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TANGENTIAL MEASURE ON THE SET OF CONVEX INFINITE CYLINDERS

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Abstract

We find a 'natural' measure on the set of convex infinite cylinders touching a convex body K in E_n. Using the curvature measures, we can calculate the probability that a convex infinite cylinder and a convex body K are tangent in some previously determined regions. As a particular case we obtain a measure on all q-planes touching K.

TANGENTIAL PROBABILITIES; CURVATURE MEASURES; HAAR MEASURE; STEINER FORMULA

1. Introduction

In recent years, several authors in integral geometry have studied the problem of 'tangential probabilities between convex bodies'. That is, in the euclidean n-dimensional space E_n, we consider two convex bodies K and K', colour one part of each of the boundaries and move one of the bodies (or both) such that they touch. The question is, what is the probability that the coloured surfaces touch?

Firey (1974) was the first to approach this problem. McMullen (1974) found the probability in the case that K and K' are convex polyhedrons. Firey gave the probability for any convex body in terms of the surface integrals. Since these integrals are only defined on the unit sphere, the coloured parts had to be inverse images of Borel sets of the unit sphere.

Schneider (1978b) avoided this restriction using, instead of the surface integrals, the 'curvature measures' which are defined on the boundary of every convex body K.

Finally, Weil (1979) showed that there exists a natural measure on $G_n(K, K')$ — all motions in E_n such that K and gK' ($g \in G_n$) are tangent — so that the sets $C_0(\beta, \beta')$ of all the motions of $G_n(K, K')$ such that β and $g\beta'$ are tangent, are

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measurable for every β, β' Borel subset of the boundaries of K and K' respectively. He also proved that this measure can be expressed in terms of the curvature measures, obtaining for polyhedrons the same result as McMullen.

In this paper we consider a similar 'tangential' problem. Using the definition of infinite convex cylinders given by Santaló (1976), let Z_q be a cylinder whose normal cross-section D is a convex body in an $(n-q)$-plane and whose generators are the q-planes orthogonal to D.

We find a measure on $Z(D)$ — all cylinders congruent to Z_q — so that this measure is concentrated on $Z(D, K)$ — all those cylinders of $Z(D)$ that are tangent to K, where K is a convex body in E_n.

This measure allows us to calculate probabilities of the following type: if η is a Borel set of the boundary of K and η' is an 'infinite sector' of the boundary of Z_q (meaning that η' is the union of all q-planes orthogonal to D through the points of $\overline{\eta'}$, where $\overline{\eta'}$ is a Borel set of the boundary of D) and supposing that K and Z_q are tangent, what is the probability that they touch in η and η'?

Moreover, if D reduces to a point, this measure gives a measure for all the q-planes touching K. This measure, here obtained as a particular case of the one on $Z(D, K)$, coincides with the measure for q-planes tangent to K found by Weil (1981).

2. Notation

Let E_n be the n-dimensional euclidean space with unit sphere Ω_n. λ_n is the Lebesgue measure of E_n and $\kappa_n = \lambda_n(\Omega_n)$. If K and K' are sets in E_n, we note by ∂K and $\partial K'$ the boundaries of K and K' respectively and by $d_n(K, K')$ the distance between K and K' in E_n.

If K is a convex body in E_n, let $W_i^n(K)$ be the *Quermassintegrale* of K $(i = 0, \cdots, n)$ defined as in Santaló (1976), p. 217. Like McMullen (1974), we shall use the intrinsic volumina of K, $V_i^n(K)$, where

$$V_j^n(K) = \frac{\binom{n}{j} W_{n-j}^n(K)}{\kappa_{n-j}}.$$

Then, if K_ε is the parallel body in the distance ε of K (see Santaló (1976), p. 220), we have the Steiner formula

$$\lambda_n(K_\varepsilon) = \sum_{j=0}^{n} \varepsilon^{n-j} \kappa_{n-j} V_j^n(K). \tag{2.1}$$

We introduce the curvature measures on convex bodies as in Schneider (1978b). If K is a convex body in E_n and β is a Borel set in K, then

$$C_\varepsilon^n(K, \beta) = \left\{ x \in E_n \;\middle|\; 0 < d_n(x, K) < \varepsilon \wedge d_n(x, K) = \min_{y \in \beta} d_n(x, y) \right\} \tag{2.2}$$

