The Hausdorff dimension of p-Cantor sets

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Abstract

In this paper we analyze Cantor type sets constructed by the removal of open intervals whose lengths are the terms of the p-sequence, $\{k^{-p}\}_{k=1}^{\infty}$. We provide sharp estimates of their Hausdorff measure and dimension. Sets of similar structure arise when studying the set of extremal points of the boundaries of the so-called random stable zonotopes.

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§1 Introduction and Notation

By a general Cantor set, we mean a compact, totally disconnected subset of the real line. Cantor sets arise in many different settings and as examples of many surprising properties. They can be constructed in different ways.

A Cantor set can be viewed as the intersection of a countable number of closed sets F_k, each of which is the finite union of closed intervals I_j^k. We will call these intervals the *intervals of step k*. The complement of these intervals in I_0, where I_0 is the smallest interval that contains the Cantor set, are called *gaps of step k*.

§1.1 Construction of a Cantor set associated to a sequence.

Let $\lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ be a sequence of non-negative numbers, such that $\sum_k \lambda_k = K < +\infty$.

Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijective map, we will say that the sequence $\{\lambda_{\sigma(k)}\}$ is a *rearrangement* of λ and we will denote this new sequence by $\sigma(\lambda)$.

We can associate to any sequence λ a Cantor set C_λ in the following way: We start with a closed interval $I_0 = [0, K]$ of length $K = \sum_{k=1}^{\infty} \lambda_k$. (Clearly by normalization we can always achieve $I_0 = [0, 1]$.) In the first step we remove from I_0 an open interval of length λ_1, getting two intervals of step 1, namely I_0^1 and I_1^1, and a gap of length λ_1. From interval I_0^1 we remove λ_2, from I_1^1 we remove λ_3, and so on. The length of I_0^1 is the sum of the lengths of all intervals which will be removed by this construction, and it is easy to see that

$$|I_0^1| = \sum_{n=1}^{\infty} \sum_{j=0}^{2^{n-1}-1} \lambda_{2^n+j}, \qquad |I_1^1| = \sum_{n=1}^{\infty} \sum_{j=2^{n-1}}^{2^n-1} \lambda_{2^n+j}.$$

In what follows, we use the notation $\text{length}(I) = \text{diam}(I) = |I|$ for any interval.

In step k of this construction we have 2^k intervals, I_0^k, I_1^k, ..., $I_{2^k-1}^k$, and for each ℓ, $0 \le \ell \le 2^k - 1$, we remove from I_ℓ^k an open interval of length $\lambda_{2^k+\ell}$, forming two intervals $I_{2\ell}^{k+1}$ and $I_{2\ell+1}^{k+1}$, so the following relation holds,

$$|I_\ell^k| = |I_{2\ell}^{k+1}| + \lambda_{2^k+\ell} + |I_{2\ell+1}^{k+1}|.$$

It is not difficult to verify that for all $k = 0, 1, \dots$ and $\ell = 0, 1, \dots, 2^k - 1$, we have

$$|I_\ell^k| = \sum_{n=k}^{\infty} \sum_{j=\ell 2^{n-k}}^{(\ell+1)2^{n-k}-1} \lambda_{2^n+j}. \tag{1}$$

We will say that the Cantor set constructed in this way using a sequence λ, is the *Cantor set associated to the sequence* λ.

It is clear from this description, that the specific order in which the gaps are chosen is related to the resulting Cantor set. As we will see in this paper, a rearrangement of the original sequence can yield a Cantor set with different Hausdorff dimension. It is also clear, that if two different sequences yield the same Cantor set, then one is a rearrangement of the other.

Remark: Let C be any Cantor set in $\mathbb{R}$ and let I_0 be the smallest interval containing C. The complement of C in I_0 is a countable union of open intervals U_i, such that $\sum_{i \in \mathbb{N}} |U_i| = |I_0|$. The following procedure will show how to define a sequence $a = \{a_k\}$, such that $C_a = C$.

Let U_{i_1} be a gap of maximal length. Define $a_1 = |U_{i_1}|$. We then choose U_{i_2} a gap of maximal length to the left of U_{i_1}, and U_{i_3} a gap of maximal length to the right of U_{i_1}. We now define $a_2 = |U_{i_2}|$ and $a_3 = |U_{i_3}|$. In the next step we define a_4 through a_7 by picking a gap of maximal length in each of the remaining intervals (i.e. $I_0 - (U_{i_1} \cup U_{i_2} \cup U_{i_3})$.) The sequence a defined in this way, satisfies that $C_a = C$.

We say that a sequence $a = \{a_k\}$ has *at least geometrical decay*, if there exist $0 < d < 1$ and $c > 0$ such that $a_k \le c d^k$ for all $k \in \mathbb{N}$. We will need in different parts of the paper, that any given sequence which tends to zero can be decomposed into finitely many or countably many subsequences, all of them having at least geometrical decay.

Lemma 1. *Let $a = \{a_n\}$ be a sequence of positive terms such that $\lim a_n = 0$. Then there exists a family of functions $\{\gamma_j : \mathbb{N} \to \mathbb{N}, \quad j = 1, 2, ..\}$ at most countable such that*

1. *γ_j is one to one and increasing for all j.*
2. *$\gamma_j(\mathbb{N}) \bigcap \gamma_{j'}(\mathbb{N}) = \emptyset$ if $j \neq j'$.*
3. *$\mathbb{N} = \bigcup_j \gamma_j(\mathbb{N})$.*
4. *For all j, the subsequence $a^{(j)} = \{a_{\gamma_j(n)}\}_{n \in \mathbb{N}}$ has at least geometrical decay.*

Proof. We define first γ_1 by

$$\gamma_1(1) = 1 \text{ and if } \gamma_1(n) \text{ is already defined then}$$

$$\gamma_1(n+1) = \min\{m \in \mathbb{N} : m > \gamma_1(n) \text{ and } a_m \leq 1/2^{n+1}\};$$

this being possible since $a_n \longrightarrow 0$. This defines γ_1 inductively. Now we assume that $\gamma_1, ..., \gamma_k$ are already defined, then if $\mathbb{N} \setminus \bigcup_{j=1}^{k} \gamma_j(\mathbb{N})$ is finite, we stop, and redefine γ_1 in such a way that $\gamma_1(\mathbb{N}) = \mathbb{N} \setminus \bigcup_{j=2}^{k} \gamma_j(\mathbb{N})$. Otherwise, we define γ_{k+1} by

$$\gamma_{k+1}(1) = \min(\mathbb{N} \setminus \bigcup_{j=1}^{k} \gamma_j(\mathbb{N})) \text{ and if } \gamma_{k+1}(n) \text{ is already defined then}$$

$$\gamma_{k+1}(n+1) = \min\{m \in \mathbb{N} \setminus (\bigcup_{j=1}^{k} \gamma_j(\mathbb{N})) : m > \gamma_{k+1}(n) \text{ and } a_m \leq 1/2^{n+1}\}.$$

