (resp. ∞) and $0 < P_0^h(C_a) < \infty$ if and only if $0 < P^h(C_a) < \infty$.*

(ii) The packing and pre-packing dimensions of the Cantor set C_a coincide.

Theorem 3.1 was used in [5] to give sufficient conditions for two dimension functions to be equivalent. Together with Theorem 3.5, we can obtain sufficient conditions for comparability.

Proposition 3.7. *Suppose f, $h \in \mathcal{D}$. If $H^h(C_a) > 0$ and $P^f(C_a) < \infty$, then $f \preceq h$.*

Proof. Let $b_n = r_n^{(a)}/n$. Since $H^h(C_a) > 0$, Theorem 3.1 implies that there exists a constant $c_1 > 0$ such that for all sufficiently large n, $h(b_n) \geq c_1/n$. Similarly, the assumption that $P^f(C_a) < \infty$ implies $P_0^f(C_a) < \infty$ by our previous theorem, and therefore there is a constant $c_2 < \infty$ with $f(b_n) \leq c_2/n$.

Now suppose $b_n \leq x < b_{n-1}$. By monotonicity, $f(x) \leq f(b_{n-1})$ and $h(x) \geq h(b_n)$. Hence

$$\frac{f(x)}{h(x)} \leq \frac{c_2 n}{c_1(n-1)} \leq 2\frac{c_2}{c_1} < \infty,$$

and therefore $f \preceq h$. □

The next result was obtained in [5, Thm. 4.4] with packing pre-measure replacing packing measure.

Corollary 3.8. *Suppose $f, h \in \mathcal{D}$. If $0 < H^g(C_a) \leq P^g(C_a) < \infty$ for $g = f$ and h, then $f \equiv h$.*

4. Classification of Cantor sets

An immediate consequence of Theorems 3.1 and 3.5 is the following elegant description of the dimension partition for Cantor sets C_a.

Theorem 4.1. *Suppose $a = \{a_n\}$ is a non-increasing, summable sequence of positive real numbers. Then*

$$\mathcal{H}_\alpha^{C_a} = \left\{ h \in \mathcal{D} : \varliminf_{n\to\infty} nh\left(\frac{r_n^{(a)}}{n}\right) = \alpha \right\} \text{ for } \alpha = 0, \infty,$$

$$\mathcal{H}_1^{C_a} = \left\{ h \in \mathcal{D} : 0 < \varliminf_{n\to\infty} nh\left(\frac{r_n^{(a)}}{n}\right) < \infty \right\},$$

and similarly for $\mathcal{P}_\beta^{C_a}, \beta = 0, 1, \infty$, but with lim sup replaced by lim inf. (See Table 1.)

In particular, note that C_a is h-regular if and only if h is a dimension function associated to a.

Table 1. Classification of functions in $\mathcal{D}$ for C_a

	P_0	P_1	P_∞
H_0	0-h Hausdorff measure 0-h Packing measure	0-h Hausdorff measure h-Packing set	0-h Hausdorff measure ∞-h Packing measure
H_1		h-regular set	h-Hausdorff set ∞-h Packing measure
H_∞			∞-h Hausdorff measure ∞-h Packing measure

For the computation of dimensions the relevant behaviour of a sequence is that of its tail; thus we introduce the following definitions.

Definition 4.2. (a) We say two sequences a, b are *tail-equivalent* if the sequences of tails $\{r_n^{(a)}\}$, $\{r_n^{(b)}\}$ are equivalent.

(b) We say a and b are *weak tail-equivalent* if there are positive integers j, k such that $r_n^{(a)} \geq r_{jn}^{(b)}/j$ and $r_n^{(b)} \geq r_{kn}^{(a)}/k$ for all n.

Obviously, if a is tail-equivalent to b, then the dimension partitions of C_a and C_b are the same. Furthermore, equivalence of sequences implies tail-equivalence which implies weak tail-equivalence; however neither implication is reversible, as the next example illustrates.

Example 4.3. (a) Let $k_1 = 1$ and inductively define $n_j = 2^{k_j+1} - 2$ and $k_{j+1} = n_j + k_j + 1$. Set $a_{k_j} = 2^{-k_j}$, $a_n = 2^{-(2k_j+1)}$ for $n = k_j + 1, \ldots, n_j + k_j$, and set $b_n = 2^{-2k_j}$ for $n = k_j + 1, \ldots, n_j + k_j$. Since $a_{k_j}/b_{k_j} = 2^{k_j}$, $\{a_n\}$ is not equivalent to $\{b_n\}$. However one can easily check that $\{a_n\}$ is tail equivalent to $\{b_n\}$.

