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GEOMETRIC METHODS IN WAVELET THEORY

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To the memory of our teacher and colleague Pucho Larotonda, who taught us much more than Mathematics.

Abstract. In this paper we present an overview of how geometric methods can be successfully used to solve problems in Analysis. We will focus on *self-similar* objects and use their structure to construct frames, Riesz bases and wavelet bases in $\mathbb{R}^d$ with a single generator function. Further, we show that the generating functions for these systems are dense in $\mathcal{L}^2(\mathbb{R}^d)$.

1. Introduction

In this article we review some geometric methods that have proven to be very successful in different contexts and we show their application to wavelet construction (sections 5 and 6).

We describe the notion of self-similarity as developed by Hutchinson [Hut81] and its characterization in terms of contraction mappings in general metric spaces. We show that when these results are applied to affine functions in the euclidean space using a fixed expanding matrix, they can produce a self-similar tiling of the space. In other words, given an expansive matrix $M \in \mathbb{R}^{d\times d}$ and an admissible lattice Γ, it is in general the case that there exists a self-similar set associated to M that tiles the plane by Γ-translates.

Tilings by lattice translates are associated to local Fourier orthonormal bases, which leads to the notion of spectral sets and to the "Fuglede Conjecture", as we describe in section 4.

When the property of tiling by Γ-translates is linked to the self-similarity of the tile by an expansive matrix, a beautiful construction of wavelet bases is obtained. This is the theory of wavelet sets (section 5).

Finally, a careful choice of the expanding matrix and the lattice in the wavelet set construction, allows to approximate any function in $\mathcal{L}^2(\mathbb{R}^d)$ by a generator of a Riesz wavelet system. We develop this density result in section 6.

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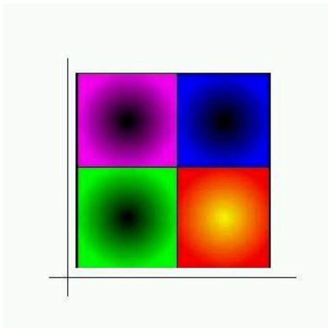

FIGURE 1. Self-Similar square

2. Self-Similarity

Throughout this section the concept of *self-similarity* in a very general way will play a fundamental role. In $\mathbb{R}^d$, a similarity is a function of the type $Ax+b$ where b is a point in $\mathbb{R}^d$ and A is a $d\times d$ isotropic matrix, i.e. all eigenvalues have the same magnitude. The name reflects the fact that this transformation *copies* objects into *similar* objects, just smaller, bigger or simply translated or rotated, depending on the matrix A and the point b.

In this sense, an object could be called *self-similar* if you can write it as the finite union of copies of itself, i.e.

$$B = \bigcup_{i=1}^{r} A_i(B) + b_i.$$

A trivial example is the unit square in $\mathbb{R}^2$ (see Figure 1).

Not so trivial, but still simple, is the well known *middle third Cantor set*, which we will denote by $C_{1/3}$. If you look at the Cantor set, you realize, that you can write it as the union of two shrunken copies of itself, one in the $[0, \frac{1}{3}]$ interval and the other one in the $[\frac{2}{3}, 1]$ interval.

Both examples are examples of compact sets in $\mathbb{R}^d$. They also share the property, that they are the (almost) disjoint union of smaller copies of themselves. This allows to virtually *see* the self-similarity.

We will also be interested in self-similar functions, or measures. Self-similar objects are interesting for us, since the self-similarity allows us to recover information of the whole object by looking only at some part, since one can think that the properties are "translated" into each part. Therefore self-similar objects may be easier to study than "arbitrary" objects. This is one of the reasons why we are interested in finding and characterizing self-similar objects. The fact that they are simple to describe and in many cases they are dense, in the sense that they approximate arbitrary objects, makes this characterization extremely useful.

One of the main tools to construct self-similar objects, will be by resort to the *fixed point theorem* or *Banach contraction principle*. Since we will try to apply it to very different scenarios, we will state it here in its most general way. The proof can be found in may textbooks, such as [Rud64].