If the process does not end in a finite number of steps then $\mathbb{N} = \bigcup_j \gamma_j(\mathbb{N})$ since every number n must be selected at most at step n. ∎

§1.2 Hausdorff measure and dimension

Definition. Let $A \subset \mathbb{R}$ be a Borel-measurable set and $\alpha > 0$, for $\delta > 0$ consider

$$\mathcal{H}_\delta^\alpha(A) = \inf\Big\{ \sum (\text{diam}(E_i))^\alpha \ : \ E_i \text{ open}, \ \cup E_i \supset A, \text{diam}(E_i) \leq \delta \Big\}.$$

Then, the *α-dimensional Hausdorff measure* $\mathcal{H}^\alpha(A)$, is defined by the relation

$$\mathcal{H}^\alpha(A) = \lim_{\delta \to 0} \mathcal{H}_\delta^\alpha(A).$$

The Hausdorff dimension of a set A,

$$\dim_H(A) = \sup\{\alpha : \mathcal{H}^\alpha(A) > 0\}.$$

Throughout this paper, the only dimension with which we will be dealing is the Hausdorff dimension, so from now on, we will omit the subscript H.

In general, the computation of the Hausdorff dimension or the Hausdorff measure of a set is very difficult, see [2], [3] and references therein. The estimates from above are usually simpler and for the lower bounds very few examples are known.

For our particular case, again, the upper bounds are relatively easily obtained, and for the lower bounds we need some sharp estimates on the size of the intervals of the construction.

§1.3 The p-sequence.

We will study the Cantor sets that are constructed using the p-sequence $\lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ and its rearrangements where $\lambda_k = k^{-p}$; we call such sets *p-Cantor sets.* Sets of similar structure (only more complicated since they are random and are in $\mathbb{R}^d$ with $d \geq 2$) arise when studying the set of extremal points of the boundaries of random stable zonotopes, which were introduced in [1] as examples of a random element in the space of compact convex subsets of $\mathbb{R}^d$ with a non-atomic distribution. Thus, in a sense, this study is intended to understand what can be expected when studying the above mentioned sets of extremal points.

Our main result is the following theorem.

Theorem 1. *Let $\lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ be defined by $\lambda_k = \left(\frac{1}{k}\right)^p$, $p > 1$. Then*

$$dim\ C_\lambda = \frac{1}{p}.$$

Moreover, if $\sigma(\lambda)$ is any rearrangement of the sequence λ, then

$$0 \leq dim\, C_{\sigma(\lambda)} \leq \frac{1}{p}.$$

Furthermore, for each $0 \leq s \leq \frac{1}{p}$, there exists a rearrangement $\sigma_s(\lambda)$ such that

$$dim\, C_{\sigma_s(\lambda)} = s \quad and \quad \mathcal{H}^s(C_{\sigma_s(\lambda)}) > 0.$$

The proof of this theorem will occupy us for most of the remainder of the paper, and we will break it up into several parts, some of which are of interest in themselves.

(i) We will first show that for any rearrangement $\sigma(\lambda)$, $\dim C_{\sigma(\lambda)} \leq \frac{1}{p}$.

(ii) We will then show, that $\dim C_\lambda = \frac{1}{p}$ and $\mathcal{H}^{\frac{1}{p}}(C_\lambda) > 0$.

(iii) Finally, for each $s, 0 \leq s \leq \frac{1}{p}$, we will exhibit a particular rearrangement $\sigma_s(\lambda)$, such that dim $C_{\sigma_s(\lambda)} = s$. In particular this rearrangement will satisfy $\mathcal{H}^s(C_{\sigma_s(\lambda)}) > 0$.

§2 Upper bound of Hausdorff dimension

Theorem 2. *Let C_λ be the Cantor set associated to the sequence $\lambda_k = k^{-p}$, $k \in \mathbb{N}$, with $p > 1$. Then*

$$\dim C_\lambda \le \frac{1}{p}.$$

Moreover, if $\sigma(\lambda)$ is any rearrangement of this sequence, then

$$\dim C_{\sigma(\lambda)} \le \frac{1}{p}.$$

Proof. Consider n large enough such that $\sum_{j=n+1}^{\infty} \lambda_j \le \delta$. Suppose that we removed from I_0 open intervals of lengths λ_1, λ_2, …, λ_n. What remains can be written as the union of closed intervals $E_j^{(n)}$, $j = 1, 2, \ldots, n+1$. Since $\sum_{j=1}^{n+1} |E_j^{(n)}| = \sum_{j=n+1}^{\infty} \lambda_j$, then $\{E_j^{(n)} : j = 1, 2, \ldots, n+1\}$ is a δ-covering of C. Using the Hölder inequality, we have

$$\begin{aligned}\sum_{j=1}^{n+1} |E_j^{(n)}|^s &\le \left(\sum_{j=1}^{n+1} |E_j^{(n)}|\right)^s (n+1)^{1-s} \\ &= \left(\sum_{j=n+1}^{\infty} \lambda_j\right)^s (n+1)^{1-s}.\end{aligned}$$

But using integral comparison, we have

$$\frac{1}{p-1}(n+1)^{1-p} = \int_{n+1}^{\infty} \left(\frac{1}{x}\right)^p dx \le \sum_{j=n+1}^{\infty} \lambda_j \le \int_{n}^{\infty} \left(\frac{1}{x}\right)^p dx = \frac{1}{p-1} n^{1-p}.$$

Therefore

$$\sum_{j=1}^{n+1} |E_j^{(n)}|^s \le \left(\frac{1}{p-1}\right)^s \frac{(n+1)^{1-s}}{n^{sp-s}}.$$

Hence, if $s \ge \frac{1}{p}$,

$$\limsup_{n\to\infty} \sum_{j=1}^{n+1} |E_j^{(n)}|^s < \infty.$$

This proves, that for any $s \ge \frac{1}{p}$, $\mathcal{H}_\delta^s(C_\lambda) < \infty$, and therefore $\dim C_\lambda \le \frac{1}{p}$.

To prove the second part of the Theorem, we can argue as follows. Let $\sigma(\lambda)$ be any rearrangement of $\lambda = \{\lambda_k\}$. Regardless in what order we remove open intervals from I_0, there will be a step m at which the first n intervals of length λ_1, λ_2, …, λ_n will be removed $(m \ge n)$. We consider the $n+1$ closed intervals $E_j^{(n)}$ which are complementary (in I_0) to these removed n-intervals. Clearly $E_j^{(n)}$, $j = 1, \ldots, n+1$ again forms a δ-covering of $C_{\sigma(\lambda)}$ and the same bounds as before hold. Thus Theorem 2 is proved. ∎

Using a similar proof as in this theorem, we can show that if $a = \{a_k\}$ is a sequence that decays geometrically or faster, i.e. $a_k \leq r^k$ with $r < 1$, then $\dim C_a = 0$.