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and a local Steiner formula holds

$$\lambda_n(C_\varepsilon^n(K,\beta)) = \sum_{j=0}^{n-1} \varepsilon^{n-j}\kappa_{n-j}\Psi_j^n(K,\beta) \tag{2.3}$$

from which the curvature measures $\Psi_0^n(K,\cdot),\cdots,\Psi_{n-1}^n(K,\cdot)$ are defined. Then using (2.1) we have

$$\Psi_j^n(K,\partial K) = V_j^n(K) \qquad j=0,\cdots,n-1. \tag{2.4}$$

Let G_n be the group of all motions and SO_n the group of all rotations in E_n, with Haar measures μ and ν respectively. We use the following representation for μ: if γ is the continuous application from $SO_n \times E_n$ in G_n,

$$\gamma: (\theta, t) \to g_{\theta,t}$$

where $g_{\theta,t}(x) = \theta x + t$ for every x in E_n, then μ is the measure in G_n induced from $\nu \otimes \lambda_n$ by γ, that is $\mu = \gamma(\nu \otimes \lambda_n)$ (Schneider (1979), pp. 94–95) and if M is a Borel set in G_n

$$\mu(M) = \int_{SO_n}\int_{T(\theta)} \lambda_n(dx)\nu(d\theta) \tag{2.5}$$

with $T(\theta) = \{x \in E_n \mid g_{\theta,x} \in M\}$.

If K and K' are two convex bodies in E_n, $G_n(K,K')$ denotes the set of all those g of G_n, such that K and gK' are tangent. Weil (1979) proved that there exists a finite Borel measure on $G_n(K,K')$, $\phi(K,K',\cdot)$, such that

$$\phi(K,K',\eta) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mu(C_\varepsilon(\eta)) \tag{2.6}$$

where η is a Borel set in G_n and

$$C_\varepsilon(\eta) = \{g \in G_n \mid 0 < d_n(K,gK') < \varepsilon \wedge \tau(K,gK')\circ g \in \eta\} \tag{2.7}$$

where $\tau(K,gK')$ is the translation by the vector $u(K,gK')$ with $u(K,gK') = t - t'$ if $t \in K$, $t' \in K'$ realize the distance between K and gK'.

Further on, if β and β' are Borel subsets of ∂K and $\partial K'$ respectively, then

$$C_0(\beta,\beta') = \{g \in G_n(K,K') \mid \beta \cap g\beta' \neq \varnothing\} \tag{2.8}$$

is $\phi(K,K',\cdot)$-measurable and

$$\phi(K,K',C_0(\beta,\beta')) = \frac{n+1}{\kappa_n}\sum_{j=0}^{n-1}\frac{\kappa_{n-j}\kappa_{j+1}}{\binom{n+1}{j+1}}\Psi_j(K,\beta)\Psi_{n-1-j}(K',\beta') \text{ (Weil (1979)).} \tag{2.9}$$

Finally we define infinite convex cylinders. Let O be a fixed point in E_n and let L_{n-q} be an $(n-q)$-plane through O. Let D be a bounded convex body in L_{n-q}.

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For each point $x \in D$ we consider the q-plane orthogonal to L_{n-q}. The union of all such L_q is the cylinder Z_q (Santaló (1976), p. 270). The q-planes L_q are the generators of Z_q, and D is a normal cross-section. Note that a cylinder is determined, up to a motion, by its cross-section D.

If $Z(D)$ is the set of all those cylinders congruent to Z_q, let us consider the continuous and surjective application α_D from $SO_n \times L_{n-q}$ in $Z(D)$:

$$\alpha_D : (\theta, t) \rightarrow \theta^{-1}(Z_q + t).$$

We now give $Z(D)$ the 'final topology' according to α_D, that is the finest topology for which α_D is continuous, and so $\gamma_D = \alpha_D(\nu \otimes \lambda_{n-q})$ is a G_n-invariant measure on $Z(D)$ (see Schneider (1979), p. 106), and if A is a measurable set of $Z(D)$

$$\gamma_D(A) = \int_{SO_n} \int_{R(\theta)} \lambda_{n-q}(dx)\nu(d\theta) \tag{2.10}$$

with $R(\theta) = \{t \in L_{n-q} \mid \alpha_D(\theta, t) \in A\}$.