Theorem 2.1 (Generalized Banach Contraction Principle)**.** *Let (X, d) be a complete metric space, and let $f : X \longrightarrow X$ be a function such that there exists a metric d_1 which is equivalent to d, for which there exist $0 < c < 1$, and $n > 0$ such that for all $x, y \in X$*

$$d_1(f^n(x), f^n(y)) \leq c\, d_1(x, y). \tag{2.1}$$

Then there exists a unique point x^* in X, such that $f(x^*) = x^*$.

2.1. **The space of compact sets in $\mathbb{R}^d$.** To describe the setting for the theory, we start defining an appropriate structure for the space of non-empty compact sets in $\mathbb{R}^d$. Precisely, we define $\mathcal{H}$ by

$$\mathcal{H} = \{K \subset \mathbb{R}^d, K \neq \emptyset, K\text{compact}\}. \tag{2.2}$$

We want to transform $\mathcal{H}$ into a metric space, and therefore we need to define an appropriate metric on $\mathcal{H}$. We would like that this metric takes the similarity of the shapes into account. For example, a point P and a segment S should not be close with the appropriate distance.

One distance that performs this task in a reasonable way, is the so-called *Hausdorff distance.*

Definition 2.2. Let $A, B \subset \mathbb{R}^d$ be non-empty compact sets, then the *Hausdorff distance* between A and B is

$$d_H(A, B) = \inf_{\varepsilon}\{A \subseteq B_\varepsilon \quad \textbf{and} \quad B \subseteq A_\varepsilon\}, \tag{2.3}$$

where

$$A_\varepsilon = \bigcup_{x \in A} B(x, \varepsilon) = \{x \in \mathbb{R}^d : d(x, A) < \varepsilon\}. \tag{2.4}$$

The following Theorem is straightforward (see for example [Hut81].

Theorem 2.3. *The space $\mathcal{H}$ of all non-empty compact subsets of $\mathbb{R}^d$ equipped with the Hausdorff distance is a complete metric space.*

3. Attractors and Self-Similar Sets

The following theorem, due to Hutchinson ([Hut81]), is a key result in the theory of self-similar sets and provides a simple way to construct them.

Theorem 3.1. *Let $s_1, \ldots, s_m$ be m contraction-mappings in $\mathbb{R}^d$, with contraction factors c_i. There exists a unique non-empty compact set $\mathcal{A}$ satisfying*

$$\mathcal{A} = \cup_{i=1}^m s_i(\mathcal{A})$$

(i.e. $\mathcal{A}$ is self-similar with respect to $s_1, ..., s_m$). Furthermore, if $\mathbf{s} : \mathcal{H} \to \mathcal{H}$ is the map defined by $\mathbf{s}(B) = \bigcup_{i=1}^m s_i(B)$, for each compact set $B_0 \neq \emptyset$, the sequence $\{B_k\}_{k \in \mathbb{N}}$ given by $B_k = \mathbf{s}(B_{k-1})$ converges to $\mathcal{A}$ in $(\mathcal{H}, d_H)$.

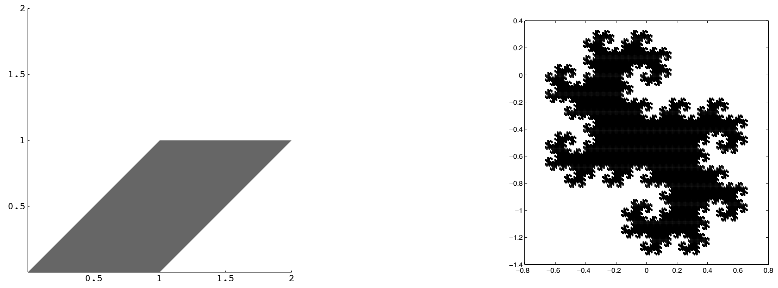

FIGURE 2. Self-Similar parallelogram and twin dragon attractors

Proof. The proof is immediate noting that

$$d_H(\mathbf{s}(A), \mathbf{s}(B)) \leq (\max_{1 \leq i \leq m} c_i)\, d_H(A, B).$$

□

Examples of Self-Similar Sets:

- Examples in $\mathbb{R}$
  - $s_1(x) = \frac{1}{2}x \quad s_2(x) = \frac{1}{2}x + \frac{1}{2} \quad \mathcal{A} = [0, 1]$
  - $s_1(x) = \frac{1}{3}x \quad s_2(x) = \frac{1}{3}x + \frac{2}{3} \quad \mathcal{A} = C_{1/3}$
- Examples in $\mathbb{R}^2$
  - $s_1(x) = \frac{1}{2}I_2 x \quad s_2(x) = \frac{1}{2}I_2 x + (\frac{1}{2}, 0)$
    $s_3(x) = \frac{1}{2}I_2 x + (0, \frac{1}{2}) \quad s_4(x) = \frac{1}{2}I_2 x + (\frac{1}{2}, \frac{1}{2})$

    $\mathcal{A} = [0, 1] \times [0, 1]$ (see Figure 1).
  - $s_1(x) = M^{-1}x \quad s_2(x) = M^{-1}(x + (1, 0))$ where

    $M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ or $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ In this case, the attractors $\mathcal{A}$ are shown in Figure 2.

Focusing on the application we have in mind, we will look at a particular case of Theorem 3.1 which is satisfied by all our examples so far.

For this, let Γ be a *full rank lattice*, i.e. $\Gamma = R\mathbb{Z}^d$ with R any invertible matrix. Let $\gamma_1, \ldots, \gamma_d$ be a set of generators for the lattice Γ, i.e., independent vectors such that

$$\Gamma = \{m_1\gamma_1 + \cdots + m_d\gamma_d : m_i \in \mathbb{Z}\}.$$

Then the rectangular parallelepiped

$$P = \{x_1\gamma_1 + \cdots + x_d\gamma_d : 0 \leq x_i < 1\}$$

is a *fundamental domain* for the group $\mathbb{R}^d/\Gamma$. A matrix $M \in \mathbb{R}^{d\times d}$ is said to be *expansive*, if all the eigenvalues have absolute value bigger than 1. If $M \in \mathbb{R}^{d\times d}$ is expansive, and Γ is a lattice such that $M\Gamma \subset \Gamma$, a set of representatives of the quotient $\Gamma/M(\Gamma)$ is called a *full set of digits* for that lattice. Note that, since $M\Gamma \subset \Gamma$, $\det M \in \mathbb{Z}$. A full set of digits always has $|\det M|$ elements. We have the following result.

Proposition 3.2. *Let $M \in \mathbb{R}^{d\times d}$ be an expansive matrix and let Γ be a lattice such that $M\Gamma \subset \Gamma$, $m = |\det M|$, and $\mathcal{D} = \{d_1, \dots, d_m\}$ be a full set of digits. There always exists a non-empty compact set $\mathcal{A} \subset \mathbb{R}^d$ satisfying*

$$\mathcal{A} = \cup_{i=1}^{m} M^{-1}(\mathcal{A} + d_i). \tag{3.1}$$

The following properties of $\mathcal{A}$ will be useful [Ban91], cf. also [GM92]. For a general description see also [CHM04]. If $X \subset \mathbb{R}^d$, we will denote by $|X|$ the d-dimensional Lebesgue measure of X.

Lemma 3.3. *Let $\mathcal{A}$ be as in* (3.1)*. Then the following statements hold.*

(a) $\mathcal{A} + \Gamma = \mathbb{R}^n$.

(b) *$\mathcal{A}$ has nonempty interior, $\mathcal{A}$ is the closure of $\mathcal{A}^\circ$, and $|\partial\mathcal{A}| = 0$.*

(c) *$|\mathcal{A} \cap (\mathcal{A} + k)| = 0$ for all $k \in \Gamma \setminus \{0\}$ if and only if $|\mathcal{A}| = |P|$. In this case, $\mathcal{A} \cap (\mathcal{A} + k) \subset \partial\mathcal{A}$ for each $k \in \Gamma \setminus \{0\}$.*

(d) $\#(\mathcal{A}^\circ \cap \Gamma) \le 1$.

In other words, part (c) above says that if $|\mathcal{A}| = |P|$, then $\mathcal{A}$ is a *tile* in the sense that the Γ-translates $\{\mathcal{A} + k\}_{k\in\Gamma}$ cover $\mathbb{R}^d$ with overlaps of measure zero.