Proposition 1. *Let $a = \{a_k\}_{k \in I\!N}$ be any sequence such that $a_k \leq r^k$ for $r < 1$. Then, the Cantor set C_a has Hausdorff dimension 0.*

Proof. We will show, that for each $\epsilon > 0$, $\dim C_a \leq \epsilon$. We argue as before: suppose that n is sufficiently large and such that $\sum_{j=n+1}^{\infty} r^j \leq \delta$. Suppose that we removed from $I_0 = [0, \sum a_k]$ open intervals of lengths a_1, a_2, …, a_n. What remains can be written as the union of closed intervals $E_j^{(n)}$, $j = 1, 2, \ldots, n+1$. Since $\sum_{j=1}^{n+1} |E_j^{(n)}| = \sum_{j=n+1}^{\infty} a_j \leq \sum_{j=n+1}^{\infty} r^j \leq \delta$, then $\{E_j^{(n)} : j = 1, 2, \ldots, n+1\}$ is a δ-covering of C_a. Using the Hölder inequality, we have for each $\epsilon > 0$,

$$
\begin{aligned}
\sum_{j=1}^{n+1} |E_j^{(n)}|^\epsilon &\leq \left(\sum_{j=1}^{n+1} |E_j^{(n)}| \right)^\epsilon (n+1)^{1-\epsilon}, \\
&\leq \left(\sum_{j=n+1}^{\infty} r^j \right)^\epsilon (n+1)^{1-\epsilon}, \\
&= \left(\frac{r^{n+1}}{1-r} \right)^\epsilon (n+1)^{1-\epsilon}.
\end{aligned}
$$

But then

$$
\limsup_{n \to \infty} \sum_{j=1}^{n+1} |E_j^{(n)}|^\epsilon < \infty,
$$

which proves that $\dim C_a \leq \epsilon$. Since this is true for every $\epsilon > 0$, we conclude that dim $C_a = 0$. ∎

Using this Proposition, together with Lemma 1, we are now able to prove the following.

Proposition 2. *Let $a = \{a_n\}$ such that $a_n > 0$ and $\sum a_n < \infty$, then there exists a rearrangement $\sigma(a)$ of a such that dim $C_{\sigma(a)} = 0$.*

Proof. Using Lemma 1, we can decompose the sequence a into at most countably many subsequences, all of then having at least geometrical decay. Let $\{\gamma_j\}$ be the family of functions given by Lemma 1 and let C_{γ_j} be the Cantor set associated to the subsequence $\{a_{\gamma_j(n)}\}$. Note that, since the sequence $\{a_{\gamma_j(n)}\}$ has at least geometric decay, dim $C_{\gamma_j} = 0$ by Corollary 1. Define now:

$$t_0 = 0$$

$$\text{and } t_j = \sum_n a_{\gamma_j(n)}, \quad j = 1, 2, \ldots.$$

Then we have $C_{\gamma_j} \subset [0, t_j]$. Define C be the union of translates

$$C = \bigcup_j \left(C_{\gamma_j} + (\sum_{k=1}^{j-1} t_k) \right).$$

The set C is a Cantor set and dim $C = 0$ (since it is the at most countable union of Cantor sets of dimension zero.) The lengths of the gaps of C correspond to the terms of the original sequence a. Then, there is a rearrangement $\sigma(a)$ of the sequence, that is associated to the Cantor set C, that is $C = C_{\sigma(a)}$. ∎

§3 Lower bound of the Hausdorff dimension.

An estimate for the lower bound of dim C_λ is much more involved.

§3.1 The Cantor set C_λ.

In this section we are going to prove that the Cantor set C_λ has positive $\frac{1}{p}$-Hausdorff measure.

Theorem 3. *Let $p>1$ and let $\lambda=\{\lambda_k\}_{k\in\mathbb{N}}$ be the sequence defined by $\lambda_k=k^{-p}$. If C_λ is the Cantor set associated to the sequence λ, then*

$$\mathcal{H}^{\frac{1}{p}}(C_\lambda)\geq\frac{1}{8}\left(\frac{2^p}{2^p-2}\right)^{\frac{1}{p}}.$$

First we prove the following lemma which is the main ingredient to the proof of the theorem. We remind the reader that I_ℓ^k stands for the ℓth interval obtained in the kth step of construction, which was described in the introduction. The length of the interval I_ℓ^k is given by equation (1).

Lemma 2. *For all $k=1,2,\ldots$ and $\ell=0,1,\ldots,2^k-1$,*

$$\lambda_{2^k+\ell+1}\frac{2^p}{2^p-2}\leq|I_\ell^k|\leq\frac{2^p}{2^p-2}\lambda_{2^k+\ell}.$$

Proof. We can rewrite the expression for $|I_\ell^k|$ in (1) as follows

$$|I_\ell^k|=\sum_{h=0}^{\infty}\sum_{j=0}^{2^h-1}\lambda_{2^{k+h}+\ell2^h+j}.$$

Then

$$\begin{aligned}|I_\ell^k|&\leq\sum_{h=0}^{\infty}\frac{2^h}{(2^h(2^k+\ell))^p}\\&=\frac{1}{(2^k+\ell)^p}\sum_{h=0}^{\infty}\left(\frac{1}{2^{p-1}}\right)^h\\&=\lambda_{2^k+\ell}\frac{2^{p-1}}{2^{p-1}-1}=\lambda_{2^k+\ell}\frac{2^p}{2^p-2}.\end{aligned}$$

The bound from below is obtained in similar way:

$$\begin{aligned}|I_\ell^k|&=\sum_{h=0}^{\infty}\sum_{j=0}^{2^h-1}\lambda_{2^{k+h}+\ell2^h+j}\\&\geq\sum_{h=0}^{\infty}\frac{2^h}{2^{hp}\left(2^k+\ell+1-\frac{1}{2^h}\right)^p}\geq\frac{1}{(2^k+\ell+1)^p}\sum_{h=0}^{\infty}\left(\frac{1}{2^{(p-1)}}\right)^h\\&=\lambda_{2^k+\ell+1}\frac{2^p}{2^p-2}.\end{aligned}$$

∎

Corollary 1. *For all k and all $0 \le \ell \le \ell' < 2^k$,*

$$1 \le \frac{|I_\ell^k|}{|I_{\ell'}^k|} \le 2^p.$$

Proof. Since

$$\frac{|I_\ell^k|}{|I_{\ell'}^k|} \le \frac{|I_0^k|}{|I_{2^k-1}^k|},$$

the result follows from Lemma 2. ∎

The next lemma is a simple algebraic property of numbers which we will use repeatedly.