3. Tangential measure on convex cylinders in E_n

In this section we consider E_n as the orthogonal direct sum $E_n = L_{n-q} \oplus L_q$ and Π_{n-q} as the orthogonal projection on L_{n-q}.

Let K be a convex body in E_n and Z_q an infinite convex cylinder whose normal cross-section D is a convex body in L_{n-q}. Note that, if Z' is congruent to Z_q (that is $Z' \in Z(D)$) then there exists (θ, t) in $SO_n \times L_{n-q}$ such that $Z' = \theta^{-1}(Z_q + t)$, and it is clear that $d_n(K, Z') = d_{n-q}(\Pi_{n-q}(\theta K), D + t)$.

If $Z(D, K)$ denotes the set of all cylinders of $Z(D)$ that are tangent to K, we want to define on $Z(D, K)$ a 'natural' measure. Following Weil (1981), consider β a Borel set in $Z(D)$ and $\varepsilon > 0$:

$$M_\varepsilon(K, \beta) = \{Z' \in Z(D) \mid 0 < d_n(K, Z') < \varepsilon \wedge \tau_{K,Z'}(Z') \in \beta\}$$

with $\tau_{K,Z'} \in G_n$ and if $Z' = \theta^{-1}(Z_q + t)$ then $\tau_{K,Z'}(Z') = \theta^{-1}(Z_q + t + u)$ where $u \in L_{n-q}$ is such that $u = y - y'$, if $y \in \Pi_{n-q}(\theta K)$, $y' \in (D + t)$ satisfy $d_{n-q}(y, y') = d_{n-q}(\Pi_{n-q}(\theta K), D + t)$.

It turns out that $M_\varepsilon(K, \beta)$ is a Borel set in $Z(D)$.

Let us now calculate $\gamma_D(M_\varepsilon(K, \beta))$. Using (2.10) we have

$$\gamma_D(M_\varepsilon(K, \beta)) = \int_{SO_n} \int_{R(\theta)} \lambda_{n-q}(dx)\nu(d\theta) \tag{3.1}$$

with

$$R(\theta) = \{t \in L_{n-q} \mid \alpha_D(\theta, t) \in M_\varepsilon(K, \beta)\}.$$

In Appendix I we prove that

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$$R(\theta)=C_\varepsilon^{n-q}(\Pi_{n-q}(\theta K)+(-D),A_\beta^{n-q}(\theta)),$$

with

$$(-D)=\{x\in L_{n-q}\mid -x\in D\}$$

and

$$A_\beta^{n-q}(\theta)=\{t\in L_{n-q}\mid \theta^{-1}(Z_q+t)\in(\beta\cap Z(D,K))\}.$$

But then, using the local Steiner formula (2.3), we obtain

$$\lambda_{n-q}(R(\theta))=\sum_{j=0}^{n-q-1}\varepsilon^{n-q-j}\kappa_{n-q-j}\Psi_j^{n-q}(\Pi_{n-q}(\theta K)+(-D),A_\beta^{n-q}(\theta)). \tag{3.2}$$

As in Schneider (1978a), we give ε the values $1,\cdots,n-q$ and obtain

$$\begin{aligned}&\Psi_j^{n-q}(\Pi_{n-q}(\theta K)+(-D),A_\beta^{n-q}(\theta))\\&=\sum_{k=1}^{n-q}a_k^j\lambda_{n-q}(C_\varepsilon^{n-q}(\Pi_{n-q}(\theta K)+(-D),A_\beta^{n-q}(\theta)))\end{aligned} \tag{3.3}$$

with constants a_k^j which do not depend on θ. As $\lambda_{n-q}(R(\theta))$ is measurable in the sense of θ, then each of the terms of the sum (3.3) is a measurable function of θ and so $\Psi_j^{n-q}(\Pi_{n-q}(\theta K)+(-D),A_\beta^{n-q}(\theta))$ depends measurably on θ. Equations (3.1) and (3.2) lead us to a local Steiner formula

$$\gamma_D(M_\varepsilon(K,\beta))=\sum_{j=0}^{n-q-1}\varepsilon^{n-q-j}\kappa_{n-q-j}\int_{SO_n}\Psi_j^{n-q}(\Pi_{n-q}(\theta K)+(-D),A_\beta^{n-q}(\theta))\nu(d\theta).$$