A long-standing open problem was the question of whether for each dilation matrix M there exists a full set of digits D such that the corresponding attractor $\mathcal{A}$ is a tile. Lagarias and Wang proved that this is the case if $n = 1, 2, 3$ or if $m = |\det(M)| > d$ [LW95], [LW96], [LW97]. Potiopa [Pot97] however showed that if $d = 4$ and

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ -1 & 0 & -1 & 1 \end{bmatrix},$$

then there is no set of digits $D = \{d_1, d_2\}$ such that the unique self similar set $\mathcal{A}$ associated to (M, D) i.e. $\mathcal{A} = M^{-1}\mathcal{A} + d_1 \cup M^{-1}\mathcal{A} + d_2$ is a tile, cf. [LW99]. Note that this matrix M has determinant 2.

4. Spectral Sets

Let $\Omega \subset \mathbb{R}^d$ a measurable set such that $0 < |\Omega| < +\infty$. The set Ω is called a *spectral set* if there exists a discrete set $\Lambda = \{\lambda_k \,:\, k \in \mathbb{Z}\} \subset \mathbb{R}^d$ such that the set of exponential functions

$$\left\{ \frac{1}{|\Omega|^{1/2}}\, e^{2\pi i \lambda_k x} \right\}_{k\in\mathbb{Z}} \tag{4.1}$$

is an orthonormal basis of $\mathcal{L}^2(\Omega)$. In this case, Λ is called the *spectrum* of Ω. Many results on spectral sets can be found in the work of Jörgensen et. al ([JP91, JP92, JP98a, JP98b, JP99, BJR99]), Wang ([Wan02, PW01]) and others.

In 1974 Fuglede ([Fug74]) proved the following theorem.

Theorem 4.1. *Ω is a spectral set with spectrum $\Gamma' = (R^{-1})^*\mathbb{Z}^d$ if and only if Ω tiles $\mathbb{R}^d$ by translations on $\Gamma = R\mathbb{Z}^d$ (the dual lattice of Γ').*

He further conjectured that his theorem was still true, if one removes the condition that the spectrum of Ω has to be a lattice. Terence Tao ([Tao04]) proved in 2003, that - at least for dimension $d \geq 5$ - this is false.

However, Fuglede's Theorem allows us to add an additional equivalence to item (c) of Lemma 3.3 above:

Proposition 4.2. *Let $M \in \mathbb{R}^{d\times d}$ be an expansive matrix and $m = |\det M|$. Let Γ be a lattice such that $M\Gamma \subset \Gamma$ and let $\mathcal{D} = \{d_1, \ldots, d_m\} \subset \mathbb{R}^d$ be a full set of digits. We have*

$$\mathcal{A} = \cup_\gamma(\mathcal{A}+\gamma) \quad \textit{if and only if} \quad \left\{\frac{1}{|\mathcal{A}|^{1/2}}\, e^{2\pi i\gamma' x}\right\}_{\gamma'\in\Gamma'} \textit{ is an o.n. basis of } \mathcal{L}^2(\mathcal{A}). \tag{4.2}$$

5. Minimal Supported (in Frequency) Wavelets

For a lattice Γ, an expansive matrix $M \in \mathbb{R}^{d\times d}$ such that $M\Gamma \subset \Gamma$, $m = |\det M|$, and $\psi \in \mathcal{L}^2(\mathbb{R}^d)$ we consider the set

$$\mathcal{F} = \left\{|\det(M)|^{j/2}\psi(M^j x - \gamma) : j \in \mathbb{Z}, \gamma \in \Gamma\right\}. \tag{5.1}$$

The question we are addressing now, is for which $\psi \in \mathcal{L}^2(\mathbb{R}^d)$, $\mathcal{F}$ is an orthonormal basis for $\mathcal{L}^2(\mathbb{R}^d)$. Such a function will be called a *wavelet*.

The following Theorem, whose proof is straightforward, characterizes those wavelets whose Fourier transform has support with smallest possible measure.