Lemma 3. *Let a, b, c be arbitrary positive numbers, let $p > 1$ and set $x = a + b + c$. Then*

$$x \le \frac{2^p}{2^p - 2}\, c \quad \Longrightarrow \quad x^{\frac{1}{p}} \ge a^{\frac{1}{p}} + b^{\frac{1}{p}}.$$

Proof.

$$\text{If} \quad x \le \frac{2^p}{2^p - 2}\, c, \quad \text{then} \quad \frac{x - c}{2} \le \frac{x}{2^p}.$$

Equivalently,

$$\left(\frac{a+b}{2}\right)^{\frac{1}{p}} \le \frac{x^{\frac{1}{p}}}{2},$$

and since by convexity of the function $x^{\frac{1}{p}}$,

$$\frac{a^{\frac{1}{p}} + b^{\frac{1}{p}}}{2} \le \left(\frac{a+b}{2}\right)^{\frac{1}{p}},$$

the result follows. ∎

Lemma 4. *For all $k \ge 1$ and $\ell = 0, 1, \ldots, 2^k - 1$,*

$$|I_\ell^k|^{\frac{1}{p}} \ge |I_{2\ell}^{k+1}|^{\frac{1}{p}} + |I_{2\ell+1}^{k+1}|^{\frac{1}{p}}.$$

Proof. By construction and from Lemma 2 we have

$$|I_\ell^k| = |I_{2\ell}^{k+1}| + |I_{2\ell+1}^{k+1}| + \lambda_{2^k+\ell} \quad \text{and}$$
$$|I_\ell^k| \le \frac{2^p}{2^p-2}\lambda_{2^k+\ell}$$

Hence, by Lemma 3 the result is obtained. ∎

Lemma 5. *Let J be an arbitrary open interval such that $J \subset I_0$ and assume that $J \cap C_\lambda \neq \emptyset$. Define $k_0 := \min\{k \in \mathbb{N} : J$ contains an interval of step $k\}$. Then we have the following bound on the length of J:*

$$4|J|^{\frac{1}{p}} \ge \sum_{\ell\,:\,I_\ell^{k_0} \cap J \neq \emptyset} |I_\ell^{k_0}|^{\frac{1}{p}}.$$

Proof. Note first, that since $J \cap C_\lambda \neq \emptyset$ and J is open, k_0 is well defined. Next observe that if I_ℓ^k and $I_{\ell+1}^k$ are consecutive intervals from step k and obtained from one interval in step $k-1$, (we shall say that these two intervals have a common "father"), then ℓ is even. If we consider two consecutive intervals of step k not having a common father, i.e. I_ℓ^k and $I_{\ell+1}^k$ with ℓ odd, let I be the minimal closed interval containing I_ℓ^k and $I_{\ell+1}^k$. Then $I - \left(I_\ell^k \cup I_{\ell+1}^k\right)$ is a gap of a previous step, i.e.

$$|I| = |I_\ell^k| + \lambda_{2^s+r} + |I_{\ell+1}^k|$$

with $s \le k-2$ and $r = [\frac{\ell}{2^{k-s}}] < \frac{\ell}{2^{k-s}}$. We want to prove that $|I|^{\frac{1}{p}} \ge |I_\ell^k|^{\frac{1}{p}} + |I_{\ell+1}^k|^{\frac{1}{p}}$. By Lemma 2 we have

$$|I_\ell^k| \le \frac{2^p}{2^p-2}\lambda_{2^k+\ell},$$

and since

$$\lambda_{2^k+\ell} = \frac{1}{(2^k+\ell)^p} \le \frac{1}{(2^{k-s})^p\,(2^s+r)^p},$$

we conclude that

$$|I_\ell^k| \le \frac{1}{(2^{k-s})^p}\frac{2^p}{2^p-2}\lambda_{2^s+r}.$$

Since $|I_{\ell+1}^k| < |I_\ell^k|$ and $k-s \ge 2$, we have

$$|I| \le \left(\frac{2}{(2^{k-s})^p}\frac{2^p}{2^p-2} + 1\right)\lambda_{2^s+r} \le \frac{2^p}{2^p-2}\lambda_{2^s+r}.$$

Thus we can apply Lemma 3 to obtain

$$|I|^{\frac{1}{p}} \geq |I_\ell^k|^{\frac{1}{p}} + |I_{\ell+1}^k|^{\frac{1}{p}}. \tag{2}$$

Let now k_0 be defined as above. First we observe that the interval J can contain at most two intervals of step k_0. We will prove the case in which J contains exactly two, the other case can be proved similarly. (Note that by definition of k_0, J must contain at least one interval of step k_0.)

Let $I_\ell^{k_0}$ and $I_{\ell+1}^{k_0}$ be the intervals in J. Then ℓ is odd and only four Cantor intervals of step k_0 can intersect J. These are $I_{\ell-1}^{k_0}$, $I_\ell^{k_0}$, $I_{\ell+1}^{k_0}$, and $I_{\ell+2}^{k_0}$. Let $\tilde{I}$ be the smallest interval containing $I_\ell^{k_0}$ and $I_{\ell+1}^{k_0}$. Using (2) and since $J \supset \tilde{I}$, we have

$$|J|^{\frac{1}{p}} \geq |\tilde{I}|^{\frac{1}{p}} \geq |I_\ell^{k_0}|^{\frac{1}{p}} + |I_{\ell+1}^{k_0}|^{\frac{1}{p}}. \tag{3}$$

Now using Corollary 1 we know that

$$2^p |I_\ell^{k_0}| \geq |I_{\ell-1}^{k_0}|,$$

that is,

$$2|J|^{\frac{1}{p}} \geq 2|I_\ell^{k_0}|^{\frac{1}{p}} \geq |I_{\ell-1}^{k_0}|^{\frac{1}{p}}. \tag{4}$$

Finally, since $|I_{\ell+1}^{k_0}| \geq |I_{\ell+2}^{k_0}|$ and $I_{\ell+1}^{k_0} \subset J$,

$$|J|^{\frac{1}{p}} \geq |I_{\ell+1}^{k_0}|^{\frac{1}{p}} \geq |I_{\ell+2}^{k_0}|^{\frac{1}{p}}. \tag{5}$$

From (3), (4), and (5) we get

$$4|J|^{\frac{1}{p}} \geq |I_{\ell-1}^{k_0}|^{\frac{1}{p}} + |I_\ell^{k_0}|^{\frac{1}{p}} + |I_{\ell+1}^{k_0}|^{\frac{1}{p}} + |I_{\ell+2}^{k_0}|^{\frac{1}{p}}.$$

∎

Lemma 6. *Let J be an arbitrary open interval in I_0. Let $k \in \mathbb{N}$ be fixed. Then*

$$4|J|^{\frac{1}{p}} \geq \sum_{\ell \,:\, I_\ell^k \subset J} |I_\ell^k|^{\frac{1}{p}}.$$

Proof. If $J \cap C_\lambda = \emptyset$, the result is trivial.