We now define the desired measure $\phi_D(K,\cdot)$ as the coefficient of ε in the sum above, that is

$$\phi_D(K,\beta)=2\int_{SO_n}\Psi_{n-q-1}^{n-q}(\Pi_{n-q}(\theta K)+(-D),A_\beta^{n-q}(\theta))\nu(d\theta).$$

Then $\phi_D(K,\cdot)$ is a finite measure on $Z(D)$, concentrated on $Z(D,K)$, and so

$$\phi_D(K,\beta)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\gamma_D(M_\varepsilon(K,\beta)). \tag{3.4}$$

4. Application of the formula (3.4)

In this paragraph we want to apply formula (3.4) to a very special set β.

Using the same notation given in the last section, let η be a Borel set in ∂K and η' an 'infinite sector' of ∂Z_q; we then want to calculate the following probability: if $Z'\in Z(D,K)$ is such that Z' and K are tangent, $Z'=\theta^{-1}(Z_q+t)$, what is the probability that η and $\theta^{-1}(\eta'+t)$ are tangent?

Therefore we have to define what is meant by an 'infinite sector' of ∂Z_q. If $\overline{\eta'}$ is a Borel set in ∂D, for every $x\in\overline{\eta'}$, let $L_q(x)$ be the q-plane orthogonal to D

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through x, then we define

$$\eta' = \bigcup_{x \in \eta'} (L_q(x)).$$

Now we set

(4.1) $A(\eta, \eta') = \{Z' \in Z(D, K) \mid \text{if } Z' = \theta^{-1}(Z_q + t) \text{ then } \eta \cap \theta^{-1}(\eta' + t) \neq \emptyset\}.$

The first question is, is $A(\eta, \eta')$ $\phi_D(K, \cdot)$-measurable? The second is, can we express $\phi_D(K, A(\eta, \eta'))$ in terms of the curvature measures of K and Z_q?

The measurability of $A(\eta, \eta')$ is just like that of $C_0(\beta, \beta')$ defined in (2.8) (see Weil (1979)).

To express $\phi_D(K, A(\eta, \eta'))$ in terms of the curvature measures, we define $\zeta_j^n(Z_q, \eta')$ analogously to the definition of $W_i^n(Z_q)$ given by Santaló (1976), p. 270:

$$\zeta_j^n(Z_q, \eta') = \begin{cases} \dfrac{n-q}{n} \Psi_j^{n-q}(D, \overline{\eta'}) & j = 0, \cdots, n-q-1 \\ 0 & j = n-q, \cdots, n. \end{cases} \tag{4.2}$$

Moreover, if $B_q(x, h)$ is the sphere with centre x and radius h in $L_q(x)$, we define

$$\begin{cases} Z_q(h) = \bigcup_{x \in D} B_q(x, h) \\ \eta'(h) = \bigcup_{x \in \overline{\eta'}} B_q(x, h). \end{cases} \tag{4.3}$$

Now, using (2.4) and following Santaló (1976), p. 271, taking into consideration the different numbering, we obtain

$$\Psi_j^n(Z_q(h), \eta'(h)) = \begin{cases} \sigma_j(h) & j = 0, \cdots, q-1 \\ \dfrac{n}{n-q} \zeta_{j-q}^n(Z_q, \eta') \kappa_q h^q + \sigma_j(h) & j = q, \cdots, n-1 \end{cases} \tag{4.4}$$

where $\sigma_j(h)$ is such that $\sigma_j(h)/h^q \to 0$ as $h \to \infty$.

Remembering $\phi(K, K', \cdot)$ defined in (2.6), we define

$$\overline{\phi_D}(K, A(\eta, \eta')) = \lim_{h \to \infty} \frac{1}{\kappa_q h^q} \phi(K, Z_q(h), C_0(\eta, \eta'(h))) \tag{4.5}$$

with $C_0(\eta, \eta'(h))$ as in (2.8).

To reach the desired result, we need the following lemma, which we prove in Appendix II.