Theorem 5.1. *Let Γ and M be as before, and let $Q \subset \mathbb{R}^d$ be such that*

- *$\{Q+\gamma' : \gamma' \in \Gamma'\}$ tiles $\mathbb{R}^d(=\widehat{\mathbb{R}^d})$ (i.e. Q is a tile for Γ', the dual lattice of Γ)*
- *$\{M^jQ : j \in \mathbb{Z}\}$ tiles $\mathbb{R}^d(=\widehat{\mathbb{R}^d})$, i.e.*

$$\cup_j M^jQ = \mathbb{R}^d \setminus \{0\} \ \textit{and} \ \ |M^jQ \cap M^kQ| = 0,\ j \neq k.$$

Then if ψ is such that $\hat{\psi} = \chi_Q$ we have

$$\mathcal{F} = \left\{|\det(M)|^{j/2}\psi(M^j x - \gamma) : j \in \mathbb{Z}, \gamma \in \Gamma\right\}$$

is an orthonormal basis of $\mathcal{L}^2(\mathbb{R}^d)$. ψ is called a Minimal Supported in Frequency wavelet (MSFW).

5.1. **Construction of Q.** Therefore, in order to obtain MSFW, we need to construct a set Q that satisfies the conditions above. Sets of this type are called *wavelet sets* and have been studied by many groups of researchers ([BL01, BMM99, BS02, BS04, DLS97, DLS98, ILP98, SW98, Wan02, Zak96]. We will illustrate the construction given by Benedetto [BL01, BS02, BS04] for a particular case, which will be useful in the next section.

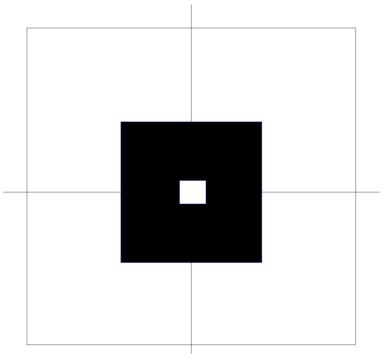

FIGURE 3. The annulus tiles by dilations by M but not by Γ translates

Assume that $M = \lambda I_d$ with $|\lambda| \geq 3$ and that $\Gamma = \gamma\mathbb{Z}^d$, for $0 < \gamma < +\infty$. (The choice of $\lambda > 3$ is only to simplify the construction.) Set $Q_0 = [0, \frac{\gamma}{2}]^d \setminus M^{-1}[0, \frac{\gamma}{2}]^d$ (see Figure 3). This set tiles by dilation by M, but not by Γ-translates. We need to *fill the hole*.

We define

$$\begin{aligned}\widetilde{T} : \mathbb{R}^d &\longrightarrow \mathbb{R}^d\\ x &\mapsto x + \gamma\xi_j \quad \text{if } x \in j^{\text{th}} - \text{quadrant}\end{aligned}$$

where ξ_j is the vertex of the cube $[-1,1]^d$ that lies in the j^{th}-quadrant.

Let us call

$$A_0 = M^{-1}[0, \frac{\gamma}{2}]^d = [0, \frac{\gamma}{(2\lambda)}]^d \tag{5.2}$$

and define

$$A_i := (M^{-1} \circ \widetilde{T})^i(A_0), i = 1, 2, \dots \tag{5.3}$$

It will be convenient to use the notation A_0^j for the intersection of the set A_0 with the j^{th}-quadrant. With this notation, note that

$$\begin{aligned} A_i &= \bigcup_{1\leq j\leq 2^d} (M^{-1} \circ \widetilde{T})^i(A_0^j) \\ &= \bigcup_{1\leq j\leq 2^d} T_{\gamma M^{-1}\xi_j + \cdots + \gamma M^{-i}\xi_j}(M^{-i}(A_0^j)). \end{aligned} \tag{5.4}$$

Here T_y denotes the usual translation by y in $\mathcal{L}^2(\mathbb{R}^d)$. Therefore we have that

$$\sum_{i=1}^{\infty} \mu(A_i) = \mu(A_0) \sum_{i=1}^{\infty} \left(\frac{1}{\lambda^d}\right)^i = \left(\frac{\gamma}{\lambda}\right)^d \frac{1}{\lambda^d - 1}. \tag{5.5}$$

We define the set

$$Q := M\left(\bigcup_{i=0}^{\infty} A_i\right) \setminus \left(\bigcup_{i=0}^{\infty} A_i\right), \tag{5.6}$$

Observe that:

- By construction, Q tiles $\mathbb{R}^d \setminus 0$ by dilations by M.