Now, if $J \cap C_\lambda \neq \emptyset$, and J is open, then there exists some Cantor interval $I_\ell^k \subset J$. Define, as before, $k_0 := \min\{k \in \mathbb{N} : I_\ell^k \subset J \text{ for some } 0 \leq \ell < 2^k - 1\}$. By Lemma 5

$$4|J|^{\frac{1}{p}} \geq \sum_{\ell : I_\ell^{k_0} \subset J} |I_\ell^{k_0}|^{\frac{1}{p}}$$

Now, if $k' \geq k_0$, using Lemma 4 inductively, we have

$$4|J|^{\frac{1}{p}} \geq \sum_{\ell : I_\ell^{k'} \subset J} |I_\ell^{k'}|^{\frac{1}{p}}$$

and if $k' < k_0$ there are no intervals $I_\ell^{k'} \subset J$, $\ell = 0, \ldots, 2^{k'} - 1$ and the inequality is obvious. ∎

Lemma 7. *Let $\{E_i\}_{i=1,\ldots,n}$ be a finite family of open intervals such that*

(1) $C_\lambda \subset \cup_{i=1}^n E_i$

(2) $E_i = (a_i, b_i)$ with $a_i \notin C_\lambda$, $b_i \notin C_\lambda$, $|E_i| < \delta, i = 1, \ldots, n$.

Then

$$\sum_{i=1}^n |E_i|^{\frac{1}{p}} \geq \frac{1}{4} \frac{1}{(2^p - 2)^{\frac{1}{p}}}.$$

Proof. Since $a_i, b_i \notin C_\lambda$, for k large enough, we can arrange that for all ℓ, $|I_\ell^k|$ is sufficiently small so that for all ℓ, $I_\ell^k \subset E_i$ for some i.

Therefore, using Lemma 6,

$$\begin{aligned} \sum_{i=1}^n |E_i|^{\frac{1}{p}} &\geq \sum_{i=1}^n \frac{1}{4} \left(\sum_{\ell \,:\, I_\ell^k \subset E_i} |I_\ell^k|^{\frac{1}{p}} \right) \\ &\geq \frac{1}{4} \sum_{\ell=0}^{2^k-1} |I_\ell^k|^{\frac{1}{p}}. \end{aligned}$$

Since $|I_\ell^k| \geq |I_{\ell+1}^k|$, we get

$$\frac{1}{4} \sum_{\ell=0}^{2^k-1} |I_\ell^k|^{\frac{1}{p}} \geq \frac{2^k}{4} |I_{2^k-1}^k|^{\frac{1}{p}},$$

and, using the estimate of Lemma 2, we get the result. ∎

We are now ready to prove Theorem 3.

Proof (of Theorem 3).

Let $F = \{F_i\}_{i \in \mathbb{N}}$ be a covering of C_λ with open intervals of length less than δ,

$$\bigcup F_i \supset C_\lambda \quad \text{and} \quad \operatorname{diam}(F_i) < \delta, \ \forall \ i.$$

Since C_λ is compact, let $\{F_{h_k} = (\alpha_k, \beta_k)\}_{k=1}^m$ be a finite subcovering of C_λ, $F_{h_k} \in F, k = 1, \ldots, m$.

Let $\varepsilon > 0$. Since $\mathbb{R} \setminus C_\lambda$ is dense, we can construct open intervals, $E_k = (a_k, b_k), k = 1, \ldots, m$ such that

$$F_{h_k} \subset E_k, \quad |E_k|^{\frac{1}{p}} < |F_{h_k}|^{\frac{1}{p}} + \frac{\varepsilon}{m} \quad \text{and} \quad a_k, b_k \notin C_\lambda.$$

Therefore,

$$\sum_{k=1}^{m} |E_k|^{\frac{1}{p}} < \sum_{k=1}^{m} |F_{h_k}|^{\frac{1}{p}} + \varepsilon.$$

Since ε is arbitrary and using Lemma 7 we have

$$\frac{1}{4} \frac{1}{(2^p - 2)^{\frac{1}{p}}} \leq \sum_{k=1}^{n} |F_{h_k}|^{\frac{1}{p}} \leq \sum_{i \in \mathbb{N}} |F_i|^{\frac{1}{p}}.$$

Since F was an arbitrary open δ-covering of C_λ this shows that

$$\mathcal{H}_\delta^{\frac{1}{p}}(C_\lambda) \geq \frac{1}{4} \frac{1}{(2^p - 2)^{\frac{1}{p}}},$$

that is,

$$\mathcal{H}^{\frac{1}{p}}(C_\lambda) \geq \frac{1}{4} \frac{1}{(2^p - 2)^{\frac{1}{p}}} \quad \text{and} \quad \dim C_\lambda \geq \frac{1}{p}.$$

∎

§3.2 The Cantor sets $C_{\tilde{\lambda}}$

In this section we are going to prove the following theorem.

Theorem 4. *Let $\lambda = \{\lambda_k = k^{-p}\}$. For each $0 \leq s \leq \frac{1}{p}$ there exists a rearrangement $\sigma_s(\lambda)$ of λ such that*

$$\mathcal{H}^s(C_{\sigma_s(\lambda)}) > 0.$$

For this, we will first find the Hausdorff dimension of some Cantor sets $C_{\tilde{\lambda}}$ where $\tilde{\lambda}$ is a particular subsequence of the original p-sequence, and then we will use this result together with the following Lemma to establish our result.

Lemma 8. *If $\tilde{\lambda}$ is a subsequence of $\lambda = \{\lambda_k = k^{-p}\}$ such that dim $C_{\tilde{\lambda}} = s$, then there exists a rearrangement $\sigma(\lambda) = \sigma_s(\lambda)$ of λ, such that*

$$dim\ C_{\sigma_s(\lambda)} = dim\ C_{\tilde{\lambda}} = s.$$

Proof. Let $\gamma = \{\gamma_k\}$ be the subsequence obtained from λ after deleting the terms of the subsequence $\tilde{\lambda}$. By Proposition 2 there exists a rearrangement $\sigma(\gamma)$ of γ, such that dim $C_{\sigma(\gamma)} = 0$. Let now $t_1 = \sum_k \gamma_k$, then we have $C_{\sigma(\gamma)} \subset [0, t_1]$. Define

$$C = C_{\sigma(\gamma)} \bigcup (C_{\tilde{\lambda}} + \{t_1\}).$$

The set C is a Cantor set and dim $C = s$ (since it is the union of one Cantor set of dimension s and one of dimension 0.) The lengths of the gaps of C correspond to the terms of the original sequence λ. Then, there is a rearrangement σ of the sequence, that is associated to the Cantor set C, that is $C = C_{\sigma(\lambda)}$. ∎

It is now clear, that in order to obtain Theorem 4, it suffices for each s to find a particular subsequence $\tilde{\lambda}$, such that dim $C_{\tilde{\lambda}} = s$.