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Lemma. Let K be a convex body in E_n, η a Borel set in ∂K, Z_q an infinite convex cylinder whose normal cross-section D is included in L_{n-q}, and η' an 'infinite sector' in ∂Z_q such that $\overline{\eta'} = \eta' \cap D$ is a Borel set in ∂D. If $Z_q(h)$ and $\eta'(h)$ are the sets defined in (4.4), $A(\eta, \eta')$, $\overline{\phi_D}(K, \cdot)$ and $\phi_D(K, \cdot)$ defined in (4.1), (4.5) and (3.4) respectively, then it holds that

$$\overline{\phi_D}(K, A(\eta, \eta')) = \phi_D(K, A(\eta, \eta')).$$

Proposition. Given K a convex body, and Z_q an infinite convex cylinder whose normal cross-section D is included in L_{n-q}, in E_n, let η be a Borel set in ∂K and η' an 'infinite sector' in ∂Z_q such that $\overline{\eta'} = \eta' \cap D$ is a Borel set in ∂D, then the measure of the set of those cylinders Z' congruent to Z_q such that K and Z' are tangent in η and η' is

$$\phi_D(K, A(\eta, \eta')) = \sum_{j=0}^{n-q-1} \frac{n(j+1)}{(n-q)\binom{n}{j}} \frac{\kappa_{n-j}\kappa_{j+1}}{\kappa_n} \Psi_j^n(K, \eta)\zeta_{n-j-1-q}^n(Z_q, \eta').$$

Proof. Using the lemma, $\phi_D(K, A(\eta, \eta')) = \overline{\phi_D}(K, A(\eta, \eta'))$, and remembering (2.9) and (4.5),

$$\overline{\phi_D}(K, A(\eta, \eta')) = \lim_{h\to\infty} \frac{1}{\kappa_q h^q} \frac{n+1}{\kappa_n} \sum_{j=0}^{n-1} \frac{\kappa_{n-j}\kappa_{j+1}}{\binom{n+1}{j+1}} \Psi_j^n(K, \eta)\Psi_{n-1-j}^n(Z_q(h), \eta'(h)).$$

Using (4.4) we now obtain the result.

Corollary 1. If K is a convex body in E_n and Z_q an infinite convex cylinder in E_n whose normal cross-section D is in L_{n-q}, and if η is a Borel set in ∂K and η' is an 'infinite sector' in ∂Z_q such that $\overline{\eta'} = \eta' \cap D$ is a Borel set in ∂D, then if $Z' = \theta^{-1}(Z_q + t)$, is such that Z' and K are tangent, the probability that η and $\theta^{-1}(\eta' + t)$ are tangent is

$$\frac{\displaystyle\sum_{j=0}^{n-q-1} \frac{n(j+1)}{\binom{n}{j}(n-q)} \frac{\kappa_{n-j}\kappa_{j+1}}{\kappa_n} \Psi_j^n(K, \eta)\zeta_{n-j-1-q}^n(Z_q, \eta')}{\displaystyle\sum_{j=0}^{n-q-1} \frac{n(j+1)}{\binom{n}{j}(n-q)} \frac{\kappa_{n-j}\kappa_{j+1}}{\kappa_n} V_j^n(K)V_{n-j-1-q}^n(Z_q)}.$$

The proof follows immediately from the proposition.

Corollary 2. If K is a convex body in E_n and η is a Borel set in ∂K, then the measure of all the q-planes being tangent to K in η, is proportional to $\Psi_{n-q-1}^n(K, \eta)$, if $\Psi_i^n(K, \cdot)$ are the curvature measures of K.

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This means that $\Psi^n_{n-q-1}(K,\cdot)$ give a 'natural' measure on the set of the q-planes tangent to K.

Proof. If D from the proposition reduces to a point, $\phi_D(K,\cdot)$ is a measure on all the q-planes tangent to K. Moreover

$$\zeta_0^n(Z_q,\eta')=1 \quad \text{and} \quad \zeta_i^n(Z_q,\eta')=0 \qquad i=1,\cdots,n-q-1.$$

Therefore

$$\phi_D(K,\eta)=\frac{n\kappa_{n-q}\kappa_{q+1}}{\binom{n}{n-1-q}\kappa_n}\Psi^n_{n-1-q}(K,\eta).$$

We conclude by exemplifying the results of the corollary in the case where $n=3$ and ∂K is a surface of class C^2. In this case, we have two possibilities for infinite convex cylinders:

(i) $q=1$ and therefore $n-q=2$ and so the normal cross-section of Z_1 is a plane convex body,

(ii) $q=2$ and therefore $n-q=1$ and Z_2 is a space section limited by two parallel planes.