(b) The sequence $\{e^{-n}\}$ is weak tail-equivalent to $\{e^{-2n}\}$, but not tail-equivalent.

We are now ready to state and prove our classification result. For notational ease we write $\mathcal{H}_\alpha^a$, $\mathcal{P}_\beta^a$ rather than $\mathcal{H}_\alpha^{C_a}$, $\mathcal{P}_\beta^{C_a}$.

Theorem 4.4. *Suppose C_a and C_b are Cantor sets associated to non-increasing, summable sequences a, b of positive numbers. The following are equivalent:*

1. *The dimension function associated to a is equivalent to the dimension function associated to b.*
2. *C_a and C_b are h-regular for precisely the same set of dimension functions h.*
3. *The dimension partitions associated with C_a and C_b coincide; i.e., $\mathcal{H}_\alpha^a \cap \mathcal{P}_\beta^a = \mathcal{H}_\alpha^b \cap \mathcal{P}_\beta^b$ for all $\alpha \leq \beta \in \{0, 1, \infty\}$.*
4. *The sequence a is weak tail-equivalent to the sequence b.*

Proof. $(2 \Rightarrow 1)$ Since C_b is h_b-regular, C_a must also be h_b-regular. Thus h_a, $h_b \in \mathcal{H}_1^a \cap \mathcal{P}_1^a$. But all functions in $\mathcal{H}_1^a \cap \mathcal{P}_1^a$ are equivalent by Corollary 3.8.

$(3 \Rightarrow 2)$ is obvious since (2) could be stated as $\mathcal{H}_1^a \cap \mathcal{P}_1^a = \mathcal{H}_1^b \cap \mathcal{P}_1^b$.

$(4 \Rightarrow 3)$ It is easy to see that if there exists an integer j such that $r_n^{(a)} \geq r_{jn}^{(b)}/j$, then $\mathcal{H}_0^a \subseteq \mathcal{H}_0^b$ and $\mathcal{H}_\infty^b \subseteq \mathcal{H}_\infty^a$.

To see that $\mathcal{P}_\infty^b \subseteq \mathcal{P}_\infty^a$, suppose there is a sequence $\{n_i\}$ with $\lim_i n_i h\left(r_{n_i}^{(b)}/n_i\right) = \infty$. Let $m_i = \left[\frac{n_i}{j}\right]$, where $[z]$ means the integer part of z. As $n_i \geq j m_i$ and r_n/n is decreasing,

$$\frac{r_{n_i}^{(b)}}{n_i} \leq \frac{r_{jm_i}^{(b)}}{jm_i} \leq \frac{r_{m_i}^{(a)}}{m_i}.$$

Since h is increasing

$$2jm_i h\left(\frac{r_{m_i}^{(a)}}{m_i}\right) \geq n_i h\left(\frac{r_{n_i}^{(b)}}{n_i}\right) \to \infty;$$

therefore, $\overline{\lim}\, nh(\frac{r_n^{(a)}}{n}) = \infty$. A similar argument proves $\mathcal{P}_0^a \subseteq \mathcal{P}_0^b$.

Consequently, if a is weak tail-equivalent to b, then $\mathcal{H}_\alpha^a = \mathcal{H}_\alpha^b$ and $\mathcal{P}_\alpha^a = \mathcal{P}_\alpha^b$ for $\alpha = 0, \infty$. This forces $\mathcal{H}_1^a = \mathcal{H}_1^b$ and $\mathcal{P}_1^a = \mathcal{P}_1^b$.

$(1 \Rightarrow 4)$ We will prove a slightly more general result; namely, if $h_a \preceq h_b$, then there is a positive integer j such that $r_n^{(a)} \geq r_{jn}^{(b)}/j$. There is no loss of generality in assuming $h_x(r_n^{(a)}/n) = h_x(r_n^{(b)}/n) = 1/n$. As $h_a \preceq h_b$ there is an integer $j > 0$ such that

$$h_b\left(\frac{r_n^{(a)}}{n}\right) \geq \frac{1}{j}\, h_a\left(\frac{r_n^{(a)}}{n}\right) = \frac{1}{jn} = h_b\left(\frac{r_{jn}^{(b)}}{jn}\right).$$

The increasingness of h_b establishes the claim. □

Corollary 4.5. *If $h_a \equiv h_b$, then C_a and C_b have the same Hausdorff and packing dimensions.*

Example 4.6. In [5, Ex. 4.6] a construction is given of a Cantor set C_a which has Hausdorff and packing dimension s, but where $P_0^s(C_a) = \infty$. Thus if C_b is any s-regular Cantor set, then C_a and C_b have the same dimensions, but not the same dimension partitions.