Let $x \geq 2$ be a fixed real number. We define a subsequence $\tilde{\lambda}_m$ by the following relation: using the fact that m can be decomposed uniquely into $m = 2^k + j$ with $k \geq 0$ and $j = 0, 1, \ldots, 2^{k-1}$, and using the notation $[x]$ to denote the greatest integer in x, we set

$$\tilde{\lambda}_m = \tilde{\lambda}_{2^k+j} = \lambda_{[x^k]+j} = \left(\frac{1}{[x^k]+j}\right)^p.$$

Since this subsequence is completely determined by x, and in order to avoid cumbersome notation, we denote by C_x the Cantor set $C_{\tilde{\lambda}}$.

Theorem 5. *With the above notation and with*

$$\alpha(p,x) := \frac{\log 2}{p \log x},$$

then

$$dim\, C_x = \alpha(p,x),$$

moreover

$$\mathcal{H}^{\alpha(p,x)}(C_x) > 0.$$

Proof. The proof of this Theorem has the same flavor as that of Theorem 2 and Theorem 3. However since we are dealing with subsequences of the original p-series, the estimates are not the same. Again we will split the proof of the Theorem into two separate statements - one for the upper bound and one for the lower bound. We use the following notation: all quantities introduced in the proofs of Theorems 3 and 4 corresponding to the sequence $\{\lambda_j\}$ will be used with the sign $\tilde{}$ to denote quantities corresponding to $\{\tilde{\lambda}_j\}$.

To shorten notation, from now on let $\alpha = \alpha(p,x) = \frac{\log 2}{p \log x}$.

Proposition 3. *Let C_x be the Cantor set associated to the sequence $\tilde{\lambda} = \{\tilde{\lambda}_{2^k+j} = \lambda_{[x^k]+j},\ k \in I\!N, j = 0, 1, \ldots, 2^k - 1\}$. Then*

$$dim\ C_x \leq \alpha.$$

Proof. The proof goes along the lines of the proof of Theorem 2, only instead of λ_j, we now have $\tilde{\lambda}_j$. Suppose that n is sufficiently large such that $\sum_{j=n+1}^{\infty} \tilde{\lambda}_j \leq \delta$. Also, to make the calculations simpler, we choose n of the form $n = 2^k - 1$. Suppose we remove from $\tilde{I}_0 = [0, \sum_j \tilde{\lambda}_j]$ open intervals of lengths $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n$. What remains can be written as the union of closed intervals $\tilde{E}_j^{(n)}$, $j = 1, 2, \ldots, n+1$. Since $\sum_{j=1}^{n+1} |\tilde{E}_j^{(n)}| = \sum_{j=n+1}^{\infty} \tilde{\lambda}_j$, then $\{\tilde{E}_j^{(n)} : j = 1, 2, \ldots, n+1\}$ is a δ-covering of C_x. Using the Hölder inequality, we have

$$\begin{aligned}
\sum_{j=1}^{n+1} |\tilde{E}_j^{(n)}|^s &\leq \left(\sum_{j=1}^{n+1} |\tilde{E}_j^{(n)}| \right)^s (n+1)^{1-s} \\
&= \left(\sum_{j=n+1}^{\infty} \tilde{\lambda}_j \right)^s (n+1)^{1-s}.
\end{aligned}$$

We now need an estimate of the quantity

$$\sum_{j=n+1}^{\infty} \tilde{\lambda}_j = \sum_{m=k}^{\infty} \sum_{j=0}^{2^m-1} \tilde{\lambda}_{2^m+j}.$$

We have

$$\begin{aligned}
\sum_{m=k}^{\infty} \sum_{j=0}^{2^m-1} \tilde{\lambda}_{2^m+j} &= \sum_{m=k}^{\infty} \sum_{j=0}^{2^m-1} \lambda_{[x^m]+j} \\
&= \sum_{m=k}^{\infty} \sum_{j=0}^{2^m-1} \frac{1}{([x^m]+j)^p} \le \sum_{m=k}^{\infty} \frac{2^m}{[x^m]^p} \\
&\le \sum_{m=k}^{\infty} \left(\frac{1}{x^m-1}\right)^p 2^m \le \sum_{m=k}^{\infty} 2^p \left(\frac{2}{x^p}\right)^m \\
&= \left(\frac{2}{x^p}\right)^k 2^p \frac{x^p}{x^p-2} \le K(p) \left(\frac{2}{x^p}\right)^k
\end{aligned}$$

where $K(p) = 4^p/(2^p-2)$ is a bound for $(2x)^p/(x^p-2)$. Since $2^k = n+1$ and $\alpha = \frac{\log 2}{p \log x}$, we have that

$$x^{pk} = (n+1)^{\frac{1}{\alpha}},$$

and

$$\begin{aligned}
\sum_{j=1}^{n+1} |\tilde{E}_j^{(n)}|^s &\le K^s(p) \frac{(n+1)^s}{x^{pks}} (n+1)^{1-s} \\
&= K^s(p)(n+1)^{1-\frac{s}{\alpha}},
\end{aligned}$$

which again, if $s \ge \alpha$ yields $\limsup_{n\to\infty} \sum_{j=1}^{n+1} |\tilde{E}_j^{(n)}|^s < \infty$, and therefore,

$$\dim C_x \le \alpha.$$

∎

The second stage, is to provide the lower bound for the Hausdorff dimension of C_x. We will again show, that actually the Hausdorff measure of C_x, $\mathcal{H}^\alpha(C_x)$, is positive, however this estimate is much harder to obtain. The theorem that we will prove is the following.

Theorem 6. *Let $p > 1$, $x > 2$ and let $\tilde{\lambda} = \{\tilde{\lambda}_k\}_{k\in\mathbb{N}}$ be the sequence defined by $\tilde{\lambda}_{2^k+j} = \left(\frac{1}{[x^k]+j}\right)^p$. If C_x is the Cantor set associated to the sequence $\tilde{\lambda}$, and $\alpha = \frac{\log 2}{p \log x}$, then there exists $c > 0$, such that*

$$\mathcal{H}^\alpha(C_x) \geq c\left(\frac{x^p}{x^p - 2}\right)^\alpha.$$

This Theorem is the analogue of Theorem 3, which was proved combining Lemmas 2, 3, 4, 5, 6 and 7. We will need to provide the analogues of these Lemmas for this new sequence. Note that Lemma 3 is independent of the sequence, so we only need to prove the other Lemmas. These are the contents of Lemma 9 (c.f. Lemma 2), Lemma 10 (c.f. Lemma 4), Lemma 11 (c.f. Lemmas 5 and 6) and Lemma 12 (c.f. Lemma 7).