In Case (i) we consider a convex body in E_3(∂K is C^2) with volume V, surface area S and integral of mean curvature M; η is a Borel set in ∂K, with surface area S_η, integral of mean curvature M_η, and area of spherical image Ω_η, where we understand by spherical image that to every point of η corresponds the end of the normal unitary vector to K at this point.

If D is the normal cross-section of Z_1, let f be the area and l the length of ∂D; and if η' is an 'infinite sector' of ∂Z_1, let u be the length of $\partial(\eta'\cap D)$ and ζ the angle between the normals (in L_2) at the extremes of $\eta'\cap D$. Now, supposing that K and Z_1 are tangent, the probability that they contact at η and η' is (Corollary 1)

$$\frac{\Omega_\eta u+M_\eta\zeta}{4\pi l+2\pi M}$$

and if D reduces to a point, the measure of all the lines touching K at η is $\frac{3}{4}M_\eta$.

In Case (ii), let K and η be as in (i), and η' one of the planes that limit Z_2, then the probability that Z_2 touches K in η and η' (supposing they are tangent) is $\Omega_\eta/4\pi$ and if Z_2 reduces to one plane, the measure of all the planes touching K at η is $\Omega_\eta/2\pi$.

Appendix I

We want to prove that

$$R(\theta)=C_\varepsilon^{n-q}(\Pi_{n-q}(\theta K)+(-D),A_\beta^{n-q}(\theta)).$$

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Let E_n be $E_n = L_{n-q} \oplus L_q$ and $D = Z_q \cap L_{n-q}$. If

$$R(\theta) = \{t \in L_{n-q} \mid \alpha_D(\theta, t) \in M_\varepsilon(K, \beta)\}$$

with

$$M_\varepsilon(K, \beta) = \{Z' \in Z(D) \mid 0 < d_n(K, Z') < \varepsilon \wedge \tau_{K,Z'}(Z') \in \beta\}$$

with $\tau_{K,Z'} \in G_n$ such that, if $Z' = \theta^{-1}(Z_q + t)$, then

$$\tau_{K,Z'}(Z') = \theta^{-1}(Z_q + t + u)$$

$u \in L_{n-q}$, $u = y - y'$ if $y \in \Pi_{n-q}(\theta K)$, $y' \in (D + t)$ satisfy the condition that

$$d_{n-q}(y, y') = d_{n-q}(\Pi_{n-q}(\theta K), D + t).$$

We first observe that

$$d_n(\theta K, Z_q + t) = d_{n-q}(\Pi_{n-q}(\theta K), D + t).$$

If $A_\beta^{n-q}(\theta) = \{t \in L_{n-q} \mid \theta^{-1}(Z_q + t) \in (\beta \cap Z(D, K))\}$ and remembering (2.2)

$$\begin{aligned}
C_\varepsilon(\Pi_{n-q}(\theta K) + (-D), A_\beta^{n-q}(\theta)) &= \Big\{t \in L_{n-q} \,\Big|\, 0 < d_{n-q}(t, \Pi_{n-q}(\theta K) + (-D)) < \varepsilon \\
&\qquad \wedge d(t, \Pi_{n-q}(\theta K) + (-D)) = \min_{y \in A_\beta^{n-q}} d(t, y)\Big\} \\
&= \Big\{t \in L_{n-q} \,\Big|\, 0 < d_{n-q}(\Pi_{n-q}(\theta K), D + t) < \varepsilon, \\
&\qquad d_{n-q}(\Pi_{n-q}(\theta K), D + t) \\
&\qquad = \min_y \{d(t, y) \mid \theta^{-1}(Z_q + y) \in (\beta \cap Z(D, K))\}\Big\} \\
&= \{t \in L_{n-q} \mid 0 < d_{n-q}(\Pi_{n-q}(\theta K), D + t) < \varepsilon, \\
&\qquad \tau_{K,Z'}(\theta^{-1}(Z_q + t)) \in \beta\} \\
&= \{t \in L_{n-q} \mid 0 < d_{n-q}(\theta K, Z_q + t) < \varepsilon, \\
&\qquad \tau_{K,Z'}(\theta^{-1}(Z_q + t)) \in \beta\} \\
&= R(\theta).
\end{aligned}$$