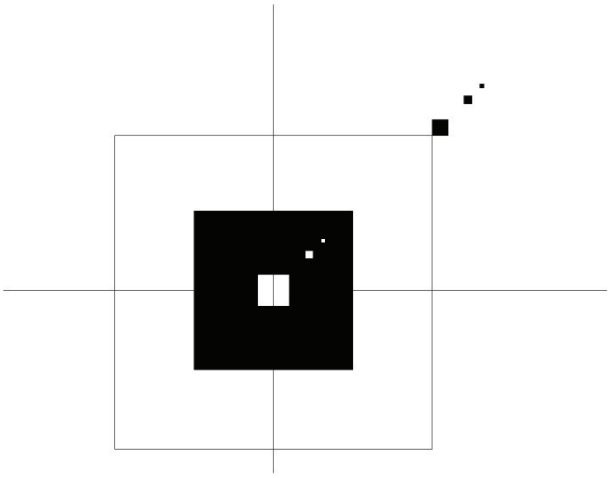

FIGURE 4. Q tiles by dilations by M **and** by Γ translates

- Furthermore Q tiles $\mathbb{R}^d$ by translations on Γ. For this, we first note that if $x \in A_n$ then

$$\frac{\gamma}{\lambda-1}\left(1-\left(\frac{1}{\lambda}\right)^n\right) \leq \|x\|_\infty \leq \frac{\gamma}{\lambda-1}\left(1-\left(\frac{1}{\lambda}\right)^n\left(\frac{1+\lambda}{2\lambda}\right)\right). \tag{5.7}$$

This fact allows us to conclude that:
1. $A_i \subset ([0,\frac{\gamma}{2}]^d \setminus A_0)$, for $i \geq 1$,
2. $A_i \cap A_j = \emptyset$ if $i \neq j$,

which allows us to rewrite Q

$$\begin{aligned} Q &= M\left(\bigcup_{i=0}^{\infty} A_i\right) \setminus \left(\bigcup_{i=0}^{\infty} A_i\right) \\ &= \left(MA_0 \setminus (\bigcup_{i=0}^{\infty} A_i)\right) \cup \left(\bigcup_{i=0}^{\infty} \tilde{T}A_i\right). \end{aligned}$$

This shows that Q is in fact Γ-congruent to $[0,\frac{\gamma}{2}]^d$.

Therefore, using Fuglede's Theorem (Theorem 4.1) we have that

$$\left\{\frac{1}{\gamma^{d/2}}e^{2\pi i\gamma k\omega}\chi_Q(\omega) : k \in \mathbb{Z}^d\right\}$$

is an orthonormal basis for $\mathcal{K}_Q$, and using that $D_M f := m^{d/2}f(M\cdot)$ is a unitary operator, we conclude that

$$\mathcal{F} = \left\{|\det(M)|^{j/2}\psi(M^j x - \gamma k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d\right\}$$

is an orthonormal basis of $\mathcal{L}^2(\mathbb{R}^d)$.

6. Riesz basis are dense

In this section we will outline, how we can use the construction of the previous section, to answer an open question posed by D. Larson. For details, we refer the reader to [CM06].

The question we are going to address now is the following:

Given $f \in \mathcal{L}^2(\mathbb{R}^d)$, and $\varepsilon > 0$, does there exist a function $\psi \in \mathcal{L}^2(\mathbb{R}^d)$, an expansive matrix $M \in \mathbb{R}^{d\times d}$, and a lattice Γ, such that

- $\|f - \psi\|_2 < \varepsilon$ and
- $\mathcal{F} = \left\{ |\det(M)|^{j/2}\psi(M^j x - \gamma k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d \right\}$ is a Riesz basis for $\mathcal{L}^2(\mathbb{R}^d)$?

Here we are relaxing the condition of being an orthonormal basis, to the Riesz basis condition.