Lemma 9. *For every fixed $x > 2$ and all $k \geq 1$ and $\ell = 0, \ldots, 2^k - 1$,*

$$\frac{x^p}{x^p - 2} \cdot \frac{1}{x^{(k+1)p}} \leq |\tilde{I}_\ell^k| \leq \frac{x^p}{x^p - 2}\left(\frac{1}{x^k - 1}\right)^p. \tag{6}$$

Proof. The proof is similar to the proof of Lemma 2. We have that

$$\begin{aligned}
\tilde{I}_\ell^k &= \sum_{h=0}^\infty \sum_{j=0}^{2^h-1} \tilde{\lambda}_{2^{k+h}+\ell 2^h + j} \leq \sum_{h=0}^\infty 2^h \tilde{\lambda}_{2^{k+h}+\ell 2^h} \\
&\leq \sum_{h=0}^\infty 2^h \frac{1}{([x^{k+h}] + \ell 2^h)^p} \leq \sum_{h=0}^\infty \frac{2^h}{[x^{k+h}]^p} \\
&\leq \sum_{h=0}^\infty \frac{2^h}{(x^{k+h} - 1)^p} \leq \frac{1}{(x^k - 1)^p} \sum_{h=0}^\infty \left(\frac{2}{x^p}\right)^h \\
&= \left(\frac{1}{x^k - 1}\right)^p \frac{x^p}{x^p - 2}.
\end{aligned}$$

The bound from below is obtained in a similar way

$$\begin{aligned}
|\tilde{I}_\ell^k| &= \sum_{h=0}^\infty \sum_{j=0}^{2^h-1} \tilde{\lambda}_{2^{k+h}+\ell 2^h + j} \\
&\geq \sum_{h=0}^\infty 2^h \tilde{\lambda}_{2^{k+h+1}} = \sum_{h=0}^\infty \frac{2^h}{[x^{k+1+h}]^p} \\
&\geq \frac{1}{x^{(k+1)p}} \sum_{h=0}^\infty \left(\frac{2}{x^p}\right)^h = \frac{x^p}{x^p - 2} \cdot \frac{1}{x^{(k+1)p}}.
\end{aligned}$$

∎

We again have the corollary whose proof is a direct application of the estimates of Lemma 9:

Corollary 2. *For all k and all $0 \le \ell \le \ell' < 2^k$,*

$$1 \le \frac{|\tilde{I}^k_\ell|}{|\tilde{I}^k_{\ell'}|} \le (2x)^p.$$

Since Lemma 9 gives an estimate which is not as precise as that of Lemma 2, we have now a weaker version of Lemma 4.

Lemma 10. *For all $k \in I\!N$ and all $\ell = 0, 1, \ldots, 2^k - 1$*

$$|\tilde{I}^k_\ell|^\alpha \ge b(k)\left(|\tilde{I}^{k+1}_{2\ell}|^\alpha + |\tilde{I}^{k+1}_{2\ell+1}|^\alpha\right) \tag{7}$$

where the sequence $b(k)$ satisfies, $0 < b(k) < 1$ and $\prod_k^\infty b(k) = \zeta$, with $\zeta = \zeta(p, x) > 0$.

Proof. From Lemma 9 we get the following estimate

$$|\tilde{I}^k_\ell| \le \frac{x^p}{x^p - 2}\left(\frac{[x^k] + \ell}{x_k - 1}\right)^p \tilde{\lambda}_{2^k+\ell},$$

and since $\ell \le 2^k - 1$,

$$|\tilde{I}^k_\ell| \le \frac{x^p}{x^p - 2}\left(\frac{x^k + 2^k - 1}{x_k - 1}\right)^p \tilde{\lambda}_{2^k+\ell}.$$

Since $|\tilde{I}^k_\ell| = |\tilde{I}^{k+1}_{2\ell}| + \tilde{\lambda}_{2^k+\ell} + |\tilde{I}^{k+1}_{2\ell+1}|$, this gives the estimate

$$|\tilde{I}^k_\ell|\left(1 - \frac{x^p - 2}{x^p}\left(\frac{x^k - 1}{x^k + 2^k - 1}\right)^p\right) \ge |\tilde{I}^{k+1}_{2\ell}| + |\tilde{I}^{k+1}_{2\ell+1}|,$$

or, regrouping,

$$\frac{|\tilde{I}^k_\ell|}{x^p}\left(\frac{x^p}{2} - \left(\frac{x^p}{2} - 1\right)\left(\frac{x^k - 1}{x^k + 2^k - 1}\right)^p\right) \ge \frac{|\tilde{I}^{k+1}_{2\ell}| + |\tilde{I}^{k+1}_{2\ell+1}|}{2},$$

$$\frac{|\tilde{I}^k_\ell|}{x^p}\left(1 + \left(\frac{x^p}{2} - 1\right)\left(1 - \left(\frac{x^k - 1}{x^k + 2^k - 1}\right)^p\right)\right) \ge \frac{|\tilde{I}^{k+1}_{2\ell}| + |\tilde{I}^{k+1}_{2\ell+1}|}{2}.$$

Denoting

$$c(k) = \left(1 + \left(\frac{x^p}{2} - 1\right)\left(1 - \left(\frac{x^k - 1}{x^k + 2^k - 1}\right)^p\right)\right),$$

taking both sides to the power α, applying convexity agruments as in Lemma 3 and remembering that $x^{p\alpha} = 2$, we get

$$\frac{|\tilde{I}^k_\ell|^\alpha}{2}(c(k))^\alpha \ge \frac{\left(|\tilde{I}^{k+1}_{2\ell}|^\alpha + |\tilde{I}^{k+1}_{2\ell+1}|^\alpha\right)}{2},$$

and therefore

$$|\tilde{I}_\ell^k|^\alpha \geq b(k)\left(|\tilde{I}_{2\ell}^{k+1}|^\alpha + |\tilde{I}_{2\ell+1}^{k+1}|^\alpha\right), \quad \text{where} \quad b(k) = \frac{1}{c(k)^\alpha}.$$

Since $c(k) > 1$, $0 < b(k) < 1$. To see that $\prod_k^\infty b(k) = \zeta > 0$ it is enough to see that $\prod_k^\infty c(k) < +\infty$. Since $x > 2$ and $p > 1$ we can write

$$\begin{aligned}
1 - \left(\frac{x^k - 1}{x^k + 2^k - 1}\right)^p &= 1 - \frac{1}{\left(1 + \frac{2^k}{x^k - 1}\right)^p} \leq 1 - \frac{1}{\left(1 + 2(\frac{2}{x})^k\right)^p} \\
&= \frac{\left(1 + 2(\frac{2}{x})^k\right)^p - 1^p}{\left(1 + 2(\frac{2}{x})^k\right)^p} \leq \frac{p\left(1 + 2(\frac{2}{x})^k\right)^{p-1} 2(\frac{2}{x})^k}{\left(1 + 2(\frac{2}{x})^k\right)^p} \leq 2p(\frac{2}{x})^k.
\end{aligned}$$

The second inequality follows by an application of the Mean Value Theorem. Using this, we see that the following series converges

$$\sum_{k=1}^\infty \left(\frac{x^p}{2} - 1\right)\left(1 - \left(\frac{x^k - 1}{x^k + 2^k - 1}\right)^p\right) \leq 2p\left(\frac{x^p}{2} - 1\right)\sum_k^\infty \left(\frac{2}{x}\right)^k < +\infty,$$

and therefore $\Pi_k^\infty c(k) < +\infty$. ∎

Here it is appropriate to mention that the proof can not be carried out for the case $x = 2$, since the estimate in Lemma 9 is too rough.