Appendix II. Proof of the lemma

Using (2.6) we first observe that

$$\text{(A.1)} \qquad \overline{\phi_D}(K, A(\eta, \eta')) = \lim_{h \to \infty} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon \kappa_q h^q} \mu(C_\varepsilon(C_0(\eta, \eta'(h))))$$

and according to Schneider (1979), p. 148 and using (4.4)

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$$\mu(C_\varepsilon(C_0(\eta,\eta'(h)))) = \sum_{p=1}^{n} \varepsilon^p \sum_{i=0}^{n-p} b_{npi} \Psi_i^n(K,\eta) \Psi_{n-i-p}^n(Z_q(h),\eta'(h))$$

(b_{npi} constants depending only on n, p and i)

$$= \sum_{p=1}^{n-q} \varepsilon^p \left(\sum_{i=0}^{n-q-p} b_{npi} \frac{n\kappa_q h^q}{n-q} \Psi_i^n(K,\eta) \zeta_{n-i-p}^n(Z_q,\eta') + \sum_{i=0}^{n-p} b_{npi} \Psi_i(K,\eta) \sigma_{n-i-p}(h) \right)$$

$$+ \sum_{p=n-q+1}^{n} \varepsilon^p \sum_{i=0}^{n-p} b_{npi} \Psi_i^n(K,\eta) \sigma_{n-i-p}(h)$$

$$= \sum_{p=1}^{n} \varepsilon^p \sum_{i=0}^{n-p} b_{npi} \Psi_i^n(K,\eta) \sigma_{n-i-p}(h) + \sum_{p=1}^{n-q} \varepsilon^p \sum_{i=0}^{n-q-p} b_{npi} \frac{n\kappa_q h^q}{n-q} \Psi_i^n(K,\eta) \zeta_{n-i-p}^n(Z_q,\eta').$$

Then

$$\frac{1}{\varepsilon\kappa_q h^q} \mu(C_\varepsilon(C_0(\eta,\eta'(h)))) = \sum_{p=1}^{n} \varepsilon^{p-1} \sum_{i=0}^{n-p} b_{npi} \Psi_i^n(K,\eta) \frac{\sigma_{n-i-p}(h)}{\kappa_q h^q} + \sum_{p=1}^{n-q} \varepsilon^{p-1} \sum_{i=0}^{n-q-p} b_{npi} \frac{n}{n-q} \Psi_i^n(K,\eta) \zeta_{n-i-p}^n(Z_q,\eta')$$

and therefore

$$\lim_{h\to\infty} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon\kappa_q h^q} \mu(C_\varepsilon(C_0(\eta,\eta'(h)))) = \lim_{\varepsilon\to 0} \lim_{h\to\infty} \frac{1}{\varepsilon\kappa_q h^q} \mu(C_\varepsilon(C_0(\eta,\eta'(h)))),$$

and remembering (A.1)

$$\overline{\phi_D}(K, A(\eta,\eta')) = \lim_{\varepsilon\to 0} \lim_{h\to\infty} \frac{1}{\kappa_q h^q} \mu(C_\varepsilon(C_0(\eta,\eta'(h)))).$$

We shall now prove that

$$\lim_{h\to\infty} \frac{1}{\kappa_q h^q} \mu(C_\varepsilon(C_0(\eta,\eta'(h)))) = \gamma_D(M_\varepsilon(K, A(\eta,\eta')))$$

which gives the desired result.