We recall that a set $\{v_k\}_{k\in\mathbb{Z}}$ is a **Riesz basis** for a Hilbert space $\mathcal{H}$, if it is complete in $\mathcal{H}$ and there exist constants $0 < A \leq B < +\infty$, such that for every $n \in \mathbb{N}$ and every finite sequence of scalars $c = (c_1, \dots, c_n)$,

$$A\sum_{i=1}^{n}|c_i|^2 \leq \|\sum_{i=1}^{n} c_i v_i\|^2 \leq B\sum_{i=1}^{n}|c_i|^2.$$

The constants A and B are called *Riesz basis bounds.*

In order to show, how the previous results can be used to give a positive answer to this question, we need the following result.

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^d$ be a set of finite measure. If $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}^d$ satisfies that $\left\{e^{2\pi i\lambda_k\omega}\chi_\Omega(\omega) : k \in \mathbb{Z}\right\}$ is a Riesz basis for $\mathcal{K}_\Omega$ with bounds A and B and $h \in \mathcal{L}^2(\mathbb{R}^d)$ satisfies that $0 < p < |h(\omega)| < P < +\infty$ then*

$$\{h(\omega)e^{2\pi i\lambda_k\omega} : k \in \mathbb{Z}\}$$

is a Riesz basis for $\mathcal{K}_\Omega$, with Riesz bounds $pA\mu(\Omega)$ and $PB\mu(\Omega)$.

If $M \in \mathbb{R}^{d\times d}$ is an invertible matrix and Ω satisfies that $\cup_{j\in\mathbb{Z}}a^j\Omega = \mathbb{R}^d$ up to a set of zero measure, with the union being almost disjoint, and $\{g_k : k \in \mathbb{Z}\}$ is a Riesz basis for $\mathcal{K}_\Omega$, then

$$\{D_M^j g_k : k, j \in \mathbb{Z}\}$$

is a Riesz basis for $\mathcal{L}^2(\mathbb{R}^d)$ with the same bounds. (Here D_M is the dilation operator defined in the previous section.)

Proof. The first assertion is immediate, and the second one follows from the fact that the dilation is a unitary operator in $\mathcal{L}^2(\mathbb{R}^d)$. □

In view of this Lemma, if we are given a function in $\mathcal{L}^2(\mathbb{R}^d)$, we need to find the *right* lattice and the *appropriate* dilation matrix. For this we proceed in the following way:

- Let $g \in \mathcal{L}^2(\mathbb{R}^d)$, such that $\|\hat{f} - \hat{g}\|_2^2 < \varepsilon/2$ and $\hat{g}$ is continuous.
- Choose $R \in \mathbb{R}$ such that $\displaystyle\int_{\mathbb{R}^d\setminus B(0,R/2)} |\hat{g}(\omega)|^2 d\omega < \delta$

- We now select $r > 0$ small enough such that:

$$r < \frac{R}{3} \tag{6.1}$$

$$\int_{B_\infty(0,r/2)} |\hat{g}(\omega)|^2 d\omega < \frac{\varepsilon^2}{16} \tag{6.2}$$

$$\text{and} \quad \mathrm{a}r^d < \frac{\varepsilon^2}{16} \quad \text{where} \quad \mathrm{a} := \max\{|\hat{g}(\omega)|^2 : \omega \in B_\infty(0, R/2)\}. \tag{6.3}$$

Let now

$$\Gamma = R\mathbb{Z}^d \quad \text{and} \quad M = \frac{R}{r} I_{d\times d}.$$

Note that by the choice of r, we have that $\frac{R}{r} \geq 3$ and we are therefore in the previously described situation.

We define the set Q as in (5.6), and for $\alpha = \frac{\varepsilon}{8(R)^{d/2}}$, the function h by

$$h(\omega) := \begin{cases} \hat{g}(\omega) & x \in Q \cap E_\alpha \\ \alpha & x \in Q \setminus E_\alpha \\ 0 & \text{else} \end{cases}.$$

where

$$E_\alpha := \{\omega \in \mathbb{R}^d : |\hat{g}(\omega)| > \alpha\}.$$

Then the function ψ, with $\hat{\psi} = h$ satisfies that

$$\mathcal{F} = \left\{ |\det(M)|^{j/2} \psi(M^j x - \gamma k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d \right\}$$

is a Riesz basis for $\mathcal{L}^2(\mathbb{R}^d)$ (see [CM06] for the details of the proof).

7. Acknowledgments

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