We can now prove the analogues of Lemma 5 and 6.

Lemma 11. *Let $\tilde{J}$ be an arbitrary open interval in $\tilde{I}_0$. Let $k \in \mathbb{N}$ be fixed and again let $\alpha = \log 2/(p \log x)$. Then there exists c, independent of k, such that*

$$c|\tilde{J}|^\alpha \geq \sum_{\ell : \tilde{I}_\ell^k \subset \tilde{J}} |\tilde{I}_\ell^k|^\alpha.$$

Proof. If $\tilde{J} \cap C_{\tilde{\lambda}} = \emptyset$, the result is trivial. Otherwise, define $k_0 := \min\{k \in \mathbb{N} : \tilde{J}$ contains an interval of step $k\}$. As before, $\tilde{J}$ can contain at most two intervals of step k_0. Again we will only prove the case in which $\tilde{J}$ contains exactly two.

Let $\tilde{I}_\ell^{k_0}$ and $\tilde{I}_{\ell+1}^{k_0}$ be the intervals in $\tilde{J}$. Then ℓ is odd and only four Cantor intervals of step k_0 can intersect $\tilde{J}$. These are $\tilde{I}_{\ell-1}^{k_0}$, $\tilde{I}_\ell^{k_0}$, $\tilde{I}_{\ell+1}^{k_0}$, and $\tilde{I}_{\ell+2}^{k_0}$. Since $\tilde{J} \supset \tilde{I}_\ell^{k_0}$ and $|\tilde{I}_\ell^{k_0}| \geq |\tilde{I}_{\ell+1}^{k_0}| \geq |\tilde{I}_{\ell+2}^{k_0}|$, we have

$$3\,|\tilde{J}|^\alpha \geq |\tilde{I}_\ell^{k_0}|^\alpha + |\tilde{I}_{\ell+1}^{k_0}|^\alpha + |\tilde{I}_{\ell+2}^{k_0}|^\alpha$$

Now using Corollary 2 we know that

$$(2x)^p|\tilde{I}_\ell^{k_0}| \geq |\tilde{I}_{\ell-1}^{k_0}|,$$

that is

$$(2x)^{p\alpha}|\tilde{J}|^\alpha \geq (2x)^{p\alpha}|\tilde{I}_\ell^{k_0}|^\alpha \geq |\tilde{I}_{\ell-1}^{k_0}|^\alpha.$$

Hence

$$(3+(2x)^{p\alpha})|\tilde{J}|^\alpha \geq |\tilde{I}_{\ell-1}^{k_0}|^\alpha + |\tilde{I}_\ell^{k_0}|^\alpha + |\tilde{I}_{\ell+1}^{k_0}|^\alpha + |\tilde{I}_{\ell+2}^{k_0}|^\alpha.$$

Now inductively we apply Lemma 10 and therefore we get:

$$(3+(2x)^{p\alpha})|\tilde{J}|^\alpha \geq b(k_0)b(k_0+1)\dots b(k-1)\sum_{\ell:\tilde{I}_\ell^k \subset \tilde{J}} |\tilde{I}_\ell^k|^\alpha.$$

Since

$$b(k_0)b(k_0+1)\dots b(k-1) \geq \prod_{j=1}^{\infty} b(j) = \zeta > 0,$$

we have

$$\frac{3+(2x)^{p\alpha}}{\zeta}|\tilde{J}|^\alpha \geq \sum_{\ell:\tilde{I}_\ell^k \subset \tilde{J}} |\tilde{I}_\ell^k|^\alpha.$$

∎

The last remaining lemma is the analog of Lemma 7.

Lemma 12. *Let $\{\tilde{E}_i\}_{i=1,\dots,n}$ be a finite family of open intervals such that*

(1) $C_x \subset \bigcup_{i=1}^n \tilde{E}_i$

(2) $\tilde{E}_i = (a_i, b_i)$ with $a_i \notin C_x$, $b_i \notin C_x$, $|\tilde{E}_i| < \delta, i = 1, \dots, n$.

Then there exists $c > 0$ such that

$$\sum_{i=1}^n |\tilde{E}_i|^\alpha \geq c\left(\frac{x^p}{x^p-2}\right)^\alpha.$$

Proof. As in the proof of Lemma 7, since $a_i, b_i \notin C_x$, if k is large enough, we have that for all ℓ, $|\tilde{I}_\ell^k|$ is sufficiently small so that $\tilde{I}_\ell^k \subset \tilde{E}_i$ for some i.

Therefore using Lemma 11, and denoting $2c = \zeta/(3+(2x)^{p\alpha})$, we have

$$\sum_{i=1}^{n} |\tilde{E}_i|^\alpha \geq \sum_{i=1}^{n} \frac{\zeta}{3+(2x)^{p\alpha}} \left(\sum_{\ell\,:\,\tilde{I}_\ell^k \subset \tilde{E}_i} |\tilde{I}_\ell^k|^\alpha \right)$$

$$\geq 2c \sum_{\ell=0}^{2^k-1} |\tilde{I}_\ell^k|^\alpha.$$

Since $|\tilde{I}_\ell^k| \geq |\tilde{I}_{\ell+1}^k|$, we get

$$\sum_{\ell=0}^{2^k-1} |\tilde{I}_\ell^k|^\alpha \geq 2^k |\tilde{I}_{2^k-1}^k|^\alpha,$$

and using the estimate of Lemma 9 and the fact that $x^{p\alpha} = 2$, we get

$$\sum_{i=1}^{n} |\tilde{E}_i|^\alpha \geq 2c2^k \left(\frac{x^p}{x^p-2} \right)^\alpha \frac{1}{x^{(k+1)p\alpha}}$$

$$= c \left(\frac{x^p}{x^p-2} \right)^\alpha.$$

∎

The proof of Theorem 6 can be completed by combining Lemmas 9, 10, 11 and 12.

§4 A generalization

In this section we will generalize the results obtained in the previous sections, to the case in which at each step instead of removing one interval from each interval of step k, we remove $\ell - 1$ intervals. The particular case $\ell = 2$ is the one considered so far. We will call the associated Cantor set C_ℓ.

Surprisingly, the same results are true. Since the proofs can be obtained by the same methodology as presented in the previous sections, we will only state one of the generalized results in this direction. Note that when carrying out the proofs, the role played by 2^k is now played by ℓ^k since everytime we had 2^k intervals at a given step k, now we have ℓ^k intervals at that same step k.

Theorem 7. *Let $\lambda = \{\lambda_k\}_{k\in I\!N}$ be defined by $\lambda_k = \left(\frac{1}{k}\right)^p$, $p > 1$. Then for $\ell \geq 2$,*

$$dim\ C_\ell = \frac{1}{p}.$$

Moreover

$$\mathcal{H}^{\frac{1}{p}}(C_\ell) > 0.$$

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