In Theorem 4.4 we proved that Cantor sets C_a and C_b have the same dimension partition if and only if the sequences a and b are weak tail-equivalent. We conclude the paper by determining when the class of sequences weak tail-equivalent to a coincides with the class of sequences tail-equivalent to a.

Recall the example of the sequences $a = \{e^{-n}\}$ and $b = \{e^{-2n}\}$ which are weak tail-equivalent, but not tail-equivalent. We can take $h_a(x) = |\log x|^{-1}$ and $h_b(x) = |\log \sqrt{x}|^{-1}$. These associated dimension functions are, of course, equivalent, but their inverse functions are neither equivalent nor doubling. As we see below, the latter is the key property.

Lemma 4.7. *If $h \equiv g$ and h^{-1} is doubling, then $h^{-1} \equiv g^{-1}$. In particular g^{-1} is doubling.*

Proof. Since $h \equiv g$ we have $c_1 h(g^{-1}(y)) \leq y \leq c_2 h(g^{-1}(y))$ for suitable constants c_1, c_2. As h^{-1} is doubling this implies that for some (probably different) positive constants, $c_1 g^{-1}(y) \leq h^{-1}(y) \leq c_2 g^{-1}(y)$, and therefore $g^{-1} \equiv h^{-1}$. □

We can now prove the following theorem.

Theorem 4.8. *Given a non-increasing, summable sequence a, let W_a denote the equivalence class of sequences weak tail-equivalent to a. The following are equivalent:*

(1) *There exists $b \in W_a$ such that h_b^{-1} is doubling.*
(2) *For every $b \in W_a$, h_b^{-1} is doubling.*
(3) *Every $b \in W_a$ is tail-equivalent to a.*

Proof. (1) $\Longleftrightarrow$ (2) follows from the previous lemma.

(2) $\Longrightarrow$ (3) Assume $b \in W_a$. Then $h_a^{-1} \equiv h_b^{-1}$ and both are doubling. These properties ensure that

$$c_1 h_a^{-1}\left(\frac{1}{n}\right) \leq h_b^{-1}\left(\frac{1}{n}\right) \leq c_2 h_a^{-1}\left(\frac{1}{n}\right)$$

and hence that there exist constants c_1', c_2' such that

$$c_1' \frac{r_n^{(a)}}{n} \leq \frac{r_n^{(b)}}{n} \leq c_2' \frac{r_n^{(a)}}{n};$$

that is, the sequence a is tail-equivalent to b.

(3) $\Longrightarrow$ (1) We will prove this by contradiction. Suppose that h_a^{-1} is not doubling. We will show there is a sequence $b \in W_a$ that is not tail-equivalent to a. Indeed, let $b = \{b_k\}$ be defined as

$$b_1 = a_1, \qquad b_{2k} = b_{2k+1} = \frac{a_k}{2}.$$

Then $r_{2n}^{(b)} = r_n^{(a)}$, and by the definition of h_a, there exist constants $c_1 \leq c_2$ such that

$$\frac{r_n^{(b)}}{r_n^{(a)}} = \frac{1}{2}\frac{r_{n/2}^{(a)}}{n/2}\frac{n}{r_n^{(a)}} \geq \frac{1}{2}\frac{h_a^{-1}(2c_1/n)}{h_a^{-1}(c_2/n)} \geq \frac{1}{2}\frac{h_a^{-1}(2c_1/n)}{h_a^{-1}(c_1/n)}, \tag{4.1}$$

where in the last inequality we used the fact that h_a^{-1} is increasing.

Since h_a^{-1} is both increasing and non-doubling, it follows that

$$\sup_n \frac{h_a^{-1}(2/n)}{h_a^{-1}(1/n)} = \infty.$$

Using this in equation (4.1) we conclude that the ratios $r_n^{(b)}/r_n^{(a)}$ are not bounded above and thus the sequences a and b are not tail-equivalent. However, as they are clearly weak tail-equivalent, $b \in W_a$. This contradicts (3). □

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