We use the expression for μ given in (2.5):

$$\text{(A.2)} \qquad \mu(C_\varepsilon(C_0(\eta,\eta'(h)))) = \int_{SO_n} \int_{T(\theta)} \lambda_n(dx)\nu(d\theta)$$

$$T(\theta) = \{x \in E_n \mid g_{\theta,x} \in C_\varepsilon(C_0(\eta,\eta'(h)))\}$$
$$= \{x \in E_n \mid 0 < d_n(K, \theta(Z_q(h)) + x) < \varepsilon \wedge \tau(K, \theta(Z_q(h) + x)) \circ g_{\theta,x} \in C_0(\eta,\eta'(h))\}.$$

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Let L_{n-q} be such that the normal cross-section of Z_q, D, is in L_{n-q}. Now, for each fixed θ we want to calculate $T(\theta)$. We consider $E_n = \theta L_{n-q} \oplus \theta L_q$, where L_q is orthogonal to L_{n-q} and

$$T_{x_1}(\theta) = \{x_2 \in \theta L_q \mid x_1 + x_2 \in T(\theta)\}$$

$$T_{x_2}(\theta) = \{x_1 \in \theta L_{n-q} \mid x_1 + x_2 \in T(\theta)\}.$$

Using Fubini's theorem,

$$\int_{T(\theta)} \lambda_n(dx) = \int_L \left(\int_{T_{x_1}(\theta)} \lambda_q(dx_2) \right) \lambda_{n-q}(dx_1)$$

where

$$L = \bigcup_{x_2 \in \theta L_q} T_{x_2}(\theta)$$

and since $L = \tilde{\Pi}_{n-q}(T(\theta))$ where $\tilde{\Pi}_{n-q}$ is the orthogonal projection on θL_{n-q} it follows that

$$\int_{T(\theta)} \lambda_n(dx) = \int_{\tilde{\Pi}_{n-q}(T(\theta))} \int_{T_{x_1}(\theta)} \lambda_q(dx_2) \lambda_{n-q}(dx_1).$$

Let us calculate $\int_{T_{x_1}(\theta)} \lambda_q(dx_2)$:

$$T_{x_1}(\theta) = \{x_2 \in \theta L_q \mid 0 < d_n(K, \theta Z_q(h) + x_1 + x_2) < \varepsilon$$
$$\wedge\ \tau(K, \theta Z_q(h) + x_1 + x_2) \circ g_{\theta, x_1 + x_2} \in C_0(\eta, \eta'(h))\}.$$

If $y \in K$ is such that

$$d_n(K, \theta Z_q(h) + x_1 + x_2) = d_n(y, \theta Z_q(h) + x_1 + x_2)$$

then there exists $p(x_1, \theta) \in \theta L_q$, such that

$$d_n(y, \theta Z_q(h) + x_1 + x_2) = d_n(y, (\theta D + x_1) + p(x_1, \theta))$$

from which it follows that

$$x_2 = p(x_1, \theta) + u \quad \text{for some } u \in L_q \text{ with } \|u\|_q \leqq h.$$

Further on, for every u with $\|u\|_q \leqq h$, $p(x_1, \theta) + u \in T_{x_1}(\theta)$. Hence

$$\int_{T_{x_1}(\theta)} \lambda_q(dx_2) = \kappa_q h^q \qquad (\forall \theta, \forall x_1),$$

so

$$\int_{T(\theta)} \lambda_n(dx) = \kappa_q h^q \int_{\tilde{\Pi}_{n-q}(T(\theta))} \lambda_{n-q}(dx_1).$$

It is clear that

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$$\tilde{\Pi}_{n-q}(T(\theta)) = R(\theta^{-1})$$

with

$$R(\theta^{-1}) = \{x_1 \in L_{n-q} \mid \alpha_D(\theta^{-1}, x_1) \in M_\varepsilon(K, A(\eta, \eta'))$$

and then

$$\int_{SO_n} \int_{\tilde{\Pi}_{n-q}(T(\theta))} \lambda_{n-q}(dx_1)\nu(d\theta) = \int_{SO_n} \int_{R(\theta)} \lambda_{n-q}(dx_1)\nu(d\theta).$$

Using (A.2) we then obtain

$$\begin{aligned} \mu(C_\varepsilon(C_0(\eta, \eta'(h)))) &= \kappa_q h^q \int_{SO_n} \int_{R(\theta)} \lambda_{n-q}(dx_1)\nu(d\theta) \\ &= \kappa_q h^q \gamma_D(M_\varepsilon(K, A(\eta, \eta')). \end{aligned}$$

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